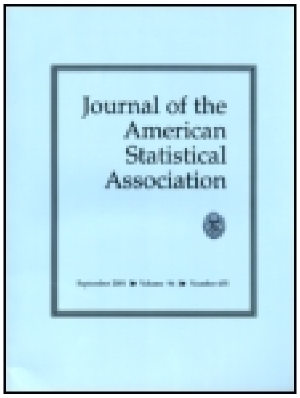

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Generalized Measures of Correlation for Asymmetry, Nonlinearity, and Beyond

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Generalized Measures of Correlation for Asymmetry, Nonlinearity, and Beyond

Shurong ZHENG, Ning-Zhong SHI, and Zhengjun ZHANG

Applicability of Pearson's correlation as a measure of explained variance is by now well understood. One of its limitations is that it does not account for asymmetry in explained variance. Aiming to develop broad applicable correlation measures, we study a pair of generalized measures of correlation (GMC) that deals with asymmetries in explained variances, and linear or nonlinear relations between random variables. We present examples under which the paired measures are identical, and they become a symmetric correlation measure that is the same as the squared Pearson's correlation coefficient. As a result, Pearson's correlation is a special case of GMC. Theoretical properties of GMC show that GMC can be applicable in numerous applications and can lead to more meaningful conclusions and improved decision making. In statistical inference, the joint asymptotics of the kernel-based estimators for GMC are derived and are used to test whether or not two random variables are symmetric in explaining variances. The testing results give important guidance in practical model selection problems. The efficiency of the test statistics is illustrated in simulation examples. In real-data analysis, we present an important application of GMC in explained variances and market movements among three important economic and financial monetary indicators. This article has online supplementary materials.

KEY WORDS: Asymmetric correlation; Economic study; Linear dependence; Nonlinear dependence; Nonparametric estimation.

1. INTRODUCTION

In almost all statistical inference problems, dealing with how random variables depend on each other plays a fundamental role in model selection. In the literature, since its introduction, Pearson's correlation coefficient has been the most dominant dependence measure used in numerous applications. It mainly depicts a symmetric and linear relationship between two variables. Its theoretical properties have been thoroughly studied. Rodgers and Nicewander (1988) presented 13 ways to look at the correlation coefficients. For many applications, it leads to meaningful and interesting interpretations of variables under study. However, it may also give misleading results in many applications. This phenomenon has been witnessed in many published articles, for example, O'Grady (1982), Ozer (1985), Drouet-Mari and Kotz (2001), and Zhang (2008), among many others. Recently, Zhang, Qi, and Ma (2011) showed that the sample-based Pearson's correlation coefficient is asymptotically independent of the quotient correlation coefficient, which is a very important property as it shows that these two correlation coefficients measure completely different dependencies between two random variables. Certainly, Pearson's correlation coefficient has its limitations in measuring variable dependencies. To overcome its limitations, various dependence measures have been proposed in the literature. We shall not detail them in the present work. We refer readers to the books by Joe (1997) and Drouet-Mari and Kotz (2001), which excellently summarize various dependence measures.

For Pearson's correlation coefficient, one of its limitations is that it does not account for asymmetry in explained variances that are often innate among nonlinearly dependent random variables. As a result, measures dealing with asymmetries are needed. In fact, studying the asymmetric dependent characteristics of random variables has drawn more and more attention, especially the studies of stock returns such as those by Zhang and Shinki (2006) and the Hong, Tu, and Zhou (2007), and the references therein. However, theoretical foundations of asymmetry in explained variances do not exist and are yet to be developed.

This article is intended to introduce effective and broadly applicable statistical tools for dealing with asymmetry and nonlinear correlations between random variables. For simplicity of illustration, we regard "linear" or "symmetric" as a special case of "nonlinear" or "asymmetric." In the case of "linear and symmetric," Pearson's correlation coefficient is an extremely important and widely used analytical tool in statistical data analysis. New dependence measures that comprise Pearson's correlation coefficient as a special case should be of the greatest interest to practitioners. We aim to develop such dependence measures. In Section 2.1, we use a well-known variance decomposition formula to introduce a pair of dependence measures of correlation: the generalized measures of correlation (GMC).

Theoretical properties of GMC are illustrated in Section 2.2. One can see that our proposed GMC has various connections to Pearson's correlation coefficient and the coefficient of determination in regression models, and they are identical to the squared Pearson's correlation coefficient when two random variables are related in a linear equation. A special case is that two random variables follow a bivariate normal distribution. More importantly, GMCs are nonzero while Pearson's correlation coefficient may have a zero value when two random variables are nonlinearly dependent. In addition, GMCs also have monotonic

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dependence properties in explained variances. One can also see that our proposed GMC may be used as an alternative statistical tool in Granger causality inference. For this purpose, we introduce two new measures: auto generalized measures of correlation (AGMC) and Granger causality generalized measures of correlation (GcGMC).

The rest of the article is structured as follows. In Section 3, we present a nonparametric method in computing our proposed GMC. The joint asymptotics of two GMC estimators are derived and they are used to test whether two explained variances are identical or not. Starting from theoretical foundations, we will analyze examples covering a wide range of dependency between random variables in Section 4. We study three types of bivariate t random variables in particular and derive their corresponding GMCs. We also calculate GMCs for a three-sectional extreme value copula that shows an asymmetric dependence and extreme dependence between two underlying random variables. Numerical illustrations of GMC are presented in Section 5. One can see that GMCs can be very useful dependence measures, especially where explained variance is concerned. In Section 6, we present real-data analysis through three important economic variables: the exchange rate of Japanese Yen against U.S. dollar, U.S. federal funds rate, and Japan deposit rate. They are indicators of both countries' economic status: for example, whether they are healthy or not. People have hoped that the comparisons may help reveal similarities and find answers (even solutions) to an economic recovery from the current international financial crisis. From a market perspective, plotting these variables shows no similarity, linear relationship, or co-monotone relationship. However, our particular GMC shall display economic changes between these two countries. Section 7 discusses potential extensions of the present article and the limitations of our proposed measures. Technical derivations are presented in the online Appendix (see Supplementary Materials posted on the journal web site).

2. GENERALIZED MEASURES OF CORRELATION: DEFINITIONS AND PROPERTIES

2.1 Generalized Measures of Correlation

In computing the coefficient of determination, R^2, in a linear regression model, the total variation in the response variable is partitioned into two component sums of squares, that is, explained variation due to regression and unexplained variation. Here, we shall introduce our GMC based on a well-known variance decomposition formula

$$\mathrm{var}(X) = \mathrm{var}(E(X|Y)) + E(\mathrm{var}(X|Y)), \tag{1}$$

whenever $E(Y^2) < \infty$ and $E(X^2) < \infty$. Note that $\mathrm{var}(E(X|Y))$ is the variance of conditional mean of X given Y, and hence $\mathrm{var}(E(X|Y))/\mathrm{var}(X)$ can certainly be interpreted as the explained variance of X by Y. We have

$$\frac{\mathrm{var}(E(X|Y))}{\mathrm{var}(X)} = 1 - \frac{E(\mathrm{var}(X|Y))}{\mathrm{var}(X)} = 1 - \frac{E[\{X - E(X|Y)\}^2]}{\mathrm{var}(X)}.$$

Similarly, we can define the explained variance of Y given X. Therefore, it is natural to introduce a pair of GMC as

$$\{\mathrm{GMC}(Y|X),\ \mathrm{GMC}(X|Y)\} = \left\{1 - \frac{E[\{Y - E(Y|X)\}^2]}{\mathrm{var}(Y)}, 1 - \frac{E[\{X - E(X|Y)\}^2]}{\mathrm{var}(X)}\right\}. \tag{2}$$

It is worth noting that in a regression model $Y = g(X) + \epsilon$, the R^2 is identical to $\mathrm{GMC}(Y|X)$ when $g(X)$ is chosen to be $E(Y|X)$. Other than this identity, the R^2 has not been used in studying asymmetric explained variances between random variables. Because the main usage of R^2 is in regression analysis, GMC seems a natural choice of measures of explained variances between random variables.

This pair of GMCs enjoys many good properties that will be illustrated in detail in the following section. One of them is that the two measures are identical when (X, Y) is a bivariate normal random vector. The GMC can depict the nonlinear or asymmetric relation between two variables. They are true measures for explained variances.

We note that in the literature, the existing generalizations of Pearson's correlation, which are mostly related to our proposed measures, include those by Holland and Wang (1987), Doksum et al. (1994), Jones (1996), and references therein. These generalizations are focusing on either measuring the strength and direction of the association locally or treating variables symmetrically. On the other hand, Doksum and Samarov (1995) studied nonparametric estimation of global functionals in regression and Pearson's correlation ratio, which is a ratio of variance of functionals of covariates and variance of the response variable. Wang (2001) considered Pearson's correlation ratio in the study of Granger causality in time series. In this article, GMC is directly motivated by Equation (1), and a pair of GMC measures is studied jointly, which is more meaningful in statistical inference. A pair of GMC measures the association between two random variables globally, and it deals with asymmetry, nonlinearity, and beyond.

In our context, $\mathrm{GMC}(Y|X) > \mathrm{GMC}(X|Y)$, $\mathrm{GMC}(Y|X) < \mathrm{GMC}(X|Y)$, and $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y)$ lead to more meaningful statistical inference. For example: (1) An economist would like to study the mutual impact of two economic indicator variables on each other. She would definitely want to know which of the two variables is more important, not just a simple answer of the two variables being correlated. For this case, when $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y) > 0$, the mutual impacts between X and Y are the same in terms of market variability, and they are equally important or unimportant; when $\mathrm{GMC}(Y|X) > \mathrm{GMC}(X|Y)$, X is more important than Y. (2) In clinics, it is often observed that a patient's symptoms manifested interactions between different types of disease. A physician would like to know which symptom appeared first and whether or not one symptom results in the development of another symptom. For this case, when $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y)$, these two symptoms may be diagnosed as concurrently occurred; when $\mathrm{GMC}(Y|X) > \mathrm{GMC}(X|Y)$, it is likely that the disease related to X occurred first and triggered the disease related to Y. (3) In high-dimensional variable screening and model building such as in gene selection, knowing the explained variances between the predictors would definitely help model builders to find the more efficient subset of variables (genes). For this case, when $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y) > 0$, a biologist may think that two selected genes are equally important or unimportant; when $\mathrm{GMC}(Y|X) > \mathrm{GMC}(X|Y)$, X is more important than Y. (4) In graphical network modeling (such as social network and genome wide association study), GMC can provide directional graphical models contrasting to the present applications of nondirectional graphical models. Examples like these are endless.

2.2 Properties of Generalized Measures of Correlation

We have the following proposition that shows that the GMC can measure the nonlinear and asymmetric relation between two variables. Please see the online supplementary materials for proofs of the propositions.

Proposition 1. Suppose both X and Y have finite second moments. Then

(i) GMC is an indicator lying between 0 and 1, that is,

$$0 \le \mathrm{GMC}(Y|X), \mathrm{GMC}(X|Y) \le 1,$$

and if X and Y are independent, then $\mathrm{GMC}(Y|X) = 0$, $\mathrm{GMC}(X|Y) = 0$.

