1.1. SVD of $A = [1, 1]$

$U = [1]$

$\Sigma = \sqrt{2}$

$V^T = [\,1/\sqrt{2}\,,\ 1/\sqrt{2}\,]$

1.2. 2. (15 pts) Let $A \in \mathbb{R}^{m \times n}$ and let σ_1 be the maximum singular value of A. For $x \in \mathbb{R}^n \backslash \{0\}$ the spectral norm of A is defined as $||A||_2 = \max_x \frac{||Ax||_2}{||x||_2}$. Prove that

$$||A||_2 = \sigma_1.$$

Proof: $\|A\|_2 = \max\limits_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$

$$\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|U\Sigma V^T x\|_2}{\|x\|_2}$$ since U is orthornormal $\|U\| = 1$

$$= \sup_{x\neq 0} \frac{\|\Sigma V^T x\|_2}{\|x\|_2}$$ let $y = V^T x \Rightarrow Vy = x$

$$= \sup_{y \neq 0} \frac{\|\Sigma y\|_2}{\|Vy\|_2}$$

$$= \sup_{y\neq 0} \frac{\left(\sum_{i=1}^r \sigma_i^2 |y_i|^2\right)^{\frac{1}{2}}}{\left(\sum_{i=1}^r |y_i|^2\right)^{\frac{1}{2}}} \leq \sigma_1.$$

For $y = [1\ 0\ \cdots\ 0]^T$, $\|\Sigma y\|_2 = \sigma_1$, and the supremum is attained, which correspond to $x = v_1$. (Hence, $Av_1 = \sigma_1 u_1$.)

Therefore, $\|A\|_2 \triangleq \sup\limits_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max\limits_{\|x\|_2 = 1} \|Ax\|_2 = \sigma_1 = \sigma_{max}(A)$

2.1. $f(x) = \sin(x) + \cos(x)$ center around 0:

$f'(x) = \cos(x) - \sin(x)$ $f(0) = 0+1 = 1$

$f''(x) = -\sin(x) - \cos(x)$ $f'(0) = 1-0 = 1 = f^5(0)$

$f^3(x) = -\cos(x) + \sin(x)$ $f''(0) = -0-1 = -1$

$f^4(x) = \sin(x) + \cos(x)$ $f^3(0) = -1+0 = -1$

$f^5(x) = \cos(x) - \sin(x)$ $f^4(0) = 0+1 = 1$

$T(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n$

degree 5 Taylor polynomial center around 0:

$T(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$

2.2. $f(x,y) = x^2 + y^2 + 2xy - 3x + 2y + 5$

$\nabla f(x,y) = (2x+2y-3,\ 2y+2x+2)$

$H_f = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$Q_f(X) = f(x_0) + \nabla f(X_0)\cdot(X-X_0) + \frac{1}{2}(X-X_0)^T H_f(X_0)(X-X_0)$

Given $X_0 = (5,10)$

$Q_f(x,y) = f(5,10) + \nabla f(5,10)\cdot\begin{pmatrix} x-5 \\ y-10 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-5 \\ y-10 \end{pmatrix}^T \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x-5 \\ y-10 \end{pmatrix}$

$f(5,10) = 5^2 + 10^2 + 2\times5\times10 - 3\times5 + 2\times10 + 5 = 235$

$\nabla f(5,10) = (10+20-3,\ 20+10+2) = (27, 32)$

$\underline{(x-5\ \ y-10)\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x-5 \\ y-10 \end{pmatrix}} = (2x-10+2y-20\ \ \ 2x-10+2y-20)\begin{pmatrix} x-5 \\ y-10 \end{pmatrix}$

$= (2x+2y-30\ \ \ 2x+2y-30)\begin{pmatrix} x-5 \\ y-10 \end{pmatrix} = 2x^2 + 2xy - 30x - 10x - 10y + 150$
$+ 2xy + 2y^2 - 30y - 20x - 20y + 300$

$= 2x^2 + 4xy + 2y^2 - 60x - 60y + 450$

Therefore, $Q_f(x,y) = 235 + (27, 32)\begin{pmatrix} x-5 \\ y-10 \end{pmatrix} + \frac{1}{2}(2x^2 + 4xy + 2y^2 - 60x - 60y + 450)$

$= 235 + 27x - 135 + 32y - 320 + x^2 + 2xy + y^2 - 30x - 30y + 225$

$= 5 - 3x + 2y + x^2 + 2xy + y^2$

3. Derivatives.

(1) $f(x) = \frac{1}{1+e^{-x}}$, $x \in \mathbb{R}$, $f: \mathbb{R}^1 \to \mathbb{R}^1$ $(\nabla f)^{1\times 1}$

$$\underset{1\times 1}{\frac{df}{dx}} = (1+e^{-x})^{-1} = -(1+e^{-x}) \cdot (-e^{-x})$$
$$= e^{-x}(1+e^{-x})$$

