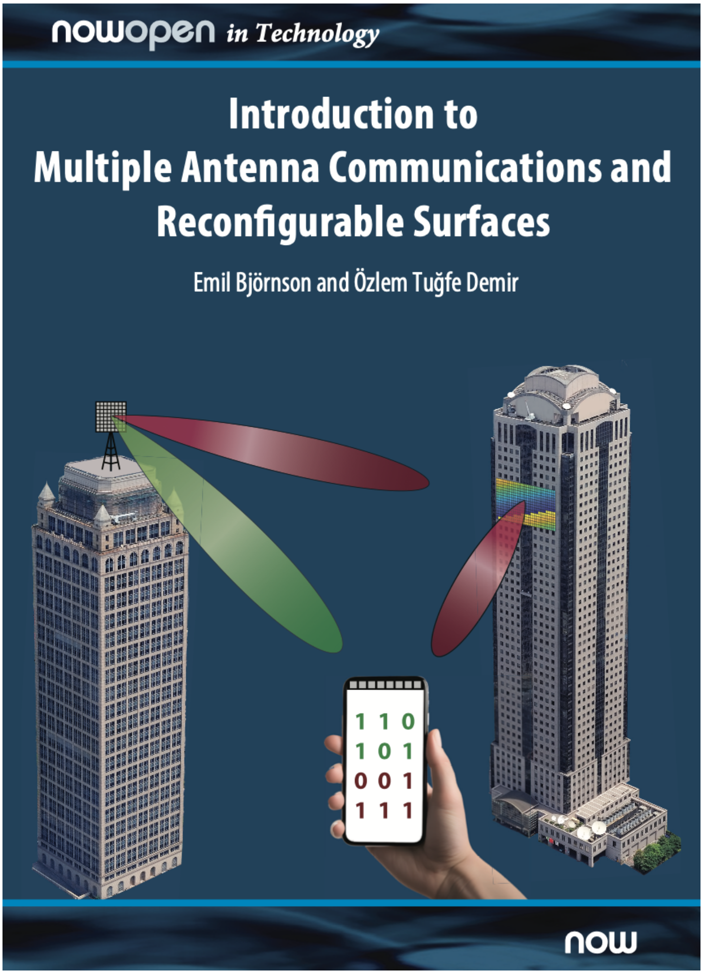
Answers to the Exercises in



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Exercises in Chapter 1

Answer 1.1. (a) 0 dBm. (b) 1 Watt. (c) P_{rx} [dBm] $= -70$ dBm. $P_{\text{rx}} = 10^{-7}$ mW. (d) SNR $= 30$ dB. The SNR is dimensionless.

Answer 1.2. (a) The downlink SNR is 0 dB and the uplink SNR is -20 dB. (b) 10 antennas are needed to increase the uplink SNR by 10 dB. (c) Reducing the bandwidth to 1 MHz in the uplink will increase the SNR to -10 dB SNR.

Answer 1.3. (a) $\alpha = 3.5$, $\Upsilon = 10^{-3}$. (b) $\alpha = 3.5$, $\Upsilon = 10^{-3}/1.5^2$.

Answer 1.4. (a) $c = \frac{48}{7}$. (b) The maximum effective area is $\frac{48}{7}\frac{\lambda^2}{4\pi}$ and is achieved in the directions $\varphi = -\pi/4$, $\theta = 0$ and $\varphi = \pi/3$, $\theta = 0$.

Answer 1.5. (a) SNR ≈ 24 dB. (b) SNR ≈ 34.1 dB. (c) SNR ≈ 32.9 dB.

Answer 1.6. (a) $\frac{\pi a}{2\lambda} + \frac{\pi a^2}{4\lambda d}$ radians and $-\frac{\pi a}{2\lambda} + \frac{\pi a^2}{4\lambda d}$ radians. (b) $\frac{\pi a}{2\lambda} + \frac{\pi}{8}$ radians. The first term is the distance-independent phase variation that a plane wave creates when impinging from the considered non-zero angle. The second term is created by the spherical curvature and is no larger than $\pi/8$, which is in line with the definition of the Fraunhofer distance.

Answer 1.7. $\tau_1 \approx \frac{d}{c} - \frac{\lambda\cos(\varphi)\cos(\theta)}{4c} + \frac{\lambda\sin(\theta)}{2c}$, $\tau_2 \approx \frac{d}{c} + \frac{\lambda\sin(\varphi)\cos(\theta)}{3c}$, and $\tau_3 \approx \frac{d}{c} - \frac{\lambda\sin(\theta)}{2c}$. The solution is not unique. Adding or subtracting a constant to all three delays will also maximize the received signal power. Moreover, $\tau_1 + l_1/f$, $\tau_2 + l_2/f$, and $\tau_3 + l_3/f$ for any integers l_1, l_2, and l_3 will again maximize the received signal power.

Answer 1.8. (a) $A_1 = A_2 = \sqrt{\frac{P}{2}}$. (b) $A_1 = \frac{\sqrt{P\beta_1}}{\sqrt{\beta_1+\beta_2}}$, $A_2 = \frac{\sqrt{P\beta_2}}{\sqrt{\beta_1+\beta_2}}$. (c) The first antenna will transmit with the highest power.

Answer 1.9. (a) $y_0 = \lambda/4$. (b) $y_0 \in [\lambda, 3\lambda/2)$.

Answer 1.10. Yes, $P_1 = P_2 \geq 100/225 \approx 0.44$ W will lead to an SINR of 10 dB at both receivers.

Answer 1.11. (a) $p\varrho$. (b) p/ϱ times larger.

Exercises in Chapter 2

Answer 2.1. (a) $\mathbf{y}_{\mathrm{proj},\mathbf{x}_1,\mathbf{x}_2} = \left(\frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} \frac{\mathbf{x}_1^{\mathrm{H}}}{\|\mathbf{x}_1\|} + \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} \frac{\mathbf{x}_2^{\mathrm{H}}}{\|\mathbf{x}_2\|} \right) \mathbf{y}$. (b) $\mathbf{P} = \sum_{i=1}^{L} \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|} \frac{\mathbf{x}_i^{\mathrm{H}}}{\|\mathbf{x}_i\|}$.

Answer 2.2. (a) $\mathbb{E}\{|\mathbf{v}^{\mathrm{H}}\mathbf{y}|^2\} = \mathbf{v}^{\mathrm{H}}\mathbf{R}\mathbf{v}$. (b) $\mathbb{E}\{|\mathbf{v}^{\mathrm{H}}\mathbf{z}|^2\} = 9|\sum_{m=1}^{M} v_m|^2 + 9\mathbf{v}^{\mathrm{H}}\mathbf{v}$.
(c) $\mathbb{V}\mathrm{ar}\{\|\mathbf{A}\mathbf{x}\|^2\} = 16M$.

Answer 2.3. (a) The expression in (2.224) is obtained after taking the Fourier transform of both sides in (2.223) and then simplify the expression until (2.224) can be identified. (b) The minimum bandwidth is $B/2$ and $P(f)$ should be constant for $f \in (-B/2, B/2]$, which is satisfied by the sinc function. (c) This pulse satisfies the Nyquist criterion since it is symmetric around $f = 0$ and $P(f) + P(f - B)$ is constant for $f \in [0, B/2]$.

Answer 2.4. (a) $v(t) = \int_0^T z(t-u)e^{-\mathrm{j}2\pi f_{\mathrm{c}} u} \partial u$. (b) It is the same expression.

Answer 2.5. (a) After expanding the convolution in (2.123) as an integral expression for $n[l]$, $\mathbb{E}\{|n[l]|^2\} = N_0$ is obtained by using the whiteness of $w(t)$ and the unit energy of the pulse. (b) Inserting the integral expression for $n[l]$ into $\mathbb{E}\{n[l]n^*[m]\}$ results in $\mathbb{E}\{n[l]n^*[m]\} = 0$ by using the whiteness of $w(t)$ and the identity in (2.227). Since the noise samples $\{n[l]\}$ are circularly symmetric Gaussian, this shows that $n[l]$ and $n[m]$ are independent for $l \neq m$.

Answer 2.6. (a) Show that $\mathbb{E}\{(\mathbf{v}-\hat{\mathbf{v}})\mathbf{z}^{\mathrm{H}}\} = \mathbb{E}\{\mathbf{v}\mathbf{v}^{\mathrm{H}}\}\mathbf{A}^{\mathrm{H}} - \mathbf{A}^{\mathrm{H}}\left(\mathbf{A}\mathbf{A}^{\mathrm{H}} + \mathbf{I}_M\right)^{-1}\mathbb{E}\left\{\mathbf{z}\mathbf{z}^{\mathrm{H}}\right\} = \mathbf{0}$. (b) Show that the exponent term in the conditional multivariate Gaussian PDF $f_{\mathbf{v}|\mathbf{z}}(\mathbf{v}|\mathbf{z})$ is equal to $-(\mathbf{v}-\hat{\mathbf{v}})^{\mathrm{H}}\left(\mathbf{A}^{\mathrm{H}}\mathbf{A} + \mathbf{I}_K\right)(\mathbf{v}-\hat{\mathbf{v}})$ by expanding the expression. (c) $\hat{\mathbf{v}} = (\mathbf{I}_M + \mathbf{C})^{-1}\mathbf{y}$.

Answer 2.7. (a) No, a common phase-shift is not affecting the magnitude. (b) It is maximized when $\tau_1 = \tau_2 + n/f_{\mathrm{c}}$ and minimized when $\tau_1 = \tau_2 + (n + 0.5)/f_{\mathrm{c}}$, for some integer n. (c) $\mathbb{E}\{|h|^2\} = \sum_{i=1}^{L} \alpha_i^2$. (d) $\mathbb{E}\{|h|^2\} = L/3$.

Answer 2.8. (a) $\Re(y) = \Re(x) + \Re(n)$ and $\Im(y) = \Im(x) + \Im(n)$ are two independent real-valued AWGN channels. (b) B samples per second for each of the two real-valued channels, which is the same as for the complex-valued AWGN channel.
(c) $C = \frac{B}{2}\log_2\left(1 + \frac{2P}{BN_0}\right)$ [bit/s]. (d) The capacity of the complex-valued AWGN channel is larger than when using one real-valued AWGN channel since the latter only uses half of the bandwidth.

Answer 2.9. This is not correct. We must double the bandwidth and transmit power simultaneously to achieve twice the capacity.

Answer 2.10. (a) h should be selected such that $|h|^2 = \Upsilon\left(\frac{1\,\mathrm{m}}{d}\right)^{\alpha}$ but can have any phase. $C = B\log_2\left(1 + \left(\frac{1\,\mathrm{m}}{d}\right)^{\alpha}\frac{\Upsilon P}{BN_0}\right)$ bit/s. (b) SNR ≈ 11.86 dB, $C \approx 40.3$ Mbit/s. (c) $C \approx 1.26$ Mbit/s. The transmit power should be scaled by a factor 4^{α} to achieve the same capacity as in (b). (d) $C \approx 76.5$ Mbit/s. The transmit power should be scaled by a factor of $2^{-\alpha}$ to achieve the same capacity as in (b).

Answer 2.11. The capacity $C = \log_2\left(1 + \frac{q|h|^2}{N_0}\right)$ is achieved for any combination of q_1, q_2 that satisfies $q_1 + q_2 = q$.

Answer 2.12. (a) $K = L$ and it is irrespective of the specific distribution of x.
(b) $K = L - 1 + \frac{\mathbb{E}\{|x|^4\}}{\sigma^4}$. It is dependent on the distribution of x. For $x \sim \mathcal{N}_{\mathbb{C}}(0, \sigma^2)$, we get $K = L + 1$.

Answer 2.13. (a) The detector decides $\mathcal{H}_1$ if $\sum_{l=1}^{L} y[l] \geq \sigma^2 \ln(\gamma) + \frac{1}{2} = \gamma'$. The detection and false alarm probabilities are $P_{\mathrm{D}} = \int_{\gamma'}^{\infty} \frac{1}{\sqrt{2\pi L\sigma^2}} e^{-\frac{(z-L)^2}{2L\sigma^2}} \partial z$ and $P_{\mathrm{FA}} = \int_{\gamma'}^{\infty} \frac{1}{\sqrt{2\pi L\sigma^2}} e^{-\frac{z^2}{2L\sigma^2}} \partial z$, respectively. (b) The detector decides $\mathcal{H}_1$ if $\sum_{l=1}^{L} y[l] \geq \sigma^2 \ln(\gamma) + \frac{1}{2} = \gamma'$, where γ' is the unique solution that satisfies $\alpha = \int_{\gamma'}^{\infty} \frac{1}{\sqrt{2\pi L\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \partial z$. The detection probability is $P_{\mathrm{D}} = \int_{\gamma'}^{\infty} \frac{1}{\sqrt{2\pi L\sigma^2}} e^{-\frac{(z-L)^2}{2L\sigma^2}} \partial z$.

