Logic in AI - Proof theory of Modal Logical Systems

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1 What is a modal logic?

In Modal logic are added modalities: $\Box, \diamond$. If A is a formula then $\Box A$ and $\diamond A$ are formulas There are many type of modalities and meaning we can give to this operator such as:

- Temporal
- Epistemic
- Doxastic

1.1 Semantic Models with Kripke

Structure: $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \rho \rangle$

- $\mathcal{W}$ set of possible
- $\rho : \mathcal{W} \to 2^{AT}$
- $\mathcal{R} \subset \mathcal{W}x\mathcal{W}$: accessability relation

There are different axioms in Normal Modal Logic such as:

- Axiom K: $\Box(A \to B) \to (\Box A \to \Box B)$ which is the kernel of the normal mode logic.
- Axiom D: $\Box A \to \diamond B$
- Axiom T: $\Box A \to A$
- Axiom 4: $\Box A \to \Box\Box A$
- Axiom 5: $A \to \Box \diamond A$

And what we can say about this axioms?

- Axiom T, forces the relaiton R in the Kripke model to be *reflexive.*
- Axiom 4, forces the relation R in the Kripke Model to be *transitive.*
- Axiom D, forces the relation R in the Kripke model to be *serial.*
- Axiom 5, forces the relation R in the Kripke model to be *symmetric.*

1.2 Logics of Knowledge

Epistemic Modal Logic where

- $\Box$ is a Knowledge
- $\diamond$ is a Belief

  this logic is useful when we want to reason inconsistencies.

- If we know something, that is true
- If we know something, we know that we know something

1.3 Proof Theory, Deductive Systems in Modal Logic

There are different deductive styles: sequent calculus, analytic tableaux, natural deduction. And there are several approaches, including: labelled deductive systems and geometric/multidimensional deductive systems. The desiderata is *cut elimination/normalization and modularity.*

Another approach is **"Multidimensional"** systems, formulaes are enriches with a spatial coordinate (index/position) that provies information within the proof. In this way we can treat modalities in analogy as quantifiers are treated in first-orders systems. Only modal operators can "change" the spatial position of the formulae.

$$\forall \leftrightarrow \Box$$

$$\exists \leftrightarrow \diamond$$

1.3.1 General Picture

According to the assumption we make on the spatial coordinate S of formulas A^S we can "tune" the system to a specific (normal) modal logic:

- S is a sequence: by seetting some constraints on the rule $\diamond E$ and $\diamond I$ one obtains all the logic from K to $S4$
- S is a set, one captures the system $S4.2$.
- S is a singleton set: one captures $S5$.

1.4 Position and modalities

Lemma 1.1. *A formula A interacts with a formula B with respect to variable x if x occurs in free in both A and B.*

We have the possibility to add or remove the $\Box$ operator through modal interaction, "playing" with the spatial coordinate and costraint.

1.4.1 Position

Definition 1.2. A position is a *sequence of uninterpreted tokens.*

We have associative concatenations and a Successor.

1.5 What adds S4.2?

- $S4.2 = S4 + \diamond\Box A \to \Box \diamond A(.2)$

This is still an enexplored country but this Logic describes Knowledge and Belief.

1.6 Beyond S4: the Logic S5

- S5 = S4 + $\phi \to \Box \diamond \phi$ or $K + T + D + \diamond A \to \Box \diamond A$.

We obtain soundness and completeness *w.r.t* Hilbert-style.

1.7 Curry Howard Correspondence and the Intuitionistic

Through BHK we can interpret proof as we can compute the proofs.