(ii) The relation of GMC and Pearson's correlation coefficient ρ_{XY} satisfies:
- If $\rho_{XY} = \pm 1$, then $\mathrm{GMC}(Y|X) = 1$ and $\mathrm{GMC}(X|Y) = 1$.
- If $\rho_{XY} \neq 0$, then $\mathrm{GMC}(X|Y) \neq 0$ and $\mathrm{GMC}(Y|X) \neq 0$.
- If $\mathrm{GMC}(Y|X) = 0$ and/or $\mathrm{GMC}(X|Y) = 0$, then $\rho_{XY} = 0$.
- $\mathrm{GMC}(Y|X) \ge \rho^2_{XY}$ and $\mathrm{GMC}(X|Y) \ge \rho^2_{XY}$.

(iii) Suppose $Y = g(X) + \epsilon$, X and ϵ are independent, and both $g(X)$ and ϵ have finite second moments, where $g(\cdot)$ is a linear or nonlinear measurable function. Then

$$\mathrm{GMC}(Y|X) = \frac{\mathrm{var}(g(X))}{\mathrm{var}(g(X)) + \mathrm{var}(\epsilon)}.$$

Particularly, if $g(x) = ax + b$ for $a \neq 0$ and b being constant, we have

$$\mathrm{GMC}(Y|X) = \rho^2_{XY}.$$

For the extreme values of GMC, we have

$$\mathrm{GMC}(Y|X) = 1 \Longleftrightarrow Y = g(X) \quad \text{a.s.}$$

Furthermore, If g is a one-to-one measurable function, then $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y) = 1$; if g is not one to one, then $\mathrm{GMC}(Y|X) = 1 > \mathrm{GMC}(X|Y) \ge 0$.

(iv) Suppose $Y_1 = g_1(X) + \epsilon_1$ and $Y_2 = g_2(X) + \epsilon_2$, where ϵ_1 and ϵ_2 are independent of X, $g_1(\cdot)$ and $g_2(\cdot)$ are linear or nonlinear measurable functions. If either (1) $\mathrm{var}(g_1(X)) = \mathrm{var}(g_2(X))$, $\mathrm{var}(\epsilon_1) < \mathrm{var}(\epsilon_2)$; or (2) $\mathrm{var}(g_1(X)) > \mathrm{var}(g_2(X))$, $\mathrm{var}(\epsilon_1) = \mathrm{var}(\epsilon_2)$, we have

$$\mathrm{GMC}(Y_1|X) > \mathrm{GMC}(Y_2|X).$$

(v) If $\mathrm{var}(Y_1) = \mathrm{var}(Y_2)$ and $\inf_f E[\{Y_1 - f(X)\}^2] = \inf_g E[\{Y_2 - g(X)\}^2]$, where f, g are measurable functions, then

$$\mathrm{GMC}(Y_1|X) = \mathrm{GMC}(Y_2|X).$$

If $\mathrm{var}(Y_1) = \mathrm{var}(Y_2)$ and $\inf_f E[\{Y_1 - f(X)\}^2] < \inf_g E[\{Y_2 - g(X)\}^2]$, then we have

$$\mathrm{GMC}(Y_1|X) > \mathrm{GMC}(Y_2|X).$$

Example 1. Considering a special example $Y = X^2$, $X \sim N(0, 1)$ where Y is a nonlinear measurable function in Proposition 1 (iii), we have generalized correlation coefficients $\mathrm{GMC}(Y|X) = 1$, $\mathrm{GMC}(X|Y) = 0$, $\mathrm{GMC}(Y|X) \neq \mathrm{GMC}(X|Y)$, but Pearson's correlation coefficient $\rho_{XY} = 0$.

Remark 1. In Proposition 1, it is clear in (i), (ii), and (iii) that GMC characterizes nonlinear or asymmetric relation between two variables, where "linear" or "symmetric" is considered as a special case of "nonlinear" or "asymmetric," respectively. If Y is perfectly nonlinearly dependent on X, $\mathrm{GMC}(Y|X)$ is 1; and if X and Y are independent, the GMCs are 0. From (i), the GMC cannot tell whether or not two random variables are positively correlated or negatively correlated. It should be regarded as a nonlinear dependence measure. In (ii), when X and Y have a nonzero Pearson's linear correlation coefficient, the GMCs of X and Y are always greater than zero, but not reversely as shown in Example 1. On the other hand, as long as one of GMCs is zero, Pearson's correlation must be zero. $\mathrm{GMC}(Y|X) \ge \rho^2_{XY}$ and $\mathrm{GMC}(X|Y) \ge \rho^2_{XY}$ indicate that GMC is capable of revealing more association information between two random variables beyond the linear relation. These properties show that GMC has more applicabilities than Pearson's correlation coefficient has. They are strong indications of the generality of GMC as measure of dependence considering that Pearson's correlation only measures linear and symmetric relation. In (iv), if the linear or nonlinear relation of Y_1 on X is stronger than that of Y_2 on X, then $\mathrm{GMC}(Y_1|X)$ is larger than $\mathrm{GMC}(Y_2|X)$. It shows monotonicity of GMC, which is an important property in defining a correlation measure and in defining a prediction criterion for model/variable selections. In (v), if X has the stronger ability to predict Y_1 than to predict Y_2, then GMC correlation $Y_1|X$ is larger than GMC correlation $Y_2|X$, that is, $\mathrm{GMC}(Y_1|X) > \mathrm{GMC}(Y_2|X)$.

The following proposition shows that the squared Pearson's correlation coefficient is identical to the GMC under the bivariate normal distribution.

Proposition 2. For the bivariate normal distribution, ρ_{XY} and $(\mathrm{GMC}(Y|X), \mathrm{GMC}(X|Y))$ are equivalent in depicting the relation of X and Y, that is, $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y) = \rho^2_{XY}$.

The following proposition calculates GMCs when marginal distributions are uniform on [0,1]. The proof of the proposition is straightforward.

Proposition 3. Suppose that X and Y have continuous distribution functions $F_X(x)$ and $F_Y(y)$, respectively. Then

$$\begin{aligned} \mathrm{GMC}(F_Y(Y)|X) &= 12E(\{E(F_Y(Y)|X)\}^2) - 3, \\ \mathrm{GMC}(F_X(X)|Y) &= 12E(\{E(F_X(X)|Y)\}^2) - 3. \end{aligned} \tag{3}$$

Formulas in Equation (3) can be compared with Spearman's correlation

$$\rho_s(X, Y) = \mathrm{cor}(F_X(X), Y_Y(Y)) = 12E(F_X(X)F_Y(Y)) - 3,$$

which does not account for asymmetry in explained variances. Examples with uniform marginals will be illustrated in Section 4.

We argue that the three propositions above clearly show that the GMC is a true measure for explained variances, and for linear or nonlinear relations between two random variables. We note that the calculation of our proposed GMC involves computing the variance of conditional expectation and the expectation of conditional variance, which may be a difficult task in deriving explicit GMC formulas. In Section 4, we shall derive explicit forms of GMC in several joint distributional models.

2.3 Generalized Measures of Correlation in Time Series

In time series study, the autocorrelation function is an important concept. Our GMC can naturally be extended to time series models. Suppose that $\{X_t, Y_t\},\ t > 0$ is a bivariate time series. We define AGMC as:

$$\begin{aligned} \mathrm{AGMC}_k(X_t) &= \mathrm{GMC}(X_t|X_{t-k}), \\ \mathrm{AGMC}_k(Y_t|X_t) &= \mathrm{GMC}(Y_t|X_{t-k}),\ k > 0. \end{aligned} \tag{4}$$

Granger causality (Granger 1969) has been widely used in economics since the 1960s. It is a powerful statistical concept of causality that is based on prediction. It is normally tested in a bivariate linear autoregressive model of two variables X_t and Y_t. For simplicity, we assume an order one bivariate linear autoregressive model. We say Y_t Granger-causes X_t if

$$E[\{X_t - E(X_t|X_{t-1})\}^2] > E[\{X_t - E(X_t|X_{t-1}, Y_{t-1})\}^2], \tag{5}$$

that is, X_t can be better predicted using the histories of both X_t and Y_t than using the history of X_t alone. Similarly, we say X_t Granger-causes Y_t if

$$E[\{Y_t - E(Y_t|Y_{t-1})\}^2] > E[\{Y_t - E(Y_t|Y_{t-1}, X_{t-1})\}^2]. \tag{6}$$

Using the fact $E(\mathrm{var}(X_t|X_{t-1})) = E[\{X_t - E(X_t|X_{t-1})\}^2]$ and

$$\begin{aligned} &E[\{E(X_t|X_{t-1}) - E(X_t|X_{t-1}, Y_{t-1})\}^2] \\ &\quad = E[\{X_t - E(X_t|X_{t-1})\}^2] - E[\{X_t - E(X_t|X_{t-1}, Y_{t-1})\}^2], \end{aligned}$$

one can see that Equation (5) is equivalent to

$$1 - \frac{E[\{X_t - E(X_t|X_{t-1}, Y_{t-1})\}^2]}{E(\mathrm{var}(X_t|X_{t-1}))} > 0. \tag{7}$$

Similarly, Equation (6) is equivalent to

$$1 - \frac{E[\{Y_t - E(Y_t|Y_{t-1}, X_{t-1})\}^2]}{E(\mathrm{var}(Y_t|Y_{t-1}))} > 0. \tag{8}$$

When both Equations (5) and (6) are true, we have a *feedback system*. Equations (7) and (8) can be extended to a more general form, and we introduce our Granger causality GMC as follows.

Definition 1. Suppose that $\{X_t, Y_t\},\ t > 0$ is a bivariate stationary time series. Define GcGMC as:

$$\begin{aligned} &\mathrm{GcGMC}(X_t|\mathcal{F}_{t-1}) \\ &= 1 - \frac{E[\{X_t - E(X_t|X_{t-1}, X_{t-2}, \ldots, Y_{t-1}, Y_{t-2}, \ldots)\}^2]}{E(\mathrm{var}(X_t|X_{t-1}, X_{t-2}, \ldots))}, \end{aligned} \tag{9}$$

$$\begin{aligned} &\mathrm{GcGMC}(Y_t|\mathcal{F}_{t-1}) \\ &= 1 - \frac{E[\{Y_t - E(Y_t|Y_{t-1}, Y_{t-2}, \ldots, X_{t-1}, X_{t-2}, \ldots)\}^2]}{E(\mathrm{var}(Y_t|Y_{t-1}, Y_{t-2}, \ldots))}, \end{aligned} \tag{10}$$

where $\mathcal{F}_{t-1} = \sigma(X_{t-1}, X_{t-2}, \ldots, Y_{t-1}, Y_{t-2}, \ldots)$.

- If $\mathrm{GcGMC}(X_t|\mathcal{F}_{t-1}) > 0$, we say Y Granger causes X.
- If $\mathrm{GcGMC}(Y_t|\mathcal{F}_{t-1}) > 0$, we say X Granger causes Y.
- If $\mathrm{GcGMC}(X_t|\mathcal{F}_{t-1}) > 0$ and $\mathrm{GcGMC}(Y_t|\mathcal{F}_{t-1}) > 0$, we say that we have a *feedback system*.
- If $\mathrm{GcGMC}(X_t|\mathcal{F}_{t-1}) > \mathrm{GcGMC}(Y_t|\mathcal{F}_{t-1})$, we say that X is more predictable than Y.
- IF $\mathrm{GcGMC}(Y_t|\mathcal{F}_{t-1}) > \mathrm{GcGMC}(X_t|\mathcal{F}_{t-1})$, we say that Y is more predictable than X.

