Automatica 139 (2022) 110190

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

A convex optimization approach to synthesizing state feedback data-driven controllers for switched linear systems ☆

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ARTICLE INFO

Article history:
Received 8 February 2020
Received in revised form 7 October 2021
Accepted 9 December 2021
Available online 26 February 2022

Keywords:
Data-driven control
Switched linear systems
Farkas lemma
Polyhedral Lyapunov functions
Chordal sparsity

ABSTRACT

This paper seeks to develop a computationally tractable framework for data-driven control of switched linear systems. Specifically, given a model structure and experimental data collected at different operating points, we seek to directly design a state-feedback controller that stabilizes a system that arbitrarily switches amongst all sub-systems that could have generated the observed data, without an explicit plant identification step. The main result of the paper shows that this robust optimization problem can be recast, through the use of duality, into a polynomial optimization form and efficiently solved, leading to a robust controller with guaranteed ℓ_∞ worst-case performance. The effectiveness of the proposed technique is illustrated with several examples, including control of the horizontal motion of a quadcopter

1. Introduction

Switched linear systems, e.g. systems switching amongst a family of linear time-invariant sub-systems (Liberzon & Morse, 1999), arise in the context of a wide domain of applications ranging from fault-tolerant control to manufacturing and DC-to-DC power converters (Morse, 1995). Additionally, they can be used as a "poor man's" models of non-linear phenomena. Given their importance, substantial research has been devoted during the past decade to the problem of synthesizing stabilizing controllers for switched systems operating in different scenarios. In particular Blanchini, Miani, and Mesquine (2009) established that a necessary and sufficient condition for stability under arbitrary switching is the existence of a common polyhedral Lyapunov function and indicated how to obtain a switched stabilizing controller by solving a bilinear matrix inequality (BMI). Alternatively, several stabilization methods, based on a common quadratic Lyapunov function (CQLF) have been proposed. As shown in Liberzon, Hespanha, and Morse (1999), existence of a CQLF is equivalent to the solvability of the Lie algebras generated by the state matrices. Necessary and sufficient quadratic stability conditions for a system switching between two modes were provided in Shorten and Narendra (2002) and Shorten, Narendra, and Mason (2003). Less conservative conditions based on multiple Lyapunov function were proposed in DeCarlo, Branicky, Pettersson, and Lennartson (2000) where quadratic Lyapunov-like functions are found in each sub-region, such that the energy is nonincreasing during the switch. For all the approaches above, interested readers are referred to the excellent survey Lin and Antsaklis (2009).

While successful, existing stabilization techniques hinge on the availability of a system model. In practical scenarios, designing controllers for switched systems typically entails first identifying a plant model along with an associated uncertainty description suitable to be used by existing controller design techniques. However, the process of identifying models for switched systems and obtaining identification error bounds by validating these models against additional data is far from trivial. Indeed, in its most general form, this identification/(in)validation step is known to be NP-hard (see for instance Ozay, Lagoa, & Sznaier, 2015; Ozay, Sznaier, & Lagoa, 2014). It is also worth noting that, even in the case of non-switching LTI systems, this two-step approach is generally conservative, since typically the error bounds provided by the identification/(in)validation steps are not tight. The situation is even worse in the case of switched systems due to the additional conservatism introduced at the identification step by the relaxations required to obtain tractable problems.

Data-driven control (DDC) methods seek to avoid the plant identification step by synthesizing a controller directly from the experimental data, based on the observation that this data can

☆ This work was partially supported by NSF, United States grants CNS-1646121, ECCS-1808381 and CNS-2038493, AFOSR, United States grant FA9550-19-1-0005, and ONR, United States grant N00014-21-1-2431. Portions of this paper were presented at the 9th IFAC Symposium on Robust Control Design, September 3–5, 2018, Florianópolis, Brazil and the 57th IEEE Conference on Decision and Control, December 17–19, 2018, Miami Beach, Florida, USA. This paper was recommended for publication in revised form by Associate Editor Yasuaki Oishi under the direction of Editor Sophie Tarbouriech.

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https://doi.org/10.1016/j.automatica.2022.110190

be used as a proxy for models. Given the advantages of obviating the identification step, combined with the increased availability of data, in the past few years data-driven control has been the subject of renewed interest, leading to a number of approaches. Earlier work based on input/output models (Campi, Lecchini, & Savaresi, 2002; Formentin, Savaresi, & Del Re, 2012; Hjalmarsson, Gevers, Gunnarsson, & Lequin, 1998; Karimi, Mišković, & Bonvin, 2002) sought a controller such that the closed-loop system approximately matched a desired reference model. While these methods worked well in many scenarios, closed-loop stability could be guaranteed only in the ideal case when an infinite number of data points are considered. An alternative to these methods, based on robust optimization and the concept of superstability, was proposed in Cheng, Sznaier, and Lagoa (2015). While this approach guaranteed closed-loop stability for all plants compatible with the observed data, it was limited to SISO systems.

In the past two years, the issue of guaranteeing closed-loop stability from finite data moved to the forefront. This led to a number of approaches to design data-driven state-feedback controllers guaranteed to stabilize the closed-loop system. For LTI systems these approaches include Berberich, Koch, Scherer and Allgöwer (2020), Berberich, Köhler, Muller and Allgower (2020), Dai, Sznaier, and Solvas (2020), De Persis and Tesi (2019, 2020), Vanwaarde, Camlibel, and Mesbahi (2020), van Waarde, Eising, Trentelman, and Camlibel (2020), based on the idea of characterizing the underlying dynamics in terms of matrices built from the observed data. Recent nonlinear DDC methods include Dai and Sznaier (2020), Guo, De Persis, and Tesi (2020), Tanaskovic, Fagiano, Novara, and Morari (2017). While successful, at the present time, none of these methods can handle switched systems. To the best of our knowledge, the only DDC control method for switched systems currently available (other than the conference versions of the present paper Dai & Sznaier, 2018a, 2018b) is Breschi and Formentin (2019). However, this method is currently limited to SISO systems and does not provide closed-loop stability guarantees.

Motivated by the issues noted above, in this paper we propose a state-feedback DDC synthesis framework for switched linear systems, capable of handling finite, noisy data records, while guaranteeing closed-loop stability of all plants compatible with the observed data (the consistency set). The key observation is that, under the assumption of ℓ_∞ bounded process noise, both the consistency set (e.g., state matrices $\mathbf{A}, \mathbf{B}$) and the set of all plants that share a common polyhedral control Lyapunov function (CPCLF) are polyhedrons in parameter space. Thus, in this context, the state-feedback DDC problem reduces to finding a CPCLF such that the former is contained in the later. As shown in the paper, by exploiting the extended Farkas' lemma, this can be recast as a polynomial optimization problem, which in turn can be reduced to a rank constrained semi-definite program (SDP), solvable using standard convex relaxations of rank. Further, this semi-definite program is endowed with chordal sparsity, which allows for obtaining an algorithm whose computational complexity scales linearly with the number of sub-systems.

The rest of the paper is organized as follows: Section 2 recalls some background results and formally states the problem under consideration. Section 3 contains the main results of the paper, combining elements from duality, polyhedral Lyapunov functions, and polynomial optimization to transform the problem into a tractable convex optimization. Section 4 briefly indicates how to extend these results to the continuous time case. Section 5 provides examples illustrating the practicality of the algorithms introduced in the previous section. Finally, Section 6 summarizes the paper and discusses open issues. Portions of this paper were presented at the 2018 RoCond (Dai & Sznaier, 2018a) and 2018 CDC (Dai & Sznaier, 2018b). This version includes complete proofs, the extension of the results to the continuous time case, an exploration of the use of an underlying sparse graph structure to alleviate the computational complexity and additional examples, including the stabilization of a simplified model of a quadcopter.

