# Summary of Mathematical Methods for Artificial Intelligence

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## 1 Complex Numbers

Complex numbers arise from the need to solve polynomial equations like $z^2 = -1$. These equations do not have solutions in the real numbers, so complex numbers are introduced as an extension.

### 1.1 Definition

A complex number $z$ is defined as:

$$z = x + iy, \quad x, y \in \mathbb{R}$$

where $i$ is the imaginary unit satisfying $i^2 = -1$. The set of complex numbers is denoted by $\mathbb{C}$.

### 1.2 Properties

For a complex number $z = x + iy$:

- The real part of $z$ is $x$ and is denoted as $\Re(z)$.
- The imaginary part of $z$ is $y$ and is denoted as $\Im(z)$.
- The complex conjugate of $z$ is $\overline{z} = x - iy$.
- The modulus of $z$ is $|z| = \sqrt{x^2 + y^2}$.

### 1.3 Trigonometric Form

A complex number can be represented in trigonometric form:

$$z = \rho(\cos\theta + i\sin\theta)$$

where $\rho = |z|$ is the modulus and $\theta$ is the argument of $z$, $\theta = \text{atan2}(y, x)$.

### 1.4 Exponential Form

Using Euler's formula, a complex number can be expressed as:

$$z = \rho e^{i\theta}$$

where $e^{i\theta} = \cos\theta + i\sin\theta$.

# 2 Fourier Series

Fourier series allow periodic functions to be expressed as a sum of sines and cosines.

### 2.1 Definition

A function $f(x)$ defined on $[-L, L]$ can be expanded in a Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where $c_n$ are the Fourier coefficients given by:

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx$$

### 2.2 Convergence

The pointwise convergence of the Fourier series is governed by the Dirichlet conditions, and the series converges to $f(x)$ at points where $f$ is continuous.

# 3 Fourier Transform

The Fourier transform generalizes the Fourier series to non-periodic functions.

### 3.1 Definition

The Fourier transform of a function $f(x)$ is given by:

$$\mathcal{F}\{f(x)\} = F(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx}\, dx$$

### 3.2 Inverse Fourier Transform

The inverse Fourier transform is given by:

$$f(x) = \mathcal{F}^{-1}\{F(k)\} = \int_{-\infty}^{\infty} F(k) e^{2\pi ikx}\, dk$$

# 4 Lebesgue Integration

Lebesgue integration is a fundamental concept in functional analysis and provides a more general framework than Riemann integration.

## 4.1 Lebesgue Spaces

Lebesgue spaces $L^p$ are spaces of functions for which the $p$-th power of the absolute value is integrable:

$$L^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \int_{\mathbb{R}} |f(x)|^p \, dx < \infty \right\}$$