(2) $f(x) = \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$, $x \in \mathbb{R}$, $f: \mathbb{R}^1 \to \mathbb{R}^1$ $(\nabla f)^{1\times 1}$

$$\underset{1\times 1}{\frac{df}{dx}} = -\frac{1}{2\sigma^2}(x-\mu)^2 e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \cdot \left(-\frac{1}{2\sigma^2}\right) \cdot 2(x-\mu)$$
$$= \frac{1}{2\sigma^4}(x-\mu)^3 \cdot \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

(3) $f(x) = \sin(x_1)\cos(x_2)$, $x \in \mathbb{R}^2$, $f: \mathbb{R}^2 \to \mathbb{R}^1$ $(\nabla f)^{1\times 2}$

$$\frac{\partial f}{\partial x_1} = \cos(x_1)\cos(x_2) \qquad \frac{\partial f}{\partial x_2} = -\sin(x_1)\sin(x_2)$$

$$\underset{1\times 2}{\frac{\partial f}{\partial x}} = \left[\frac{\partial f}{\partial x_1} \quad \frac{\partial f}{\partial x_2}\right] = \left[\cos(x_1)\cos(x_2) \quad -\sin(x_1)\sin(x_2)\right]$$

(4) $f(x) = x x^T$, $x \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^{n\times n}$

$\underset{n\times 1}{[x]}\,\underset{1\times n}{[x^T]}$ Resulting ∇f: $n\times n\times n$

$$f(x) = \begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & & \vdots \\ \vdots & & \boxed{x_ix_j} \ddots & \vdots \\ x_nx_1 & \cdots & \cdots & x_n^2 \end{bmatrix}$$

$$f(x_1+h, x_2, \cdots, x_n) = \begin{bmatrix} (x_1+h)^2 & (x_1+h)x_2 & \cdots & (x_1+h)x_n \\ (x_1+h)x_2 & x_2^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ (x_1+h)x_n & \cdots & \cdots & x_n^2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x_1} = \lim_{h\to 0} \frac{f(x_1+h, x_2, \cdots, x_n) - f(x_1, x_2, \cdots x_n)}{h} = \lim_{h\to 0} \frac{1}{h}\begin{bmatrix} (x_1+h)^2 - x_1 & hx_2 & \cdots & hx_n \\ hx_2 & & & \\ \vdots & & 0 & \\ hx_n & & & \end{bmatrix}$$

$$= \lim_{h\to 0}\begin{bmatrix} \frac{(x_1+h)^2 - x_1}{h} & x_2 & \cdots & x_n \\ x_2 & & & \\ \vdots & & 0 & \\ x_n & & & \end{bmatrix}$$

Similar process can be done to find $\frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \cdots, \frac{\partial f}{\partial x_n}$, where each partial derivative of x_i with respect to f is a $n \times n$ matrix.

Therefore, the derivative of f is a vector including n $n \times n$ matrix

$$\nabla f^{n\times n\times n} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right] = \left[\lim_{h\to 0}\begin{bmatrix} \frac{(x_1+h)^2 - x_1}{h} & x_2 \cdots x_n \\ \vdots & \\ x_n & 0 \end{bmatrix}, \cdots\cdots, \lim_{h\to 0}\begin{bmatrix} 0 & x_n \\ & \vdots \\ x_1 \cdots & \frac{(x_n+h)^2 - x_n}{h} \end{bmatrix}\right]$$

(5) $f(x) = \sin(\log(x^T x))$, $x \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}^1$ $(\nabla f)^{1\times n}$

$$\underset{1\times n}{\frac{df}{dx}} = \cos(\log(x^T x)) \cdot \frac{2x^T}{x^T x}$$

(6) $f(z) = \log(1+z)$, where $z = x^T x$, $x \in \mathbb{R}^n$, $f: \mathbb{R}^1 \to \mathbb{R}^1$

$z: \mathbb{R}^n \to \mathbb{R}^1$

$(\nabla f)^{1\times n}$

$$\underset{1\times n}{\frac{df}{dx}} = \frac{df}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{1}{1+z} \cdot 2x^T = \frac{2x^T}{1+x^T x}$$

(7) $f(x) = \underset{1\times n}{x^T}\underset{n\times n}{A}\underset{n\times 1}{x}$, where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $f: \mathbb{R}^n \to \mathbb{R}^1$