2. Preliminaries

2.1. Notation

$\mathbb{R}, (\mathbb{R}^+)$	Set of (non-negative)real numbers
$\mathbb{N}$	Set of non-negative integers
$\mathbb{1}$	A vector of 1 s
$\mathbf{I}_n$	$n \times n$ identity matrix
$\mathbf{x}, \mathbf{X}$	A vector in $\mathbb{R}^n$, a matrix in $\mathbb{R}^{m\times n}$
$\lvert\mathbf{x}\rvert$	A vector with elements $\lvert\mathbf{x}_i\rvert$
$\mathbf{X} \geq 0$	$\mathbf{X}$ is element-wise non-negative (e.g. $\mathbf{X}(i,j) \geq 0$)
$\mathbf{X} \succeq 0$	$\mathbf{X}$ is positive semi-definite
$\sigma_i(\mathbf{X})$	The ith largest singular value of the matrix $\mathbf{X}$
$\lVert\mathbf{x}\rVert_\infty$	ℓ_∞-norm of the vector $\mathbf{x} \in \mathbb{R}^n : \lVert\mathbf{x}\rVert_\infty \doteq \sup_i \lvert\mathbf{x}(i)\rvert$
$\lVert\mathbf{X}\rVert_\infty$	ℓ_∞ induced-norm of the matrix $\mathbf{X} \in \mathbb{R}^{m\times n} : \lVert\mathbf{X}\rVert_\infty \doteq \sup_i \sum_{j=1}^n \lvert\mathbf{X}(i,j)\rvert$
$\otimes$	Matrix Kronecker product
$\mathbf{vec}(\mathbf{X})$	Matrix vectorizing operation $\mathbf{vec}(\mathbf{X}) = \left[\mathbf{X}(:,1)^T, \ldots \mathbf{X}(:,n)^T\right]^T$

2.2. Properties of the Kronecker product

In this paper we will make extensive use of the following equality, whose proof can be found for instance in Horn and Johnson (2012).

$$\mathbf{vec}(\mathbf{AXB}) = (\mathbf{B}^T \otimes \mathbf{A})\mathbf{vec}(\mathbf{X}) \tag{1}$$

2.3. Stability of switched systems

Definition 1. Given a switched system of the form:

$$\boldsymbol{x}_{k+1} = \mathbf{A}_i \boldsymbol{x}_k, \ i = 1, \ldots, N \tag{2}$$

let $\mathcal{P}$ denote a compact polyhedron containing the origin in its interior, and $\mathcal{V}$ its associated gauge function, e.g.

$$\mathcal{V}(\mathbf{x}) \doteq \inf\{\mu \in \mathbb{R}^+ : \mathbf{x} \in \mu\mathcal{P}\}$$

Then $\mathcal{V}(\boldsymbol{x})$ is a polyhedral Lyapunov function for (2) if and only if along its trajectories

$$\mathcal{V}(\boldsymbol{x}_{k+1}) - \mathcal{V}(\boldsymbol{x}_k) < 0, \quad \mathcal{V}(\boldsymbol{x}_k) > 0, \quad \forall \boldsymbol{x}_k \neq 0$$

for any arbitrary switching sequence.

Theorem 1 (*Blanchini et al., 2009*)**.** *The origin is an asymptotically stable equilibrium point of (2) for any arbitrary switching sequence (e.g. the system is switching stable) if and only if there exist a full column rank matrix* $\mathbf{V}$ *and N matrices* $\mathbf{H}_i$ *with* $\lVert\mathbf{H}_i\rVert_\infty < 1$ *such that*

$$\mathbf{V}\mathbf{A}_i = \mathbf{H}_i\mathbf{V} \tag{3}$$

Similarly, the continuous time system

$$\dot{\boldsymbol{x}}_t = \mathbf{A}_i \boldsymbol{x}_t, \ i = 1, \ldots, N \tag{4}$$

is switching stable if and only if there exist a full column rank matrix $\mathbf{V}$*, N scalars* $\tau_i > 0$*, and N matrices* $\mathbf{H}_i$ *such that*

$$\begin{aligned} &\mathbf{V}\mathbf{A}_i = \mathbf{H}_i\mathbf{V} \text{ and} \\ &\lVert\mathbf{I} + \tau_i \mathbf{H}_i\rVert_\infty < 1 \end{aligned} \tag{5}$$

In these cases, $\mathcal{V}(\mathbf{x}) = \|\mathbf{V}\mathbf{x}\|_\infty$ is a polyhedral Lyapunov function for (2) (discrete time case) and for the Euler auxiliary system (Blanchini & Miani, 2008) of (4) (continuous time case). These facts guarantee switched stability of both (2) and, under a non-Zenoness assumption, of (4).

2.4. Extended Farkas lemma

Lemma 1 (Henrion, Tarbouriech, & Kucera, 1999)**.** *Consider two polyhedrons of the form $\mathbf{P}_N \doteq \{\boldsymbol{x} : \mathbf{N}\boldsymbol{x} \leq \boldsymbol{n}\}$ and $\mathbf{P}_M \doteq \{\boldsymbol{x} : \mathbf{M}\boldsymbol{x} \leq \boldsymbol{m}\}$, where $\mathbf{M}, \mathbf{N}$ are matrices and $\boldsymbol{m}, \boldsymbol{n}$ are vectors. Then $\mathbf{P}_N \subseteq \mathbf{P}_M$ if and only if there exists a matrix $\mathbf{Y}$ (Farkas multiplier) with non-negative entries such that*

$$\begin{aligned} \mathbf{Y}\mathbf{N} &= \mathbf{M} \\ \mathbf{Y}\boldsymbol{n} &\leq \boldsymbol{m} \end{aligned} \tag{6}$$

2.5. Polynomial optimization problems

The main result of this paper shows that the data-driven control problem can be reduced to a (non-convex) quadratically constrained quadratic problem (QCQP) of the form:

$$\min_{\mathbf{x}} \mathbf{v_x}^T \mathbf{Q}_0 \mathbf{v_x} \; s.t. \; \mathbf{v_x}^T \mathbf{Q}_k \mathbf{v_x} \geq 0, \; k = 1, \ldots, N \tag{7}$$

for suitable symmetric matrices $\mathbf{Q}_i, i = 0, \ldots, N$, where $\mathbf{v_x}^T = [1, x_1, x_2, \ldots, x_n]$. This problem is a special case of polynomial optimization problems of the form:

$$p^* = \min_{\boldsymbol{x} \in \mathcal{K}} \quad p(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}} \tag{8}$$

where $\boldsymbol{x}^{\boldsymbol{\alpha}}$ denotes the monomial $\prod_{i=1}^n x_i^{\alpha_i}$ with $\alpha_i \in \mathbb{N}$, $\boldsymbol{\alpha} \doteq [\alpha_1, \ldots, \alpha_n]$ and the semi-algebraic set $\mathcal{K} = \{\boldsymbol{x} \in \mathbb{R}^n : g_k(\boldsymbol{x}) \geq 0,\ k = 1, \ldots, N\}$ is defined by a set of polynomial constraints of the form $g_k(\boldsymbol{x}) = \sum_\alpha g_{k,\alpha} \boldsymbol{x}^\alpha \geq 0$. As shown in Lasserre (2001), problem (8) is equivalent to the following optimization over the set $\mathcal{P}(\mathcal{K})$ of probability measures μ supported on $\mathcal{K}$:

$$p^* = \min_{\mu \in \mathcal{P}(\mathcal{K})} \int p(\boldsymbol{x}) d\mu = \min_{\mu} \sum_{\boldsymbol{\alpha}} p_{\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}} \tag{9}$$

$$\text{subject to } m_{\boldsymbol{\alpha}} \doteq \int_K \boldsymbol{x}^{\boldsymbol{\alpha}} d\mu \tag{10}$$

where $m_{\boldsymbol{\alpha}}$ denotes the $\boldsymbol{\alpha}$th moment of the measure μ. Under mild conditions (Lasserre, 2001), condition (10) is equivalent to a semi-definite set of constraints of the form

$$\mathbf{M}(\boldsymbol{m}) \succeq 0, \; \mathbf{L}(g_k \boldsymbol{m}) \succeq 0, k = 1, \ldots N \tag{11}$$

Here $\boldsymbol{m}$ denotes the moment sequence $\{m_{\boldsymbol{\alpha}}\}$ ordered by the index $\boldsymbol{\alpha}_i$, arranged in a grevlex order. The entries of the (infinite dimensional) moment $\mathbf{M}(\boldsymbol{m})$ and localization matrices $\mathbf{L}(g_k \boldsymbol{m})$, are given by

$$\begin{aligned} \mathbf{M}(\boldsymbol{m})(i,j) &= m_{\boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j} \\ \mathbf{L}(g_k \boldsymbol{m})(i,j) &= \sum_{\boldsymbol{\beta}} g_{k,\boldsymbol{\beta}} \, m_{\boldsymbol{\beta} + \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_j}, \quad k = 1, \ldots, N \end{aligned} \tag{12}$$

where $g_{k,\boldsymbol{\beta}}$ are the coefficients of the kth polynomial that defines the set K. Thus, Problem (9)–(10) is convex in $m_{\boldsymbol{\alpha}}$, albeit infinite dimensional. A sequence of finite dimensional convex relaxations with cost $p_{\boldsymbol{m}}^d \uparrow p^*$ can be obtained by replacing the matrices in (11) by truncated matrices $\mathbf{M}_d(\boldsymbol{m})$, $\mathbf{L}_d(g_k \boldsymbol{m})$ containing moments of order up to 2d. Further, if for some d the solution to the problem above satisfies (Lasserre, 2009, Theorem 3.11)

$$\operatorname{rank}[\mathbf{M}_d(m_{\boldsymbol{\alpha}})] = \operatorname{rank}\left[\mathbf{M}_{d - \max(\frac{\deg(g_k(\mathbf{x}))}{2})}(m_{\boldsymbol{\alpha}})\right] \tag{13}$$

then the relaxation is exact, that is $p_{\mathbf{m}}^d = p^*$.

