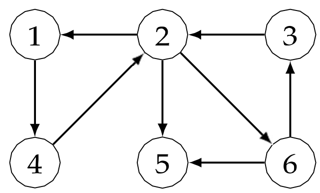
Graphs and matrices

Counting paths

The powers of an adjacency matrix of a graph have an interesting property. When V is an adjacency matrix of an unweighted graph, the matrix V^n contains the numbers of paths of n edges between the nodes in the graph.

For example, for the graph



the adjacency matrix is

$$V = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Now, for example, the matrix

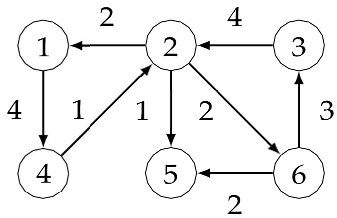
$$V^4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix},$$

contains the numbers of paths of 4 edges between the nodes. For example, $V^4[2,5] = 2$, because there are two paths of 4 edges from node 2 to node 5: $2 \to 1 \to 4 \to 2 \to 5$ and $2 \to 6 \to 3 \to 2 \to 5$.

Shortest paths

Using a similar idea in a weighted graph, we can calculate for each pair of nodes the minimum length of a path between them that contains exactly n edges. To calculate this, we have to define matrix multiplication in a new way, so that we do not calculate the numbers of paths but minimize the lengths of paths.

As an example, consider the following graph:



Let us construct an adjacency matrix where ∞ means that an edge does not exist, and other values correspond to edge weights. The matrix is

$$V = \begin{bmatrix} \infty & \infty & \infty & 4 & \infty & \infty \\ 2 & \infty & \infty & \infty & 1 & 2 \\ \infty & 4 & \infty & \infty & \infty & \infty \\ \infty & 1 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 3 & \infty & 2 & \infty \end{bmatrix}.$$

Instead of the formula

$$AB[i,j] = \sum_{k=1}^{n} A[i,k] \cdot B[k,j]$$

we now use the formula

$$AB[i,j] = \min_{k=1}^{n} A[i,k] + B[k,j]$$

for matrix multiplication, so we calculate a minimum instead of a sum, and a sum of elements instead of a product. After this modification, matrix powers correspond to shortest paths in the graph.

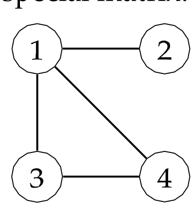
For example, as

$$V^4 = \begin{bmatrix} \infty & \infty & 10 & 11 & 9 & \infty \\ 9 & \infty & \infty & \infty & 8 & 9 \\ \infty & 11 & \infty & \infty & \infty & \infty \\ \infty & 8 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 12 & 13 & 11 & \infty \end{bmatrix},$$

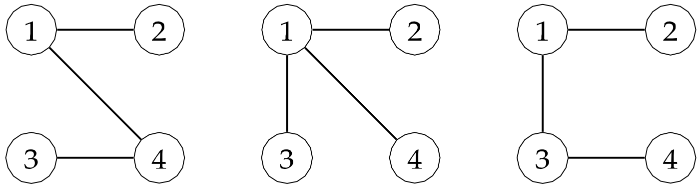
we can conclude that the minimum length of a path of 4 edges from node 2 to node 5 is 8. Such a path is $2 \rightarrow 1 \rightarrow 4 \rightarrow 2 \rightarrow 5$.

Kirchhoff's theorem

Kirchhoff's theorem provides a way to calculate the number of spanning trees of a graph as a determinant of a special matrix. For example, the graph



has three spanning trees:



To calculate the number of spanning trees, we construct a **Laplacean matrix** L, where $L[i,i]$ is the degree of node i and $L[i,j] = -1$ if there is an edge between nodes i and j, and otherwise $L[i,j] = 0$. The Laplacean matrix for the above graph is as follows:

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

It can be shown that the number of spanning trees equals the determinant of a matrix that is obtained when we remove any row and any column from L. For example, if we remove the first row and column, the result is

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\right) = 3.$$

The determinant is always the same, regardless of which row and column we remove from L.

Note that Cayley's formula in Chapter 22.5 is a special case of Kirchhoff's theorem, because in a complete graph of n nodes

$$\det\left(\begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}\right) = n^{n-2}.$$