1. MLE and MoM.

①. $X_1, \cdots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$

$$L(\underline{x}, p) = P_p(\underline{X_n} = \underline{x}) = P_p(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$$
$$= \prod_{i=1}^{n} P_p(X_i = x_i)$$
$$= \prod_{i=1}^{n} f_p(x_i)$$

for Bernoulli samples, $f_p(x_i) = p^{x}(1-p)^{1-x}$, $x \in N$

$$MLE(p) = p_{MLE} = \underset{p \in \Theta}{\arg\max}\ L(\underline{X_n}, p)$$

$$= \underset{p \in \Theta}{\arg\max}\ \underbrace{\log L(\underline{X_n}, p)}_{l(p)}$$

$$= \underset{p \in \Theta}{\arg\max} \sum_{i=1}^{n} \log f_p(x_i)$$

solve for p: $l'(p) = \frac{d}{dp} \sum_{i=1}^{n} \log f_p(x_i)$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial p} \log(p^{x_i}(1-p)^{1-x_i})$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial p} \{x_i \log p + (1-x_i)\log(1-p)\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{x_i}{p} - \frac{1-x_i}{1-p} \right\}$$

$$= \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\sum_{i=1}^{n} x_i - \cancel{p\sum_{i=1}^{n} x_i} - np + \cancel{p\sum_{i=1}^{n} x_i} = 0$$

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Therefore, $p_{MLE} = \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$.

②.

prior: $p = f_x(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

$Y = \sum_{i=1}^{n} X_i$

* Since Beta distribution is a conjugate prior to binomial likelihood, we know that $Y \sim \text{Binomial}(n, p)$; and posterior has the same form as prior.

$$P_Y(y) = \binom{n}{y} p^y (1-p)^{n-y} \qquad P(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}$$

$$P(p|Y) \propto P(Y|p)\, P(p) \propto \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha+y-1} (1-p)^{\beta+n-y-1}$$

$$\Downarrow$$

$$\text{Beta}(\alpha+y,\ \beta+n-y).$$

$$\therefore f_{Y,p}(y,p) = p(Y|p)\,P(p) = \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha+y-1} (1-p)^{\beta+n-y-1}$$

③. $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

let $\vec{\theta} = (\mu, \sigma^2)$

$$L(\vec{\theta}) = \prod f(x_i) = \prod \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot e^{-\frac{\Sigma(x_i-\mu)^2}{2\sigma^2}}$$

$$l(\vec{\theta}) = n \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{\Sigma(x_i-\mu)^2}{2\sigma^2}$$

for μ_{MLE}:

$$\frac{\partial l(\vec{\theta})}{\partial \mu} = 0 + \frac{2\Sigma(x_i-\mu)}{2\sigma^2} = \frac{\Sigma(x_i-\mu)}{\sigma^2} \overset{set}{=} 0$$

$$\Rightarrow \Sigma(x_i-\mu) = 0$$

$$\Sigma x_i - n\mu = 0$$

$$\mu_{MLE} = \frac{\Sigma x_i}{n} = \bar{x}$$

for σ^2_{MLE}:

$$\frac{\partial l(\vec{\theta})}{\partial \sigma^2} = -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{\Sigma(x_i-\mu)^2}{2\sigma^4} \overset{set}{=} 0$$

$$\frac{n}{2}\cdot\frac{1}{\sigma^2} = \frac{\Sigma(x_i-\mu)^2}{2\sigma^4}$$

$$n\sigma^2 = \Sigma(x_i-\mu)^2$$

$$\Rightarrow \sigma^2_{MLE} = \frac{1}{n}\Sigma(x_i-\mu)^2$$

$$= \frac{1}{n}\Sigma(x_i-\bar{x})^2$$

④. $X_1, X_2, \cdots, X_n \overset{iid}{\sim} Exp(\lambda)$

$f(x) = \lambda e^{-\lambda x}$

$\mu_1 = E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}$

Thus, $\hat{\mu}_1 = \frac{1}{\hat{\lambda}} = \overline{X}$, and therefore, $\lambda_{MoM} = \frac{1}{\overline{X}}$

⑤. $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \beta(\theta, 1)$

$f(x; a, b) = \frac{\Gamma(a+b)\, x^{a-1} (1-x)^{b-1}}{\Gamma(a)\,\Gamma(b)}$, where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$.

$f(x; \theta, 1) = \frac{\Gamma(\theta+1)\, x^{\theta-1}}{\Gamma(\theta)\,\Gamma(1)} = \theta x^{\theta-1}$

(a)

for MLE: $l'(\theta) = \frac{d}{d\theta} \sum_{i=1}^{n} \log f(x; \theta, 1)$

$= \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \{\log\theta + (\theta-1)\log x_i\}$

$= \sum_{i=1}^{n} (\frac{1}{\theta} + \log x_i)$

$= \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i \overset{set}{=} 0$

$\theta_{MLE} = \frac{-n}{\sum_{i=1}^{n} \log x_i}$

(b)

for MoM: $E(X) = \frac{\theta}{\theta+1} = \overline{X}$

$\theta\overline{X} + \overline{X} = \theta$

$\theta(\overline{X} - 1) = -\overline{X}$

$\theta_{MoM} = \frac{\overline{X}}{1-\overline{X}}$

2. Let x_i represent result of a handshake pair $\{i,j\}$ (1 as Yes)
Let Y represents every person makes a handshake with X_i.
X_i follows a Bernolli $(\frac{1}{10})$ samples.

$$Y = x_1 + x_2 + \cdots + x_n = \sum_{i=1}^{n} x$$

$$E(X_i) = \frac{1}{10} \text{ for all } i.$$

$$E(Y) = E(X_1 + \cdots + X_n) = \frac{n}{10}$$

Here, X_i can be any person, which means there are $(n-1)$ possible ~~start~~ of handshakes, and considering $\{i,j\}$ won't be counted twice,
The ~~total~~ hand shakes will be $\frac{(n-1)n}{2 \cdot 10}$.
In this way, we can use Y as a representative of every possible person's hand shakes.