It is easy to show that $0 \le \mathrm{GcGMC}(X_t|\mathcal{F}_{t-1}) \le 1$ and $0 \le \mathrm{GcGMC}(Y_t|\mathcal{F}_{t-1}) \le 1$. Other theoretical properties of GcGMC can be derived along the line of GMC. This work will make a single article too overloaded, and we shall put the study of GcGMC in a separate project. We present the following proposition that relates GcGMC to Grainger's causality .

Proposition 4. If $E(\mathrm{var}(X_t|X_{t-1})) = E(\mathrm{var}(Y_t|Y_{t-1}))$ and the strength of that X_t Granger causes Y_t is stronger than the strength of that Y_t Granger causes X_t, that is,

$$\begin{aligned} &E[\{E(Y_t|Y_{t-1}) - E(Y_t|Y_{t-1}, X_{t-1})\}^2] \\ &\qquad > E[\{E(X_t|X_{t-1}) - E(X_t|X_{t-1}, Y_{t-1})\}^2], \end{aligned}$$

then

$$\mathrm{GcGMC}(Y_t|Y_{t-1}, X_{t-1}) > \mathrm{GcGMC}(X_t|X_{t-1}, Y_{t-1}).$$

3. NONPARAMETRIC ESTIMATORS OF GMC

The GMC in Equation (2) involves evaluations of conditional means and variances, which is not as easy as computing Pearson's correlation coefficient in practice. In the literature, there have been quite a few developments in estimating conditional variances such as those by Fan and Yao (1998) and Hansen (2004), among others. We propose to use nonparametric kernel-based methods to estimate GMC. The construction of our estimators for each conditional variance in GMC is similar to the existing methods in the literature. The main task here is to establish the joint asymptotics of the estimators.

Throughout Section 3 and proofs of Lemma 1, Theorems 1 and 2, we denote $f^X(x)$ and $f^Y(y)$ as the density functions of X and Y, respectively. $\{(X_i, Y_i),\ i = 1, \ldots, n\}$ is a random sample of (X, Y). Denote $R^1 = (-\infty, +\infty)$, $s_x = \inf\{x:\ f^X(x) > 0,\ x \in R^1\}$, $S^x = \sup\{x:\ f^X(x) > 0,\ x \in R^1\}$, $s_y = \inf\{y:\ f^Y(y) > 0,\ y \in R^1\}$, $S^y = \sup\{y:\ f^Y(y) > 0,\ y \in R^1\}$. For notational convenience, we drop the limits in all integrals. The lower and upper limits are s_x and S^x, respectively, in all integrals with respect to dx, and the lower and upper limits are s_y and S^y, respectively, in all integrals with respect to dy.

3.1 The Estimators

First, GMC can be expressed as

$$\mathrm{GMC}(Y|X) = 1 - \frac{E[\{Y - E(Y|X)\}^2]}{\mathrm{var}(Y)} = \frac{\int \frac{(\phi^{Y|X}(x))^2}{f^X(x)}dx - \mu_Y^2}{\sigma_Y^2}$$

and

$$\mathrm{GMC}(X|Y) = \frac{\int \frac{(\phi^{X|Y}(y))^2}{f^Y(y)}dy - \mu_X^2}{\sigma_X^2},$$

where $\mu_X = EX$, $\mu_Y = EY$, $\sigma_X^2 = \mathrm{var}(X)$, $\sigma_Y^2 = \mathrm{var}(Y)$, $\phi^{Y|X}(x) = \int yf(x,y)dy$, $\phi^{X|Y}(y) = \int xf(x,y)dx$, and $f(x,y)$ is the joint density of X and Y. Let the kernel densities of (X, Y) be

$$\hat{f}(x,y) = \frac{1}{nh^2}\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right)K\left(\frac{y - Y_i}{h}\right),$$

$$f_n^X(x) = \frac{1}{nh}\sum_{i=1}^n K\left(\frac{x - X_i}{h}\right), \quad \text{and}$$

$$f_n^Y(y) = \frac{1}{nh}\sum_{i=1}^n K\left(\frac{y - Y_i}{h}\right),$$

where $K(.)$ is a kernel function and h is the bandwidth. Then the Nadaraya–Watson estimator is

$$\hat{E}(Y|X = x) = \frac{\frac{1}{nh}\sum_{i=1}^n Y_i K((x - X_i)/h)}{\frac{1}{nh}\sum_{i=1}^n K((x - X_i)/h)} = \frac{\phi_n^{Y|X}(x)}{f_n^X(x)},$$

where $\phi_n^{Y|X}(x) = \frac{1}{nh}\sum_{i=1}^n Y_i K(\frac{x - X_i}{h})$. Similarly, we obtain

$$\hat{E}Y = \int y f_n^Y(y)dy = \bar{Y} + h \cdot E_K^1,$$

$$\hat{E}Y^2 = \int y^2 f_n^Y(y)dy = \frac{1}{n}\sum_{i=1}^n Y_i^2 + h^2 E_K^2 + 2h\bar{Y}E_K^1,$$

$$\widehat{\mathrm{var}}(Y) = S_Y^2 + h^2 \mathrm{var}_K,$$

and

$$\hat{E}[\{\hat{E}(Y|X)\}^2] = \int [\hat{E}(Y|X = x)]^2 f_n^X(x)dx = \int \frac{(\phi_n^{Y|X}(x))^2}{f_n^X(x)}dx,$$

where $E_K^i = \int z^i K(z)dz$, $\bar{Y}$ and S_Y^2 are the sample mean and sample variance of $Y_1, \ldots, Y_n$, and $\mathrm{var}_K = \int z^2 K(z)dz - (\int zK(z)dz)^2$. Because $\hat{E}[\{Y - \hat{E}(Y|X)\}^2] = \hat{E}Y^2 - \hat{E}[\{\hat{E}(Y|X)\}^2]$, then we have

$$\begin{aligned}
&\frac{\hat{E}[\{Y - \hat{E}(Y|X)\}^2]}{\widehat{\mathrm{var}}(Y)} \\
&= \frac{\frac{1}{n}\sum_{i=1}^n Y_i^2 + h^2 \cdot E_K^2 + 2h \cdot \bar{Y} \cdot E_K^1 - \hat{E}[\{\hat{E}(Y|X)\}^2]}{S_Y^2 + h^2 \cdot \mathrm{var}_K} \\
&= 1 + \frac{\bar{Y}^2 - h^2 \cdot \mathrm{var}_K + h^2 \cdot E_K^2 + 2h \cdot \bar{Y} \cdot E_K^1 - \hat{E}[\{\hat{E}(Y|X)\}^2]}{S_Y^2 + h^2 \cdot \mathrm{var}_K} \\
&= 1 + \frac{\bar{Y}^2 + h^2 \cdot (E_K^1)^2 + 2h \cdot \bar{Y} \cdot E_K^1 - \hat{E}[\{\hat{E}(Y|X)\}^2]}{S_Y^2 + h^2 \cdot \mathrm{var}_K} \\
&= 1 + \frac{(\bar{Y} + h \cdot E_K^1)^2 - \hat{E}[\{\hat{E}(Y|X)\}^2]}{S_Y^2 + h^2 \cdot \mathrm{var}_K} \\
&= 1 + \frac{(\bar{Y} + h \cdot E_K^1)^2 - \int \frac{(\phi_n^{Y|X}(x))^2}{f_n^X(x)}dx}{S_Y^2 + h^2 \mathrm{var}_K}.
\end{aligned}$$

Then the kernel estimators of GMC are

$$\widetilde{\mathrm{GMC}}(Y|X) = \frac{\int \frac{(\phi_n^{Y|X}(x))^2}{f_n^X(x)}dx - (\bar{Y} + hE_K^1)^2}{S_Y^2 + h^2 \mathrm{var}_K}$$

and

$$\widetilde{\mathrm{GMC}}(X|Y) = \frac{\int \frac{(\phi_n^{X|Y}(y))^2}{f_n^Y(y)}dy - (\bar{X} + hE_K^1)^2}{S_X^2 + h^2 \mathrm{var}_K},$$

where $\phi_n^{X|Y}(y)$ is defined similarly to $\phi_n^{Y|X}(x)$.

3.1.1 Choice of Bandwidth h. The bandwidth h can be selected using the method proposed by Sain, Baggerly, and Scott (1994). The method has been written in the R package "ks". Moreover, cross-validation may be used to choose h, which is a widely adopted procedure in the literature. Because $E(Y|X) = \min_{l(X)}(Y - l(X))^2$ and $E(X|Y) = \min_{l(Y)}(X - l(Y))^2$, we choose the optimal h as

$$h_{\mathrm{optimal}} = \mathrm{argmin}_{h>0}\left[\frac{\omega_1}{n}\sum_{k=1}^n (Y_k - \hat{E}_h^{-k}(Y|X = X_k))^2 + \frac{\omega_2}{n}\sum_{k=1}^n \left(X_k - \hat{E}_h^{-k}(X|Y = Y_k)\right)^2\right],$$

where ω_1 and ω_2 are weights, and $(\hat{E}_h^{-k}(Y|X = x), \hat{E}_h^{-k}(X|Y = y))$ are Nadaraya-Watson estimators computed based on data (X_i, Y_i), $i = 1, \ldots, k-1, k+1, \ldots, n$. Here ω_1 and ω_2 can be chosen as the inverse of the square root of sample variances of $Y_1, \ldots, Y_n$ and $X_1, \ldots, X_n$, that is, $\omega_1 = 1/S_Y$ and $\omega_2 = 1/S_X$.

3.2 Asymptotics of GMC Estimators

Lemma 1. Under Assumptions (a)–(c) in the online Appendix, we have

$$\int \frac{\left(\phi_n^{Y|X}(x)\right)^2}{f_n^X(x)}dx - \int \frac{(\phi^{Y|X}(x))^2}{f^X(x)}dx = T_n^{Y|X} + O_p(h^2) + O_p\left(\frac{1}{nh}\right) \quad (11)$$

$$\int \frac{\left(\phi_n^{X|Y}(y)\right)^2}{f_n^Y(y)}dy - \int \frac{(\phi^{X|Y}(y))^2}{f^Y(y)}dy = T_n^{X|Y} + O_p(h^2) + O_p\left(\frac{1}{nh}\right) \quad (12)$$

where

$$T_n^{Y|X} = \int \frac{2(\phi_n^{Y|X}(x) - E\phi_n^{Y|X}(x))\phi^{Y|X}(x)}{f^X(x)}dx - \int \frac{(f_n^X(x) - Ef_n^X(x)) \cdot (\phi^{Y|X}(x))^2}{(f^X(x))^2}dx$$

$$T_n^{X|Y} = \int \frac{2(\phi_n^{X|Y}(y) - E\phi_n^{X|Y}(y))\phi^{X|Y}(y)}{f^Y(y)}dy - \int \frac{(f_n^Y(y) - Ef_n^Y(y)) \cdot (\phi^{X|Y}(y))^2}{(f^Y(y))^2}dy.$$