Answer 2.14. $\bar{\boldsymbol{\chi}} = [0, \sqrt{7}, 0, 3\sqrt{7}/2, 3\sqrt{7}/2, 0, \sqrt{7}]^{\mathrm{T}}$.

Answer 2.15. It holds that $\|\boldsymbol{\chi}\|^2 = \|\bar{\boldsymbol{\chi}}\|^2$ since $\boldsymbol{\chi} = \mathbf{F}_S^{\mathrm{H}} \bar{\boldsymbol{\chi}}$, which establishes Parseval's relation.

Answer 2.16. (a) $\mathbf{C}_{\bar{a}} = \sqrt{S}\mathbf{F}_S \mathbf{D}_a \mathbf{F}_S^{\mathrm{H}}$. (b) After obtaining $\bar{\mathbf{c}} = \mathbf{F}_S \mathbf{D}_a \mathbf{F}_S^{\mathrm{H}} \bar{\mathbf{b}} = \frac{\mathbf{C}_{\bar{a}} \bar{\mathbf{b}}}{\sqrt{S}}$, the claim is verified. (c) The DFT of the sequence $a[k]b[k]$ is

$$\begin{cases} \sqrt{10} & \nu = 4, \\ 0 & \nu = 0, \dots, 3, 5, \dots, 9, \end{cases}$$

which is equal to $(\bar{a} \circledast \bar{b})[\nu]/\sqrt{S}$.

Answer 2.17. (a) $f_{\mathrm{c}} + f_1$, $f_{\mathrm{c}} - f_1$, $-(f_{\mathrm{c}} + f_1)$, and $-(f_{\mathrm{c}} - f_1)$. (b) $\frac{(f_{\mathrm{c}}+f_1)}{c}$, $\frac{(f_{\mathrm{c}}-f_1)}{c}$, $\frac{-(f_{\mathrm{c}}+f_1)}{c}$, and $\frac{-(f_{\mathrm{c}}-f_1)}{c}$. (c) The temporal frequencies are unaffected, but the spatial frequencies are increased by a factor c/v.

Exercises in Chapter 3

Answer 3.1. (a) Letting $b = 1/B$ and using L'Hôpital's rule, one can show that the capacity goes to zero when $b \to \infty$, i.e., $B \to 0$. This operating regime is called the bandwidth-limited region. (b) Letting $b = 1/B$, the capacity goes to $\log_2(e)\frac{P\beta}{N_0}$ when $b \to 0$, i.e., $B \to \infty$. This operating regime is called the power-limited region.

Answer 3.2. (a) $\frac{\partial C(P,B)}{\partial P} = \frac{\log_2(e)B}{BN_0/\beta+P}$. The capacity grows the fastest at $P = 0$. For $P \to \infty$, $C \to \infty$ at a logarithmic rate. (b) $\frac{\partial^2 C(P,B)}{\partial P^2} = -\frac{\log_2(e)B}{(BN_0/\beta+P)^2} < 0$, thus the capacity is a (strictly) concave function of P. (c) $\frac{\partial C(P,B)}{\partial B} = \log_2(e)\left(\ln\left(1+\frac{P\beta}{BN_0}\right) - \frac{\frac{P\beta}{BN_0}}{1+\frac{P\beta}{BN_0}}\right) > 0$, which verify that the capacity always increases with the bandwidth B for any value of the SNR. From the second-order derivative of the capacity with respect to B, i.e., $\frac{\partial^2 C(P,B)}{\partial B^2} = -\log_2(e)\frac{\frac{P^2\beta^2}{B^3N_0^2}}{\left(1+\frac{P\beta}{BN_0}\right)^2} < 0$, the capacity grows the fastest at $B = 0$. As $B \to \infty$, the capacity goes to $\log_2(e)\frac{P\beta}{N_0}$. (d) $\frac{\partial^2 C(P,B)}{\partial B^2} = -\log_2(e)\frac{\frac{P^2\beta^2}{B^3N_0^2}}{\left(1+\frac{P\beta}{BN_0}\right)^2} < 0$, thus the capacity is a (strictly) concave function of B.

Answer 3.3. (a) Increasing the bandwidth to $2B$ cannot double the capacity while increasing it to $6B$ will at least double the capacity. (b) No, to double the capacity, we must find a positive root of the equation $f(c) = \log_2\left(1+\frac{1}{c}\right) - \frac{2}{c} = 0$, which does not exist. (c) The initial selection of $x = P\beta/(BN_0)$ should satisfy the relation $1 + x \leq e^{\frac{x}{2}}$ in order to double the capacity with a scalar $c > 1$. This relation is satisfied for $x = 7$ in (a), and there is a value of c that doubles the capacity. However, it is not satisfied for $x = 1$ in (b), and no value of $c > 1$ can double the capacity.

Answer 3.4. (a) $C = B\log_2\left(1+\frac{P_{\mathrm{rx}}}{BN_0}\right)$. (b) $C = 100\,\mathrm{Mbit/s}$. (c) To get 8 times higher capacity, 255 antennas are needed at the base station.

Answer 3.5. (a) The capacity is $C = \log_2\left(1+M\frac{q}{N_0}\right)$ bit/symbol and it is achieved if $x \sim \mathcal{N}_{\mathbb{C}}(0,q)$. (b) $M = 63$ antennas are needed. (c) An SNR of $q/N_0 = 6.3$ is needed. (d) The capacity is $\log_2\left(1+4M\frac{q}{N_0}\right)$ bit/symbol. (e) The sign of the entries does not matter when computing $\|\mathbf{h}\|^2$. The capacity is still $\log_2\left(1+M\frac{q}{N_0}\right)$ bit/symbol.

Answer 3.6. (a) $d \lessapprox 508\,\mathrm{m}$. (b) We need $M \approx 240$ antennas.

Answer 3.7. (a) $\mathbf{w} = c\mathbf{C}^{-1}\mathbf{h}$ for some non-zero $c \in \mathbb{C}$. (b) An infinite SNR is achievable, which is unrealistic. In practice, the noise covariance matrix is always non-singular.

Answer 3.8. (a) The data rate is $\log_2\left(1+\frac{q_1|h_1|^2+q_2|h_2|^2}{N_0}\right)$. If $|h_1| \geq |h_2|$, $q_1 = q$ and $q_2 = 0$ will maximize the data rate. If $|h_1| < |h_2|$, $q_1 = 0$ and $q_2 = q$ will maximize the data rate. The resulting maximum data rate is $\log_2\left(1+\frac{q\max\left(|h_1|^2,|h_2|^2\right)}{N_0}\right)$. (b) The MISO channel capacity is $\log_2\left(1+\frac{q\|\mathbf{h}\|^2}{N_0}\right) = \log_2\left(1+\frac{q\left(|h_1|^2+|h_2|^2\right)}{N_0}\right)$, which is higher than (a), except when either $h_1 = 0$ or $h_2 = 0$.

Answer 3.9. (a) The MISO channel capacity is $C \approx 575\,\mathrm{Mbit/s}$, which is achieved by the MRT vector $\mathbf{p} = [3,1,-4]^{\mathrm{T}}/\sqrt{26}$. (b) We should select $\mathbf{p} = [1,1,-1]^{\mathrm{T}}/\sqrt{3}$ and the resulting data rate is $\approx 545\,\text{Mbit/s}$. (c) There is a reduction in data rate when the precoding vector can only adjust the phase/sign to the channel but not vary the amplitude to better utilize the stronger channel components.

Answer 3.10. The channel capacity is equal to the capacity of the channel from $\mathbf{x}$ to the whitened signal $\dot{\mathbf{y}} = \mathbf{C}^{-1/2}\mathbf{y}$, which is obtained by Theorem 3.1 with $N_0 = 1$ and the effective channel matrix $\dot{\mathbf{H}} = \mathbf{C}^{-1/2}\mathbf{H}$.

Answer 3.11. (a) The channel capacity is $C = \log_2(1+17) + \log_2(1+13/5)$ bit/symbol with the covariance matrix $\begin{bmatrix} \frac{13}{30}N_0 & 0 \\ 0 & \frac{17}{30}N_0 \end{bmatrix}$. (b) The channel capacity is the same as in (a) with the covariance matrix

$$\frac{N_0}{900}\begin{bmatrix} 65 & 65-130\mathrm{j} & 0 & 0 \\ 65+130\mathrm{j} & 325 & 0 & 0 \\ 0 & 0 & 425 & -51\sqrt{5}+68\sqrt{5}\mathrm{j} \\ 0 & 0 & -51\sqrt{5}-68\sqrt{5}\mathrm{j} & 85 \end{bmatrix}.$$

Answer 3.12. (a) $\frac{q_1^{\text{opt}}}{N_0} = \frac{q_2^{\text{opt}}}{N_0} = 1$ and the resulting capacity is $C = 2\log_2(1+2\cdot 1) \approx 3.17$ bit/symbol. (b) $\frac{q_1^{\text{opt}}}{N_0} = 2$, $\frac{q_2^{\text{opt}}}{N_0} = 0$, and the resulting capacity is $C = \log_2(1+2\cdot 1) \approx 1.58$ bit/symbol. (c) $\frac{q_1^{\text{opt}}}{N_0} = 2$, $\frac{q_2^{\text{opt}}}{N_0} = 0$, and the resulting capacity is $C = \log_2(1+2\cdot 4) \approx 3.17$ bit/symbol.

Answer 3.13. (a) $\frac{q}{N_0} = 9$. (b) $\frac{q}{N_0} = 4.5$. (c) $\frac{q}{N_0} = 3$. Thanks to the multiplexing gain of two, the required $\frac{q}{N_0}$ has been reduced by 3 and 1.5 compared to the SISO and SIMO channels in (a) and (b).

Answer 3.14. (a) The capacity is $\log_2(1+4\varrho)$ and is achieved by using the precoding vector $[1,1]^{\mathrm{T}}/\sqrt{2}$ and a data signal $x \sim \mathcal{N}_{\mathbb{C}}(0,q)$, where $q = \varrho N_0$. A beamforming gain is achieved compared to the corresponding SISO channel. (b) The capacity is $2\log_2(1+\varrho)$ and is achieved by transmitting an independent signal on each antenna: $x_k \sim \mathcal{N}_{\mathbb{C}}(0,q/2)$ from antenna k, for $k = 1,2$, where $q = \varrho N_0$. A multiplexing gain is achieved compared to the corresponding SISO channel. (c) $\varrho \geq 2$.

Answer 3.15. (a) $C = K\log_2\left(1+\frac{qM}{KN_0}\right)$. (b) $\frac{\partial C(M,K)}{\partial K} = \log_2(e)\left(\ln\left(1+\frac{\varrho M}{K}\right) - \frac{\frac{\varrho M}{K}}{1+\frac{\varrho M}{K}}\right)$ where $\varrho = q/N_0$. The capacity is an increasing function of K. (c) C grows the fastest at $K = 0$; that is, going from 0 to 1 transmit antenna gives the largest increase in capacity. (d) If $K = M$, the capacity grows linearly with K. (e) When $\varrho = q/N_0$ is close to zero, $C(M,K) \approx \log_2(e)\varrho M$ which is independent of K but grows linearly with M.

Answer 3.16. (a) $\lambda_1 = \lambda_{\text{sum}}$, $\lambda_2 = \ldots = \lambda_M$. (b) $\lambda_1 = \ldots = \lambda_M = \lambda_{\text{sum}}/M$.

Answer 3.17. (a) $d_i = 1/\sqrt{\mathbf{e}_i^{\mathrm{H}}(\mathbf{H}\mathbf{H}^{\mathrm{H}})^{-1}\mathbf{e}_i}$. (b) The channel gains are $d_1^2, \ldots, d_M^2$. (c) The SVD gives 10 and the approach from (b) gives 18/5, which is smaller.