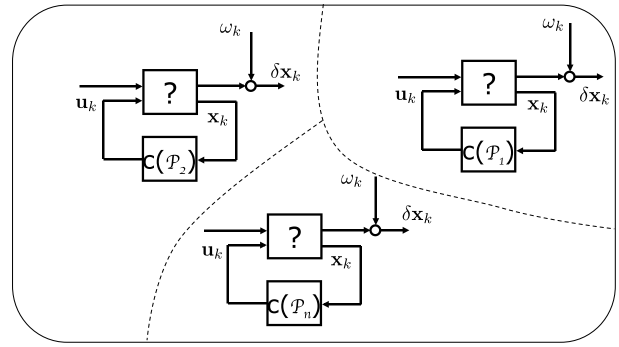

Fig. 1. Setup for switched data-driven control synthesis.

Remark 1. In the case of QCQP of the form (7), the lowest order relaxation of (9)–(11) corresponds to $d = 1$. In this case the objective is given by $p(\boldsymbol{m}) = \text{Trace}(\mathbf{Q}_o \mathbf{M}_1)$ and each localizing matrix reduces to a scalar of the form $\mathbf{L}_k(\boldsymbol{m}) = \text{Trace}(\mathbf{Q}_k \mathbf{M}_1)$. If the solution to this relaxation satisfies $\text{rank}[\mathbf{M}_1] = 1$, it can be easily shown that it is indeed exact. We will exploit this property in Section 3 to obtain a computationally tractable algorithm to synthesize data-driven controllers.

2.6. Exploiting sparsity in polynomial optimization

In this section we briefly recall some results on sparse polynomial optimization. The starting point is to associate to the optimization problem (8) the so called *correlative sparsity* graph $\mathcal{G} \doteq (\mathcal{V}, \mathcal{E})$, where each vertex corresponds to a variable x_i and where there is an edge between x_i, x_j if these variables appear in the same constraint or the product $x_i x_j$ appears in the objective function. Let $\mathcal{G}_c$ denote the chordal completion of $\mathcal{G}$, and $\mathcal{C}_i$ a set of maximal cliques of $\mathcal{G}_c$. Then, as shown in Lasserre (2006) and Waki, Kim, Kojima, and Muramatsu (2006) it is possible to construct a hierarchy of semidefinite programs of smaller size by projecting the constraints (11) onto the cliques $\mathcal{C}_i$, that is, replacing the matrices in (12) by matrices of the form $\mathbf{E}_i^T \mathbf{M}_d(\boldsymbol{m}) \mathbf{E}_i$ and $\mathbf{E}_i^T \mathbf{L}_d(g_k \boldsymbol{m}) \mathbf{E}_i$, $i = 1, \ldots, n_c$, where the matrix $\mathbf{E}_i$ selects the moments corresponding to variables in $\mathcal{C}_i$. As shown in Waki et al. (2006), for QCQP problems, the sparse and dense moment-based relaxations of (7) corresponding to $d = 1$ achieve the same optimal value. However, for a problem with N_v variables and where the correlative sparsity graph has n_c cliques of roughly the same size, the $d = 1$ sparse relaxation has computational complexity roughly $O(n_c (\frac{N_v}{n_c})^6)$ compared against $O(N_v^6)$ for the dense one.

2.7. Statement of the problem

Consider the system shown in Fig. 1, composed of N unknown sub-systems. The goal of this paper is to synthesize a switched state feedback controller that stabilizes, under arbitrary switching, all possible plants compatible with the observed experimental input/output data and some minimal a-priori information about the system. This problem can be formally stated as:

Problem 1. Consider a switched system composed of N LTI sub-systems, each described by a state space model of the form:

$$\begin{aligned} \delta\mathbf{x}_k &= \mathbf{A}_{i_k} \mathbf{x}_k + \mathbf{B}_{i_k} \mathbf{u}_k + \boldsymbol{\omega}_k \\ \text{where } \delta\mathbf{x}_k &= \begin{cases} \mathbf{x}_{k+1} & \textit{for discrete time systems} \\ \dot{\mathbf{x}}_k & \textit{for continuous time systems} \end{cases} \end{aligned} \tag{14}$$

and where $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{u}_k \in \mathbb{R}^m$, and $\boldsymbol{\omega}_k \in \mathbb{R}^n$, denote the state, input and process noise, and where i_k denotes the active sub-system at time k. Given experimental data $\{\mathbf{u}_k, \mathbf{x}_k, \delta\mathbf{x}_k, i_k\}_{k=0}^{K}$, collected from an experiment where each sub-system is sufficiently excited, find a switched state feedback controller $\mathbf{F}_i$, $i = 1, \ldots, N$ such that $\mathbf{x} = 0$ is a globally asymptotically stable equilibrium point of the closed-loop system

$$\delta\mathbf{x}_k = (\mathbf{A}_{i_k} + \mathbf{B}_{i_k}\mathbf{F}_{i_k})\mathbf{x}_k \tag{15}$$

for any arbitrary switching sequence $i_k \in \{1, \ldots, N\}$ and for all pairs $(\mathbf{A}_{i_k}, \mathbf{B}_{i_k})$ compatible with the observed experimental data.

The main result of this paper, presented in the next section, shows that, for the case of ℓ_∞ bounded noise, the problem above can be recast, through the use of duality, into a polynomial optimization form, which in turn can be relaxed to a SDP using the results from Section 2.5.

3. A convex formulation

In this section we present the main result of the paper: a convex reformulation of Problem 1. As briefly mentioned in the introduction, the key idea is to consider two sets in the parameter space: (i) the set of all plants compatible with a-priori information and experimental data, and (ii) the set of all plants compatible with the a-priori information that admit a CPCLF (the jointly stabilizable set). Problem 1 then reduces to finding a CPCLF (and associated controller) such that the consistency set is included in the jointly stabilizable set. For notational simplicity, we will address the discrete time case first and then indicate how to solve the continuous time case by exploiting the discrete time results.

3.1. Reformulation as a polynomial optimization via duality

Given experimental data $(\mathbf{x}_k, \mathbf{u}_k, \delta\mathbf{x}_k, i_k)$, where i_k denotes the label of the sub-system active at time k, and a bound ϵ on the ℓ_∞ norm of the process noise (e.g. $\|\boldsymbol{\omega}_k\|_\infty \le \epsilon$), define the consistency sets $\mathcal{P}_i$, $i = 1, \ldots, N$ as the set of all pairs $(\mathbf{A}_i, \mathbf{B}_i)$ compatible with the information collected when the ith system was active. From (14) it follows that each set $\mathcal{P}_i$ is a polyhedron of the form:

$$\begin{aligned}\mathcal{P}_i \doteq \{\mathbf{A}_i, \mathbf{B}_i : \|\mathbf{x}_{t_k^{(i)}+1} - \mathbf{A}_i\mathbf{x}_{t_k^{(i)}} - \mathbf{B}_i\mathbf{u}_{t_k^{(i)}}\|_\infty \le \epsilon,\\ k = 1, \ldots, K_i\}\end{aligned} \tag{16}$$

where $t_k^{(i)}$, $k = 1, \ldots, K_i$ denotes the times at which the ith system was active and K_i indicates the number of samples generated by this sub-system. Direct application of (1) leads, after some manipulations to eliminate the infinity norm, to the equivalent expression:

$$\begin{aligned}\mathcal{P}_i \doteq \Bigg\{ \boldsymbol{a}_i, \boldsymbol{b}_i : \begin{bmatrix} \mathbf{x}_{t_k^{(i)}}^T \otimes \mathbf{I}_n & \mathbf{u}_{t_k^{(i)}}^T \otimes \mathbf{I}_n \\ -\mathbf{x}_{t_k^{(i)}}^T \otimes \mathbf{I}_n & -\mathbf{u}_{t_k^{(i)}}^T \otimes \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_i \\ \boldsymbol{b}_i \end{bmatrix} \\ \le \begin{bmatrix} \epsilon + \mathbf{x}_{t_k^{(i)}+1} \\ \epsilon - \mathbf{x}_{t_k^{(i)}+1} \end{bmatrix}, \; k = 1, \ldots K_i \Bigg\}\end{aligned} \tag{17}$$

where $\mathbf{a}_i \doteq \mathbf{vec}(\mathbf{A}_i)$ and $\mathbf{b}_i \doteq \mathbf{vec}(\mathbf{B}_i)$. From Theorem 1 it follows that Problem 1 is equivalent to:

Problem 2. Find a full column rank matrix $\mathbf{V}$, and N matrices $\mathbf{F}_i$ such that, for all pairs $(\mathbf{A}_i, \mathbf{B}_i) \in \mathcal{P}_i$, there exists a matrix function $\mathbf{H}_i(\mathbf{A}_i, \mathbf{B}_i, \mathbf{F}_i)$ such that

$$\begin{aligned}&\mathbf{V}\,(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i) = \mathbf{H}_i(\mathbf{A}_i, \mathbf{B}_i, \mathbf{F}_i)\mathbf{V}\\ &\|\mathbf{H}_i(\mathbf{A}_i, \mathbf{B}_i, \mathbf{F}_i)\|_\infty < 1, \; \text{for } i = 1, \ldots, N.\end{aligned} \tag{18}$$

Note that feasibility of the problem above is not guaranteed, even if the underlying system is switched stabilizable, if the data is not "rich" enough, e.g. the consistency set is too large. To rule out this situation, we will make the following assumption:

Assumption 1 (*Data Richness*). Enough data have been collected such that all systems in the consistency set are switched stabilizable by a single switched state feedback controller $\mathbf{F}_i$.