Theorem 1. Let Assumptions (a)–(c) in the online Appendix be fulfilled. If $nh^2 \to \infty$, $nh^4 \to 0$, then we have

$$\sqrt{n}(\mathbf{A}^T\mathbf{\Sigma}\mathbf{A})^{-\frac{1}{2}} \times \begin{pmatrix} \widetilde{\mathrm{GMC}}(Y|X) - \mathrm{GMC}(Y|X) - \left(\int \frac{(\phi^{Y|X}(x))^2}{f^X(x)}dx - \mu_Y^2\right)\left(\frac{1}{\sigma_Y^2 + h^2\mathrm{var}_K} - \frac{1}{\sigma_Y^2}\right) \\ \widetilde{\mathrm{GMC}}(X|Y) - \mathrm{GMC}(X|Y) - \left(\int \frac{(\phi^{X|Y}(y))^2}{f^Y(y)}dy - \mu_X^2\right)\left(\frac{1}{\sigma_X^2 + h^2\mathrm{var}_K} - \frac{1}{\sigma_X^2}\right) \end{pmatrix} \Longrightarrow N(\mathbf{0}_{2\times1}, \mathbf{I}_{2\times2})$$

where

$$\mathrm{var}_K = \int z^2K(z)dz - \left(\int zK(z)\right)^2,$$

$$\mathbf{\Sigma} = \mathrm{cov}\left(\frac{\int\left(2Y_i - \frac{\phi^{Y|X}(x)}{f^X(x)}\right)\frac{\phi^{Y|X}(x)}{f^X(x)}\frac{1}{h}K\left(\frac{x-X_i}{h}\right)dx}{\sigma_Y^2 + h^2\mathrm{var}_K}, \frac{Y_i}{\sqrt{\sigma_Y^2 + h^2\mathrm{var}_K}}, Y_i^2, \frac{\int\left(2X_i - \frac{\phi^{X|Y}(y)}{f^Y(y)}\right)\frac{\phi^{X|Y}(y)}{f^Y(y)}\frac{1}{h}K\left(\frac{y-Y_i}{h}\right)dy}{\sigma_X^2 + h^2\mathrm{var}_K}, \frac{X_i}{\sqrt{\sigma_X^2 + h^2\mathrm{var}_K}}, X_i^2\right)$$

and

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -\frac{2\mu_Y}{\sqrt{\sigma_Y^2+h^2\mathrm{var}_K}} + \frac{2\left(\int\frac{(\phi^{Y|X}(x))^2}{f^X(x)}dx - \mu_Y^2\right)\mu_Y}{(\sigma_Y^2+h^2\mathrm{var}_K)^{\frac{3}{2}}} & 0 \\ -\frac{\left(\int\frac{(\phi^{Y|X}(x))^2}{f^X(x)}dx - \mu_Y^2\right)}{(\sigma_Y^2+h^2\mathrm{var}_K)^2} & 0 \\ 0 & 1 \\ 0 & -\frac{2\mu_X}{\sqrt{\sigma_X^2+h^2\mathrm{var}_K}} + \frac{2\left(\int\frac{(\phi^{X|Y}(y))^2}{f^Y(y)}dy - \mu_X^2\right)\mu_X}{(\sigma_X^2+h^2\mathrm{var}_K)^{\frac{3}{2}}} \\ 0 & -\frac{\left(\int\frac{(\phi^{X|Y}(y))^2}{f^Y(y)}dy - \mu_X^2\right)}{(\sigma_X^2+h^2\mathrm{var}_K)^2} \end{pmatrix}.$$

With the established joint asymptotics, we can make large sample inferences on explained variances. In the literature, ever since Hotelling (1953) stated that the best present-day usage in dealing with correlation coefficients is Fisher's Z-transformation test of linear (in)dependence of two random variables, the correlation coefficient has been widely used in numerous applications. As discussed in Section 2.1, the relationship between $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ reveals how the variables are associated with each other. As a result, one would want to test a particular relation of interest. Without loss of generality, we consider the following testing problem:

$$H_0 : \mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y) \quad \text{versus} \quad H_1 : \mathrm{GMC}(Y|X) \neq \mathrm{GMC}(X|Y). \tag{13}$$

The hypotheses $H_0 : \mathrm{GMC}(Y|X) \geq \mathrm{GMC}(X|Y)$ versus $H_1 : \mathrm{GMC}(Y|X) < \mathrm{GMC}(X|Y)$ and $H_0 : \mathrm{GMC}(Y|X) \leq \mathrm{GMC}(X|Y)$ versus $H_1 : \mathrm{GMC}(Y|X) > \mathrm{GMC}(X|Y)$ are similarly considered.

Theorem 2. Let Assumptions (a)–(c) in the online Appendix be fulfilled. If $nh^2 \to \infty$ and $nh^4 \to 0$, then the test statistic for the testing problem in Equation (13) has the following asymptotic distribution under H_0

$$\sqrt{n}((1, -1)\mathbf{A}^T\mathbf{\Sigma}\mathbf{A}(1, -1)^T)^{-\frac{1}{2}}(\widetilde{\mathrm{GMC}}(Y|X) - \widetilde{\mathrm{GMC}}(X|Y) - C_0) \to N(0, 1),$$

where

$$C_0 = \left(\int \frac{(\phi^{Y|X}(x))^2}{f^X(x)}dx - \mu_Y^2\right)\left(\frac{1}{\sigma_Y^2 + h^2\mathrm{var}_K} - \frac{1}{\sigma_Y^2}\right) - \left(\int \frac{(\phi^{X|Y}(y))^2}{f^Y(y)}dy - \mu_X^2\right)\left(\frac{1}{\sigma_X^2 + h^2\mathrm{var}_K} - \frac{1}{\sigma_X^2}\right)$$

and the matrices $\mathbf{A}$ and $\mathbf{\Sigma}$ are the same as those in Theorem 1.

Proof of Theorem 2 is easily obtained by Theorem 1 and the delta method.

We note that when Theorems 1 and 2 are used to make statistical inference for $\mathrm{GMC}(X|Y)$ and $\mathrm{GMC}(Y|X)$, the unknown $\mu_X, \mu_Y, \sigma_X, \sigma_Y$, $\phi^{X|Y}, \phi^{X|Y}$, f^X, f^Y, $\mathbf{\Sigma}$ and $\mathbf{A}$ can be replaced by their consistent estimators $\hat{\mu}_X, \hat{\mu}_Y, \hat{\sigma}_X, \hat{\sigma}_Y$, $\phi_n^{X|Y}, \phi_n^{X|Y}$, f_n^X, f_n^Y, $\hat{\mathbf{\Sigma}}$ and $\hat{\mathbf{A}}$, respectively. The matrix $\mathbf{A}^T\mathbf{\Sigma}\mathbf{A}$ has an order of approximately $1/h$, and hence the convergence of $\widetilde{\mathrm{GMC}}(Y|X)$ and $\widetilde{\mathrm{GMC}}(X|Y)$ is actually $(nh)^{-1/2}$. In Section 5, we use examples to demonstrate sample performances of the established theoretical results.

4. DERIVATIONS OF GMCS IN SEVERAL JOINT DISTRIBUTIONS

From the previous section, we see that the GMCs between two bivariate normal random variables are identical. In the literature, there exist many other parametric families of bivariate distribution functions that have also been used in many applications. Some of these bivariate distributions posses the property of having identical GMCs, while some of them do not. It will be very useful if we can present GMCs for each known family as GMCs are important population characteristics. However, it may be too ambitious a task to include every bivariate distribution in a single article. We choose four families of bivariate distributions to illustrate the derivations of GMC. Two families posses the property of having identical GMCs, while the other two families do not. The purpose of our mathematical derivation of GMCs is to provide some guidance in application and to show what we can get. One can see that for some families, we can get explicit formulas for GMCs, while for other different families, we can not. Also, the derivations for cases of nonidentical GMCs are much more complicated than the derivations for cases of identical GMCs in our chosen examples.

We first study the bivariate t-distributions that are also widely used in various applications besides bivariate normal distributions. In the following proposition, we illustrate three types of bivariate t-distributions with two having identical GMCs and one having different GMCs.

Proposition 5. Suppose that $\{(X_i, Y_i), i = 1, \dots, m\}$ is an independent sequence of bivariate normal vectors. Define

$$\bar{X} = \frac{1}{m}\sum_{i=1}^{m} X_i,\ \bar{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i,\ S_1^2 = \frac{1}{m-1}\sum_{i=1}^{m}(x_i - \bar{X})^2,$$
$$S_2^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \bar{Y})^2.$$

Case 1: Suppose $\mathrm{var}(X) = \sigma_1^2$, $\mathrm{var}(Y) = \sigma_2^2$, $\mathrm{cov}(X, Y) = \sigma_1\sigma_2\rho$ with $|\rho| < 1$. Define

$$T_1 = \frac{\sqrt{m}\cdot\bar{X}}{S_1}, \quad T_2 = \frac{\sqrt{m}\cdot\bar{Y}}{S_2}.$$

Then (T_1, T_2) follows a bivariate t-distribution with degrees of freedom $df = m - 1$ (Siddiqui 1967), and we have $\mathrm{GMC}(T_1|T_2) = \mathrm{GMC}(T_2|T_1)$.

Case 2: Suppose $\sigma_1 = \sigma_2$. Define

$$T_1 = \frac{\sqrt{m}\cdot\bar{X}}{S}, \quad T_2 = \frac{\sqrt{m}\cdot\bar{Y}}{S}.$$

where $S^2 = \frac{(m-1)(S_1^2+S_2^2)}{2m-1}$. Then (T_1, T_2) follows a bivariate t-distribution with degrees of freedom $df = m - 1$ (Patil and Liao 1970), and we have $\mathrm{GMC}(T_1|T_2) = \mathrm{GMC}(T_2|T_1)$.

Case 3: Let $X_i, i = 1, \dots, m_1 + m_2 \overset{\text{iid}}{\sim} N(0, \sigma^2)$. Let

$$\bar{X}_1 = \frac{1}{m_1}\sum_{i=1}^{m_1} X_i,\ \bar{X}_* = \frac{1}{m_2}\sum_{i=m_1+1}^{m_1+m_2} X_i,$$
$$\bar{X}_2 = \frac{1}{m_1+m_2}(m_1\bar{X}_1 + m_2\bar{X}_*)$$
$$S_1^2 = \frac{1}{m_1-1}\sum_{i=1}^{m_1}(X_i - \bar{X}_1)^2,$$
$$S_*^2 = \frac{1}{m_2-1}\sum_{i=m_1+1}^{m_1+m_2}(X_i - \bar{X}_*)^2,$$
$$S_2^2 = \frac{(m_1-1)S_1^2 + (m_2-1)S_*^2}{m_1+m_2-2},$$
$$T_1 = \frac{\sqrt{m_1}\cdot\bar{X}_1}{S_1},\ T_2 = \frac{\sqrt{m_1+m_2}\cdot\bar{X}_2}{S_2}.$$

Then (T_1, T_2) follows a bivariate t-distribution with degrees of freedom $(df1 = m_1 - 1,\ df2 = m_1 + m_2 - 2)$, respectively (Bulgren, Dykstra, and Hewett 1974). When $m_1 > 2$, $m_2 \geq 2$, (T_1, T_2) is a bivariate t-distributed random vector, and we have $\mathrm{GMC}(T_1|T_2) \neq \mathrm{GMC}(T_2|T_1)$.