Answer 3.18. (a) $\mathcal{I}(\bar{\mathbf{x}};\mathbf{y}) = \mathcal{I}(\bar{x}_1;\mathbf{y}) + \mathcal{I}(\bar{x}_2;\mathbf{y}|\bar{x}_1) + \mathcal{I}(\bar{x}_3;\mathbf{y}|\bar{x}_1,\bar{x}_2) + \ldots + \mathcal{I}(\bar{x}_K;\mathbf{y}|\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_{K-1})$. (b) The equality holds since $\mathcal{I}(\bar{x}_i;\mathbf{y}|\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_{i-1}) = \mathcal{I}(\bar{x}_i;\mathbf{y}_i|\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_{i-1})$ and the error $\bar{x}_i - \mathbf{w}_i^{\mathrm{H}}\mathbf{y}_i$ is independent of $\mathbf{x}_i$. (c) The achievable data rate by the LMMSE-SIC receiver processing is equal to $\mathcal{I}(\bar{\mathbf{x}};\mathbf{y})$ showing that the LMMSE-SIC receiver processing is information-theoretically optimal.

Exercises in Chapter 4

Answer 4.1. After expressing $d_M - d_1$ from (4.13), use $d_1 \gg M\Delta$ and $\sqrt{1+x^2} \approx 1 + \frac{x^2}{2}$ where $x = \sqrt{\frac{2(M-1)\Delta\sin(\varphi)}{d_1}}$ to obtain (4.17).

Answer 4.2. (a) $C = B\log_2\left(1 + \frac{PM}{BN_0}\frac{\lambda^2}{(4\pi)^2}\frac{1}{d^2}\right)$ bit/s. (b) $d \approx 795.77$ m. (c) $M = 1023$ antennas are needed. The total effective area is $\approx 0.81\,\mathrm{m}^2$. (d) $M = 102300$ antennas are needed. The total effective area is the same as in (c).

Answer 4.3. (a) 10. (b) 0.2.

Answer 4.4. The beamforming gain for a ULA with $M = 10$ cosine antennas is shown in Figures A.4.1 and A.4.2 using rectangular and polar plot, respectively. The half-power beamwidth becomes slightly smaller. The first-null beamwidth remains unchanged, whereas the amplification beamwidth becomes larger.

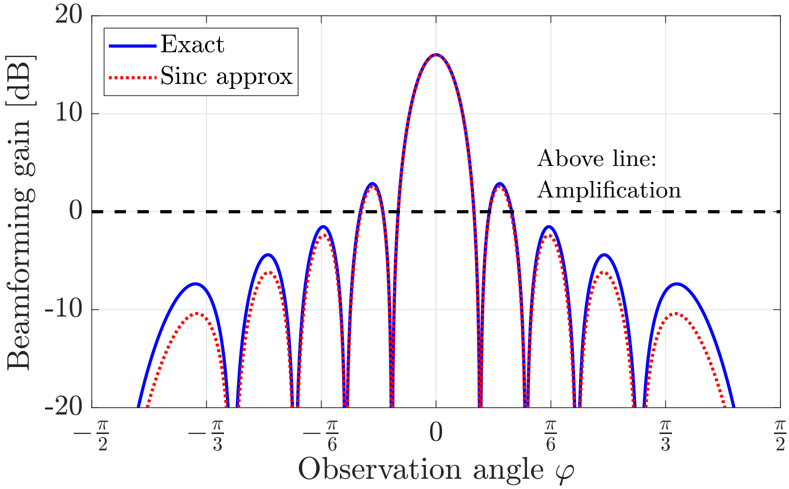

Figure A.4.1: Beamforming gain shown using a rectangular plot.

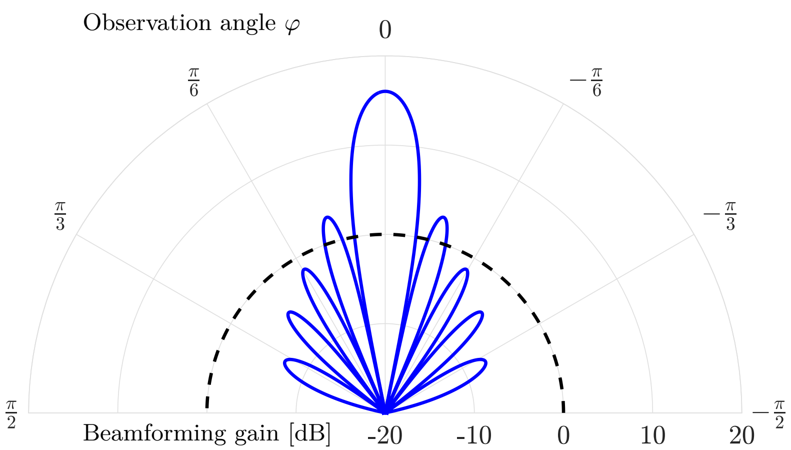

Figure A.4.2: Beamforming gain shown using a polar plot.

Answer 4.5. (a) $\frac{1}{M}\frac{\sin^2\left(M\frac{\pi(\sin(\varphi)-1)}{2}\right)}{\sin^2\left(\frac{\pi(\sin(\varphi)-1)}{2}\right)}$. (b) $4/\sqrt{M}$. c) The beamwidth is smaller when transmitting in the broadside direction. The beamwidth for end-fire beamforming decreases with $\sqrt{M}$.

Answer 4.6. (a) $C \approx 5.36$ bit/symbol. (b) The achievable data rate is approximately 1.295 bit/symbol. (c) The achievable data rate is zero.

Answer 4.7. The M beams using the angles given in (4.184) can be constructed by starting with the main beam $\varphi_{\text{beam}} = \arcsin(a/M)$ and including the $M-1$ null directions to this beam. Then, using the condition in (4.63) with properly defined angles, it can be shown that these M beams are mutually orthogonal.

Answer 4.8. (a) $M\text{sinc}^2\left(\frac{M\Delta 0.158}{\lambda}\right)$. (b) At least $M = 6$ antennas are needed. (c) At least $M = 3$ antennas are needed.

Answer 4.9. $\Delta_\lambda \approx \frac{1}{0.692} \approx 1.44$ wavelengths.

Answer 4.10. $C = 4\log_2\left(1 + \frac{q\beta}{N_0}\right)$.

Answer 4.11. $\mathbf{a}(\varphi,\theta) = \begin{bmatrix} 1 \\ e^{-\mathrm{j}\pi\frac{\Delta\sin(\varphi)\cos(\theta)}{\lambda}}e^{-\mathrm{j}\pi\frac{\sqrt{3}\Delta\sin(\theta)}{\lambda}} \\ e^{\mathrm{j}\pi\frac{\Delta\sin(\varphi)\cos(\theta)}{\lambda}}e^{-\mathrm{j}\pi\frac{\sqrt{3}\Delta\sin(\theta)}{\lambda}} \end{bmatrix}$.

Answer 4.12. (a) 22 dBi. (b) 20.77 dBi. (c) No, the results will not change.

Answer 4.13. $M = 79$ and $P = \frac{10^{3.8}}{4M} = \frac{10^{3.8}}{316} \approx 19.97\,\text{W} \approx 43\,\text{dBm}$.

Answer 4.14. (a) $C \approx 600.73$ Mbit/s. (b) $K = 16$ isotropic antennas are needed. (c) $M = 4$ isotropic antennas are needed. (d) Yes, by setting $M = K = 8$.

Answer 4.15. (a) The first-null beamwidth in the horizontal plane ($\theta = \theta_{\text{beam}}$) is $\arcsin\left(\frac{1}{\cos(\theta_{\text{beam}})L_{\text{H},\lambda}} + \frac{\sin(\varphi_{\text{beam}})}{\cos(\theta_{\text{beam}})}\right) + \arcsin\left(\frac{1}{\cos(\theta_{\text{beam}})L_{\text{H},\lambda}} - \frac{\sin(\varphi_{\text{beam}})}{\cos(\theta_{\text{beam}})}\right)$. The first-null beamwidth in the vertical plane ($\varphi = \varphi_{\text{beam}}$) is $\arcsin\left(\frac{1}{L_{\text{V},\lambda}} + \sin(\theta_{\text{beam}})\right) + \arcsin\left(\frac{1}{L_{\text{V},\lambda}} - \sin(\theta_{\text{beam}})\right)$. (b) The first-null beamwidth in the horizontal plane ($\theta = \pi/10$) is $2\arcsin\left(\frac{1}{\cos(\pi/10)\cdot 5}\right) \approx 0.42$ rad, which is larger than the beamwidth of the beamforming in the direction ($\varphi_{\text{beam}} = 0, \theta_{\text{beam}} = 0$). The first-null beamwidth in the vertical plane ($\varphi = 0$) is 1.13 rad, which is larger than the beamwidth of the beamforming in the direction ($\varphi_{\text{beam}} = 0, \theta_{\text{beam}} = 0$).

Answer 4.16. (a) $\Delta = \frac{2d}{M}$. (b) The separation is Δ.

Answer 4.17. (a) $d \approx 252.63$ m. (b) $d \approx 505.26$ m. (c) $d \approx 626.86$ m.

Answer 4.18. $\hat{\phi} = -\arg\left(\mathbf{a}^{\text{H}}(\varphi)\sum_{l=1}^{L_{\text{p}}}\mathbf{y}[l]\right)$, $\hat{\varphi} = \underset{\varphi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}{\arg\max}\left|\mathbf{a}^{\text{H}}(\varphi)\sum_{l=1}^{L_{\text{p}}}\mathbf{y}[l]\right|$, and $\hat{\beta} = \frac{\left|\mathbf{a}^{\text{H}}(\hat{\varphi})\sum_{l=1}^{L_{\text{p}}}\mathbf{y}[l]\right|^2}{L_{\text{p}}^2M^2q}$.

Answer 4.19. (a) No, the joint ML estimates of φ and θ are not unique. (b) No, for any ULA with M antennas, the joint ML estimates of φ and θ are not unique. Defining $b = \sin(\varphi)\cos(\theta) \in [-1,1]$, we can perform a one-dimensional search over b to find its optimum value $b^\star$ corresponding to the ML estimates of φ and θ. For a given value of $\hat{\theta} \in (-\pi/2, \pi/2)$ as long as $b^\star/\cos(\hat{\theta}) \in [-1,1]$, we can obtain the corresponding $\hat{\varphi}$. Hence, there is no unique solution in general.

Exercises in Chapter 5

Answer 5.1. (a) As $L \to \infty$, h converges to 0 in probability. The variance is zero and h is not Gaussian distributed. (b) The distribution of h converges to $\mathcal{N}_{\mathbb{C}}(0,\beta)$. (c) Since the variance of h goes to infinity, there is no limit and h is not Gaussian distributed.

Answer 5.2. (a) $\mathbb{E}\{h_m h_n^*\} = \beta \int_{-\pi/2}^{\pi/2} e^{-\mathrm{j}2\pi \frac{(m-n)\Delta \sin(\theta)}{\lambda}} \frac{1}{\pi} \partial\theta$.
(b) $\mathbb{E}\{h_m h_n^*\} = \beta J_0 \left(\frac{2\pi(m-n)\Delta}{\lambda}\right)$.

Answer 5.3. The coherence time becomes $T_{\mathrm{c}} = \frac{\lambda}{8\upsilon}$.

Answer 5.4. (a) The outage probability is 1 for $R \geq \log_2\left(1+\frac{q}{N_0}\right)$ and p for $0 < R < \log_2\left(1+\frac{q}{N_0}\right)$. (b) The outage probability is 1 for $R \geq \log_2\left(1+\frac{2q}{N_0}\right)$ and $2(1-p)p+p^2$ for $\log_2\left(1+\frac{q}{N_0}\right) < R < \log_2\left(1+\frac{2q}{N_0}\right)$. For $0 < R \leq \log_2\left(1+\frac{q}{N_0}\right)$, the outage probability is p^2.

Answer 5.5. (a) $P_{\mathrm{out}}(R) = \begin{cases} \sqrt{\frac{N_0(2^R-1)}{q}} & R \in \left[0, \log_2\left(1+\frac{q}{N_0}\right)\right], \\ 1 & R > \log_2\left(1+\frac{q}{N_0}\right). \end{cases}$.
(b) $P_{\mathrm{out}}(R) = \begin{cases} \sqrt{\frac{N_0(2^R-1)}{Mq}} & R \in \left[0, \log_2\left(1+\frac{Mq}{N_0}\right)\right], \\ 1 & R > \log_2\left(1+\frac{Mq}{N_0}\right). \end{cases}$. (c) For the setups considered in (a) and (b), respectively, the ϵ-outage capacities are $C_{\epsilon,(a)} = \log_2\left(1+\frac{q\epsilon^2}{N_0}\right)$ and $C_{\epsilon,(b)} = \log_2\left(1+\frac{Mq\epsilon^2}{N_0}\right)$. The graph is given in Figure A.5.1.