The (unknown) functional dependence of $\mathbf{H}_i$ on $(\mathbf{A}_i, \mathbf{B}_i, \mathbf{F}_i)$, together with the fact that (18) must hold for all pairs $(\mathbf{A}_i, \mathbf{B}_i)$ in the corresponding consistency set renders Problem 2 extremely challenging. Thus, in the sequel, we will replace (18) with the stronger condition:

$$\|\mathbf{V}\,(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i)\,\mathbf{V}^\dagger\|_\infty \le \lambda < 1, \; i = 1, \ldots, N \tag{19}$$

where $\mathbf{V}^\dagger$ denotes the left inverse of $\mathbf{V}$ and λ is the rate of convergence of the switched system. This allows for recasting Problem 2 as the following robust optimization:

Problem 3. Find a full column rank matrix $\mathbf{V}$, and N matrices $\mathbf{F}_i$ such that, for all pairs $(\mathbf{A}_i, \mathbf{B}_i) \in \mathcal{P}_i$, (19) holds.

Remark 2. It can be shown that (19) and the fact that $\mathbf{V}$ has full column rank, together with a discrete time version of Theorem 4.19 in Khalil (2002), imply that $\|\mathbf{V}\mathbf{x}\|_\infty$ is an input to state (ISS) Lyapunov function. Hence the switched control law $\mathbf{u}_k = F_{i_k}\mathbf{x}_k$ renders the system (14) ISS for any arbitrary switching sequence.

As we show in the sequel, while this relaxation provides only sufficient conditions[1] for the existence of a switched gain $\mathbf{F}_i$ that solves the DDC problem, it has the advantage of reducing to a polynomial optimization problem that can be efficiently solved using the techniques outlined in Section 2.

For given $\mathbf{V}$, $\mathbf{F}_i$, λ, define the polyhedron

$$\mathcal{P}_{\mathbf{V},\mathbf{F}_i} \doteq \{\mathbf{A}_i, \mathbf{B}_i : \|\mathbf{V}(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger\|_\infty \le \lambda < 1\} \tag{20}$$

in this context, Problem 3 can be restated as finding a matrix $\mathbf{V}$ and N gains $\mathbf{F}_i$ such that

$$\mathcal{P}_i \subseteq \mathcal{P}_{\mathbf{V},\mathbf{F}_i}, \; i = 1, \ldots N \tag{21}$$

The main idea is of the paper is to enforce (21) through the use of duality and the extended Farkas' Lemma (Henrion et al., 1999). However, pursuing this approach requires a characterization of $\mathcal{P}_{\mathbf{V},\mathbf{F}_i}$ that does not involve the $\|.\|_\infty$.

Theorem 2. *Denote by $t_k^{(i)}$, $k = 1, \ldots, K_i$ the time instants where the ith sub-system is active. Let:*

$$\mathcal{X}_i \doteq \begin{bmatrix} \mathbf{x}_{t_1^{(i)}}^T \otimes \mathbf{I}_n \\ \vdots \\ \mathbf{x}_{t_K^{(i)}}^T \otimes \mathbf{I}_n \end{bmatrix}, \mathcal{U}_i \doteq \begin{bmatrix} \mathbf{u}_{t_1^{(i)}}^T \otimes \mathbf{I}_n \\ \vdots \\ \mathbf{u}_{t_K^{(i)}}^T \otimes \mathbf{I}_n \end{bmatrix}, \boldsymbol{\xi}_i \doteq \begin{bmatrix} \mathbf{x}_{t_k^{(i)}+1} \\ \vdots \\ \mathbf{x}_{t_K^{(i)}+1} \end{bmatrix} \tag{22}$$

Given a (full column rank) matrix $\mathbf{V} \in \mathbb{R}^{p\times n}$, there exist switched feedback gains $\mathbf{F}_i$ such that

$$\|\mathbf{V}(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger\|_\infty \le \lambda \tag{23}$$

for all pairs $(\mathbf{A}_i, \mathbf{B}_i) \in \mathcal{P}_i$, $i = 1, \ldots, N$ if and only if there exist N matrices $\mathbf{F}_i \in \mathbb{R}^{m\times n}$ and 2^pN matrices $\mathbf{Y}_{\mathbf{S},i} \in \mathbb{R}^{p\times 2nK_i}$, $\mathbf{Y}_{\mathbf{S},i} \ge 0$ such that

$$\mathbf{Y}_{\mathbf{S},i}\begin{bmatrix} \mathcal{X}_i & \mathcal{U}_i \\ -\mathcal{X}_i & -\mathcal{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{S}\big((\mathbf{V}^\dagger)^T \otimes \mathbf{V}\big) & \mathbf{S}\big((\mathbf{F}_i\mathbf{V}^\dagger)^T \otimes \mathbf{V}\big) \end{bmatrix}$$

[1] Indeed these conditions are necessary and sufficient when there exist a CPCLF generated by a matrix $\mathbf{V} \in \mathbb{R}^{n\times n}$. Condition (19) allows for considering more general, "tall" matrices.

$$\mathbf{Y}_{\mathbf{S},i}\begin{bmatrix}\boldsymbol{\xi}_i+\epsilon\mathbb{1}\\-\boldsymbol{\xi}_i+\epsilon\mathbb{1}\end{bmatrix}\leq\lambda\mathbb{1} \tag{24}$$

holds for all matrices $\mathbf{S}\in\mathbb{R}^{p\times p^2}$ *of the form:*

$$\mathbf{S}=\mathbf{s}\otimes\mathbf{I}_p \tag{25}$$

where $\mathbf{s}\in\mathbb{R}^{1\times p}$ *is a vector with elements* $s_j=\pm1$.

Proof. Begin by noting that $(\mathbf{A}_i,\mathbf{B}_i)\in\mathcal{P}_i$ satisfies (23) if and only $\mathcal{P}_i\subseteq\mathcal{P}_{\mathbf{V},\mathbf{F}_i}$ defined in (20). The main idea of the proof is to use the extended Farkas' Lemma to enforce this inclusion. To this effect, we first need to eliminate the $\|.\|_\infty$ from (20). For generic vectors $\mathbf{v},\mathbf{s}\in\mathbb{R}^p$, with $\|\mathbf{s}\|_\infty=1$ we have that $|\mathbf{s}^T\mathbf{v}|\leq\|\mathbf{s}\|_\infty\|\mathbf{v}\|_1=\|\mathbf{v}\|_1$, with the inequality saturated when $s_j=\text{sign}(v_j)$. Applying this inequality to each row of the matrix $\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger$ we have that

$$\begin{aligned}&\|\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger\|_\infty\leq\lambda \iff\\&\sum_j|(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)_{\ell j}|\leq\lambda \iff\\&\sum_j s_j(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)_{\ell j}\leq\lambda \text{ for all } s_j=\pm1\end{aligned} \tag{26}$$

