① $\Delta = (1-\frac{1}{n})^n$

take log: $\log\Delta = n \cdot \ln(1-\frac{1}{n})$

$= \lim_{n\to\infty} \frac{\ln(1-\frac{1}{n})}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{(1-\frac{1}{n})\cdot(-\frac{1}{n^2})}{-\frac{1}{n^2}}$

$= -1$

$\Delta = e^{\log\Delta} = e^{-1} = \frac{1}{e}$

$(1-\frac{1}{n})^{n\ln n} = \Delta^{\ln n} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$

Therefore, as n goes to infinity, $(1-\frac{1}{n})^{n\ln n}$ goes to zero.

② m edges on labeled graphs with n nodes:

total # of edges: $\binom{n}{2}$. Therefore, m edges in graph has $\binom{\binom{n}{2}}{m}$ possible situations.

③ Let W be a set of all graphs with m edges. by ②, the # of W is $\binom{\binom{n}{2}}{m}$

We need to prove Pr(sampling any graph w in W conditioned on having m edges) is uniform.

Let X be the event that sampled graph has m edges.

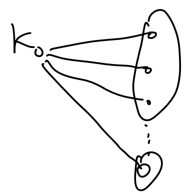
- By Bayes' Rule: $P(w|X) = P(X|w) \cdot P(w) / P(X)$
- $P(X|w)$ is the probability of observing m edges given that w is the actual graph. Since $w \in W$, we are 100% sure that the sampled graph has m edges, so $P(X|w) = 1$.
- $P(w) = p^m (1-p)^{\binom{n}{2}-m}$

· $P(X) = \Pr(\text{A sample has exactly } m \text{ edges}) = \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}$

Therefore, $P(w|X) = \dfrac{1 \cdot p^m (1-p)^{\binom{n}{2}-m}}{\binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}} = \dfrac{1}{\binom{\binom{n}{2}}{m}}$

Since there are in total $\binom{\binom{n}{2}}{m}$ graphs with m edges by ②, this means it is equally likely to sample a graph among all graphs that have m edges.

④ $p = \frac{c}{n}$



Define $Y_i = \begin{cases} 1 & \text{if } \deg(i) = 0 \\ 0 & \text{o.w.} \end{cases}$

$Y_i \sim \text{Bernorlli}(p)$

Let X be # vertices of degree k

$X = \sum_{i=1}^{n} Y_i \sim \text{Bin}(n, p)$, and let $p = \frac{c}{n}$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$= \frac{n(n-1)\cdots(n-k+1)}{k!} \left(\frac{c}{n}\right)^k \left(1-\frac{c}{n}\right)^{n-k}$

$\lim_{n\to\infty} P(X=k) = \lim_{n\to\infty} \frac{c^k}{k!} \left(1-\frac{c}{n}\right)^n$

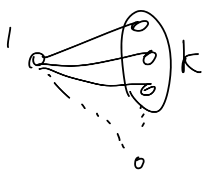
$= \frac{c^k}{k!} e^{-c}$

Therefore, the # of vertices of degree k $= nP(X=k)$

$= \frac{c^k e^{-c}}{k!} n$

⑤ Since adding each edge is independent among each other,
probability of adding one edge $= p_1 + p_2 - p_1 p_2$
Therefore, due to independency of edges
we can have $p = p_1 + p_2 - p_1 p_2$

⑥ Degree of a node in $G(n, 0.1)$ follows Binomial distribution



degree of any node is $Bin(n-1, p)$

Let X represent the degree of any node

$$P(X = k) = \binom{n-1}{k} 0.1^k (1-0.1)^{n-k}$$

$$= \binom{n-1}{k} 0.1^k \cdot 0.9^{n-k}$$

$$E(X) = (n-1)p = 0.1(n-1)$$

$$Var(X) = (n-1)p(1-p) = 0.09(n-1)$$

$$\sigma = \sqrt{0.09(n-1)} = 0.3\sqrt{n-1}$$

$$P\left(-z^*_{0.99} < \frac{X - E(X)}{\sigma} < +z^*_{0.99}\right) = 0.99$$

by z table, $z^*_{0.99} = 2.58$

so the range of degree of node can be calculated as

$$-2.58 < \frac{X - E(X)}{\sigma} < 2.58$$

$$-2.58 < \frac{X - 0.1(n-1)}{0.3\sqrt{n-1}} < 2.58$$

$$-0.774\sqrt{n-1} + 0.1(n-1) < X < 0.774\sqrt{n-1} + 0.1\sqrt{n-1}$$