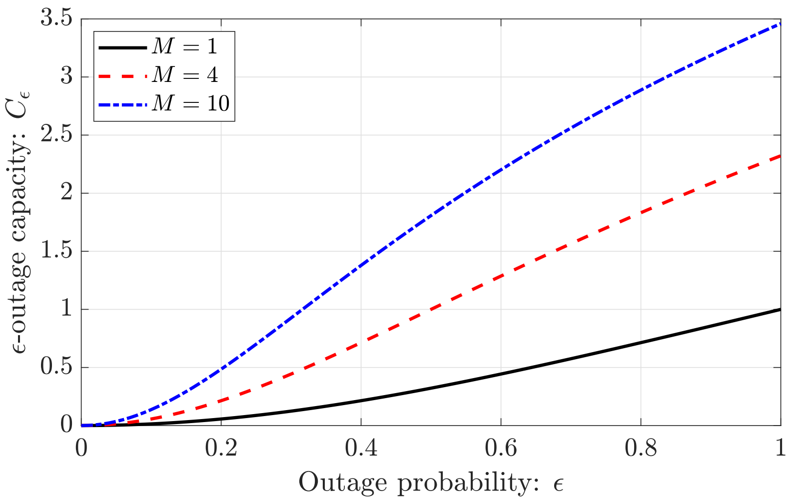

Figure A.5.1: The ϵ-outage capacity C_ϵ of the considered channel with $M=1$, $M=4$, and $M=10$ receive antennas.

Answer 5.6. (a) The outage probability is
$P_{\mathrm{out}}(R) = \begin{cases} 2\frac{N_0(2^R-1)}{q} - \frac{N_0^2(2^R-1)^2}{q^2}, & 0 < R \leq \log_2\left(1+\frac{q}{N_0}\right), \\ 1, & R > \log_2\left(1+\frac{q}{N_0}\right), \end{cases}$ and the ϵ-outage capacity is $C_\epsilon = \log_2\left(1+\frac{q\left(1-\sqrt{1-\epsilon}\right)}{N_0}\right)$. (b) The outage probability is

$P_{\text{out}}(R) = \begin{cases} \left(\frac{N_0\left(2^R-1\right)}{q}\right)^{3/2}, & 0 < R \leq \log_2\left(1+\frac{q}{N_0}\right), \\ 1, & R > \log_2\left(1+\frac{q}{N_0}\right), \end{cases}$ and the ϵ-outage capacity is $C_\epsilon = \log_2\left(1+\frac{q\epsilon^{2/3}}{N_0}\right)$.

Answer 5.7. (a) The receiver should select the antenna with the largest channel gain $|h_m|^2$. Let $\bar{h}^2 = \max_{m\in\{1,\ldots,M\}} |h_m|^2$, then the maximum capacity is $C_{\mathsf{AS}} = \left(1+\frac{q\bar{h}^2}{N_0}\right)$ for the antenna selection (AS). (b) $P_{\text{out}}(R) = \left(1-e^{-\frac{N_0\left(2^R-1\right)}{q\beta}}\right)^M$. (c) The outage probability is higher with antenna selection than with MRC. (d) The high-SNR slope of the outage probability when using antenna selection is $-M$.

Answer 5.8. Writing $\sum_{m=0}^{M-1} \frac{\left(\frac{2^R-1}{\text{SNR}}\right)^m}{m!} = e^{\frac{2^R-1}{\text{SNR}}} - \sum_{m=M}^{\infty} \frac{\left(\frac{2^R-1}{\text{SNR}}\right)^m}{m!}$ in the outage probability expression and using L'Hôpital's rule, the limit of the slope in decibel scale is obtained as $-M$, where the diversity order of M is obtained by taking the negative of it.

Answer 5.9. When the channels are fully correlated in this way, the diversity order is reduced from $M = 2$ to 1.

Answer 5.10. The variance of $\frac{\|\mathbf{h}\|^2}{\mathbb{E}\{\|\mathbf{h}\|^2\}}$ for an i.i.d. Rayleigh fading channel is given as $1/M$ from (5.58). Since $1/M$ goes to zero as M goes to infinity, i.i.d. Rayleigh fading channel provides channel hardening.

Answer 5.11. (a) The conditional capacity is given as $C_h = \frac{1}{L}\log_2\left(1+\frac{Lq|h|^2}{N_0}\right)$. (b) The capacity with repetition scheme reduces with L. Hence, the optimal value of L is 1. (c) The low-SNR approximation is $C_h \approx \log_2(e)\frac{q|h|^2}{N_0}$, which is independent of L.

Answer 5.12. (a) We can obtain an equivalent SIMO system by sending the same symbol consecutively using the columns of the DFT matrix. Then using (5.86), obtain the upper bound $\left(\frac{2^{MR}-1}{\text{SNR}}\right)^M \frac{1}{M!}$ and show that the full transmit diversity order of M is achieved. (b) The SNR is maximized using the unit-norm receive combining vector where the mth entry is $v_m = \frac{e^{\mathrm{j}\arg(h_m)}}{\sqrt{M}}$. An upper bound on the outage probability can be approximated as $\left(\frac{N_0\left(2^R-1\right)}{q\beta}\right)^M$ in the high-SNR region; thus, the full diversity order of M is achieved.

Answer 5.13. (a) The low-SNR approximation of the outage probability with Alamouti code is $P_{\text{out,MISO}}(R) \approx 1 - e^{-\frac{2RN_0}{q\log_2(e)\beta}} \sum_{m=0}^{1} \frac{\left(\frac{2RN_0}{q\log_2(e)\beta}\right)^m}{m!}$. (b) The low-SNR approximation of the outage probability for the SISO channel is $P_{\text{out,SISO}}(R) \approx 1 - e^{-\frac{RN_0}{q\log_2(e)\beta}}$. Since $P_{\text{out,SISO}} - P_{\text{out,MISO}}$ is (approximately) smaller than zero for any $R > 0$ at the low-SNR region, it is better to transmit from only one of the antennas rather than applying Alamouti code.

Answer 5.14. (a) $C_{\mathsf{SIMO}} = \log_2\left(1+\frac{qM}{N_0}\right)$. There is a beamforming gain that improves the capacity by M times at low SNR and adds an extra $\log_2(M)$ to the capacity at high SNR. (b) $C_{\mathsf{SIMO}} = \mathbb{E}\left\{\log_2\left(1+\frac{q\|\mathbf{h}\|^2}{N_0}\right)\right\}$. There are both beamforming and diversity gains, which improve the capacity by M times at low SNR and $\mathbb{E}\{\log_2(\|\mathbf{h}\|^2/|h|^2)\}$ times at high SNR. (c) The ergodic capacity can never be larger than the capacity of a corresponding non-fading channel, but the difference between the SIMO capacities in (a) and (b) vanishes as $M \to \infty$.

Answer 5.15. (a) The ergodic capacity is $p \log_2\left(1+\frac{q}{N_0}\right)$. (b) The ergodic capacity is $p \log_2\left(1+\frac{Mq}{N_0}\right)$. (c) The ergodic capacity is $C = \sum_{m=1}^{M} \binom{M}{m} p^m (1-p)^{M-m} \log_2\left(1+\frac{mq}{N_0}\right)$.

Answer 5.16. The ergodic capacity is
$C = \max_{\mathbf{R}_x \in \mathbb{C}^{K\times K}:\, \mathrm{tr}(\mathbf{R}_x)=q} \mathbb{E}\left\{\log_2\left(\det\left(\mathbf{I}_M + \frac{1}{N_0}\mathbf{C}^{-1/2}\mathbf{H}\mathbf{R}_x\mathbf{H}^{\mathrm{H}}\mathbf{C}^{-1/2}\right)\right)\right\}$.

Answer 5.17. The ergodic capacity is $C = \sum_{m=1}^{r} \log_2\left(\frac{q}{KN_0}\right) + \sum_{m=1}^{r} \mathbb{E}\left\{\log_2\left(\lambda_m\right)\right\}$.

Answer 5.18. (a) The multiplexing gain is L_{p}. The pre-log factor in front of the ergodic capacity is $L_{\mathrm{p}}\left(1-\frac{L_{\mathrm{p}}}{L_{\mathrm{c}}}\right)$. (b) The optimal pilot length and multiplexing gain is $L_{\mathrm{p}}^{\star} = \frac{L_{\mathrm{c}}}{2}$.

Answer 5.19. (a) The variance of $\frac{\|\mathbf{h}\|^2}{\mathbb{E}\{\|\mathbf{h}\|^2\}}$ cannot be lower than $1/N_{\mathrm{cl}}$ for any M. Hence, the variance does not approach zero as $M \to \infty$, and the clustered multipath propagation channel with a finite number of clusters does not provide channel hardening. (b) As $M \to \infty$, the variance converges to zero. Hence, the clustered multipath propagation channel provides channel hardening when $N_{\mathrm{cl}} \to \infty$.

Answer 5.20. At least four multipath clusters are needed to achieve a full-rank channel matrix. As an example, this can be achieved if the cluster angles are selected corresponding to the columns of the DFT matrix $\mathbf{F}_4$ from (5.193) as $(\varphi_1, \theta_1) = (0, 0)$, $(\varphi_2, \theta_2) = \left(\frac{\pi}{3}, 0\right)$, $(\varphi_3, \theta_3) = \left(-\frac{\pi}{3}, 0\right)$, and $(\varphi_4, \theta_4) = \left(\pm\frac{\pi}{2}, 0\right)$.

Answer 5.21. (a) The spatial correlation matrix is
$\mathbf{R}_h = \sum_{i=1}^{N_{\mathrm{cl}}} \frac{\beta_i}{2\Delta_\varphi} \int_{\varphi_i-\Delta_\varphi}^{\varphi_i+\Delta_\varphi} \mathbf{a}(\varphi)\mathbf{a}^{\mathrm{H}}(\varphi)\partial\varphi$. (b) The rank is approximately
$\mathrm{rank}(\mathbf{R}_h) \approx \frac{M}{2} \sum_{i=1}^{N_{\mathrm{cl}}} \left|\sin\left(\varphi_i + \Delta_\varphi\right) - \sin\left(\varphi_i - \Delta_\varphi\right)\right|$.

Answer 5.22. (a) $\breve{\mathbf{H}} = \mathbf{D}_{\mathrm{r}}^{1/2}\mathbf{U}_{\mathrm{r}}\mathbf{W}\mathbf{U}_{\mathrm{t}}^{*}\mathbf{D}_{\mathrm{t}}^{1/2}$. The entries of $\breve{\mathbf{H}}$ are independent since $\mathbf{U}_{\mathrm{r}}\mathbf{W}\mathbf{U}_{\mathrm{t}}^{*}$ has i.i.d. entries. (b) The (m,k)th entry of $\breve{\mathbf{H}}$ has the variance $\beta\lambda_{\mathrm{r},m}\lambda_{\mathrm{t},k}$. The MK variances are determined by $M+K$ parameters, so only some beamspace matrices can be expressed this way.

Answer 5.23. (a) The spatial correlation matrix is given as $\mathbf{R}_{\mathrm{ULA}} = \beta\mathbf{I}_2$. (b) The spatial correlation matrix is

$$\mathbf{R}_{\mathrm{UPA}} = \beta \begin{bmatrix} 1 & 0 & 0 & \mathrm{sinc}\left(\sqrt{2}\right) \\ 0 & 1 & \mathrm{sinc}\left(\sqrt{2}\right) & 0 \\ 0 & \mathrm{sinc}\left(\sqrt{2}\right) & 1 & 0 \\ \mathrm{sinc}\left(\sqrt{2}\right) & 0 & 0 & 1 \end{bmatrix}.$$

(c) We get i.i.d. fading with a ULA with half-wavelength spacing but not a UPA since the antennas have different spacings along the diagonals.

Exercises in Chapter 6

Answer 6.1. (a) $\partial\mathcal{R} = \left\{(R_1, R_2) : R_1, R_2 \geq 0, \frac{R_1}{2} + R_2 = 10\,\text{Mbit/s}\right\}$. (b) $R_2^{\max} = 2.5\,\text{Mbit/s}$.

