1. MLE and MoM.

① $X_1, \cdots, X_n \overset{iid}{\sim} \text{Bernoulli } (p)$

$$L(\underline{x}, p) = P_p(\underline{X}_n = \underline{x}) = P_p(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$$
$$= \prod_{i=1}^{n} P_p(X_i = x_i)$$
$$= \prod_{i=1}^{n} f_p(x_i)$$

for Bernoulli samples, $f_p(x_i) = p^x (1-p)^{1-x}$, $x \in N$

$$MLE(p) = \hat{p}_{MLE} = \underset{p \in \Theta}{\arg \max} \, L(\underline{X}_n, p)$$

$$= \underset{p \in \Theta}{\arg \max} \, \underline{\log L(\underline{X}_n, p)} \quad \hookleftarrow \ell(p)$$

$$= \underset{p \in \Theta}{\arg \max} \, \sum_{i=1}^{n} \log f_p(x_i)$$

solve for $p$: $\ell'(p) = \frac{d}{dp} \sum_{i=1}^{n} \log f_p(x_i)$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial p} \log \left( p^{x_i} (1-p)^{1-x_i} \right)$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial p} \left\{ x_i \log p + (1-x_i) \log (1-p) \right\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{x_i}{p} - \frac{1-x_i}{1-p} \right\}$$

$$= \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n - \sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i - np + p\sum_{i=1}^{n} x_i = 0$$

$$\hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Therefore, $\hat{p}_{MLE} = \hat{p} = \frac{\sum_{i=1}^{n} x_i}{n}$.

② 

prior: $p = f_x(\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$

$Y = \sum_{i=1}^{n} X_i$

\* Since Beta distribution is a conjugate prior to binomial likelihood, we know that $Y \sim \text{Binomial}(n, p)$; and posterior has the same form as prior.

$$P_Y(y) = \binom{n}{y} p^y (1-p)^{n-y} \qquad P(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}$$

$$P(p|Y) \propto P(Y|p) \, P(p) \propto \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha+y-1} (1-p)^{\beta+n-y-1}$$

$$\Downarrow$$

$$\text{Beta}(\alpha+y, \; \beta+n-y)$$

$$\therefore f_{Y,p}(y,p) = P(Y|p) \, P(p) = \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha+y-1}(1-p)^{\beta+n-y-1}$$

③ $X_1, X_2, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

Let $\vec{\theta} = (\mu, \sigma^2)$

$$L(\vec{\theta}) = \prod f(x_i) = \prod \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \cdot e^{-\frac{\sum(x_i-\mu)^2}{2\sigma^2}}$$

$$\ell(\vec{\theta}) = n \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{\sum(x_i-\mu)^2}{2\sigma^2}$$

for $\mu_{MLE}$:

$$\frac{\partial \ell(\vec{\theta})}{\partial \mu} = 0 + \frac{2\sum(x_i-\mu)}{2\sigma^2} = \frac{\sum(x_i-\mu)}{\sigma^2} \overset{set}{=} 0$$

$$\Rightarrow \sum(x_i-\mu) = 0$$

$$\sum x_i - n\mu = 0$$

$$\mu_{MLE} = \frac{\sum x_i}{n} = \bar{x}$$

for $\sigma^2_{MLE}$:

$$\frac{\partial \ell(\vec{\theta})}{\partial \sigma^2} = -\frac{n}{2}\frac{2\pi}{2\pi\sigma^2} + \frac{\sum(x_i-\mu)^2}{2\sigma^4} \overset{set}{=} 0$$

$$\frac{n}{2}\cdot\frac{1}{\sigma^2} = \frac{\sum(x_i-\mu)^2}{2\sigma^4}$$

$$n\sigma^2 = \sum(x_i-\mu)^2$$

$$\Rightarrow \sigma^2_{MLE} = \frac{1}{n}\sum(x_i-\mu)^2$$

$$= \frac{1}{n}\sum(x_i-\bar{x})^2$$

④. $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \text{Exp}(\lambda)$

$f(x) = \lambda e^{-\lambda x}$

$\mu = \bar{E}(X) = \int_0^\infty x \, \lambda \, e^{-\lambda x} dx = \frac{1}{\lambda}$

Thus, $\hat{\mu}_1 = \frac{1}{\lambda} = \bar{x}$, and therefore, $\lambda_{M \cdot M} = \frac{1}{\bar{x}}$

⑤. $X_1, X_2, \cdots, X_n \overset{iid}{\sim} \beta(\theta, 1)$

$f(x; a, b) = \dfrac{\Gamma(a+b) \, x^{a-1} \, (1-x)^{b-1}}{\Gamma(a) \, \Gamma(b)}$, where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$.