Next, write the sum of the entries of the ℓ^{th} row of the matrix with elements $s_j(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)_{\ell j}$ as

$$(\mathbf{s}\otimes\mathbf{I}_\ell)\mathbf{vec}(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)$$

where $\mathbf{I}_\ell$ denotes the ℓ^{th} row of $\mathbf{I}$. Combining the expressions above leads to:

$$\begin{aligned}&\|\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger\|_\infty\leq\lambda \iff\\&\mathbf{S}\,\mathbf{vec}(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)\leq\lambda\mathbb{1}\end{aligned} \tag{27}$$

for all matrices $\mathbf{S}\in\mathbb{R}^{p\times p^2}$ of the form (25). Applying the equality (1) to express $\mathbf{vec}(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)$ in terms of $\mathbf{a}_i\doteq\mathbf{vec}(\mathbf{A}_i)$ and $\mathbf{b}_i\doteq\mathbf{vec}(\mathbf{B}_i)$ leads to the following alternative representation of $\mathcal{P}_{\mathbf{V},\mathbf{F}_i}$:

$$\begin{aligned}\mathcal{P}_{\mathbf{V},\mathbf{F}_i}\doteq\Big\{&\boldsymbol{a}_i,\boldsymbol{b}_i:\mathbf{S}\big((\mathbf{V}^\dagger)^T\otimes\mathbf{V}\big)\boldsymbol{a}_i+\mathbf{S}\big((\mathbf{F}_i\mathbf{V}^\dagger)^T\otimes\mathbf{V}\big)\boldsymbol{b}_i\\&\leq\lambda\mathbb{1}\text{ for all matrices }\mathbf{S}\in\mathbb{R}^{p\times p^2}\text{of the form (25)}\Big\}\end{aligned}$$

The desired result follows from applying Farkas' Lemma to enforce that for all i, the polyhedra $\mathcal{P}_i$ defined in (17) satisfy $\mathcal{P}_i\subseteq\mathcal{P}_{\mathbf{V},\mathbf{F}_i}$. □

Corollary 1. *Problem 3 is equivalent to the following quadratically constrained quadratic feasibility problem: Find matrices* $\mathbf{V}\in\mathbb{R}^{p\times n}$, $\mathbf{Z}\in\mathbb{R}^{n\times p}$, *N matrices* $\mathcal{F}_i\in\mathbb{R}^{m\times p}$, 2^pN *matrices* $\mathbf{Y}_{\mathbf{S},i}\in\mathbb{R}^{p\times 2nK_i}$ *and a scalar* $0\leq\lambda<1$ *such that*

$$\begin{aligned}&\mathbf{Y}_{\mathbf{S},i}\begin{bmatrix}\mathcal{X}_i&\mathcal{U}_i\\-\mathcal{X}_i&-\mathcal{U}_i\end{bmatrix}=\begin{bmatrix}\mathbf{S}(\mathbf{Z}^T\otimes\mathbf{V})&\mathbf{S}(\mathcal{F}_i^T\otimes\mathbf{V})\end{bmatrix}\\&\mathbf{Y}_{\mathbf{S},i}\geq0\\&\mathbf{Z}\mathbf{V}=\mathbf{I}_n\\&\mathbf{Y}_{\mathbf{S},i}\begin{bmatrix}\boldsymbol{\xi}_i+\epsilon\mathbb{1}\\-\boldsymbol{\xi}_i+\epsilon\mathbb{1}\end{bmatrix}\leq\lambda\mathbb{1}\end{aligned} \tag{28}$$

holds for all matrices $\mathbf{S}\in\mathbb{R}^{p\times p^2}$ *of the form (25).*

Proof. Follows from Theorem 2 by defining $\mathbf{Z}\doteq\mathbf{V}^\dagger$, $\mathcal{F}_i\doteq\mathbf{F}_i\mathbf{V}^\dagger$ and noting that all the constraints in (28) are at most quadratic in the variables, and that, since $\mathbf{V}$ has full column rank, $\mathbf{F}_i$ can always be recovered from $\mathcal{F}_i$. □

From Corollary 1 it follows that the DDC control problem can be solved using the polynomial optimization techniques described in Section 2.5, by considering a sequence of SDPs of increasing dimension until a flat extension is achieved (Curto & L.A., 1998). Given the computational complexity entailed in considering higher-order relaxations, as outlined in Remark 1, an alternative is to minimize the rank of the moment matrix $\mathbf{M}$ corresponding to the $d=1$ relaxation of (28), through the use of a re-weighted Trace minimization (Fazel, Hindi, & Boyd, 2003). When this minimization results in a rank-1 solution $\mathbf{M}=\mathbf{v}\mathbf{v}^T$ then the desired gains can be obtained directly from $\mathbf{v}$.

3.2. A lower computational complexity formulation

While Theorem 2 and its Corollary provide a necessary and sufficient condition for the existence of a solution to Problem 3, the resulting optimization problem has a large number of constraints due to the need to consider 2^p matrices $\mathbf{S}$ with all possible sign vectors in $\mathbb{R}^p$. Thus, while these results are of theoretical interest, from a practical standpoint, their use is limited to relatively low order systems. In this section we discuss a reformulation of (24)–(25) that provides a solution to Problem 3 with substantially lower computational complexity. Recall that a set $\mathcal{X}_a\in\mathbb{R}^{n_x}\times\mathbb{R}^{n_u}$ is said to represent a set $\mathcal{X}\in\mathbb{R}^{n_x}$, if the projection of $\mathcal{X}_a$ is $\mathcal{X}$, in other words, if and only if for every $\mathbf{x}\in\mathcal{X}$ there exists some $\mathbf{u}\in\mathbb{R}^{n_u}$ such that $(\mathbf{x},\mathbf{u})\in\mathcal{X}_a$. As shown in Ben-Tal, Ghaoui, and Nemirovski (2009), $\mathcal{X}\doteq\{\mathbf{x}:\sum|x_i|\leq\lambda\}$ admits the equivalent representation: $\mathcal{X}_a\doteq\{(\mathbf{x},\mathbf{u}):\sum u_i\leq\lambda \text{ and } -u_i\leq x_i\leq u_i\}$. Using this idea, by adding p^2 variables $\mu_{\ell j}$, the second constraint in (26) can be written as:

$$|(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)_{\ell j}|\leq\mu_{\ell j}\text{ and} \tag{29a}$$

$$\sum_j\mu_{\ell j}\leq\lambda \tag{29b}$$

Combining this observation with Farkas' Lemma leads to the following result:

Theorem 3. *Problem 3 is equivalent to the existence of N matrices* $\mathbf{Y}_i\in\mathbb{R}^{2p^2\times 2nK_i}$, $\mathcal{F}_i\in\mathbb{R}^{m\times p}$, *matrices* $\mathbf{V}\in\mathbb{R}^{p\times n}$, $\mathbf{Z}\in\mathbb{R}^{n\times p}$, *a matrix* $\boldsymbol{\mu}\in\mathbb{R}^{p\times p}$, *and a scalar* $0\leq\lambda<1$ *such that the following conditions hold:*

$$\begin{aligned}&\mathbf{Y}_i\begin{bmatrix}\mathcal{X}_i&\mathcal{U}_i\\-\mathcal{X}_i&-\mathcal{U}_i\end{bmatrix}=\begin{bmatrix}\mathbf{Z}^T\otimes\mathbf{V}&\mathcal{F}_i^T\otimes\mathbf{V}\\-\mathbf{Z}^T\otimes\mathbf{V}&-\mathcal{F}_i^T\otimes\mathbf{V}\end{bmatrix}\\&\mathbf{Y}_i\geq0\\&\mathbf{Z}\mathbf{V}=\mathbf{I}_n\\&\mathbf{Y}_i\begin{bmatrix}\boldsymbol{\xi}_i+\epsilon\mathbb{1}\\-\boldsymbol{\xi}_i+\epsilon\mathbb{1}\end{bmatrix}\leq\begin{bmatrix}\mathbf{vec}(\boldsymbol{\mu})\\\mathbf{vec}(\boldsymbol{\mu})\end{bmatrix}\\&\boldsymbol{\mu}\mathbb{1}\leq\lambda\mathbb{1}\end{aligned} \tag{30}$$

Proof. Proceeding as in the proof of Theorem 2 it can be easily shown that the first four conditions in (30) are necessary and sufficient for the entries of $\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger$ to satisfy (29a) for all pairs $(\mathbf{A}_i,\mathbf{B}_i)\in\mathcal{P}_i$, while the last inequality enforces that

$$\begin{aligned}\|\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger\|_\infty&=\max_\ell\sum_j\big|(\mathbf{V}(\mathbf{A}_i+\mathbf{B}_i\mathbf{F}_i)\mathbf{V}^\dagger)_{\ell j}\big|\\&\leq\max_\ell\sum_j\mu_{\ell j}\leq\lambda\end{aligned}$$

□

As before, the polynomial feasibility problem (30) can be recast into a rank minimization and solved using a re-weighed heuristic approach, leading to Algorithm 1. Briefly, this heuristic seeks to minimize rank[$\mathbf{M}$] by iteratively minimizing the linearization of log determinant[$\mathbf{M}$] around the present solution (Fazel et al., 2003). The main difference between our approach and Fazel et al. (2003) is the use of the second largest singular value of $\mathbf{M}$ as a regularizer when updating $\mathbf{W}$, as opposed to a constant δ as in Fazel et al. (2003). Empirical evidence indicates that this adaptive heuristic works better than the original one when seeking rank-1 solutions. It is worth emphasizing that, in this context, Theorem 3 leads to substantial decrease in the computational complexity by avoiding the need for an exponential number of sign matrices $\mathbf{S}$. Indeed, while (28) involves (for each subsystem) $2^{p+1}pnK$ variables related to $\mathbf{Y}$ and $2^p p(n^2+nm)+p$ constraints, (30) requires only $p^2(4nK+1)$ variables $\mathbf{Y}$, $\boldsymbol{\mu}$ and $2p^2(n^2+nm+1)$ constraints. Thus, if using a standard interior point algorithm for solving the resulting semi-definite program, the complexity of using (28) scales as $\mathcal{O}(2^{4p}p^4)$ versus $\mathcal{O}(p^8)$ when using (30).