Answer 6.2. (a) $(R_1, R_2) = (6, 6)\,\text{Mbit/s}$. (b) $(R_1, R_2) = (1, 23.5)\,\text{Mbit/s}$. (c) $(R_1, R_2) = (3, 19.5)\,\text{Mbit/s}$.

Answer 6.3. Taking the partial derivative of $\sum_{k=1}^{K} \xi_k B \log_2\left(1 + \frac{P\beta_k}{\xi_k B N_0}\right) + \delta(1 - \sum_{k=1}^{K} \xi_k)$ with respect to ξ_k (where δ is the Lagrange multiplier), equating it to zero, and imposing the constraint $\xi_1 + \ldots + \xi_K = 1$, $\xi_k = \frac{\beta_k}{\sum_{i=1}^{K} \beta_i}$ is obtained.

Answer 6.4. (a) The capacity region is the square shown in Figure A.6.1.

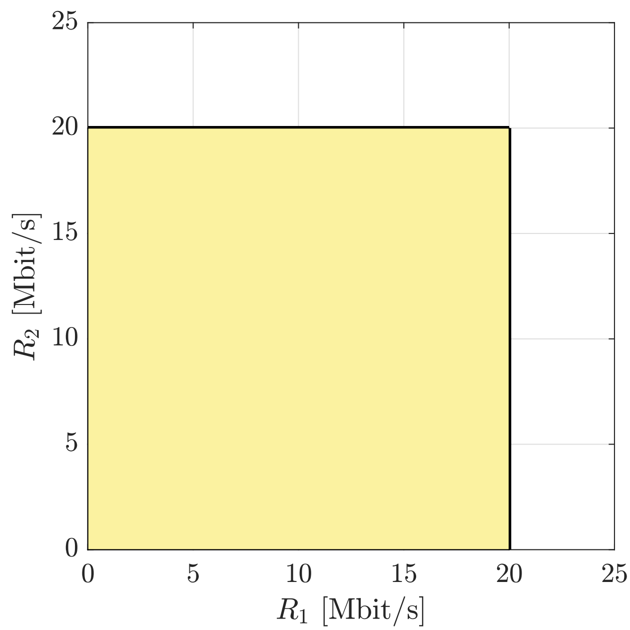

Figure A.6.1: The capacity region obtained in Exercise 6.4(a).

(b) The capacity region is shown in Figure A.6.2.

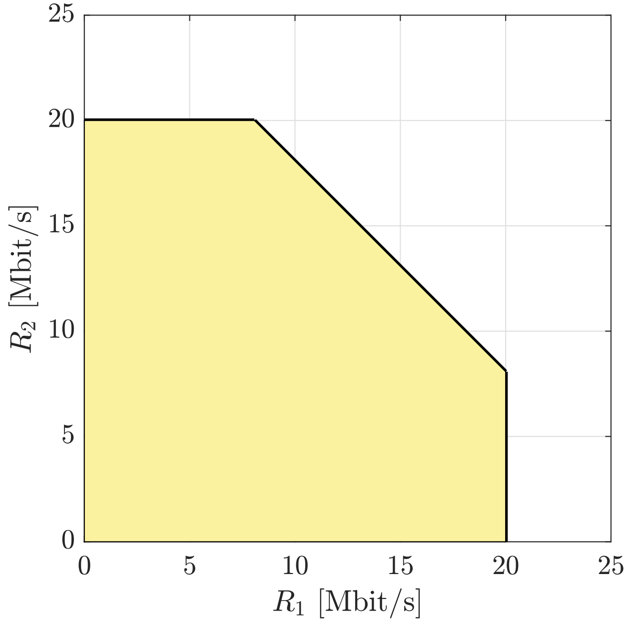

Figure A.6.2: The capacity region obtained in Exercise 6.4(b).

(c) The sum rate is 100 Mbit/s or more for $M \geq 32$.

Answer 6.5. (a) The data decoding order is user 2, user 1, and user 3, respectively. There is no time-sharing. (b) The data decoding order is user 1, user 2, and user 3, respectively. There is no time-sharing. (c) The data decoding order is sometimes user 2, user 1, and user 3, respectively. At other times, according to the time-sharing, it is user 1, user 2, and user 3, respectively.

Answer 6.6. Suppose each optimal power coefficient is strictly less than the maximum uplink power. Apply the scaling in Step 4 of Algorithm 6.1 and obtain a new set of uplink powers with at least one equal to the maximum uplink power. Verify that the new power coefficients increase the minimum rate, contradicting the initial assumption.

Answer 6.7. Having a non-zero value of $\dot{\mathbf{w}}_k$ will only degrade the rate of user k. Hence, LMMSE combining is a linear combination of the channels.

Answer 6.8. From Exercise 6.7, we know that any receive combining vector can be expressed as $\mathbf{w}_k = \alpha_{k,1}\mathbf{h}_1 + \alpha_{k,2}\mathbf{h}_2$ for some $\alpha_{k,1}, \alpha_{k,2} \in \mathbb{C}$, for $k = 1, 2$. If $\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2 = 0$, i.e., orthogonal channels, MRC will be the optimal combiner, which leads to $\alpha_k^{\mathrm{MRC}} = \alpha_{k,k}$

and $\alpha_k^{\mathrm{ZF}} = 0$. Otherwise, for any given $\alpha_{1,1}, \alpha_{1,2}, \alpha_{2,1}, \alpha_{2,2}$, setting

$$\begin{aligned}
\alpha_1^{\mathrm{ZF}} &= -\alpha_{1,2} \frac{\|\mathbf{h}_1\|^2\|\mathbf{h}_2\|^2 - |\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2|^2}{\mathbf{h}_2^{\mathrm{H}}\mathbf{h}_1} \\
\alpha_1^{\mathrm{MRC}} &= \alpha_{1,1} + \alpha_{1,2} \frac{\|\mathbf{h}_2\|^2}{\mathbf{h}_2^{\mathrm{H}}\mathbf{h}_1} \\
\alpha_2^{\mathrm{ZF}} &= -\alpha_{2,1} \frac{\|\mathbf{h}_1\|^2\|\mathbf{h}_2\|^2 - |\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2|^2}{\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2} \\
\alpha_2^{\mathrm{MRC}} &= \alpha_{2,2} + \alpha_{2,1} \frac{\|\mathbf{h}_1\|^2}{\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2}
\end{aligned}$$

will result in $\mathbf{w}_k = \alpha_{k,1}\mathbf{h}_1 + \alpha_{k,2}\mathbf{h}_2 = \alpha_k^{\mathrm{MRC}}\mathbf{w}_k^{\mathrm{MRC}} + \alpha_k^{\mathrm{ZF}}\mathbf{w}_k^{\mathrm{ZF}}$, which proves the claim.

Answer 6.9. The same rate expression $R_1^{\mathrm{ZF}} = B\log_2\left(1 + \frac{P_1^{\mathrm{ul}}\left(\|\mathbf{h}_1\|^2\|\mathbf{h}_2\|^2 - |\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_2|^2\right)}{BN_0\|\mathbf{h}_2\|^2}\right)$ is obtained either using $\mathbf{w}_1^{\mathrm{ZF}} = \mathbf{W}^{\mathrm{ZF}}[1\ 0]^{\mathrm{T}} = \mathbf{H}\left(\mathbf{H}^{\mathrm{H}}\mathbf{H}\right)^{-1}[1\ 0]^{\mathrm{T}}$ or $\mathbf{w}_1^{\text{ZF-alternative}} = \left(\mathbf{I}_M - \frac{\mathbf{h}_2}{\|\mathbf{h}_2\|}\frac{\mathbf{h}_2^{\mathrm{H}}}{\|\mathbf{h}_2\|}\right)\mathbf{h}_1$.

Answer 6.10. (a)$R_1^{\text{grouping}} = R_2^{\text{grouping}} = B\log_2\left(1 + \frac{MP\beta}{4BN_0}\right)$ and the sum rate is $2B\log_2\left(1 + \frac{MP\beta}{4BN_0}\right)$. (b) $R_1^{\text{multi-user MIMO}} = R_2^{\text{multi-user MIMO}} = B\log_2\left(1 + \frac{MP\beta}{2BN_0}\right)$ and the sum rate is $2B\log_2\left(1 + \frac{MP\beta}{2BN_0}\right)$, which is higher than the sum rate obtained in (a).

Answer 6.11. (a) The power of the received interfering signal is $\frac{P\beta_2}{4}\frac{\sin^2\left(4\frac{\pi(\sin(\varphi_1)-\sin(\varphi_2))}{2}\right)}{\sin^2\left(\frac{\pi(\sin(\varphi_1)-\sin(\varphi_2))}{2}\right)}$. (b) The relation should be $\sin(\varphi_1) = \sin(\varphi_2) + \frac{n}{2}$ for $n = \pm 1, 2, 3$. (c) $\varphi_1 = 0$, $\varphi_2 = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$, $\varphi_3 = \arcsin\left(\frac{-1}{2}\right) = -\frac{\pi}{6}$, $\varphi_4 = \arcsin(1) = \frac{\pi}{2}$. (d) ZF precoding can be utilized.

Answer 6.12. (a) The fixed-point algorithm is given in Algorithm A.6.1, where $\omega_k = 4$, for $k = 1, \ldots, K_{\mathrm{p}}$ and $\omega_k = 1$, for $k = K_{\mathrm{p}} + 1, \ldots, K$.

Algorithm A.6.1 Solving the problem in (6.155).

1: **Initialization:** Select arbitrary $P_k^{\mathrm{dl}} \in (0, P]$, for $k = 1, \ldots, K$, and the solution accuracy $\epsilon > 0$
2: **while** $\max\limits_{i\in\{1,\ldots,K\}} \frac{\mathrm{SINR}_i\left(P_1^{\mathrm{dl}},\ldots,P_K^{\mathrm{dl}}\right)}{\omega_i} - \min\limits_{i\in\{1,\ldots,K\}} \frac{\mathrm{SINR}_i\left(P_1^{\mathrm{dl}},\ldots,P_K^{\mathrm{dl}}\right)}{\omega_i} > \epsilon$ **do**
3: $\quad P_k^{\mathrm{dl}} \leftarrow \frac{\min_{i\in\{1,\ldots,K\}} \frac{\mathrm{SINR}_i\left(P_1^{\mathrm{dl}},\ldots,P_K^{\mathrm{dl}}\right)}{\omega_i}}{\frac{\mathrm{SINR}_k\left(P_1^{\mathrm{dl}},\ldots,P_K^{\mathrm{dl}}\right)}{\omega_k}} P_k^{\mathrm{dl}}$, for $k = 1, \ldots, K$
4: $\quad P_k^{\mathrm{dl}} \leftarrow \frac{P}{\sum_{i=1}^{K} P_i^{\mathrm{dl}}} P_k^{\mathrm{dl}}$, for $k = 1, \ldots, K$
5: **end while**
6: **Output:** $P_1^{\mathrm{dl}}, \ldots, P_K^{\mathrm{dl}}$

(b) $$P_k^{\mathrm{dl}} = \begin{cases} \frac{4P[(\mathbf{H}^{\mathrm{T}}\mathbf{H}^*)^{-1}]_{kk}}{\sum_{i=1}^{K_{\mathrm{p}}} 4[(\mathbf{H}^{\mathrm{T}}\mathbf{H}^*)^{-1}]_{ii} + \sum_{i=K_{\mathrm{p}}+1}^{K}[(\mathbf{H}^{\mathrm{T}}\mathbf{H}^*)^{-1}]_{ii}}, & k = 1, \ldots, K_{\mathrm{p}}, \\ \frac{P[(\mathbf{H}^{\mathrm{T}}\mathbf{H}^*)^{-1}]_{kk}}{\sum_{i=1}^{K_{\mathrm{p}}} 4[(\mathbf{H}^{\mathrm{T}}\mathbf{H}^*)^{-1}]_{ii} + \sum_{i=K_{\mathrm{p}}+1}^{K}[(\mathbf{H}^{\mathrm{T}}\mathbf{H}^*)^{-1}]_{ii}}, & k = K_{\mathrm{p}} + 1, \ldots, K. \end{cases}$$

Answer 6.13. (a) $R_{\text{sum}}^{\text{FDMA}} = 10\log_2(1+8) \approx 31.7\,\text{Mbit/s}$.

