CS 365 HW 2 Math Part

Caiwei Zheng U97031910

1 Probability.

a. X, Y r.v. $X \perp Y$. $f_X = f_Y = f$

let $Z = \max(X, Y)$. Prove $f_Z(x) = 2f(x)P(X \le x)$

To find pdf of Z, we can find cdf of Z first and then take the derivative of cdf to find pdf of Z.

$$F_Z(x) = F_{\max(X,Y)}(x)$$
$$= P(\max(X,Y) \le x)$$
$$= P(X \le x, Y \le x)$$

Since X and Y are independent.

$$P(X \le x, Y \le x) = P(X \le x) \cdot P(Y \le x)$$
$$= F_X(x) \cdot F_Y(x)$$

Then, take the derivative of $F_Z(x)$.

$$f_Z(x) = \frac{d}{dx}(F_Z(x)) = \frac{d}{dx}(F_X(x) \cdot F_Y(x))$$
$$= F_X(x) \cdot f_Y(x) + f_X(x) \cdot F_Y(x)$$
$$= P(X \le x) \cdot f(x) + f(x) \cdot P(Y \le x)$$

Since X and Y have same common density function, the cummulative density function for both variables are the same. Thus, $F_X(x) = F_Y(x)$

$$P(X \le x) = P(Y \le x)$$

Therefore, the above calculations can be ~~forward~~ by

$$f_Z(x) = P(X \le x) f(x) + f(x) P(X \le x)$$
$$= 2f(x) P(X \le x)$$

Q.E.D.

6. $U \sim unif(0,1)$, find dist'n of $\lfloor 100U \rfloor + 1$.

$$E(U) = \frac{1}{2}(0+1) = \frac{1}{2}$$

$$Var(U) = \frac{1}{12}(0-1)^2 = \frac{1}{12}$$

Since U is a uniform random variable in a closed interval $[0,1]$, $\lfloor 100U \rfloor$ maps the interval $[0,1]$ to $[0,100]$ as when $u=0$, $\lfloor 100u \rfloor = \lfloor 0 \rfloor = 0$; and when $u=1$, $\lfloor 100U \rfloor = \lfloor 100 \rfloor = 100$.

Adding 1 to the result maps $[0,100]$ to $[1,101]$.

Therefore, the distribution of $\lfloor 100U \rfloor + 1$ is a discrete uniform distribution in the interval $[1,101]$

Based on propeties of uniform distribution,

$$E(\lfloor 100U \rfloor + 1) = \frac{1}{2}(1+101) = 51$$

$$Var(\lfloor 100U \rfloor + 1) = \frac{1}{12}(1-101)^2 = \frac{10000}{12} = \frac{2500}{3}.$$

c. $U \sim \text{unif}(0,1)$, $0 < q < 1$, $X = 1 + \lfloor \frac{\log U}{\log q} \rfloor$.

Prove $X \sim \text{Geo}(p)$, and find p.

$E(U) = \frac{1}{2}(0+1) = \frac{1}{2}$

$Var(U) = \frac{1}{12}(0-1)^2 = \frac{1}{12}$

$F_U(x) = P(U \le x) = \frac{x-0}{1-0} = x$ for $0 \le x \le 1$

0 for $x < 0$

1 for $x > b$

$$F_{\frac{\log U}{\log q}}(x) = P\left(\frac{\log U}{\log q} \le x\right)$$
$$= P(\log U \le \log q^x)$$
$$= P(U \le q^x)$$

For pmf, $P(U = x) = F_U(x) - F_U(x-1)$

Similarly, $P\left(\frac{\log U}{\log q} = x\right) = P(U \le q^x) - P(U \le q^{x-1})$

$= q^x - q^{x-1} = q^{x-1}(1-q)$, for $x > 0$.

Therefore, $P_{\frac{\log U}{\log q}}(x)$ follows a geometric distribution as $P\left(\frac{\log U}{\log q} = x\right) = q^{x-1}(1-q)$ is the probability mass function of a geometric distribution with parameter p by definition. Moreover, when we take lower bound of $\frac{\log U}{\log q}$, we shift the range of $\frac{\log U}{\log q}$ to positive integers greater than or equal to $\lfloor \frac{\log U}{\log q} \rfloor$. When we add 1 to $\lfloor \frac{\log U}{\log q} \rfloor$, we further shift the set of random variable one unit to the positive side. The shift itself makes no difference to the type of distribution, so $X = 1 + \lfloor \frac{\log U}{\log q} \rfloor$ follows a geometric distribution with the geometric parameter equaling q.