Algorithm 1 Reweighted Trace based DDC Design.

Initialize: $iter = 0, \mathbf{W}^{(0)} = I, \lambda < 1, \epsilon$
Collect: $\mathbf{x}_{t_k^{(i)}}, \mathbf{u}_{t_k^{(i)}}, \mathbf{x}_{t_k^{(i)}+1}$
Build: $\mathcal{X}_i, \mathcal{U}_i, \boldsymbol{\xi}_i$
Variables: $\mathbf{Y}_i$, $\boldsymbol{\mu}$ and $\boldsymbol{m}$: the moment sequence of $\mathbf{v} \doteq \left[1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathbf{Z})^T, \mathbf{vec}(\mathcal{F}_i)^T\right]^T$, i.e. $\mathbf{V}, \mathbf{Z}, \mathcal{F}_i$ depend on $\boldsymbol{m}$. We further define the moment matrix $\mathbf{M} \doteq \mathbf{v}\mathbf{v}^T$.
repeat
Solve $\min\limits_{\mathbf{Y}_i, \boldsymbol{\mu}, \boldsymbol{m}}$ Trace($\mathbf{W}^{(iter)}\mathbf{M}$) subject to:

$$\mathbf{M} \succeq 0 \quad (a)$$
$$\mathbf{M}(1,1) = 1 \quad (b)$$
$$\mathbf{Z}\mathbf{V} = \mathbf{I}_n \quad (c)$$
$$\boldsymbol{\mu}\mathbb{1} \le \lambda\mathbb{1} \quad (d)$$
and, for all $i = 1, \ldots, N$
$$\mathbf{Y}_i \ge 0 \quad (e)$$
$$\mathbf{Y}_i \begin{bmatrix} \mathcal{X}_i & \mathcal{U}_i \\ -\mathcal{X}_i & -\mathcal{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^T \otimes \mathbf{V} & \mathcal{F}_i^T \otimes \mathbf{V} \\ -\mathbf{Z}^T \otimes \mathbf{V} & -\mathcal{F}_i^T \otimes \mathbf{V} \end{bmatrix} \quad (f)$$
$$\mathbf{Y}_i \begin{bmatrix} \boldsymbol{\xi}_i + \epsilon\mathbb{1} \\ -\boldsymbol{\xi}_i + \epsilon\mathbb{1} \end{bmatrix} \le \begin{bmatrix} \mathbf{vec}(\boldsymbol{\mu}) \\ \mathbf{vec}(\boldsymbol{\mu}) \end{bmatrix} \quad (g)$$
Update
$$\mathbf{W}^{(iter+1)} = (\mathbf{M}^{(iter)} + \sigma_2(\mathbf{M}^{(iter)})I)^{-1}$$
$iter = iter + 1$
until rank[$\mathbf{M}$] = 1.
Factor $\mathbf{M} = \mathbf{v}\mathbf{v}^T$.
Return $\mathbf{Y}_i, \boldsymbol{\mu}, \mathbf{v} = \left[1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathbf{Z})^T, \mathbf{vec}(\mathcal{F}_i)^T\right]^T$.

3.3. Further computational complexity reduction exploiting sparsity

A large portion of the computational burden in Algorithm 1 stems from the large dimension of the matrix $\mathbf{M} \in \mathbb{R}^{q\times q}$, with $q = 2pn + Npm + 1$, appearing both in the objective and semi-definite constraints. Since the computational complexity of standard interior point methods scales as (number of variables)^3, it follows that the computational complexity of Algorithm 1 scales at least cubically with the number of systems. In this section we briefly outline how to reduce this computational complexity by appealing to an underlying sparsity. Note that the only elements of the matrix $\mathbf{M}$ explicitly appearing in the same constraint in (30) are those corresponding to products among variables in the sets

$$\mathcal{C}_o = \{1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathbf{Z})^T\}$$
$$\mathcal{C}_i = \{1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathcal{F}_i)^T\}$$

Indeed, it can be easily seen that $\mathcal{C}_j$ are maximal cliques in the associated correlative sparsity graph. Thus, from Section 2.6 it follows that the constraint $\mathbf{M} \succeq 0$ can be replaced by the $N+1$ smaller constraints $\mathbf{M}_j \succeq 0$, where $\mathbf{M}_j$ only contains moments of the variables in $\mathcal{C}_j$, $j = 0, \ldots, N$. Further, from the results in Miller, Zheng, Roig-Solvas, Sznaier, and Papachristodoulou (2019) on rank minimization in the presence of chordal sparsity, it follows that the objective function in Algorithm 1 can also be decomposed into a sum of terms, each depending only on $\mathbf{M}_i$. These observations lead to Algorithm 2, whose computational complexity scales linearly with the number of sub-systems.

Algorithm 2 Sparse Reweighted Trace based DDC.

Initialize: $iter = 0, \mathbf{W}_i^{(0)} = I, i = 0, \ldots, N, \lambda < 1, \epsilon$
Collect: $\mathbf{x}_{t_k^{(i)}}, \mathbf{u}_{t_k^{(i)}}, \mathbf{x}_{t_k^{(i)}+1}$
Build: $\mathcal{X}_i, \mathcal{U}_i, \boldsymbol{\xi}_i$
Variables: $\mathbf{Y}_i$, $\boldsymbol{\mu}$, $\boldsymbol{m}_i$: the moment sequence of $\mathbf{v}_i \doteq \left[1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathbf{Z})^T\right]^T$ for $i = 0$ and $\mathbf{v}_i \doteq \left[1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathcal{F}_i)^T\right]^T$ for $i = 1, \ldots, N$, i.e. $\mathbf{V}, \mathbf{Z}, \mathcal{F}_i$ depend on $\boldsymbol{m}_i$. We further define the moment matrix $\mathbf{M}_i = \mathbf{v}_i\mathbf{v}_i^T$.
repeat
Solve $\min\limits_{\mathbf{Y}_i, \boldsymbol{\mu}, \boldsymbol{m}_i}$ $\sum_i h_i$ Trace($\mathbf{W}_i^{(iter)}\mathbf{M}_i$) subject to:

$$\mathbf{M}_i \succeq 0 \quad (a)$$
$$\mathbf{M}_i(1,1) = 1 \quad (b)$$
$$\mathbf{Z}\mathbf{V} = \mathbf{I}_n \quad (c)$$
$$\boldsymbol{\mu}\mathbb{1} \le \lambda\mathbb{1} \quad (d)$$
and, for all $i = 1, \ldots, N$
$$\mathbf{Y}_i \ge 0 \quad (e)$$
$$\mathbf{Y}_i \begin{bmatrix} \mathcal{X}_i & \mathcal{U}_i \\ -\mathcal{X}_i & -\mathcal{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^T \otimes \mathbf{V} & \mathcal{F}_i^T \otimes \mathbf{V} \\ -\mathbf{Z}^T \otimes \mathbf{V} & -\mathcal{F}_i^T \otimes \mathbf{V} \end{bmatrix} \quad (f)$$
$$\mathbf{Y}_i \begin{bmatrix} \boldsymbol{\xi}_i + \epsilon\mathbb{1} \\ -\boldsymbol{\xi}_i + \epsilon\mathbb{1} \end{bmatrix} \le \begin{bmatrix} \mathbf{vec}(\boldsymbol{\mu}) \\ \mathbf{vec}(\boldsymbol{\mu}) \end{bmatrix} \quad (g)$$
Update
$$\mathbf{W}_i^{(iter+1)} = (\mathbf{M}_i^{(iter)} + \sigma_2(\mathbf{M}_i^{(iter)})I)^{-1}$$
$$\mathbf{W}_i^{(iter+1)} = \frac{\mathbf{W}_i}{\|\mathbf{W}_i\|_2}$$
$iter = iter + 1$
until rank[$\mathbf{M}_i$] = 1.
Factor $\mathbf{M}_i = \mathbf{v}_i\mathbf{v}_i^T$.
Return $\mathbf{Y}_i, \boldsymbol{\mu}, \mathbf{v}_i = \left[1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathbf{Z})^T\right]^T, i = 0.$

$\mathbf{v}_i = \left[1, \mathbf{vec}(\mathbf{V})^T, \mathbf{vec}(\mathcal{F}_i)^T\right]^T, i = 1, \ldots, N.$

4. The continuous time case

In this section we briefly address the continuous time DDC by reducing it to a discrete time equivalent. To this effect we need the following assumption:

Assumption 2 (*Non-zenoness*)**.** The number of switches is finite on every finite interval.

