Instructions

- The homework is due on **Tuesday 4/25 at 5pm ET** before the lecture starts.
- There are 15 points available for extra credit.
- No extension will be provided, unless for serious documented reasons.
- Start early!
- Study the material taught in class, and feel free to do so in small groups, but the solutions should be a product of your own work.
- This is not a multiple choice homework; reasoning, and mathematical proofs are required before giving your final answer.

1 SVD again [20 points]

1. (5 pts) Find the SVD of $A = [1, 1]$ without the use of computing devices/software.
2. (15 pts) Let $A \in \mathbb{R}^{m \times n}$ and let σ_1 be the maximum singular value of A. For $x \in \mathbb{R}^n \backslash \{0\}$ the spectral norm of A is defined as $||A||_2 = \max_x \frac{||Ax||_2}{||x||_2}$. Prove that

$$||A||_2 = \sigma_1.$$

2 Taylor polynomial approximation [10 points]

1. (5 pts) Let $f(x) = \sin(x) + \cos(x)$. Compute the degree 5 Taylor polynomial for f at $x = 0$.
2. (5 pts) Compute the quadratic approximation of the function $f(x, y) = x^2 + y^2 + 2xy - 3x + 2y + 5$ at the point $x = 5, y = 10$.

3 Derivatives [35 points]

Compute the derivative $\frac{df}{dx}$ for the following functions. It will be helpful to identify n, m where $f : \mathbb{R}^n \to \mathbb{R}^m$, and the dimensions of the derivative first.

(a) [5pts] $f(x) = \frac{1}{1+e^{-x}}, x \in \mathbb{R}$

(b) [5 pts] $f(x) = \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big), x \in \mathbb{R}$

(c) [5 pts] $f(x) = \sin(x_1)\cos(x_2), x \in \mathbb{R}^2$.

(d) [5 pts] $f(x) = xx^T, x \in \mathbb{R}^n$.

(e) [5 pts] $f(x) = \sin(\log(x^T x)), x \in \mathbb{R}^n$.

(f) [5 pts] $f(z) = \log(1+z)$ where $z = x^T x, x \in \mathbb{R}^n$

(g) [5 pts] $f(x) = x^T A x$ where $x \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$.

4 Optimization [15 points]

1. (7.5 pts) Consider the univariate function $f(x) = x^3 + 6x^2 - 3x - 5$. Find its stationary points and indicate whether they are maximum, minimum or saddle points.

2. (7.5 pts) Explain how to solve the least squares loss in a linear model using (i) gradient descent and (ii) SVD. Discuss the pros and cons.

5 Coding [35 points]

Check the Jupyter notebook on our Git repo.