Consider Bloch equations that evolves the magnetization vector X as

$$\dot{X} = (\omega\Omega_z + A(t)\cos\theta(t)\Omega_x + A(t)\sin\theta(t)\Omega_y)X, \tag{1}$$

where $\omega \in [-B, B]$ is the offset and $A(t), \theta(t)$ are the amplitude and phase of the rf-field. Lets consider only a phase modulated rf-field and normalize $A(t) = 1$. Then the initial magnetization x_0 evolves to final state X_F as

$$X_F = U_n(\omega, \theta_n) \dots U_k(\omega, \theta_k) \dots U_1(\omega, \theta_1) X_0, \tag{2}$$

where

$$U_k(\omega, \theta_k) = \exp(\Delta t(\omega \Omega_z + \cos\theta_k \Omega_x + \sin\theta_k \Omega_y)),$$

and we consider piece-wise constant phases θ_k over time intervals of length Δt. Let the desired target state be called Y_{n+1}. Then we seek to maximize,

$$J = \frac{1}{N} \sum_j Y'_{n+1} U_n(\omega_j, \theta_n) \dots U_k(\omega_j, \theta_k) \dots U_1(\omega_j, \theta_1) X_0, \tag{3}$$

where we sample the bandwidth $[-B, B]$ in N frequencies ω_j. For broadband excitation $X_0 = (0, 0, 1)'$ and $Y_{n+1} = (1, 0, 0)'$. For broadband inversion, $X_0 = (0, 0, 1)'$ and $Y_{n+1} = (0, 0, -1)'$. We rewrite,

$$J = \frac{1}{N} \sum_j \underbrace{Y'_{n+1} U_n(\omega_j, \theta_n) \dots U_{k+1}(\omega_j, \theta_{k+1})}_{Y'_{k+1}(\omega_j)} U_k(\omega_j, \theta_k) \underbrace{U_{k-1}(\omega_j, \theta_{k-1}) \dots U_1(\omega_j, \theta_1) X_0}_{X_{k-1}(\omega_j)}. \tag{4}$$

$$J = \frac{1}{N} \sum_j Y'_{k+1}(\omega_j)(I + \Delta t(\omega_j \Omega_z + \cos\theta_k \Omega_x + \sin\theta_k \Omega_y)) X_{k-1}(\omega_j), \tag{5}$$

where we expanded $U_k(\omega_j, \theta_k)$ for small Δt and we seek to find θ_k that maximizes the above expression. Writing only the terms that depend on θ_k, we get, the term

$$\begin{aligned}
J(\theta_k) &= \frac{1}{N} \sum_j Y'_{k+1}(\omega_j)(\cos\theta_k \Omega_x + \sin\theta_k \Omega_y) X_{k-1}(\omega_j), && (6)\\
&= \frac{1}{N} \sum_j Y'_{k+1}(\omega_j)[\begin{bmatrix} \cos\theta_k \\ \sin\theta_k \\ 0 \end{bmatrix} \times X_{k-1}(\omega_j)], && (7)\\
&= \frac{1}{N} \begin{bmatrix} \cos\theta_k, & \sin\theta_k, & 0 \end{bmatrix} \underbrace{\sum_j X_{k-1}(\omega_j) \times Y_{k+1}(\omega_j)}_{e}, && (8)
\end{aligned}$$

where $\times$ denotes cross-product of vectors. Then we simply choose $\tan\theta_k = \frac{e_2}{e_1}$, where $e = (e_1, e_2, e_3)'$.

Starting with initial guess on $\theta_1, \ldots, \theta_n$, say $\theta_i = 0$, we sequentially maximize θ_k for $k = 1, \ldots, n$ and then repeat till J reaches a desired value or saturates. Observe optimization of θ_k is in closed form. We just calculate e and update θ_k and proceed to update θ_{k+1}.