Recently, the study of copula functions has become a major phenomenon in constructing joint distribution functions and modeling real data. In the literature, the bivariate Gumbel-Hougaard copula is widely used in many applications, especially in finance and insurance. It is easy to show that the GMCs of a pair of random variables following a bivariate Gumbel-Hougaard copula are identical. In an effort to model multivariate extremal dependence, Zhang (2009) introduced a three-sectional copula that partitions the probability space into three parts. We give a brief summary of the three-sectional copula here. Suppose that X and Y are two loss random variables. Among the three parts, one part is related to computing the probability that the loss of Y is a times smaller the loss of X, one part is related to computing the probability that the loss of Y is b times larger than the loss of X, and the third part is related to computing the probability of the ratio of the loss of Y and the loss of X is between a and b with $a < b$. Zhang (2009) demonstrated that the three-sectional copula performs as good as the Gumbel-Hougaard copula in modeling bivariate extreme dependence. However, the three sectional copula is able to account for either symmetry and asymmetry in explained variances by varying parameter values. In this article, we further extend the three-sectional copula to a model that gives a larger difference between the two GMCs with a price of adding a new parameter.

Suppose that U_1 and U_2 are independent uniform random variables on [0,1]. Define

$$\xi_1 = (-1/\log(U_1))^{1/\beta}; \xi_2 = (-1/\log(U_2))^{1/\beta},$$
$$\eta_1 = \max((1-\alpha_1)\xi_1,\ \alpha_1\xi_2);\ \eta_2 = \max(\alpha_2\xi_1,\ (1-\alpha_2)\xi_2), \tag{14}$$

and

$$X = \exp\left(-\frac{\alpha_1^\beta + (1-\alpha_1)^\beta}{\eta_1^\beta}\right);\ Y = \exp\left(-\frac{\alpha_2^\beta + (1-\alpha_2)^\beta}{\eta_2^\beta}\right), \tag{15}$$

where $\beta \geq 1$, $0 \leq \alpha_1, \alpha_2 \leq 1$, and $\alpha_1 + \alpha_2 < 1$.

Let $f_1(x) = \frac{x^\beta}{x^\beta + (1-x)^\beta}$ and $f_2(x) = \frac{(1-x)^\beta}{x^\beta + (1-x)^\beta}$. Then $f_1(x)$ is monotonically increasing in [0,1] and $f_2(x)$ is monotonically decreasing in [0,1]. We have the following proposition.

Proposition 6. Under the model in Equation (15), we have

$$\mathrm{GMC}(Y|X) = 12E(\{E(Y|X)\}^2) - 3,$$
$$\mathrm{GMC}(X|Y) = 12E(\{E(X|Y)\}^2) - 3, \tag{16}$$

where

$$E(\{E(Y|X)\}^2) = \frac{(f_2(a)+f_2(b))^2}{(f_2(b)+1)^2}\frac{f_2(b)}{2f_2(a)+f_2(b)} + \frac{f_1^2(a)}{(f_2(b)+1)^2}\frac{f_1(b)}{4f_1(a)-f_1(b)} + \frac{2(f_2(a)+f_2(b))f_1(a)f_1(b)f_2(b)}{(f_2(b)+1)^2(2f_1(a)f_2(b)+f_2(a)f_1(b))}$$

and

$$E(\{E(X|Y)\}^2) = \frac{(f_1(b)+f_1(a))^2}{(f_1(a)+1)^2}\frac{f_1(a)}{2f_1(b)+f_1(a)} + \frac{f_2^2(b)}{(f_1(a)+1)^2}\frac{f_2(a)}{4f_2(b)-f_2(a)} + \frac{2(f_1(b)+f_1(a))f_2(b)f_2(a)f_1(a)}{(f_1(a)+1)^2(2f_2(b)f_1(a)+f_1(b)f_2(a))}$$

with $a = 1 - \alpha_1$ and $b = \alpha_2$.

Table 1. Values of $(\alpha_1, \alpha_2, \beta)$ satisfying $\mathrm{GMC}(Y|X) = \mathrm{GMC}(X|Y)$

$\beta \geq$		α_1 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0.1	1	16.4021	26.8361	56.0792	$+\infty$	27.0396	12.9395	7.9043	1
	0.2	16.4021	1	26.8361	56.0792	$+\infty$	27.0396	12.9355	1	
	0.3	26.8361	26.8361	1	56.0792	$+\infty$	27.0395	1		
	0.4	56.0792	56.0792	56.0792	1	$+\infty$	1			
α_2	0.5	$+\infty$	$+\infty$	$+\infty$	$+\infty$	1				
	0.6	27.0396	27.0396	27.0396	1					
	0.7	12.9355	12.9355	1						
	0.8	7.9043	1							
	0.9	1								

It is easy to see that $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ are functions of a, b, and β. They are not identical. They are equal, or the difference is negligible, that is, $|\mathrm{GMC}(Y|X) - \mathrm{GMC}(X|Y)| < 10^{-10}$, for some chosen values of α_1, α_2, and β. Table 1 is a numerical illustration of choices of $(\alpha_1, \alpha_2, \beta)$ satisfying $|\mathrm{GMC}(Y|X) - \mathrm{GMC}(X|Y)| < 10^{-10}$. For example, when $(\alpha_1, \alpha_2) = (0.1, 0.1)$ and $\beta \geq 1$, we have $|\mathrm{GMC}(Y|X) - \mathrm{GMC}(X|Y)| < 10^{-10}$, that is, $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ are almost equal. Similarly, when $(\alpha_1, \alpha_2) = (0.1, 0.2)$ and $\beta \geq 16.4021$, we have $|\mathrm{GMC}(Y|X) - \mathrm{GMC}(X|Y)| < 10^{-10}$, that is, $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ are almost equal.

5. SIMULATION EXAMPLES

In this section, we use the following simulation procedure:

1. Simulate a bivariate random sample from a prespecified joint distribution using software package R v2.9 – v2.14. The choice of bandwidth h is selected using the R package "ks" developed based on that by Sain, Baggerly, and Scott (1994).
2. Empirical Type I errors under the null hypothesis and empirical detecting powers under the alternative hypothesis are calculated based on 1000 repeated samples for the testing problem in Equation (13) via Theorems 1 and 2. For the empirical Type I errors and empirical detecting powers, we only report results for the size of test chosen at 5%. Our extensive simulation examples show that the similar results hold with the size of test being at 10%. For coverage probabilities, we only report results for confidence level at 95%. Besides reporting empirical Type I errors and empirical detecting powers, we also report Monte Carlo standard errors of estimates. The Monte Carlo standard errors being close to 1 suggest that the estimators perform well.

Example 2. Empirical Type I errors. In this example, we simulate standard bivariate normal samples and bivariate t_9 samples (Case 2, with degrees of freedom $df = 9$). Table 2 displays empirical Type I errors with the size of test being 5% for selected (n, ρ) combinations. We note that our extensive simulations with different (n, ρ) combinations give empirical Type I errors similar to those in the table.

The Monte Carlo standard errors of $\sqrt{n}((1, -1)\hat{\mathbf{A}}^T \hat{\boldsymbol{\Sigma}}\hat{\mathbf{A}}(1, -1)^T)^{-1/2}\ (\widehat{\mathrm{GMC}}(Y|X) - \widehat{\mathrm{GMC}}(X|Y) - \hat{C}_0)$ in Theorem 2 are 1.11, 1.08, and 1.03 in the bivariate normal case, and 1.13, 1.08, and 1.05 in the bivariate t case for $n = 50, 100, 150$, respectively, with ρ being 0.40.

Example 3. Time series example. We simulate bivariate sample from the following bivariate time series model:

$$X_i = 0.3X_{i-1} + 0.2X_{i-2} + \epsilon_i^X$$
$$Y_i = 0.5Y_{i-1} - 0.1Y_{i-2} + \epsilon_i^Y$$

where ϵ_i^X and ϵ_i^Y follow a bivariate normal distribution with correlation coefficient $\rho = 0.4$. The empirical Type I errors from selected simulation sample sizes are presented in Table 3. The Monte Carlo standard errors of $\sqrt{n}((1, -1)\hat{\mathbf{A}}^T \hat{\boldsymbol{\Sigma}}\hat{\mathbf{A}}(1, -1)^T)^{-1/2}(\widehat{\mathrm{GMC}}(Y|X) - \widehat{\mathrm{GMC}}(X|Y) - \hat{C}_0)$ in Theorem 2 are 0.96, 1.03, and 1.02 for $n = 60, 100, 150$, respectively.

Example 4. Empirical detecting powers. In this example, we simulate bivariate t samples (Case 3) with the degrees of freedom $(\mathrm{df1} = 2, \mathrm{df2} = 21)$ and bivariate samples from Equation (14). The left panel in Figure 1 reveals empirical powers for varying sample sizes n from 50 to 750 and the size of test is chosen at 5%; the middle panel in Figure 1 plots empirical powers for $(\alpha_1, \alpha_2) = (0.4, 0.1)$ with the sample size n changing from 25 to 500; the right panel in Figure 1 plots $\alpha_2 = 0.1$ and changing α_1 values while the sample sizes are fixed at either $n = 100$ or $n = 300$.

Tables 2 and 3 show that Type I errors are controlled within their corresponding nominal levels. Figure 1 demonstrate that when two GMCs are not identical, empirical powers are increasing along with increasing sample sizes. Moreover, Figure 1

Table 2. Simulation results of Example 2 based on 1000 replications. The size of test is chosen at 5%. The correlation coefficient ρ and the sample size n are varying

$n = 50$			$\rho = 0.4$		
ρ	Normal	t_9	n	Normal	t_9
0.10	0.054	0.060	50	0.059	0.042
0.25	0.058	0.058	70	0.057	0.043
0.40	0.048	0.060	100	0.046	0.055
0.60	0.045	0.041	120	0.047	0.053
0.80	0.046	0.046	150	0.052	0.051

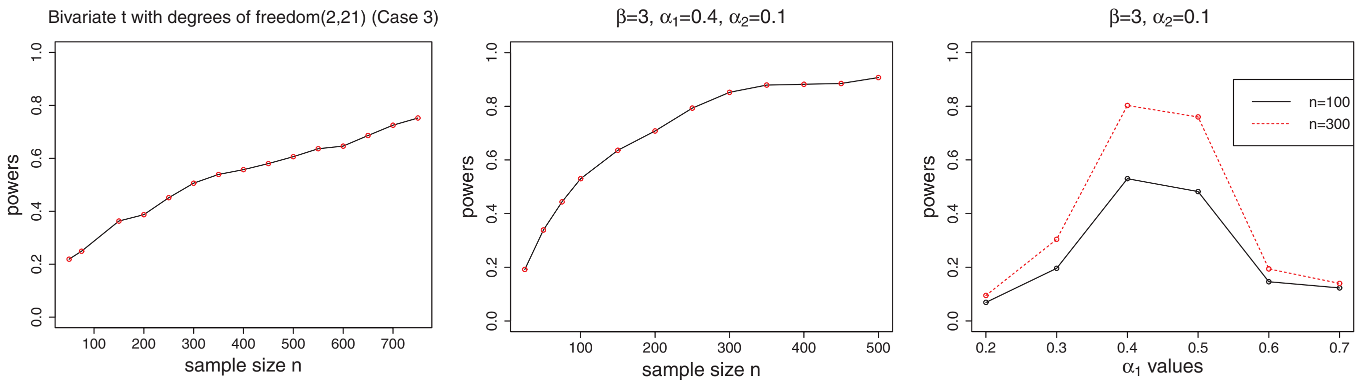

Figure 1. The demonstration of sample performance of bivariate t distribution (Case 3) with changing sample sizes (left panel) and Model (14) with changing sample sizes and varying α_1 values (middle and right panels). The size of test is at 5%. The online version of this figure is in color.