$\Rightarrow (\nabla f)^{1\times n}$

Let $w = Ax \Rightarrow f(x) = x^T w$

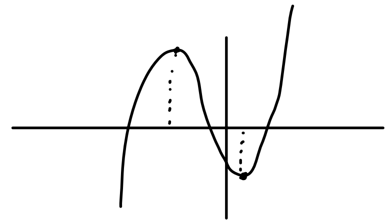
$$\underset{1\times n}{\frac{\partial f}{\partial x}} = x^T \frac{dw}{dx} + w^T \frac{\partial x^T}{\partial x}$$

$$= x^T A + w^T$$

$$= x^T A + x^T A^T$$

$$= x^T (A + A^T)$$

4. Optimization.



4.1. $f(x) = x^3 + 6x^2 - 3x - 5$

$$\frac{df}{dx} = 3x^2 + 12x - 3 = 0$$

$$x^2 + 4x - 1 = 0$$

$$(x+2)^2 = 5$$

$x + 2 = \sqrt{5}$ $\quad$ $x + 2 = -\sqrt{5}$

$x_1 = \sqrt{5} - 2$ $\quad$ $x_2 = -2 - \sqrt{5}$ $\leftarrow$ two stationary points.

when $x = \sqrt{5} - 2$ $f(x) \approx -5.361$, which is a local minimum.

when $x = -2 - \sqrt{5}$, $f(x) \approx 39.361$, which is a local maximum.

4.2. Given a linear system as $y = Ax + e$, where x is the input, y is the output and e is the noise term. A as the system parameter is a known $p \times q$ matrix. To minimize the error term, we can minimize $\hat{y} - Ax$, where $\hat{y}$ is the observation.

$$\hat{x} = \underset{x:\, \hat{y} = Ax + e}{\arg\min} \|e\|^2$$

(1) using gradient descent.

$f(x) = \|e\|^2$, $\quad e = \hat{y} - Ax$

$$\frac{\partial f}{\partial e} = 2e^T \qquad \frac{\partial e}{\partial x} = -A$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial e}\frac{\partial e}{\partial x} = -2(\hat{y}^T - x^T A^T)A$$

Set $\frac{\partial f}{\partial x} = 0 \Rightarrow -2(\hat{y}^T - x^T A^T)A = 0$

$$\hat{y}^T A = x^T \underbrace{A^T A}_{\substack{\text{if } A \text{ is full rank,} \\ A^T A \text{ has inverse}}}$$

$$\hat{y}^T A (A^T A)^{-1} = x^T$$

$$x^* = (A^T A)^{-1} A^T \hat{y}$$

(2) using SVD. (same as HW8 3.2)

To find $\min \|\hat{y} - Ax\|^2$

$$\|\hat{y} - Ax\|^2 = \|\hat{y} - (U\Sigma V^T)x\|^2$$

$$= \|U^T(\hat{y} - AVV^Tx)\|^2 \quad \text{as } U^T \text{ is orthonormal, } \|U^T\|=1$$

$$= \|U^T\hat{y} - U^TU\Sigma V^Tx\|^2$$

$$= \|U^T\hat{y} - \Sigma V^Tx\|^2 \quad \text{Let } z = V^Tx$$

$$= \underbrace{\sum_{i=1}^{r}(u_i^T\hat{y}_i - \sigma_i z_i)^2} + \sum_{i=r+1}^{p}(u_i^T\hat{y}_i)^2$$

To minimize $\|\hat{y} - Ax\|^2$, we only need to minimize

$$\sum_{i=1}^{r}(u_i^T\hat{y}_i - \sigma_i z_i)^2$$

Set it as zero $\Rightarrow$ $z_i = \dfrac{u_i^T\hat{y}_i}{\sigma_i}$, for $i = 1, 2, \cdots, r$.

which make $\|\hat{y} - Ax\|^2$ as minimum $\sum_{i=r+1}^{p}(u_i^T\hat{y}_i)^2$.

Therefore, we can find actual x^*.

$$z = V^Tx^* = \frac{U^T\hat{y}}{\Sigma}$$

$$x^* = \frac{U^T\hat{y}V}{\Sigma} = \sum_{i=1}^{q}\left(\frac{u_i^T\hat{y}}{\sigma_i}\right)v_i$$

Discussion:

- When we have large data, for example matrix A is $100{,}000 \times 1{,}000$ where there is 100,000 rows of observation, it is inefficient to calculate A^TA in results given by gradient descent as well as inverting a 1000×1000 matrix. Therefore, using gradient is computational heavier.
- On the other hand, SVD use the idea of approximation, which calculates only first k singular value. This can make the computation faster when dealing with large data.