(b) $R_{\text{sum}}^{\text{MRC}} = 20\log_2\left(1 + \frac{4}{\frac{1}{4}\frac{\sin^2\left(4\frac{\pi(\sin(0)-\sin(\pi/8))}{2}\right)}{\sin^2\left(\frac{\pi(\sin(0)-\sin(\pi/8))}{2}\right)}+1}\right) \approx 39.7\,\text{Mbit/s}$. It is higher than the rate achieved with FDMA. (c) $R_{\text{sum}}^{\text{FDMA}} = 10\log_2(1+16) \approx 40.9\,\text{Mbit/s}$.

(d) $R_{\text{sum}}^{\text{MRC}} = 20\log_2\left(1 + \frac{8}{\frac{1}{8}\frac{\sin^2\left(8\frac{\pi(\sin(0)-\sin(\pi/8))}{2}\right)}{\sin^2\left(\frac{\pi(\sin(0)-\sin(\pi/8))}{2}\right)}+1}\right) \approx 55.2\,\text{Mbit/s}$. It is higher than the rate achieved with FDMA. The gap between FDMA and multi-user MIMO is higher with an increased number of antennas.

Answer 6.14. (a) $R_k^{\text{MRC}} = B\mathbb{E}\left\{\log_2\left(1 + \frac{P_k^{\text{ul}}\|\mathbf{h}_k\|^2}{\sum_{i=1,i\neq k}^{K} P_i^{\text{ul}}\frac{|\mathbf{h}_i^{\text{H}}\mathbf{h}_k|^2}{\|\mathbf{h}_k\|^2}+BN_0}\right)\right\}$,
$R_k^{\text{ZF}} = B\mathbb{E}\left\{\log_2\left(1 + \frac{P_k^{\text{ul}}}{BN_0\left[\left(\mathbf{H}^{\text{H}}\mathbf{H}\right)^{-1}\right]_{kk}}\right)\right\}$.
(b) $R_k^{\text{MRC}} \geq B\log_2\left(1 + \frac{P_k^{\text{ul}}\beta_k(M-1)}{\sum_{i=1,i\neq k}^{K} P_i^{\text{ul}}\beta_i+BN_0}\right)$,
$R_k^{\text{ZF}} \geq B\log_2\left(1 + \frac{P_k^{\text{ul}}\beta_k(M-K)}{BN_0}\right)$. The lower bounds to the ergodic rates both increase with M. (c) $R_k^{\text{MRC}} \geq B\log_2\left(1 + \frac{P\beta(M-1)}{P\beta(K-1)+BN_0}\right)$,
$R_k^{\text{ZF}} \geq B\log_2\left(1 + \frac{P\beta(M-K)}{BN_0}\right)$. As M goes to infinity, the ratio of the lower bounds to the ergodic rates achieved with ZF combining and MRC goes to 1 while their difference converges to $B\log_2\left(1 + \frac{P\beta(K-1)}{BN_0}\right)$.

Answer 6.15. (a) $R_k^{\text{MRT}} = B\mathbb{E}\left\{\log_2\left(1 + \frac{P_k^{\text{dl}}\|\mathbf{h}_k\|^2}{\sum_{i=1,i\neq k}^{K} P_i^{\text{dl}}\frac{|\mathbf{h}_k^{\text{T}}\mathbf{h}_i^*|^2}{\|\mathbf{h}_i\|^2}+BN_0}\right)\right\}$,
$R_k^{\text{ZF}} = B\mathbb{E}\left\{\log_2\left(1 + \frac{P_k^{\text{dl}}}{BN_0\left[\left(\mathbf{H}^{\text{T}}\mathbf{H}^*\right)^{-1}\right]_{kk}}\right)\right\}$.
(b) $R_k^{\text{MRT}} \geq B\log_2\left(1 + \frac{P_k^{\text{dl}}\beta_k(M-1)}{\sum_{i=1,i\neq k}^{K} \frac{P_i^{\text{dl}}\beta_k(M-1)}{M}+BN_0}\right)$,
$R_k^{\text{ZF}} \geq B\log_2\left(1 + \frac{P_k^{\text{dl}}\beta_k(M-K)}{BN_0}\right)$. The lower bounds to the ergodic rates both increase with M. (c) $R_k^{\text{MRT}} \geq B\log_2\left(1 + \frac{P\beta\frac{M-1}{K}}{\frac{P\beta(K-1)(M-1)}{KM}+BN_0}\right)$,
$R_k^{\text{ZF}} \geq B\log_2\left(1 + \frac{P\beta\left(\frac{M}{K}-1\right)}{BN_0}\right)$. As M goes to infinity, the ratio of the lower bounds to the ergodic rates achieved with ZF precoding and MRT goes to 1 while their difference converges to $B\log_2\left(1 + \frac{P\beta\frac{K-1}{K}}{BN_0}\right)$.

Answer 6.16. (a) $\hat{\mathbf{h}}_1 = \frac{\beta_1\sqrt{\frac{2P}{B}}}{\beta_1\frac{2P}{B}+N_0}\begin{bmatrix}\mathbf{y}[1] & \mathbf{y}[2]\end{bmatrix}\frac{\boldsymbol{\phi}_1^*}{\|\boldsymbol{\phi}_1\|}$, $\hat{\mathbf{h}}_2 = \frac{\beta_2\sqrt{\frac{2P}{B}}}{\beta_2\frac{2P}{B}+N_0}\begin{bmatrix}\mathbf{y}[1] & \mathbf{y}[2]\end{bmatrix}\frac{\boldsymbol{\phi}_2^*}{\|\boldsymbol{\phi}_2\|}$.
(b) $R_1 = \left(1 - \frac{2}{L_c}\right)B\mathbb{E}\left\{\log_2\left(1 + \frac{P_1^{\text{ul}}\|\hat{\mathbf{h}}_1\|^2}{P_1^{\text{ul}}\frac{\beta_1 BN_0}{2\beta_1 P+BN_0}+P_2^{\text{ul}}\beta_2+BN_0}\right)\right\}$,

$R_2 = \left(1 - \frac{2}{L_c}\right) B\mathbb{E}\left\{\log_2\left(1 + \frac{P_2^{\mathrm{ul}}\left\|\hat{\mathbf{h}}_2\right\|^2}{P_2^{\mathrm{ul}}\frac{\beta_2 BN_0}{2\beta_2 P + BN_0} + P_1^{\mathrm{ul}}\beta_1 + BN_0}\right)\right\}.$

Answer 6.17. (a) $P_{\mathrm{out},k}(R_k) = \Pr\left\{\left\|\breve{\mathbf{h}}_k\right\|^2 < \frac{BN_0\left(2^{\frac{R_k}{B}}-1\right)}{P_k^{\mathrm{ul}}}\right\}.$

(b) $P_{\mathrm{out},k}(R_k) \leq \left(\frac{2^{\frac{R_k}{B}}-1}{\mathrm{SNR}_k}\right)^{M-K+1}\frac{1}{(M-K+1)!}$, where $\mathrm{SNR}_k = \frac{P_k^{\mathrm{ul}}\beta_k}{BN_0}$. Since the upper bound is proportional to $\mathrm{SNR}_k^{-(M-K+1)}$, the diversity order is $(M-K+1)$.

Answer 6.18. (a) $R_k^{\mathrm{ul}} = B\log_2\left(\det\left(\mathbf{I}_M + \left(\mathbf{C}_k^{\mathrm{ul}}\right)^{-1/2}\mathbf{H}_k\mathbf{P}_k^{\mathrm{ul}}\mathbf{Q}_k^{\mathrm{ul}}\left(\mathbf{P}_k^{\mathrm{ul}}\right)^{\mathrm{H}}\mathbf{H}_k^{\mathrm{H}}\left(\mathbf{C}_k^{\mathrm{ul}}\right)^{-1/2}\right)\right)$
where $\mathbf{C}_k^{\mathrm{ul}} = \sum_{i=1,i\neq k}^{K}\mathbf{H}_i\mathbf{P}_i^{\mathrm{ul}}\mathbf{Q}_i^{\mathrm{ul}}\left(\mathbf{P}_i^{\mathrm{ul}}\right)^{\mathrm{H}}\mathbf{H}_i^{\mathrm{H}} + N_0\mathbf{I}_M$.
(b) $R_k^{\mathrm{dl}} = B\log_2\left(\det\left(\mathbf{I}_N + \left(\mathbf{C}_k^{\mathrm{dl}}\right)^{-1/2}\mathbf{H}_k^{\mathrm{T}}\mathbf{P}_k^{\mathrm{dl}}\mathbf{Q}_k^{\mathrm{dl}}\left(\mathbf{P}_k^{\mathrm{dl}}\right)^{\mathrm{H}}\mathbf{H}_k^{*}\left(\mathbf{C}_k^{\mathrm{dl}}\right)^{-1/2}\right)\right)$
where $\mathbf{C}_k^{\mathrm{dl}} = \sum_{i=1,i\neq k}^{K}\mathbf{H}_k^{\mathrm{T}}\mathbf{P}_i^{\mathrm{dl}}\mathbf{Q}_i^{\mathrm{dl}}\left(\mathbf{P}_i^{\mathrm{dl}}\right)^{\mathrm{H}}\mathbf{H}_k^{*} + N_0\mathbf{I}_N$.

Exercises in Chapter 7

Answer 7.1. (a) $\bar{h}[\nu] = \frac{1-e^{-\mathrm{j}2\pi(T+1)\nu/S}}{1-e^{-\mathrm{j}2\pi\nu/S}}$ for $\nu = 0, \ldots, S-1$.
(b) $\left|\bar{h}[\nu]\right|^2 = \frac{\sin^2\left((T+1)\frac{\pi\nu}{S}\right)}{\sin^2\left(\frac{\pi\nu}{S}\right)}$ for $\nu = 0, \ldots, S-1$. (c) $\nu^\star = 8$ and the first-null coherence bandwidth is 240 kHz. (d) $\nu^\star = 4$ and the first-null coherence bandwidth is 120 kHz. The coherence bandwidth is smaller than in (c), so there is an inverse relationship with T.

Answer 7.2. The identity can be proved using the Frobenius norm property in (5.88).

Answer 7.3. (a) $\tau_{\text{spread}} = 260$ ns. (b) $\eta = 75$ ns. (c) $T = 7$, $h[0] = \alpha_1 e^{-\mathrm{j}150\pi}$, $h[1] = \alpha_1 e^{-\mathrm{j}150\pi} + \alpha_2 e^{-\mathrm{j}450\pi}$, $h[2] = \alpha_2 e^{-\mathrm{j}450\pi}$, $h[3] = h[4] = 0$, $h[5] = h[6] = \alpha_3 e^{-\mathrm{j}1710\pi}$, and $h[7] = 0.4 \cdot \alpha_3 e^{-\mathrm{j}1710\pi}$.

Answer 7.4. (a) 30 kHz. (b) $B = 120$ MHz.

Answer 7.5. (a) $\mathbb{E}\left\{\bar{h}[\nu]\bar{h}^*[\nu']\right\} = \frac{\beta}{4}\frac{1-e^{-\mathrm{j}2\pi 4(\nu-\nu')/32}}{1-e^{-\mathrm{j}2\pi(\nu-\nu')/32}}$. (b) The correlation among the subcarriers decays the further they are from each other.

Answer 7.6. (a) The total estimation variance is SN_0/q. (b) The total estimation variance is $(T+1)N_0/q$. (c) The variance is $S/(T+1) = 100$ times larger in (a) than in (b) since we are not utilizing that the many subcarrier channels are created by a smaller number of channel taps.

Answer 7.7. (a) The horizontal and vertical first-null beamwidths are 0.51 and 0.36 radians, respectively. (b) The beam has a horizontal width of around 25.5 m and a vertical width of around 18 m. (c) The horizontal first-null beamwidth is 0.51 radians. The vertical first-null beamwidth of 1.59 radians, which is roughly four times larger than in (a). The maximum beamforming gain is 64 for the array in Figure 7.18 and 16 for the alternative UPA.

Answer 7.8. The achievable rate is $\sum_{n=1}^{N_{\text{RF}}} \log_2\left(1 + \frac{q\beta_0}{N_0}\frac{M^2}{e^{-(n-1)/\Gamma}N_{\text{RF}}}\right)$ bit per subcarrier symbol.

Answer 7.9. (a) The solution is $\beta_1 = 0$ and $\beta_2 = \ldots = \beta_{N_{\text{cl}}} = 0$ for all three architectures. (b) Analog beamforming has the same solution as in (a). With hybrid beamforming, we want $\beta_1 = \ldots = \beta_{N_{\text{RF}}} = \beta/N_{\text{RF}}$ while the remaining variables are zero. With digital beamforming, we want $\beta_1 = \ldots = \beta_M = \beta/M$.