(middle and right panels) displays that Equation (14) is a flexible copula model for a wide range of dependence and explained variances between random variables. In Figure 1 (right panel), with the sample size being fixed, empirical powers give indications of how close are two GMCs for different coefficient combinations. Furthermore, Example 3 suggests that GMCs are also suitable for time series data, which is a very important property in practice. In Section 6, we will apply GMC to economic time series data analysis.

One can see from these examples that dealing with non-identical GMCs is a challenging task, which was also witnessed in Section 4 where mathematical derivations of GMCs for several cases of bivariate distributions were presented. Nevertheless, our simulation examples show that our proposed nonparametric estimators for GMCs are still able to efficiently estimate the values of GMCs and detect whether GMCs are identical or not with a sufficiently large sample size.

Next, we present an example that demonstrates the robustness of our estimation procedures.

Example 5. In this example, we simulate bivariate samples from a mixture of two bivariate normal distributions, that is,

$$(X, Y) \sim \lambda \cdot N(\mathbf{0}_2, \mathbf{\Sigma}) + (1-\lambda) \cdot N(\mathbf{0}_2, \mathbf{\Sigma}_1),$$

where $\mathbf{0}_2 = (0, 0)^T$,

$$\mathbf{\Sigma} = \begin{pmatrix} 1.00 & 0.60 \\ 0.60 & 1.00 \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma}_1 = \begin{pmatrix} 1.00 & 0.25 \\ 0.25 & 1.00 \end{pmatrix},$$

and $\lambda \in [0, 1]$ assigns the weights in the mixture. We perform the following two experiments. The simulation results are reported in Table 4.

(a) For each given $\lambda \in \{0, 0.25, 0.50, 0.75, 0.90\}$, $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ are theoretically computed based on the model. With the sample size being either $n = 50$ or $n = 100$, the asymptotic confidence intervals of $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ are obtained, respectively, in each replication. The total number of replications is 1000. The coverage rates of the true $\mathrm{GMC}(Y|X)$ and $\mathrm{GMC}(X|Y)$ by their estimated confidence intervals (at a 95% level) are evaluated.

(b) There are 5% of outlying values uniformly drawn from two-dimensional region $\{(x, y) : 2 \le x, y \le 4\}$. With the sample size fixed at $n = 60$, we evaluate the coverage rates based on 1000 replications in two ways: one with the outliers in the sample, and another one with the outliers being "removed" using the outlier removal function in the R package "mvoutlier" (see Filzmoser, Maronna, and Werner 2008).

Table 3. Simulation results of Example 3 based on 1000 replications. The size of test is chosen at 5%. The sample size n is varying

n	60	80	100	120	150
Type I errors	0.058	0.048	0.051	0.049	0.049

Table 4 shows that the coverage rates are very close to the nominal confidence level without outliers in the samples, and are acceptable with outliers presented in the samples. The Monte Carlo standard errors of

$$\sqrt{n}(\hat{\mathbf{A}}^T \hat{\mathbf{\Sigma}} \hat{\mathbf{A}})^{-1/2} \Bigg(\widetilde{\mathrm{GMC}}(Y|X) - \mathrm{GMC}(Y|X) - \left(\int \frac{(\phi^{Y|X}(x))^2}{f^X(x)} dx - \mu_Y^2 \right) \left(\frac{1}{\sigma_Y^2 + h^2 \mathrm{var}_K} - \frac{1}{\sigma_Y^2} \right), \widetilde{\mathrm{GMC}}(X|Y) - \mathrm{GMC}(X|Y) - \left(\int \frac{(\phi^{X|Y}(y))^2}{f^Y(y)} dy - \mu_X^2 \right) \times \left(\frac{1}{\sigma_X^2 + h^2 \mathrm{var}_K} - \frac{1}{\sigma_X^2} \right) \Bigg)^T$$

in Theorem 1 are also close to 1, which again shows that the estimators perform well. To illustrate, we consider three cases: (a) $n = 50$, $\lambda = 0.5$ and "without outliers"; (b) $n = 100$, $\lambda = 0.5$ and "without outliers"; and (c) $n = 60$, $\lambda = 0.5$ and "outliers removed." The Monte Carlo standard errors are $(1.10, 1.14)^T$, $(1.08, 1.11)^T$, and $(1.22, 1.20)^T$, respectively, in cases (a), (b), and (c).

This example demonstrates that GMCs and their corresponding estimators perform well when data are drawn from a mixture of two bivariate normal random vectors with (or without) outliers presented in the sample. However, there may be other examples that will break down the adopted estimation procedure. A detailed study of robust estimation of GMC is beyond the scope of the present article, and indeed it is a future research direction.

Table 4. Simulation results of coverage rates with 1000 replications and the nominal confidence level chosen at 95%

	Without outliers				With outliers and $n = 60$			
	$n = 50$		$n = 100$		Outliers removed		With outliers	
λ	GMC($Y\|X$)	GMC($X\|Y$)	GMC($Y\|X$)	GMC($X\|Y$)	GMC($Y\|X$)	GMC($X\|Y$)	GMC($Y\|X$)	GMC($X\|Y$)
0.00	0.946	0.938	0.945	0.943	0.913	0.904	0.901	0.902
0.25	0.945	0.934	0.948	0.949	0.896	0.882	0.916	0.911
0.50	0.938	0.943	0.944	0.952	0.894	0.884	0.923	0.919
0.75	0.944	0.954	0.952	0.952	0.909	0.902	0.946	0.940
0.90	0.955	0.954	0.953	0.949	0.915	0.905	0.961	0.960

6. REAL DATA ANALYSIS

United States and Japan are two of the largest economies in the world. The relationship between these two economic powers is very strong and mutually advantageous. These two countries have both suffered massive banking and financial crises. Japan's crisis began in 1989, and was followed by a long period of slow growth and deflation. The United States' crisis hit in 2008, and it is continuing with depressed economic growth. Comparing the United States to Japan has drawn a great deal of attention among politicians, economists, investors, and researchers. In making comparisons, researchers have focused on illustrating national GDP, imports, exports, S&P500 index, Nikkei index, exchange rates, and other market variables. People have hoped that the comparisons may help reveal similarities and find answers (even solutions) to an economic recovery from the current international financial crisis.

This section aims to reveal an uncanny relationship via our proposed GMC using U.S. and Japanese economic variables. Particularly, we consider monthly average exchange rates from the Japanese Yen against the U.S. dollar (JPY/USD), monthly U.S. federal funds rates, and monthly Japan deposit rates, respectively. They are very important economic indicators. Our data source is International Monetary Fund (IMF), and the data are available at *http://www.imf.org*. The time range is from January 1957 to April 2009 (see Figure 2). From a market perspective, plotting these variables shows no similarity, linear relationship, or co-monotone relationship. However, our peculiar GMC shall display economic changes between these two countries.

Considering that the Bureau of Economic Analysis (BEA) estimates of gross domestic product (GDP) are among the most widely scrutinized indicators of U.S. economic activity, and BEA releases a comprehensive revision about every 5 years, we calculate GMC using a 5-year window (60 months) and the following procedure:

- Suppose data are $\{(x_1, y_1), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1}), \ldots, (x_{n+59}, y_{n+59})\}$.
  1. $i = 1$
  2. Use $\{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{i+59}, y_{i+59})\}$ to compute GMC$(X_i|Y_i)$ and GMC$(Y_i|X_i)$
  3. $i = i + 1$ and repeat (2) until $i = n$.

One can see that this procedure will generate two dynamic GMC curves showing economic status changes over time. For notational convenience, we shall use brief letters "E" to stand for the exchange rates of Japanese Yen against U.S. dollar, "J" to mean Japan deposit rates, and "U" to indicate U.S. federal funds rates. For example, GMC(E|U) stands for the proportion of variation of the exchange rates explained by U.S. federal funds rates. The interpretation of the remaining notation is similar.

We further note that GMCs calculated using the procedure above are different from values calculated from a local dependence measure. Recall that a local dependence measure reveals the changes of strength and direction in a neighborhood of the given point value, say (x_0, y_0). Here the procedure depends on the time, that is, in each GMC calculation, we use a subsample from the entire sample. If the size of the chosen subsample is also relatively large, GMCs calculated from the subsample

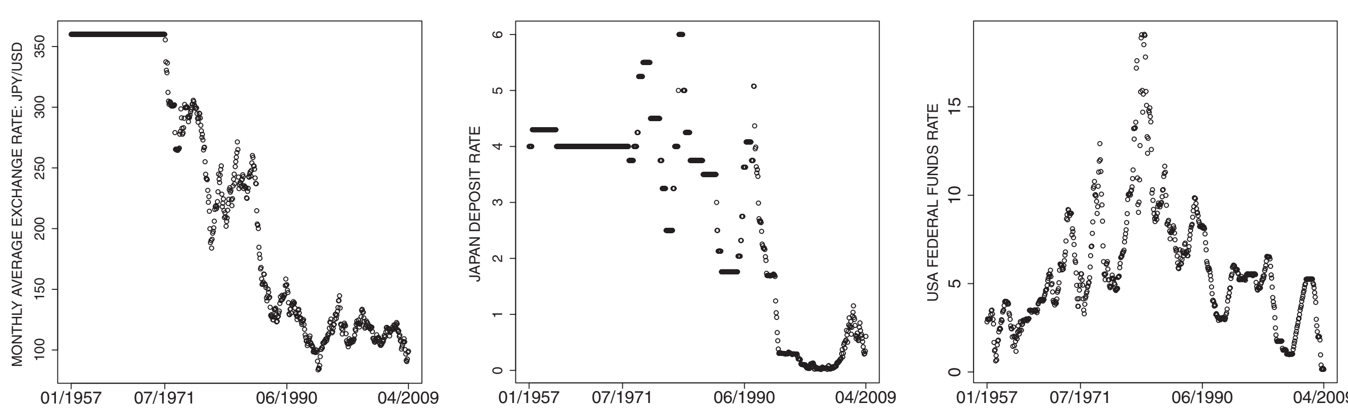

Figure 2. Plots of monthly average exchange rate of JPY/USD (left panel), monthly Japan deposit rates (middle panel), and U.S. federal funds rates (right panel).

