1 Probability.

a. Construct a random variable X that represents the result of flipping a coin, with head equaling 1 and tail equaling -1.

Therefore, $P(X=1) = P(X=-1) = 0.5$.

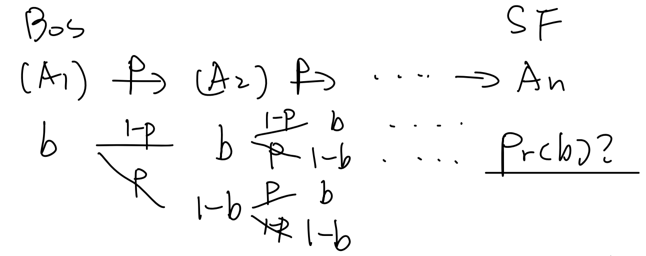
$E(X) = 1 \cdot 0.5 + (-1) \cdot 0.5 = 0$

$Var(X) = E(X^2) = 1^2 * 0.5 + (-1)^2 * 0.5 = 1$

By Chebyshev's inequality, $P(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}$

when $t = 1$, the equality holds as $P(|X-0| \geq 1) \leq \frac{1}{1^2} = 1$.

b.



Let S_i represents San Francisco receives "b" at A_i, for $i \in [1, n]$.

$P(S_2) = 1-p$

$P(S_3) = P(S_3|S_2) \cdot P(S_2) + P(S_3|\overline{S_2}) P(\overline{S_2})$ By Total Law of Probability

$= (1-p) \cdot (1-p) + p(1-p)$

$= 1 - 2p + p^2 + p - p^2 = 1 - p.$

$P(S_4) = P(S_4|S_3) P(S_3) + P(S_4|\overline{S_3}) \cdot P(\overline{S_3})$

$= (1-p) \cdot (1-p) + p(1-p) = 1-p$

$\vdots$

$P(S_n) = P(S_n|S_{n-1}) P(S_{n-1}) + P(S_n|\overline{S_{n-1}}) P(\overline{S_{n-1}}) = 1-p$

Therefore, the probability that San Francisco will receive the right value b instead of $1-b$ is $1-p$.

c. According to the question, let $P(A)=P(B)=p$. [prob of truth].
$P(\bar{A})=P(\bar{B})=1-p$ [prob of lies].

<u>Find $P(A \mid B \text{ confirms})$</u>

By Baye's Theorem, $P(A \mid B \text{ confirms}) = \dfrac{P(B \text{ confirms} \mid A)P(A)}{P(B \text{ confirms})}$

<u>Case</u>	<u>Actual</u>	<u>Actual</u>		
①	A true	B lie	⇒ By what B says, A must be lie	X
②	A true	B true	⇒ By what B says, A is true	✓
③	A lie	B true	⇒ By what B says, A is true	X
④	A lie	B lie	⇒ By what B says, A is lie	✓

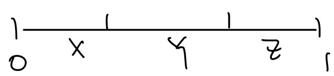
$P(B \text{ Confirms} \mid A) = p$

$$P(B \text{ confirms}) = P(B \text{ confirms} \mid A)P(A) + P(B \text{ confirms} \mid \bar{A})P(\bar{A})$$
$$= p \cdot p + (1-p)^2$$
$$= 1-2p$$

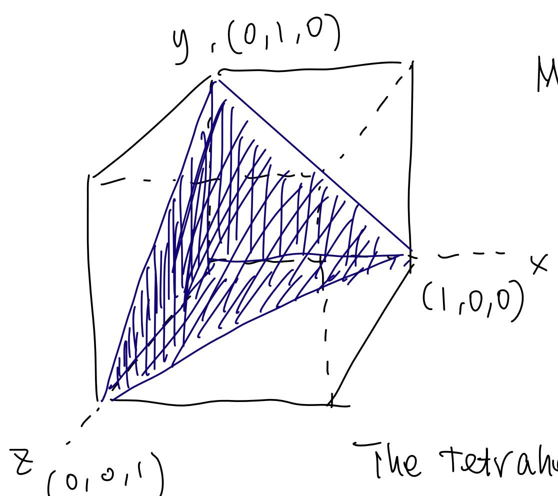
Therefore, $P(A \mid B \text{ confirms}) = \dfrac{p \cdot p}{1-2p} = \dfrac{p^2}{1-2p}$

Given that B confirms the statement by A is true, the probability of A telling truth equals to $\dfrac{p^2}{1-2p}$.

d.



$X, Y, Z \sim \text{unif}(0,1)$



Map X, Y, Z into a 3-D space.
Since we went $X+Y+Z \le 1$,
when $X=0$, $Y+Z \le 1$
$Y=0$, $X+Z \le 1$
$Z=0$, $X+Y \le 1$, where we get a tetrahedron inside the cube.

The tetrahedron formed the maximum likelihood for $X+Y+Z$. Therefone,

$$P(X+Y+Z \le 1) = P(\text{area of tetrahedron}) / P(\text{area of cube}) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{1 \times 1 \times 1} = \frac{1}{6}$$