$f(x; \theta, 1) = \dfrac{\Gamma(\theta+1) \, x^{\theta-1}}{\Gamma(\theta) \, \Gamma(1)} = \theta \, x^{\theta-1}$

(a)

for MLE: $\ell'(\theta) = \dfrac{d}{d\theta} \sum_{i=1}^n \log f(x; \theta, 1)$

$= \sum_{i=1}^n \dfrac{\partial}{\partial \theta} \left( \log \theta + (\theta-1) \log x_i \right)$

$= \sum_{i=1}^n \left( \dfrac{1}{\theta} + \log x_i \right)$

$= \dfrac{n}{\theta} + \sum_{i=1}^n \log x_i \overset{Set}{=\!=} 0$

$\theta_{MLE} = \dfrac{-n}{\sum_{i=1}^n \log x_i}$

(b)

for M.M: $E(X) = \dfrac{\theta}{\theta+1} = \bar{x}$

$\theta \bar{x} + \bar{x} = \theta$

$\theta(\bar{x} - 1) = -\bar{x}$

$\theta_{M \cdot M} = \dfrac{\bar{x}}{1 - \bar{x}}$

2. Let $X_i$ represent result of a handshake pair $\{i,j\}$ (1 as Yes)
Let $Y$ represents every person makes a handshake with $X_i$.
$X_i$ follows a Bernolli $(\frac{1}{10})$ samples.
$$Y = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} x$$
$E(X_i) = \frac{1}{10}$ for all $i$.
$$E(Y) = E(X_1 + \cdots + X_n) = \frac{n}{10}$$

By Chernoff bounds for independent Bernoulli random variables,
$$P(Y \geq (1+\delta)\mu) \leq e^{-\delta^2 \mu /(2+\delta)}$$
$$P(Y \leq (1-\delta)\mu) \leq e^{-\delta^2 \mu /2}$$

Let $\delta = 0.05$, then we have
$$P(Y \geq 1.05 \frac{n}{10}) \leq e^{-0.05^2 \cdot \frac{n}{10} / 2.05} = e^{-\frac{n}{8200}}$$
$$P(Y \leq 0.95 \frac{n}{10}) \leq e^{-0.05^2 \cdot \frac{n}{10} / 2} = e^{-\frac{n}{8000}}$$

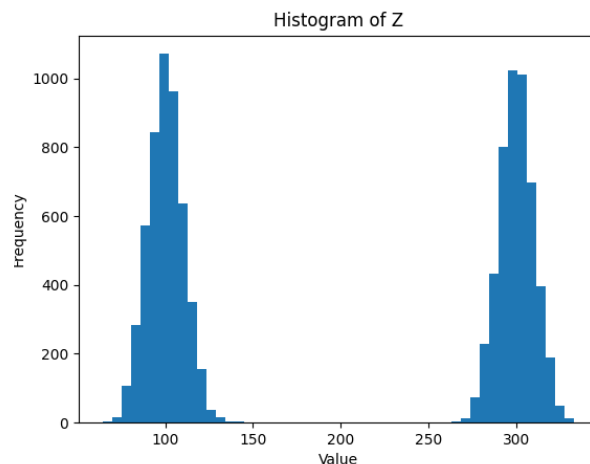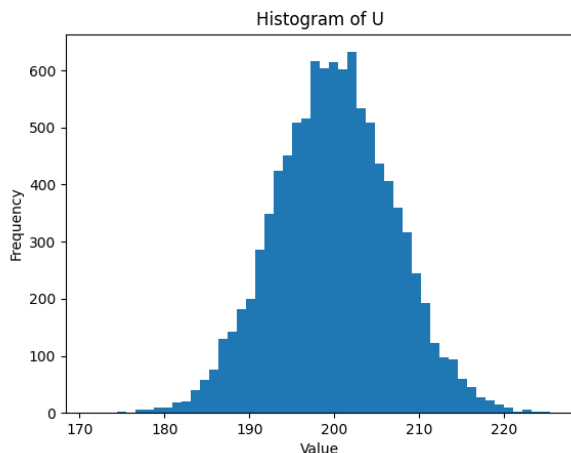Since $e^{-\frac{n}{8200}}$ and $e^{-\frac{n}{8000}}$ goes to 0 as $n \to \infty$, we have shown that for handshakes per person on average, when $n \to \infty$, $P(0.95 \, n/10 \leq Y \leq 1.05 \, n/10) \to 1$
Therefore as $n \to \infty$, every person from the party shook hands in the range $[0.95 \, n/10, \; 1.05 \, n/10]$.

3.  $X \sim N(100, 10^2)$     $X \perp\!\!\!\perp Y$ ,     $U = \frac{1}{2}(X+Y)$
    $Y \sim N(300, 10^2)$

① Simulation Histogram :



② $E(U) = E\left(\frac{1}{2}(X+Y)\right) = \frac{1}{2}[E(X)+E(Y)] = \frac{1}{2}(100+300) = 200$

$E(Z) = \frac{1}{2}E(Z_1) + \frac{1}{2}E(Z_2)$    as  $Z_1$ and $Z_2$ are independent.

$\quad = \frac{1}{2}\cdot 100 + \frac{1}{2}\cdot 300 = 200$

③ $Var(U) = Var\left(\frac{1}{2}(X+Y)\right) = \frac{1}{4} Var(X+Y)$

Since $X, Y$ both follow normal distribution.

$\qquad X+Y \sim N(\mu_x+\mu_y, \; \sigma_x^2 + \sigma_y^2)$

$\therefore Var(U) = \frac{1}{4}(10^2+10^2) = 50$

$Var(Z) = E(Z^2) - (E(Z))^2$

Second moment of $Z$ :

$\quad E(Z^2) = \int z^2\left(\frac{1}{2}f_1(z) + \frac{1}{2}f_2(z)\right)dz$

$\qquad = \frac{1}{2}E(Z_1^2) + \frac{1}{2}E(Z_2^2)$

$\qquad = \frac{1}{2}(\sigma_1^2 + \mu_1^2) + \frac{1}{2}(\sigma_2^2 + \mu_2^2)$

$\qquad = \frac{1}{2}(100^2+100) + \frac{1}{2}(300^2+100)$

$\qquad = 50100$

Therefore, $Var(Z) = 50100 - 200^2 = 10100$