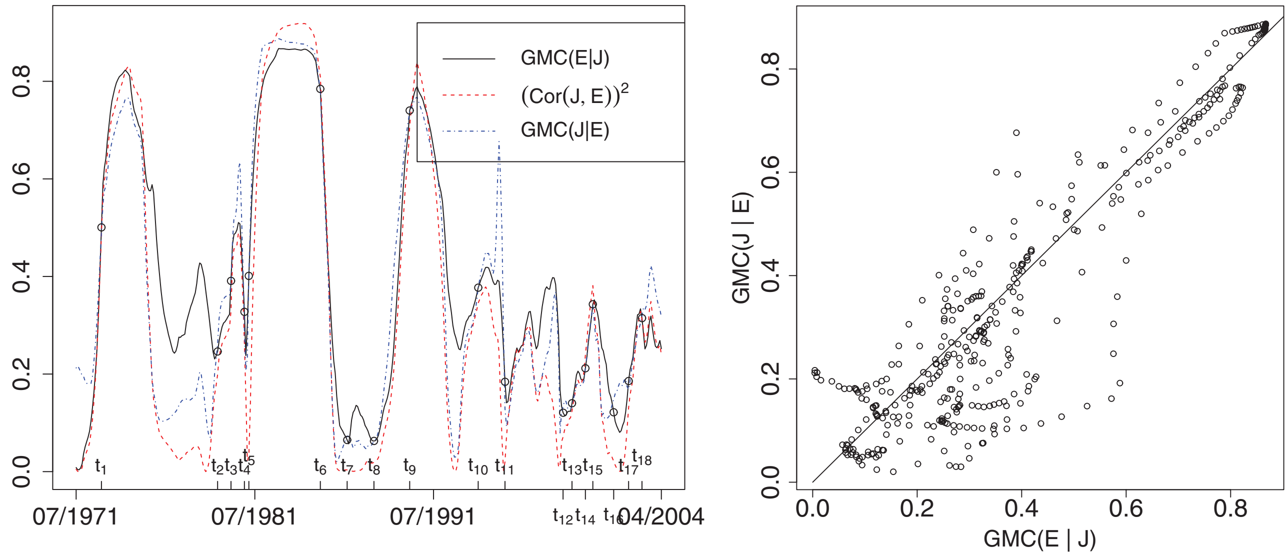

Figure 3. Plots of GMC and the square of ordinary correlation coefficients between JPY/USD exchange rates and Japan deposit rates over time. Time points that GMC((E|J) and GMC(J|E) change relative positions are $t_1 = 12/1972$, $t_2 = 06/1979$, $t_3 = 03/1980$, $t_4 = 12/1980$, $t_5 = 03/1983$, $t_6 = 03/1985$, $t_7 = 09/1986$, $t_8 = 03/1988$, $t_9 = 03/1990$, $t_{10} = 01/1994$, $t_{11} = 07/1995$, $t_{12} = 10/1998$, $t_{13} = 04/1999$, $t_{14} = 01/2000$, $t_{15} = 06/2000$, $t_{16} = 08/2001$, $t_{17} = 06/2002$, $t_{18} = 03/2003$. The right panel plots pairwise GMC with a straight line of 45 degrees. The online version of this figure is in color.

should be close to GMCs calculated from the entire data. On the other hand, if the time series show nonstationarity, correlations calculated from subsamples should be more meaningful. Clearly, in this study the three time series are nonstationary. Due to this observation and the economic cycle, we calculate GMCs using subsamples. Moreover, we also calculate GMCs using the entire dataset.

When computing GMC using Japan deposit rates, we use data from July 1971 to April 2009 because the Japan deposit rate did not show any changes before July 1971. When computing GMC using the exchange rates, we use data from July 1971 to April 2009 since the exchange rates before July 1971 did not show any changes.

Figure 3 displays GMC(E|J), the squared Pearson's correlation $\mathrm{cor}^2(E, J)$, and GMC(J|E) from July 1971 to April 2004. One can immediately see that the overall trends among the three curves are similar. The variations of $\mathrm{cor}^2(E, J)$ look smaller than the variations of GMC, which may indicate that the relationship between the two market variables is not linearly dependent. The curves of GMC reflect the history of Japan economy. The first valley of GMC curves occurred between 1972 and 1981 (see $t_1 = 12/1972$ and $t_4 = 12/1980$ in Figure 3). During this time period, the United States and Japan experienced three major economic conflicts: the first one led to the U.S. import surcharge of August 1971; the second one damaged Japanese confidence in its American connection and had an immediate impact on the political career of the then Prime Minister, Takeo Fukuda, due to major U.S. pressure on Japan during 1977–1978 to boost its domestic growth rate; the third one concerned the reemerging issue of security relations between the two countries (see Bergsten 1982, for more details). Notice that during this period, GMC(E|J) is larger than GMC(J|E), that is, the strength of explained variance in exchange rates by the deposit rate is stronger than the strength of explained variance in deposit rates by the exchange rate. This phenomenon coincided with the damaged Japanese confidence in its American connection. It also clearly suggests that GMC can provide more information than Pearson's correlation can provide. The second valley occurred between 1986 and 1990 in which Japan experienced one of the great bubble economies in history (see $t_7 = 09/1986$ and $t_9 = 03/1990$ in Figure 3). It began after the Japanese agreed, in the so-called Plaza Accord with the United States in 1985, to increase substantially the value of the Yen (which doubled by 1988; see Asher 1996, for more details). After the 1989 Japan economic crisis, the dynamic variations in computed empirical GMC are smaller than those in the proceeding time periods, which may indicate that other economic, social, or political factors play a role in the variations of these two economic variables. We note that in the plot there are time points where GMC(E|J) and GMC(J|E) changed relative positions. These points tell which economic variable is more influential in explained variances during a particular time period, which may also reflect the foreign relations between two countries. Based on the right panel of the figure, we can see that the deposit rate has more impacts in Japan economy growth than the exchange rate has.

Figure 4 compares GMC(E|U), the squared Pearson's correlation $\mathrm{cor}^2(E, U)$, and GMC(U|E) from July 1971 to April 2004. The overall trends among the three curves are similar to those in Figure 3. One can see that the influence of the federal funds rates on the exchange rates is more significant than the Japan deposit rates on the exchange rates. There are fewer points where GMC((E|U) and GMC(U|E) changed relative positions. We note that from October 1992 to August 1998, GMC(U|E) is larger than GMC(E|U), that is, the strength of explained variance in the federal funds rates by the exchange rates is much stronger than the strength of explained variance in the exchange rates by the federal funds rates. This empirical finding again coincides with the economic status during that time period as Griswold (1998, p. 1) stated: "From 1992 and 1997, the U.S. trade deficit almost tripled, while at the same time U.S. industrial production

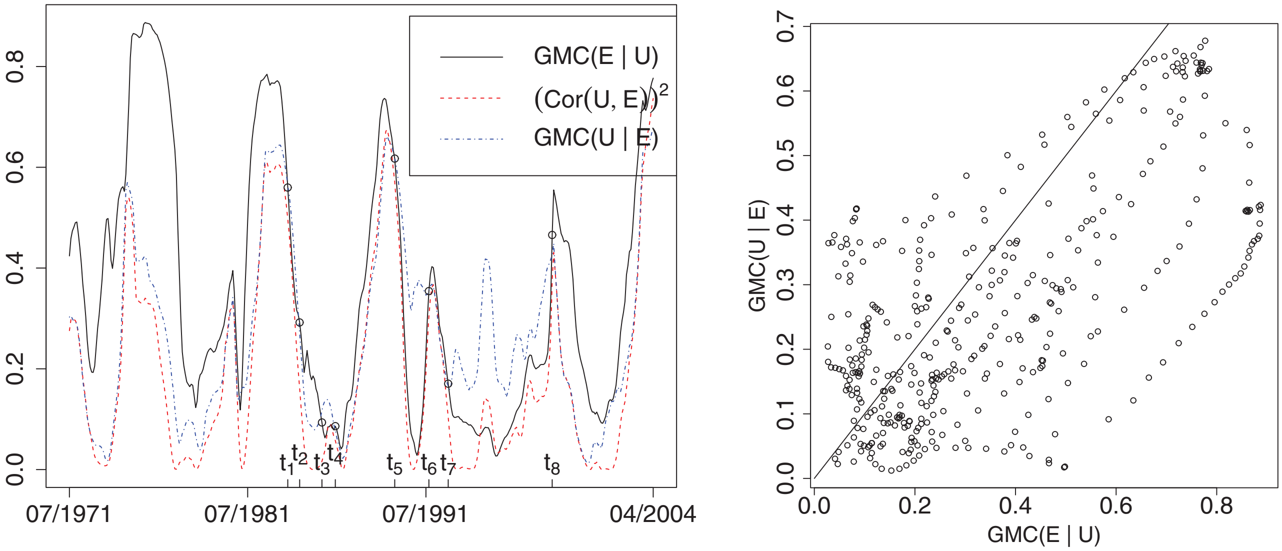

Figure 4. Plots of GMC and the square of ordinary correlation coefficients between JPY/USD exchange rates and U.S. federal funds rates over time. Time points where GMC((E|U) and GMC(U|E) change relative positions are $t_1 = 10/1983$, $t_2 = 06/1984$, $t_3 = 09/1985$, $t_4 = 06/1986$, $t_5 = 10/1989$, $t_6 = 09/1991$, $t_7 = 10/1992$, $t_8 = 08/1998$. The right panel plots pairwise GMC with a straight line of 45 degrees. The online version of this figure is in color.

increased by 24% and manufacturing output by 27%. Trade deficits do not cost jobs. In fact rising trade deficits correlate with falling unemployment rates. Far from being a drag on economic growth, the U.S. economy has actually grown faster in years in which the trade deficit has been rising than in years in which the deficit has shrunk. Trade deficits may even be good news for the economy because they signal global investor confidence in the United States and rising purchasing power among domestic consumers." This is more evidence that GMC is superior in explaining asymmetry of market movements while the ordinary correlation certainly cannot achieve this purpose.

Figure 5 shows GMC(J|U), the squared Pearson's correlation $\mathrm{cor}^2(J, U)$, and GMC(U|J) from July 1971 to April 2004. We see that the U.S. federal funds rates and Japan deposit rates mutually influence each other. The number of points that GMC(J|U) and GMC(U|J) changed relative positions is more than those in Figures 3 and 4. We note that between 1977 and 1981, GMC(J|U) is much larger than GMC(U|J), which tells that the U.S. money regulation policy had a major impact on Japan deposit rates. During that time period, U.S. President Jimmy Carter tried to combat economic weakness and unemployment by increasing government spending, and he established voluntary wage and

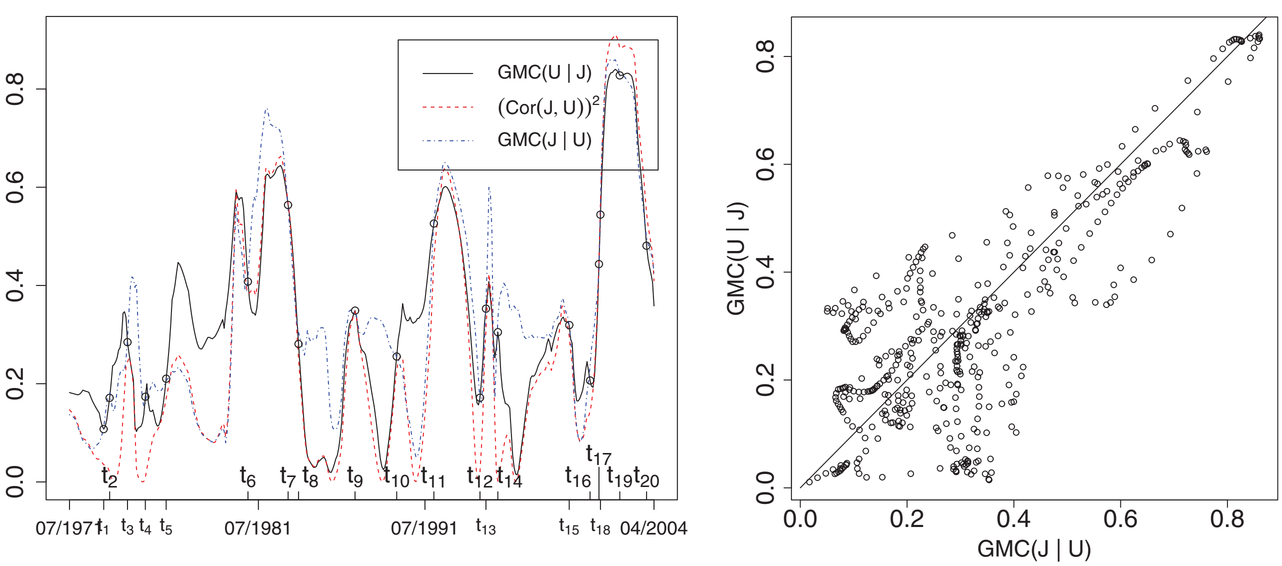

Figure 5. Plots of GMC and the square of ordinary correlation coefficients between U.S. federal funds rates and Japan deposit rates over time. Time points where GMC((J|U) and GMC(U|J) change relative positions are $t_1 = 07/1973$, $t_2 = 11/1973$, $t_3 = 11/1974$, $t_4 = 11/1975$, $t_5 = 01/1977$, $t_6 = 08/1981$, $t_7 = 11/1983$, $t_8 = 06/1984$, $t_9 = 08/1987$, $t_{10} = 12/1989$, $t_{11} = 01/1992$, $t_{12} = 08/1994$, $t_{13} = 12/1994$, $t_{14} = 08/1995$, $t_{15} = 08/1999$, $t_{16} = 10/2000$, $t_{17} = 04/2001$, $t_{18} = 05/2001$, $t_{19} = 06/2002$, $t_{20} = 12/2003$. The right panel plots pairwise GMC with a straight line of 45 degrees. The online version of this figure is in color.