Answer 7.10. (a) $2S \cdot \log_2\left(1 + \frac{2q\beta}{N_0}\right)$ bit per OFDM symbol.
(b) $\bar{\boldsymbol{\chi}}[\nu] \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}, \mathrm{diag}(q/2, q/2))$.

Answer 7.11. The fraction is $\frac{MK\kappa+1}{MK(\kappa+1)}$ and is a monotonically increasing function of κ.

Answer 7.12. (a) $\varphi = \arcsin\left(\left(1 + 0.1\frac{\nu}{S}\right)\frac{\sqrt{3}}{2}\right)$. (b) No. (c) 5.4 degrees.

Answer 7.13. The squared Frobenius norm is $\sum_{i=\min(r_\nu, N_{\text{RF}})+1}^{r_\nu} s_i^4$, where r_ν is the rank of $\bar{\mathbf{H}}[\nu]$ and s_i is its ith singular value.

Exercises in Chapter 8

Answer 8.1. (a) Since the number of non-zero eigenvalues of $\hat{\mathbf{R}}_L$ is equal to the number of non-zero singular values of $\mathbf{Y}_L$, where the latter is at most L, the rank of $\hat{\mathbf{R}}_L$ is at most L. Hence, it is a rank-deficient matrix. (b) When $\mathbf{w} = \tilde{\mathbf{U}}\tilde{\mathbf{x}}/\|\tilde{\mathbf{x}}\|^2$ then $|\mathbf{w}^{\mathrm{H}}\mathbf{a}(\varphi,\theta)| = 1$ and the objective function $\mathbf{w}^{\mathrm{H}}\hat{\mathbf{R}}_L\mathbf{w}$ in (8.17) becomes zero, which is the optimal solution since $\mathbf{w}^{\mathrm{H}}\hat{\mathbf{R}}_L\mathbf{w}$ is lower bounded by zero. (c) The modified Capon spectrum is $P(\varphi,\theta) = \frac{1}{\sum_{m=1}^{L}\frac{|\bar{x}_m|^2}{s_m^2/L+\epsilon}+\frac{\|\tilde{\mathbf{x}}\|^2}{\epsilon}}$, where $\bar{x}_m$ denotes the mth entry of $\bar{\mathbf{x}}$. Its value differs for the array response vectors that satisfy $\tilde{\mathbf{x}} = \tilde{\mathbf{U}}^{\mathrm{H}}\mathbf{a}(\varphi,\theta) \neq \mathbf{0}$ according to $\bar{\mathbf{x}}$ and $\tilde{\mathbf{x}}$.

Answer 8.2. (a) $\mathbf{a}(z) = [1, z, z^2, \ldots, z^{M-1}]^{\mathrm{T}}$. (b) The DOA estimates can be found by the smallest values of $\mathbf{a}^{\mathrm{H}}(\varphi)\hat{\mathbf{U}}_{\mathrm{n}}\hat{\mathbf{U}}_{\mathrm{n}}^{\mathrm{H}}\mathbf{a}(\varphi)$, which is the denominator of the MUSIC spectrum in (8.46). Noting that $\mathbf{a}^{\mathrm{H}}(\varphi) = \mathbf{a}^{\mathrm{T}}(z^{-1})$, the aim is to find the complex values z on the unit circle that results in as small values of $\mathbf{a}^{\mathrm{T}}(z^{-1})\hat{\mathbf{U}}_{\mathrm{n}}\hat{\mathbf{U}}_{\mathrm{n}}^{\mathrm{H}}\mathbf{a}(z)$ as possible. The complex roots appear in reciprocal pairs, i.e., if z is a root, then $(z^*)^{-1}$ is also a root. For each pair of roots, the projection onto the closest angle on the unit circle becomes the same. As a result, for K sources, the projections of K complex roots (which appear in reciprocal pairs) onto the unit circle are found, which are closest to the unit circle in the complex plane. The DOA estimates $\hat{\varphi}_k$ are obtained from these K angles given by $-2\pi\frac{\Delta\sin(\hat{\varphi}_k)}{\lambda}$.

Answer 8.3. (a) No, the azimuth angle pairs $(0,\pi)$ and $(\pi/2, -\pi/2)$ create ambiguity since the corresponding array response vectors are identical. (b) We can unambiguously estimate the DOA for all the angles $\varphi \in [0, 2\pi)$ since each angle corresponds to a unique array response vector.

Answer 8.4. (a) $\mathbf{p}^\star[l] = \left(\mathbf{A}^{\mathrm{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{y}[l]$, $l = 1, \ldots, L$. (b) After inserting $\mathbf{p}^\star[l]$ into the objective function, it becomes $\sum_{l=1}^{L}\left(\mathbf{y}^{\mathrm{H}}[l]\mathbf{y}[l] - \mathbf{y}^{\mathrm{H}}[l]\mathbf{A}\left(\mathbf{A}^{\mathrm{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{y}[l]\right)$. Omitting the constant term $\sum_{l=1}^{L}\mathbf{y}^{\mathrm{H}}[l]\mathbf{y}[l]$, the DOAs that minimize the objective function are the ones that maximize $\sum_{l=1}^{L}\mathbf{y}^{\mathrm{H}}[l]\mathbf{A}\left(\mathbf{A}^{\mathrm{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{y}[l]$. (c) When $K = 1$, the function to be maximized becomes $\frac{1}{M}\sum_{l=1}^{L}\left|\mathbf{a}^{\mathrm{H}}(\varphi_1,\theta_1)\mathbf{y}[l]\right|^2$, which is L/M times of the spectrum $P(\varphi,\theta)$ of the conventional beamforming in (8.7).

Answer 8.5. (a) The MUSIC spectrum for the whitened signal $\mathbf{y}^{\mathrm{w}}[l] = \mathbf{C}^{-1/2}\mathbf{y}[l]$ is $\frac{1}{\mathbf{a}^{\mathrm{H}}(\varphi,\theta)\mathbf{C}^{-1/2}\hat{\mathbf{U}}_{\mathrm{n}}^{\mathrm{w}}\left(\hat{\mathbf{U}}_{\mathrm{n}}^{\mathrm{w}}\right)^{\mathrm{H}}\mathbf{C}^{-1/2}\mathbf{a}(\varphi,\theta)}$ where $\hat{\mathbf{U}}_{\mathrm{n}}^{\mathrm{w}} \in \mathbb{C}^{M\times(M-K)}$ be the matrix whose columns are the eigenvectors of $\hat{\mathbf{R}}_L^{\mathrm{w}} = \frac{1}{L}\sum_{l=1}^{L}\mathbf{y}^{\mathrm{w}}[l]\left(\mathbf{y}^{\mathrm{w}}[l]\right)^{\mathrm{H}}$ corresponding to the smallest $(M-K)$ eigenvalues. (b) The MUSIC spectrum is $\frac{1}{\mathbf{a}^{\mathrm{H}}(\varphi,\theta)\mathbf{M}^{\mathrm{H}}\hat{\mathbf{U}}_{\mathrm{n}}\hat{\mathbf{U}}_{\mathrm{n}}^{\mathrm{H}}\mathbf{M}\mathbf{a}(\varphi,\theta)}$, where $\hat{\mathbf{U}}_{\mathrm{n}} \in \mathbb{C}^{M\times(M-K)}$ be the matrix whose columns are the eigenvectors of $\hat{\mathbf{R}}_L = \frac{1}{L}\sum_{l=1}^{L}\mathbf{y}[l]\mathbf{y}[l]^{\mathrm{H}}$ corresponding to the smallest $(M-K)$ eigenvalues.

Answer 8.6. (a) The ML cost function is $(\mathbf{r}-\mathbf{m}(x,y))^{\mathrm{T}}\,\mathbf{C}^{-1}\,(\mathbf{r}-\mathbf{m}(x,y))$, where

$$\mathbf{r}=\begin{bmatrix}\mathbf{r}_{\mathrm{TDOA}}\\ \mathbf{r}_{\mathrm{DOA}}\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}\mathbf{C}_{\mathrm{TDOA}} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{\mathrm{DOA}}\end{bmatrix},$$

$$\mathbf{m}(x,y)=\begin{bmatrix}\sqrt{(x_2-x)^2+(y_2-y)^2}-\sqrt{(x_1-x)^2+(y_1-y)^2}\\ \vdots\\ \sqrt{(x_{\overline{M}}-x)^2+(y_{\overline{M}}-y)^2}-\sqrt{(x_1-x)^2+(y_1-y)^2}\\ \arctan\left(\frac{y-y_{\overline{M}+1}}{x-x_{\overline{M}+1}}\right)\\ \vdots\\ \arctan\left(\frac{y-y_M}{x-x_M}\right)\end{bmatrix}.$$

(b) The minimum number of receivers is $M=3$ with $\overline{M}=2$. (c) The minimum number of receivers is $M=4$ with $\overline{M}=2$ or $\overline{M}=3$.

Answer 8.7. (a)

$$\mathbf{b}=\begin{bmatrix}r_1^2-x_1^2-y_1^2\\ \vdots\\ r_M^2-x_M^2-y_M^2\end{bmatrix},\quad \mathbf{A}=\begin{bmatrix}-2x_1 & -2y_1 & 1\\ \vdots & \vdots & \vdots\\ -2x_M & -2y_M & 1\end{bmatrix},$$

$$\mathbf{w}=\begin{bmatrix}n_1^2+2n_1\sqrt{(x_1-x)^2+(y_1-y)^2}\\ \vdots\\ n_M^2+2n_M\sqrt{(x_M-x)^2+(y_M-y)^2}\end{bmatrix}.$$

(b) No, the mean is equal to $\mathbb{E}\{n_m^2\}>0$. If the noise is sufficiently small so that $\mathbb{E}\{n_m^2\}\approx 0$, then we can approximate $\mathbf{w}$ as a zero-mean vector.

Answer 8.8. The ML cost function is $\frac{\|\mathbf{r}_{\mathrm{LOS}}-\mathbf{d}_{\mathrm{LOS}}(x,y)\|^2}{2\sigma_{\mathrm{d}}^2}+\frac{\mathbf{1}_{\overline{M}}^{\mathrm{T}}(\mathbf{r}_{\mathrm{NLOS}}-\mathbf{d}_{\mathrm{NLOS}})}{\sigma_b^2}$ where $\mathbf{1}_{\overline{M}}$ is the $\overline{M}$-dimensional all ones vector, $\mathbf{r}_{\mathrm{NLOS}}=[r_1,\ldots,r_{\overline{M}}]^{\mathrm{T}}$, $\mathbf{r}_{\mathrm{LOS}}=[r_{\overline{M}+1},\ldots,r_M]^{\mathrm{T}}$, $\mathbf{r}=[\mathbf{r}_{\mathrm{NLOS}}^{\mathrm{T}},\mathbf{r}_{\mathrm{LOS}}^{\mathrm{T}}]^{\mathrm{T}}$, and

$$\mathbf{d}_{\mathrm{NLOS}}(x,y)=\begin{bmatrix}\sqrt{(x_1-x)^2+(y_1-y)^2}\\ \vdots\\ \sqrt{(x_{\overline{M}}-x)^2+(y_{\overline{M}}-y)^2}\end{bmatrix},$$

$$\mathbf{d}_{\mathrm{LOS}}(x,y)=\begin{bmatrix}\sqrt{(x_{\overline{M}+1}-x)^2+(y_{\overline{M}+1}-y)^2}\\ \vdots\\ \sqrt{(x_M-x)^2+(y_M-y)^2}\end{bmatrix}.$$

Answer 8.9. $\sum_{l=1}^{L}\Re(y[l]h^*)$ is the sufficient statistics for the Neyman-Pearson detector.

Answer 8.10. (a) $\mathrm{SNR}=LM^2\varrho$. (b) $L=4$ and $M=5$.

Answer 8.11. (a) At least $M=13$ antennas are needed. (b) At least $L=160$ symbols are needed.

Answer 8.12. The sufficient statistics is $\frac{P_{\mathrm{r},1}\sigma^{-4}}{1+P_{\mathrm{r},1}L\sigma^{-2}}\left|\mathbf{1}_L^{\mathrm{H}}\mathbf{y}_1\right|^2+\frac{P_{\mathrm{r},2}\sigma^{-4}}{1+P_{\mathrm{r},2}L\sigma^{-2}}\left|\mathbf{1}_L^{\mathrm{H}}\mathbf{y}_2\right|^2$, where $\mathbf{y}_m=[y_m[1],\ldots,y_m[L]]^{\mathrm{T}}\in\mathbb{C}^L$ and $\mathbf{n}_m=[n_m[1],\ldots,n_m[L]]^{\mathrm{T}}\sim\mathcal{N}_{\mathbb{C}}(\mathbf{0},\sigma^2\mathbf{I}_L)$, for $m=1,2$.