By Chernoff bounds for independent Bernoulli random variables,

$$P(Y \geq (1+\delta)\mu) \leq e^{-\delta^2 \mu/(2+\delta)}$$

$$P(Y \leq (1-\delta)\mu) \leq e^{-\delta^2 \mu/2}$$

Let $\delta = 0.05$, then we have

$$P(Y \geq 1.05 \frac{n}{10}) \leq e^{-0.05^2 \cdot \frac{n}{10} / 2.05} = e^{-\frac{n}{8200}}$$

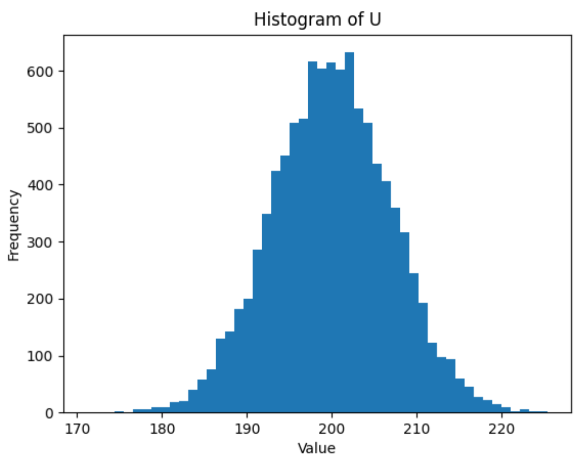
$$P(Y \leq 0.95 \frac{n}{10}) \leq e^{-0.05^2 \cdot \frac{n}{10} / 2} = e^{-\frac{n}{8000}}$$

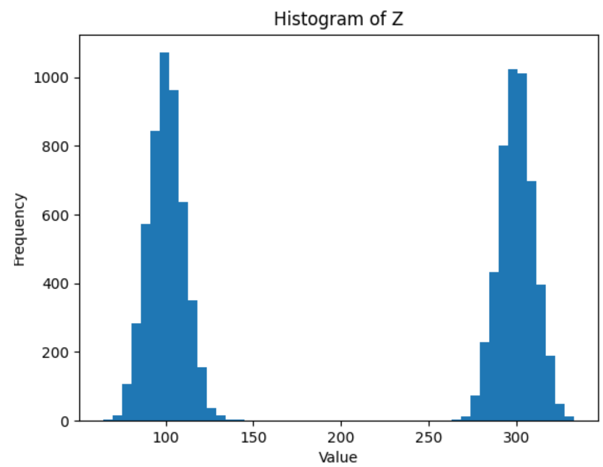
Since $e^{-\frac{n}{8200}}$ and $e^{-\frac{n}{8000}}$ goes to 0 as $n \to \infty$, we have shown that for handshakes per person on average, when $n \to \infty$, $P(0.95\, n/10 \leq Y \leq 1.05\, n/10) \to 1$
Therefore as $n \to \infty$, every person from the party shook hands in the range $[0.95\, n/10,\ 1.05\, n/10]$.

3. $X \sim N(100, 10^2)$ $\quad X \perp\!\!\!\perp Y, \quad U = \frac{1}{2}(X+Y)$

$Y \sim N(300, 10^2)$

① Simulation Histogram:





② $E(U) = E(\frac{1}{2}(X+Y)) = \frac{1}{2}[E(X)+E(Y)] = \frac{1}{2}(100+300) = 200$

$E(Z) = \frac{1}{2}E(Z_1) + \frac{1}{2}E(Z_2)$ as Z_1 and Z_2 are independent.

$= \frac{1}{2} \cdot 100 + \frac{1}{2} \cdot 300 = 200$

③ $Var(U) = Var(\frac{1}{2}(X+Y)) = \frac{1}{4} Var(X+Y)$

Since X, Y both follow normal distribution.

$X+Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

$\therefore Var(U) = \frac{1}{4}(10^2 + 10^2) = 50$

$Var(Z) = E(Z^2) - (E(Z))^2$

Second moment of Z:

$E(Z^2) = \int z^2 (\frac{1}{2} f_1(z) + \frac{1}{2} f_2(z)) dz$

$= \frac{1}{2} E(Z_1^2) + \frac{1}{2} E(Z_2^2)$

$= \frac{1}{2}(\sigma_1^2 + \mu_1^2) + \frac{1}{2}(\sigma_2^2 + \mu_2^2)$

$= \frac{1}{2}(100^2 + 100) + \frac{1}{2}(300^2 + 100)$

$= 50100$

Therefore, $Var(Z) = 50100 - 200^2 = 10100$