Start by noting that, if $\mathbf{V}$ is a full column rank matrix such that:

$$\|\mathbf{V}\big((\mathbf{I}_n + \tau\mathbf{A}_i) + \tau\mathbf{B}_i\mathbf{F}_i\big)\mathbf{V}^\dagger\|_\infty \le \lambda < 1,\ i = 1, \ldots, N \quad (31)$$

then

$$\mathbf{H}_i \doteq \frac{\mathbf{V}\big((\mathbf{I}_n + \tau\mathbf{A}_i) + \tau\mathbf{B}_i\mathbf{F}_i\big)\mathbf{V}^\dagger - \mathbf{I}_p}{\tau}$$

together with Assumption 2, certifies closed-loop switched stability. This follows from the fact that condition (5) guarantees that the closed-loop vector field at time t points in a direction of descent of $\|\mathbf{V}\mathbf{x}_t\|$. Let $\mathbf{A_e} \doteq \mathbf{I}_n + \tau\mathbf{A}$, $\mathbf{B_e} \doteq \tau\mathbf{B}$. In this context, the continuous time DDC problem reduces to finding a scalar τ, a matrix $\mathbf{V}$ and gains $\mathbf{F}_i$ such that for all pairs $(\mathbf{A}_e, \mathbf{B}_e)$ in the consistency set, $\|\mathbf{V}(\mathbf{A}_e+\mathbf{B}_e\mathbf{F}_i)\mathbf{V}^\dagger\|_\infty < 1$. Next, use the fact that

$$\begin{aligned}&\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \boldsymbol{\omega}_t \iff \\ &\tau\dot{\mathbf{x}}_t + \mathbf{x}_t = (\mathbf{I}_n + \tau\mathbf{A})\mathbf{x}_t + \tau\mathbf{B}\mathbf{u}_t + \tau\boldsymbol{\omega}_t \iff \\ &\tau\dot{\mathbf{x}}_t + \mathbf{x}_t = \mathbf{A}_e\mathbf{x}_t + \mathbf{B}_e\mathbf{u}_t + \tau\boldsymbol{\omega}_t\end{aligned}$$

to rewrite the consistency set as:

$$\begin{aligned}\mathcal{P}_i = \Bigg\{ \boldsymbol{a}_i, \boldsymbol{b}_i : &\begin{bmatrix} \mathbf{x}^T_{t_k^{(i)}} \otimes \mathbf{I}_n & \mathbf{u}^T_{t_k^{(i)}} \otimes \mathbf{I}_n \\ -\mathbf{x}^T_{t_k^{(i)}} \otimes \mathbf{I}_n & -\mathbf{u}^T_{t_k^{(i)}} \otimes \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_i \\ \boldsymbol{b}_i \end{bmatrix} \\ &\le \begin{bmatrix} \tau\epsilon + \tau\dot{\mathbf{x}}_{t_k^{(i)}} + \mathbf{x}_{t_k^{(i)}} \\ \tau\epsilon - \tau\dot{\mathbf{x}}_{t_k^{(i)}} - \mathbf{x}_{t_k^{(i)}} \end{bmatrix}, \ k = 1, \dots, K_i \Bigg\}\end{aligned}$$

Application of Farkas' lemma leads to the following results, whose proof, omitted for brevity, follows along the lines of the proofs of Theorems 2 and 3:

Theorem 4. *The continuous version of Problem 3 is equivalent to the existence of N matrices* $\mathbf{Y}_i \in \mathbb{R}^{2p^2\times 2nK}$, $\mathcal{F}_i \in \mathbb{R}^{m\times p}$, *matrices* $\mathbf{V} \in \mathbb{R}^{p\times n}$, $\mathbf{Z} \in \mathbb{R}^{n\times p}$, *a matrix* $\boldsymbol{\mu} \in \mathbb{R}^{p\times p}$ *and scalars* $0 \le \lambda < 1$, $\tau > 0$ *such that the following conditions hold:*

$$\begin{aligned}&\mathbf{Y}_i \begin{bmatrix} \mathcal{X}_i & \mathcal{U}_i \\ -\mathcal{X}_i & -\mathcal{U}_i \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^T \otimes \mathbf{V} & \mathcal{F}_i^T \otimes \mathbf{V} \\ -\mathbf{Z}^T \otimes \mathbf{V} & -\mathcal{F}_i^T \otimes \mathbf{V} \end{bmatrix} \\ &\mathbf{Y}_i \ge 0 \\ &\mathbf{Z}\mathbf{V} = \mathbf{I}_n \\ &\mathbf{Y}_i \begin{bmatrix} \boldsymbol{\xi}_i + \tau\epsilon\mathbb{1} \\ -\boldsymbol{\xi}_i + \tau\epsilon\mathbb{1} \end{bmatrix} \le \begin{bmatrix} \mathbf{vec}(\boldsymbol{\mu}) \\ \mathbf{vec}(\boldsymbol{\mu}) \end{bmatrix} \\ &\boldsymbol{\mu}\mathbb{1} \le \lambda\mathbb{1}\end{aligned} \tag{32}$$

(32) leads to polynomial optimization problems with an additional variable τ, that can be solved using a slightly modified version of Algorithms 1 and 2. An alternative is to simply perform a line search on τ until a feasible solution is found.

5. Illustrative examples

In this section, we present an academic example that illustrates the proposed approach and its performance as a function of noise level and number of data points, and a practically motivated one, control of the horizontal dynamics of a quadcopter. To reduce the computational burden, in all cases we used a square matrix $\mathbf{V}$. All the optimization problems were solved using a combination of CVX (Grant & Boyd, 2014) and Sedumi (Sturm, 1999) in MATLAB.

Example 1 (*Data-driven Control of Discrete Systems*). This example illustrates the effectiveness of Algorithms 1 and 2 using a Monte Carlo approach. We generated 100 switching systems using the drss command. Each switching system contained three subsystems of size $n = 3$, $m = 2$. All system matrices have eigenvalues constrained to $(-1.1,\ 1.1)$. This enabled the subsystems to be slightly unstable while keeping the trajectories at reasonable scales. For benchmarking purposes, we used the model-based method proposed in Blanchini et al. (2009) to verify whether the generated systems were switching stabilizable. Specifically, we checked feasibility of Eq. (4) in Blanchini et al. (2009). To apply the proposed data-driven design, we generated trajectories starting from an initial state and input uniformly distributed in $(-1,\ 1)$. Finally, the state was corrupted with a random noise $\boldsymbol{\omega}_k$, where $\|\boldsymbol{\omega}_k\|_\infty \le \epsilon$. We considered different noise levels $\epsilon = [0,\ 0.05,\ 0.1,\ 0.15]$ and collected $K = 120$ samples for each experiment. Applying Algorithm 1 with $\lambda = 0.99$ led to Table 1. Note that for the noiseless case, the relaxation of (30) described in Algorithm 1 successfully found a stabilizing controller in 99% of the cases. As the noise increases, the success rate decreases, since the consistency set becomes larger and thus a controller that robustly stabilizes this set may not exist. This issue can be alleviated by collecting more samples. Table 2 shows the effect of the number of samples on the design with fixed $\epsilon = 0.1$, $\lambda = 0.99$. Next we show that exploiting sparsity leads to a faster algorithm. Applying Algorithm 2 on the same data with $K = 120$, $\epsilon = 0.1$, and choosing $h = [10, 1, 1, 1]$, $\lambda = 0.99$ reduced the mean computational time to 49.7481s from 55.1688s for Algorithm 1.

Table 1
Number of successful designs.

ϵ	0	0.05	0.1	0.15
# of success	99	92	85	70

Table 2
Number of successful designs.

K	60	120	180	240
# of success	58	85	89	91

Remark 3. In addition to the reduction in computational time, a second advantage of exploiting the sparse chordal decomposition is the ability to find a faster closed–loop system, by selecting a smaller λ which may be infeasible for Algorithm 1. This stems from the fact that, when using the chordal decomposition, only the submatrices of the moment matrix corresponding to cliques in the chordal graph are subject to the rank 1 constraint (Miller et al., 2019).