2. Bayes rule.

$$\boxed{\text{Bayes' Rule: } P(A|B) = \frac{P(B|A)\cdot P(A)}{P(B)}}$$

(a) $n=2$

By Bayes' Rule,

$$P(N=1 \mid Z=0.05) = \frac{P(Z=0.05 \mid N=1)\, P(N=1)}{P(Z=0.05)}$$

For $P(Z=0.05)$, it means the probability of having all N iid uniform RV $\{X_i\}_{i=1,\dots,N}$ greater than 0.05.

Simply considering $N=1$:

$$P(Z=0.05 \mid N=1) = P(X > 0.05)$$
$$= 1 - P(X \leq 0.05)$$

By cummulative density function of uniform distribution,

$$F(x) = P(X \leq x) = \begin{cases} x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x < 0 \\ 1 & \text{for } x > 1 \end{cases}$$

Therefore, $\underline{P(Z=0.05 \mid N=1) = 1 - 0.05 = 0.95}$

Then, for $P(Z=0.05)$, we can use Law of total probability.

when $n=2$, $\underline{P(N=1) = P(N=2) = \frac{1}{2}}$

$$P(Z=0.05) = P(Z=0.05 \mid N=1)\, P(N=1) + P(Z=0.05 \mid N=2)\, P(N=2)$$
$$= 0.95 \cdot \tfrac{1}{2} + P(Z=0.05 \mid N=2) \cdot \tfrac{1}{2}$$

When there are 2 R.V.s generated from $\text{unif}\sim(0,1)$.

$P(Z=0.05 \mid N=2) = P(X_1 > 0.05 \text{ and } X_2 > 0.05)$.

Since X_1 and X_2 are independent and identical R.V.s.

$$= P(X_1 > 0.05) \cdot P(X_2 > 0.05)$$
$$= (1 - P(x \leq 0.05))^2$$
$$= 0.95^2$$

Thus, $\underline{P(Z=0.05) = (0.95 + 0.95^2)\tfrac{1}{2}}$

Above all, $P(N=1 \mid Z=0.05) = \dfrac{0.95 \cdot \cancel{\tfrac{1}{2}}}{(0.95 + 0.95^2)\cdot \cancel{\tfrac{1}{2}}} = 0.5128$

(b) $n = 10$.

Similarly,

By Bayes' Rule,

$$P(N=1 \mid Z=0.05) = \frac{P(Z=0.05 \mid N=1)\,P(N=1)}{P(Z=0.05)}$$

In this case, still, $P(Z=0.05 \mid N=1) = 0.95$

$P(N=1) = P(N=10) = 0.5$.

However, $P(Z=0.05) = P(Z=0.05 \mid N=1)P(N=1) + P(Z=0.05 \mid N=10)P(N=10)$

$= 0.95 \times \frac{1}{2} + P(Z=0.05 \mid N=10) \times \frac{1}{2}$

Considering the second term:

$P(Z=0.05 \mid N=10) = P(X_1 > 0.05, X_2 > 0.05, \cdots, X_{10} > 0.05)$

Due to the reason that X_i from $i=1$ to 10 are all iid,

$$= \prod_{i=1}^{10} P(X_i > 0.05)$$

$$= \prod_{i=1}^{10} \left(1 - P(X_i \le 0.05)\right)$$

Still, by cdf of uniform distribution,

$$= \prod_{i=1}^{10} (1 - 0.05)$$

$$= 0.95^{10}$$

Therefore, $P(Z=0.05) = (0.95 + 0.95^{10}) \cdot \frac{1}{2}$

In summary, $P(N=1 \mid Z=0.05) = \dfrac{0.95 \cdot \cancel{0.5}}{(0.95 + 0.95^{10}) \cdot \cancel{0.5}}$

$$= \frac{1}{1 + 0.95^{9}}$$

$$\approx 0.6134$$