Lab 7

Asymptotic analysis and G(n,p)

Asymptotic analysis

In mathematical analysis, asymptotic analysis is a method of describing limiting behavior.

Formally, given functions *f*(*x*) and *g*(*x*), we define a binary relation

$$f(x) \sim g(x)$$

If and only if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$$

Or similarly

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x)(1 + o(1))$$

Example: $x^2 - 2x \sim (x+1)x$

Exercise

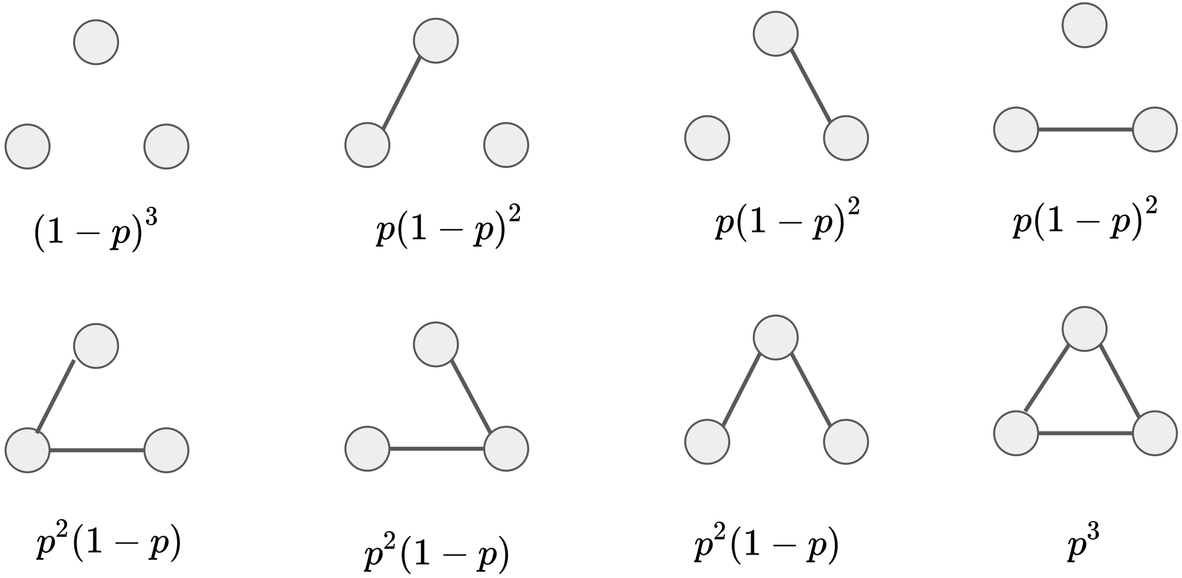
1. In the limit as n goes to infinity, how does $\left(1-\frac{1}{n}\right)^{n\ln n}$ behave?

- Take log and use L'Hôpital's rule
- Take log and use Taylor series of ln(1+x)

2. What is $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{n}$

G(n,p) model

A labeled graph is constructed by connecting labeled nodes randomly. Each edge is included in the graph with probability p, independently from every other edge.



Existence of triangles

Let X be the number of triangles in G(n,p).

Expected number of triangles

$$E[X] = \binom{n}{3} p^3$$

To bound the probability of triangle existence

$$\Pr[X = 0] \leq \Pr[|X - E[X]| \geq E[X]] \leq \frac{Var[X]}{E[X]^2}$$

Existence of triangles

To get the variance, define $\triangle_{i,j,k}$ be the indicator random variable that equals to 1 if a triangle exists with vertices i, j and k. Then,

$$E[X^2] = E\left[\left(\sum_{i,j,k\in[n]} \triangle_{i,j,k}\right)\left(\sum_{x,y,z\in[n]} \triangle_{x,y,z}\right)\right] = \sum_{i,j,k\in[n]} \sum_{x,y,z\in[n]} E[\triangle_{i,j,k}\triangle_{x,y,z}]$$

Case 1: i, j, k share at most one vertex with x, y, z, for such combinations,

$$\sum_{case\ 1} E[\triangle_{i,j,k}\triangle_{x,y,z}] = \sum_{case\ 1} E[\triangle_{i,j,k}]E[\triangle_{x,y,z}]$$
$$\leq \sum_{i,j,k\in[n]} E[\triangle_{i,j,k}] \sum_{x,y,z\in[n]} E[\triangle_{x,y,z}] = E[X]^2$$

Case 2, i, j, k and x, y, z share 2 nodes, for these combinations,

$$\sum_{case\ 2} E[\triangle_{i,j,k}\triangle_{x,y,z}] = \binom{n}{4}p^5$$

Case 3, i, j, k and x, y, z are the same, for these combinations

$$\sum_{case\ 3} E[\triangle_{i,j,k}\triangle_{x,y,z}] = \sum_{i,j,k\in[n]} E[\triangle_{i,j,k}] = E[X]$$

Combine all three cases, $Var[X] = E\left[X^2\right] - E[X]^2 \leq E[X] + o(1)$

Finally, $\Pr[X = 0] \leq \frac{Var[X]}{E[X]^2} \leq \frac{6}{n^3p^3}$

Diameter

See textbook Section 8.2