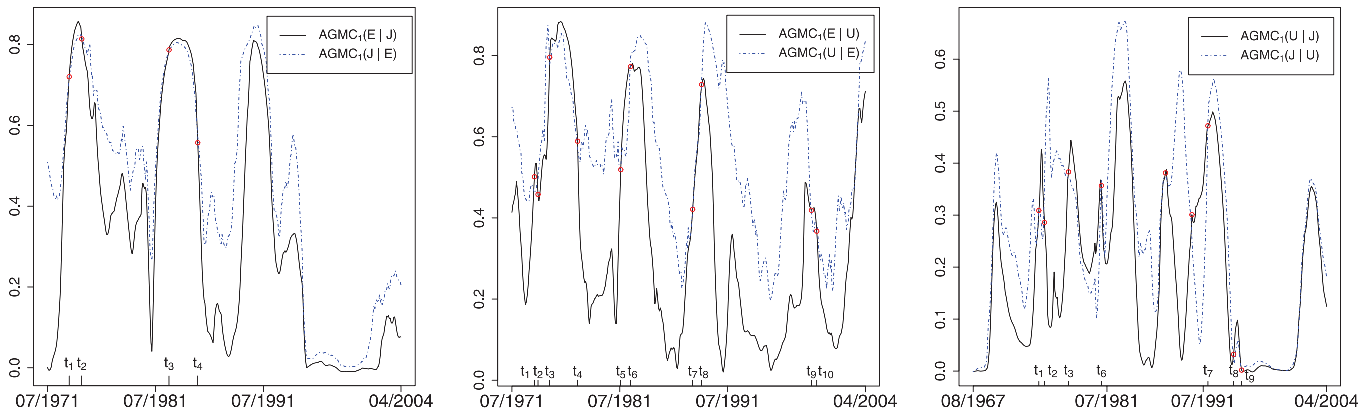

Figure 6. Plots of AGMC. The left panel is AGMC between JPY/USD exchange rates and Japan deposit rates over time. Time points where $\mathrm{AGMC}_1((\mathrm{E}|\mathrm{J})$ and $\mathrm{AGMC}_1(\mathrm{J}|\mathrm{E})$ change relative positions are $t_1 = 07/1973$, $t_2 = 09/1974$, $t_3 = 10/1982$, $t_4 = 06/1985$. The middle panel is AGMC between JPY/USD exchange rates and U.S. federal funds rates over time. Time points where $\mathrm{AGMC}_1((\mathrm{E}|\mathrm{U})$ and $\mathrm{AGMC}_1(\mathrm{U}|\mathrm{E})$ change relative positions are $t_1 = 08/1973$, $t_2 = 12/1973$, $t_3 = 01/1975$, $t_4 = 08/1977$, $t_5 = 08/1981$, $t_6 = 07/1982$, $t_7 = 04/1988$, $t_8 = 02/1989$, $t_9 = 04/1999$, $t_{10} = 10/1999$. The right panel is AGMC between U.S. federal funds rates and Japan deposit rates over time. Time points where $\mathrm{AGMC}_1(\mathrm{J}|\mathrm{U})$ and $\mathrm{AGMC}_1(\mathrm{U}|\mathrm{J})$—change relative positions are $t_1 = 06/1974$, $t_2 = 01/1975$, $t_3 = 07/1977$, $t_4 = 12/1980$, $t_5 = 08/1987$, $t_6 = 05/1990$, $t_7 = 01/1992$, $t_8 = 09/1994$, $t_9 = 07/1995$. The online version of this figure is in color.

price guidelines to control inflation. But the most important element in the war against inflation was the Federal Reserve Board, which clamped down hard on the money supply beginning in 1979. By refusing to supply all the money an inflation-ravaged economy wanted, the Fed caused interest rates to rise. As a result, consumer spending and business borrowing slowed abruptly. The economy soon fell into a deep recession (Source: U.S. Economy in 1970s from U.S. Department of State). Before the 1989 Japan economic crisis (time t_{10}), GMC(J|U) is much larger than GMC(U|J) (t_9 to t_{10}), while after the economic crisis, GMC(J|U) is much smaller than GMC(U|J) (t_{10} to t_{11}), that is, the explained variances in Japan economy had been delayed by the U.S. economy. This is not surprising as at the beginning of an economic recession, the economy is weak, and it can hardly have an immediate economic bounce and recovery, that is, the influence from other economic variable is weak.

We note that by (ii) of Proposition 1, the proposed measures are theoretically not lower than the squared Pearson's correlation coefficients. In Figures 3–5, we see that there are still some points at which the squared sample correlation coefficients are larger than the estimated GMCs. The reason is that Pearson's correlation coefficient is estimated by using empirical distribution and the proposed measures are estimated by using kernel density. If a kernel density approach is used to estimate the squared Pearson's correlation coefficient, the lines of $\mathrm{GMC}(X|Y)$ and $\mathrm{GMC}(Y|X)$ will be higher than the line of the squared Pearson's correlation coefficient in the three figures.

Figure 6 compares AGMC. From the left and the middle panels, one can see that the lagged (1 month) impact from the exchange rate on U.S. federal funds rate and Japan deposit rate is higher than either the lagged-1 impact on the exchange rate from U.S. federal funds rate or the lagged-1 impact from Japan deposit. This phenomenon may suggest that the exchange rates can be used to help researchers build more reliable prediction models. The right panel in the figure shows that U.S. economy is more influential than Japan.

We now use all data to evaluate GMC. We obtain

- GMC(J|U) has an estimate of $\widehat{\mathrm{GMC}}(\mathrm{J}|\mathrm{U}) = 0.641$ and a 95% CI is $(0.573, 0.709)$. GMC(U|J) has an estimate of $\widehat{\mathrm{GMC}}(\mathrm{U}|\mathrm{J}) = 0.560$ and a 95% CI is $(0.483, 0.638)$. The p-value of the testing problem $H_0 : \mathrm{GMC}(\mathrm{U}|\mathrm{J}) = \mathrm{GMC}(\mathrm{J}|\mathrm{U})$ is 0.0008, and hence $\mathrm{GMC}(\mathrm{U}|\mathrm{J}) \neq \mathrm{GMC}(\mathrm{J}|\mathrm{U})$ is significant.
- GMC(E|J) has an estimate of $\widehat{\mathrm{GMC}}(\mathrm{E}|\mathrm{J}) = 0.725$ and a 95% CI is $(0.647, 0.801)$. GMC(J|E) has an estimate of $\widehat{\mathrm{GMC}}(\mathrm{J}|\mathrm{E}) = 0.598$ and a 95% CI is $(0.530, 0.666)$. The p-value of the testing problem $H_0 : \mathrm{GMC}(\mathrm{E}|\mathrm{J}) = \mathrm{GMC}(\mathrm{J}|\mathrm{E})$ is 0.0005, and hence $\mathrm{GMC}(\mathrm{E}|\mathrm{J}) \neq \mathrm{GMC}(\mathrm{J}|\mathrm{E})$ is significant.
- GMC(E|U) has an estimate of $\widehat{\mathrm{GMC}}(\mathrm{E}|\mathrm{U}) = 0.686$ and a 95% CI is $(0.618, 0.755)$. GMC(U|J) has an estimate of $\widehat{\mathrm{GMC}}(\mathrm{U}|\mathrm{E}) = 0.585$ and a 95% CI is $(0.530, 0.641)$. The p-value of the testing problem $H_0 : \mathrm{GMC}(\mathrm{U}|\mathrm{E}) = \mathrm{GMC}(\mathrm{E}|\mathrm{U})$ is 0.002, and hence $\mathrm{GMC}(\mathrm{U}|\mathrm{E}) \neq \mathrm{GMC}(\mathrm{E}|\mathrm{U})$ is significant.

This real-data analysis clearly shows that our proposed GMC is more informative in explaining variations and movements in economic and financial monetary indicators. Our empirical findings show that there are some economic similarities between the United States and Japan; however, the economic development dynamics between these two economic powers are asymmetric, and the universal truth is still that the United States has more impacts in the world economy. As a result, our findings may be helpful in making monetary regulation policies.

7. CONCLUSIONS

We have demonstrated that our proposed GMC is superior in characterizing the asymmetry of explained variances. GMC contains the ordinary correlation coefficient as a special case when two random variables are related in a linear equation or they are bivariate normally distributed. Theoretical foundations of GMC show that when two random variables are correlated through a measurable function, at least one of GMC takes the extreme value 1 while the ordinary correlation coefficient can still be 0. GMC also obeys monotone properties. These properties are strong evidence that GMC is a true nonlinear dependence measure, especially in explained variances. It may be safe to say that GMC can be applied to many research areas where Pearson's correlation coefficient is either applicable or not applicable.

Our definitions of GMC are mainly for bivariate random variables. It is possible to extend the definitions to cases of multivariate random variables, that is, we shall deal with the explained variance in X_1 by $X_2, X_3, \ldots, X_k$. In an attempt to relate GMC to Granger causality between two time series, we introduced GcGMC. We shall conduct a full study of properties of GcGMC in a separate project. GcGMC will be applied to bivariate time series study. We expect to obtain more meaningful results and discover things previously not revealed.

We note that the computation of Pearson's correlation coefficient is easy while GMC involves conditional expectations, and hence it may be computationally challenging in practice. Based on our simulation Example 5 and other extensive simulations, we found as long as sample size is 60 or larger our nonparametric estimators give good approximated values of GMC. Our nonparametric estimators are kernel-based estimators, which is a standard procedure in nonparametric statistics, and therefore they can easily be implemented in any existing software packages.

It is worth noting that Little and Rubin (1987) and Liu (1994) proposed a similar indicator to illustrate the fraction of the missing information in the data augmentation. To the best of our knowledge, the properties of their indicator as the correlation have not been discussed in detail. Analogous to their indicator, our GMC can be thought of as an idea of increasing dimension to measure the asymmetry.

SUPPLEMENTARY MATERIALS

Appendix: Contains the proofs of all the properties discussed in the text.

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