An important feature of the proposed controller is its ability to reject persistent disturbances. Assume that the closed-loop switched system is affected by a ℓ_∞ bounded disturbance $\boldsymbol{\omega}_k$, that is:

$$\mathbf{x}_{k+1} = (\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i)\mathbf{x}_k + \boldsymbol{\omega}_k \tag{33}$$

then

$$\begin{aligned}||\mathbf{V}\mathbf{x}_{k+1}||_\infty &\le ||\mathbf{V}(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i)\mathbf{x}_k||_\infty + ||\mathbf{V}\boldsymbol{\omega}_k||_\infty \\ &= ||(\mathbf{V}(\mathbf{A}_i + \mathbf{B}_i\mathbf{F}_i))\mathbf{V}^\dagger\mathbf{V}\mathbf{x}_k||_\infty + ||\mathbf{V}\boldsymbol{\omega}_k||_\infty \\ &\le \lambda||\mathbf{V}\mathbf{x}_k||_\infty + ||\mathbf{V}\boldsymbol{\omega}_k||_\infty\end{aligned} \tag{34}$$

Thus, if $\|\mathbf{V}\mathbf{x}_k\|_\infty \le \frac{\|\mathbf{V}\boldsymbol{\omega}_k\|_\infty}{1-\lambda}$, then

$$||\mathbf{V}\mathbf{x}_{k+1}||_\infty \le (\frac{\lambda}{1-\lambda} + 1)||\mathbf{V}\boldsymbol{\omega}_k||_\infty = \frac{||\mathbf{V}\boldsymbol{\omega}_k||_\infty}{1-\lambda} \tag{35}$$

It follows that the set

$$\mathcal{S} \doteq \left\{ x : ||\mathbf{V}\mathbf{x}_k||_\infty \le \nu \doteq \frac{||\mathbf{V}\boldsymbol{\omega}_k||_\infty}{1-\lambda} \right\} \tag{36}$$

is positively invariant. Therefore, any trajectory starting in $\mathcal{S}$ is uniformly bounded by $\|\mathbf{x}_k\|_\infty \le M$, where $M = max\|\mathbf{x}_k\|_\infty$ subject to $\|\mathbf{V}\mathbf{x}_k\|_\infty \le \nu$. This feature is illustrated in Fig. 2, showing the value of $\|\mathbf{V}\mathbf{x}\|_\infty$ for a sample closed-loop trajectory from one of the random systems used in this example, corresponding to a design with $\lambda = 0.5$, excited with a random disturbance $\boldsymbol{\omega}_k$ with components ± 0.1.

Example 2 (*Data-driven Control of Continuous Time System*). In this example we consider the horizontal motion of a Quanser Qball-X4 quadrotor. The motion dynamics in the X direction are described by the following continuous model:

$$\begin{bmatrix} \dot{X} \\ \ddot{X} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{4Ksin(\theta)}{M} \\ 0 & 0 & -\omega \end{bmatrix} \begin{bmatrix} X \\ \dot{X} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} u_Z \tag{37}$$

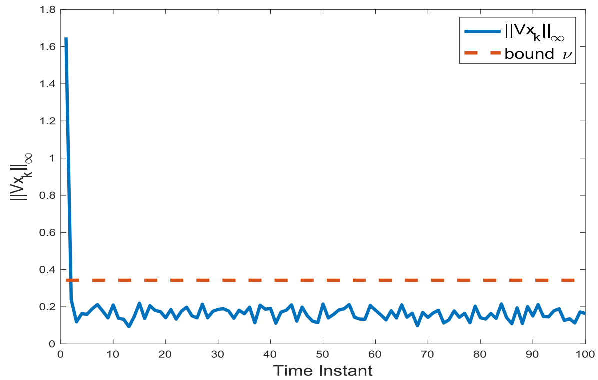

Fig. 2. Sample evolution of $\|\mathbf{V}\mathbf{x}\|_\infty$ in the presence of a persistent disturbance.

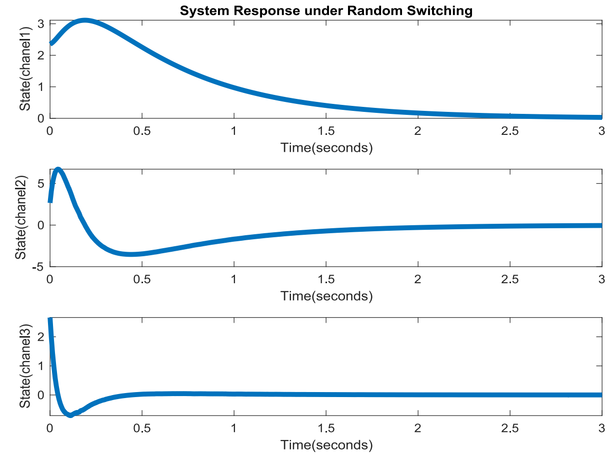

Fig. 3. Quadcopter state trajectories corresponding to a random initial condition and random switching.

where $K = 120$ N, $M = 1.4$ kg, $\omega = 15$ rad/s are parameters given in Educate (2007) and θ is the pitch angle. Note that when θ changes, we have different model representing the flight status of the quadcopter, hence it can be viewed as a switched system. We consider the case where the quadcopter has two flying modes with $\theta = 10^\circ$ and 20°.

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 59.54 \\ 0 & 0 & -15 \end{bmatrix} \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix} \tag{System 1}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 117.3 \\ 0 & 0 & -15 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix} \tag{System 2}$$

The experimental data was generated by exciting the switched system with a random input $\mathbf{u}_k$, where $\|\mathbf{u}_k\|_\infty \le 1$ and random process noise $\boldsymbol{\omega}_k$, where $\|\boldsymbol{\omega}_k\|_\infty \le 0.1$. Applying the continuous time version of Algorithm 1 with $\lambda = 0.99$ and $\tau = 0.04$ leads, in 48 s, to the following controller:

$$\mathbf{F}_1 = \begin{bmatrix} -0.2558 & -0.1898 & -0.7513 \end{bmatrix} \tag{Gain 1}$$

$$\mathbf{F}_2 = \begin{bmatrix} -0.2564 & -0.1892 & -1.8518 \end{bmatrix} \tag{Gain 2}$$

The ℓ_∞ norms of the corresponding closed-loop sub-systems are $\|\mathbf{V}(\mathbf{A}_{e1} + \mathbf{B}_{e1}\mathbf{F}_1)\mathbf{Z}\|_\infty = 0.9765$ and $\|\mathbf{V}(\mathbf{A}_{e2} + \mathbf{B}_{e2}\mathbf{F}_2)\mathbf{Z}\|_\infty = 0.9735$, respectively, hence guaranteeing switched stability. A sample closed-loop trajectory starting from a random initial condition and under random switching is shown in Fig. 3.

To explore this example further, note that since our algorithm solves a feasibility problem, the above solution is only one workable controller in the set. This opens up the possibility of using the additional degrees of freedom to satisfy additional performance criteria. As an example, it may be of interest to find a single controller that stabilizes all plants, by enforcing that all $\mathbf{F}_i = \mathbf{F}_o$. Applying this idea to the example above leads to the following time invariant controller:

$$\mathbf{F} = \begin{bmatrix} -0.6657 & -0.4771 & -2.1659 \end{bmatrix} \tag{Gain 1}$$

that guarantees closed-loop switched stability. Interestingly, this static gain stabilizes the nonlinear model (37) for $\theta \in [10^\circ, 20^\circ]$.

6. Conclusion

This paper presented a framework for synthesizing data-driven switched state feedback controllers for both continuous and discrete time switched systems. The main idea is to exploit necessary and sufficient conditions for stability under arbitrary switching, given in terms of the existence of a common polyhedral Lyapunov function. While in principle this leads to a very challenging non-convex min–max optimization problem, the main result of the paper shows that it can be reduced, via Farkas' Lemma, to a polynomial optimization. In turn, by exploiting tools from the theory of moments, this optimization can be reduced to a rank-constrained SDP for which efficient convex relaxations are readily available. Further, as noted in the paper, this rank constrained SDP exhibits chordal sparsity, a feature that can be exploited to mitigate the computational complexity growth with the number of sub-systems.

The resulting controller is guaranteed to render the closed-loop system ISS for all plants in the consistency set and arbitrary switching sequences. To the best of the author's knowledge, the proposed method is the first data-driven one capable of handling unknown switched systems and certify closed-loop switched stability. In addition, the proposed controller provides uniform worst-case bounds on the peak value of the magnitude of the state in the presence of bounded disturbances. These features were illustrated by synthesizing a purely data-driven controller to stabilize a quadcopter switching between two different pitch angles.

Perhaps the main limitation of the proposed method is the fact that it is currently limited to state feedback controllers and process noise models. Efforts are underway to extend the framework to handle measurement noise and output feedback by combining the framework presented here with the data-driven observers proposed in Dai and Sznaier (2019).

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