Answer 8.13. The sufficient statistics is $\sum_{l=1}^{L}\frac{P_{\mathrm{r}}|x[l]|^2}{P_{\mathrm{r}}|x[l]|^2+\sigma^2}|y[l]|^2$.

Answer 8.14. (a) The sufficient statistics becomes $|\mathbf{1}_L^{\mathrm{H}}\mathbf{C}^{-1}\mathbf{y}|^2$. This operation is equivalent to firstly whitening the received signal as $\mathbf{C}^{-1/2}\mathbf{y}=\sqrt{P_{\mathrm{r}}}\mathbf{C}^{-1/2}\mathbf{1}_L h+\mathbf{C}^{-1/2}\mathbf{n}$, taking the inner product of the resulting signal with the vector in front of h, i.e., $\mathbf{C}^{-1/2}\mathbf{1}_L$, and checking the value of the norm square of it. (b) The sufficient statistics is $\mathbf{y}^{\mathrm{H}}\mathbf{C}^{-1}\mathbf{y}-\mathbf{y}^{\mathrm{H}}\left(P_{\mathrm{r}}\mathbf{I}_L+\mathbf{C}\right)^{-1}\mathbf{y}$. This operation is equivalent to checking the difference between norm squares of the whitened signal $\mathbf{C}^{-1/2}\mathbf{y}$ under hypothesis $\mathcal{H}_0$ and the whitened signal $(P_{\mathrm{r}}\mathbf{I}_L+\mathbf{C})^{-1/2}\mathbf{y}$ under hypothesis $\mathcal{H}_1$.

Answer 8.15. (a) $\mathrm{SNR}_{\mathrm{t}}=\frac{P\sigma_{\mathrm{RCS}}\left|\mathbf{w}^{\mathrm{H}}\mathbf{h}_{\mathrm{r}}\right|^2\left|\mathbf{h}_{\mathrm{t}}^{\mathrm{T}}\mathbf{p}\right|^2}{\sigma^2}$. (b) $\mathbf{w}=\frac{\mathbf{h}_{\mathrm{r}}}{\|\mathbf{h}_{\mathrm{r}}\|}$. (c) $\mathbf{p}=\frac{2\sqrt{2}}{3}e^{\mathrm{j}\phi_{\mathrm{t}}}\frac{\mathbf{h}_{\mathrm{t}}^*}{\|\mathbf{h}_{\mathrm{t}}\|}+\frac{1}{3}e^{\mathrm{j}\phi_{\mathrm{u}}}\frac{\mathbf{h}_{\mathrm{u}}^*}{\|\mathbf{h}_{\mathrm{u}}\|}$, where ϕ_{t} and ϕ_{u} are arbitrary phase-shifts.

Exercises in Chapter 9

Answer 9.1. (a) $N \geq 4143$. (b) $N \geq 42$.

Answer 9.2. (a) $\mathbb{E}\left\{|h|^2\right\} = N\beta_r\beta_t$. (b)$|h|^2 = N^2\beta_r\beta_t$. (c) The channel gain in (b) is larger than the average channel gain in (a). The beamforming gain from the reflection is missing in (a), but the aperture gain remains.

Answer 9.3. (a) $N\beta_\mathrm{r}\beta_\mathrm{t}$. (b) $N\beta_\mathrm{r}\beta_\mathrm{t} + N(N-1)\frac{\pi^2}{16}\beta_\mathrm{r}\beta_\mathrm{t}$.

Answer 9.4. (a) $N^2\beta_\mathrm{r}\beta_\mathrm{t} = \frac{N^2\lambda^4}{(64\pi d_\mathrm{t} d_\mathrm{r})^2}$. (b) $N = 64$, $NA_\mathrm{m} = 0.04\,\mathrm{m}^2$. (c) $N = 6359$, $NA_\mathrm{m} \approx 0.04\,\mathrm{m}^2$.

Answer 9.5. (a) $\widehat{\check{\mathbf{h}}} = \frac{1}{\sqrt{q}}\mathbf{D}_\psi^{-1}\check{\mathbf{y}}$ where $\mathbf{D}_\psi = \mathrm{diag}(e^{\mathrm{j}\psi_1}, \ldots, e^{\mathrm{j}\psi_N})$. (b) $\widehat{\check{\mathbf{h}}} = \frac{1}{\sqrt{qN}}\mathbf{F}_N^\mathrm{H}\check{\mathbf{y}}$. (c) In (a), we have $\widehat{\check{\mathbf{h}}} = \check{\mathbf{h}} + \frac{1}{\sqrt{q}}\mathbf{D}_\psi^{-1}\mathbf{n}$ with the variance of the noise terms being equal to $\frac{N_0}{q}$. In (b), we have $\widehat{\check{\mathbf{h}}} = \check{\mathbf{h}} + \frac{1}{\sqrt{qN}}\mathbf{F}_N^\mathrm{H}\mathbf{n}$ with the variance of the noise terms being equal to $\frac{N_0}{qN}$.

Answer 9.6. (a) $\mathbf{h}_\mathrm{t} = \sqrt{\beta_\mathrm{t}}\begin{bmatrix}\mathbf{a}_{N_\mathrm{H}}(\varphi_\mathrm{i}, 0)\\ \vdots \\ \mathbf{a}_{N_\mathrm{H}}(\varphi_\mathrm{i}, 0)\end{bmatrix}$, $\mathbf{h}_\mathrm{r} = \sqrt{\beta_\mathrm{r}}\begin{bmatrix}\mathbf{a}_{N_\mathrm{H}}(\varphi_\mathrm{o}, 0)\\ \vdots \\ \mathbf{a}_{N_\mathrm{H}}(\varphi_\mathrm{o}, 0)\end{bmatrix}$.

(b) $h = \underbrace{\begin{bmatrix}1\\ e^{\mathrm{j}\psi_1}\\ \vdots \\ e^{\mathrm{j}\psi_N}\end{bmatrix}^\mathrm{T}}_{=\boldsymbol{\psi}^\mathrm{T}}\underbrace{\begin{bmatrix}h_\mathrm{s}\\ N_\mathrm{V} h_{\mathrm{r},1} h_{\mathrm{t},1}\\ \vdots \\ N_\mathrm{V} h_{\mathrm{r},N_\mathrm{H}} h_{\mathrm{t},N_\mathrm{H}}\end{bmatrix}}_{=\check{\mathbf{h}}}$, where $h_{\mathrm{r},n}$ and $h_{\mathrm{t},n}$ denote the channel from and to any element in the nth column of the surface, for $n = 1, \ldots, N_\mathrm{H}$. The $e^{\mathrm{j}\psi_n}$ represents the same phase-shift assigned to all the metaatoms in the nth column. (c) Given the received signal

$$\underbrace{\begin{bmatrix}y[1]\\ \vdots \\ y[L_\mathrm{p}]\end{bmatrix}}_{=\check{\mathbf{y}}} = \underbrace{\begin{bmatrix}\boldsymbol{\psi}[1] & \ldots & \boldsymbol{\psi}[L_\mathrm{p}]\end{bmatrix}^\mathrm{T}}_{=\boldsymbol{\Psi}}\check{\mathbf{h}}\sqrt{q} + \underbrace{\begin{bmatrix}n[1]\\ \vdots \\ n[L_\mathrm{p}]\end{bmatrix}}_{=\check{\mathbf{n}}},$$

the ML estimator is $\widehat{\check{\mathbf{h}}} = \frac{1}{\sqrt{q}}\boldsymbol{\Psi}^{-1}\check{\mathbf{y}}$, where the matrix $\boldsymbol{\Psi} \in \mathbb{C}^{L_\mathrm{p}\times(N_\mathrm{H}+1)}$ must be invertible. This requires that $L_\mathrm{p} = N_\mathrm{H}+1$. Longer pilot transmission is possible if we replace the matrix inverse $\boldsymbol{\Psi}^{-1}$ by the $\left(\boldsymbol{\Psi}^\mathrm{H}\boldsymbol{\Psi}\right)^{-1}\boldsymbol{\Psi}^\mathrm{H}$. (d) $\frac{N+1}{N_\mathrm{H}+1}$ times smaller total variance.

Answer 9.7. (a) $a = \frac{q_2}{\beta_\mathrm{t} q_1 + N_0}$. (b) $\mathrm{SNR} = \frac{\beta_\mathrm{r}\beta_\mathrm{t}\frac{q_1}{N_0}\frac{q_2}{N_0}}{\beta_\mathrm{r}\frac{q_2}{N_0} + \beta_\mathrm{t}\frac{q_1}{N_0} + 1}$. (c) $N \geq \sqrt{\frac{\frac{q_1}{N_0}\frac{q_2}{N_0}}{\beta_\mathrm{r}\frac{q_2}{N_0}\frac{q}{N_0} + \beta_\mathrm{t}\frac{q_1}{N_0}\frac{q}{N_0} + \frac{q}{N_0}}}$. (d) $N \geq 354$.

Answer 9.8. (a) $\left\|\check{\mathbf{h}}\right\|_1^2 \geq \left\|\check{\mathbf{h}}\right\|^2$. They are equal when there is no reconfigurable surface or when $|h_{\mathrm{r},n}h_{\mathrm{t},n}|$ is only non-zero for one value of n and $h_\mathrm{s} = 0$. (b) The channel gain with MRT is $(N+1)\left\|\check{\mathbf{h}}\right\|^2$, which is larger than $\|\check{\mathbf{h}}\|_1^2$.

Answer 9.9. The main steps of the derivation are as follows:

$$\begin{aligned}(p * g_\mathrm{s} * p)(t) &= \sum_{i=1}^{L_\mathrm{s}} \alpha_{\mathrm{s},i}(p(t) * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t+\eta-\tau_{\mathrm{s},i}) * p(t)) \\ &= \sum_{i=1}^{L_\mathrm{s}} \alpha_{\mathrm{s},i}(p(t) * p(t) * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t+\eta-\tau_{\mathrm{s},i})) \\ &= \sum_{i=1}^{L_\mathrm{s}} \alpha_{\mathrm{s},i} e^{-\mathrm{j}2\pi f_\mathrm{c}(\tau_{\mathrm{s},i}-\eta)} \mathrm{sinc}(B(t+\eta-\tau_{\mathrm{s},i})).\end{aligned}$$

$$\begin{aligned}&(p * g_{\mathrm{r},n} * \vartheta_{n;\psi_n} * g_{\mathrm{t},n} * p)(t) = \sum_{j=1}^{L_\mathrm{r}}\sum_{i=1}^{L_\mathrm{t}} \alpha_{\mathrm{r},n,j}\alpha_{\mathrm{t},n,i}(p(t) * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t+\eta-\tau_{\mathrm{r},n,j}) \\ &\quad * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t-\tau_{\psi_n}) * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t-\tau_{\mathrm{t},n,i}) * p(t)) \\ &= \sum_{j=1}^{L_\mathrm{r}}\sum_{i=1}^{L_\mathrm{t}} \alpha_{\mathrm{r},n,j}\alpha_{\mathrm{t},n,i}(p(t) * p(t) * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t+\eta-\tau_{\mathrm{r},n,j}) \\ &\quad * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t-\tau_{\psi_n}) * e^{-\mathrm{j}2\pi f_\mathrm{c} t}\delta(t-\tau_{\mathrm{t},n,i})) \\ &= \sum_{j=1}^{L_\mathrm{r}}\sum_{i=1}^{L_\mathrm{t}} \alpha_{\mathrm{r},n,j}\alpha_{\mathrm{t},n,i} e^{-\mathrm{j}2\pi f_\mathrm{c}(\tau_{\mathrm{r},n,j}+\tau_{\mathrm{t},n,i}+\tau_{\psi_n}-\eta)} \mathrm{sinc}\big(B(t+\eta-\tau_{\mathrm{r},n,j}-\tau_{\mathrm{t},n,i}-\tau_{\psi_n})\big).\end{aligned}$$

Answer 9.10. (a) $\lim_{f\to 0}\Gamma_{0n} = -1$, whose amplitude and phase are 1 and π, respectively. (b) $\lim_{f\to\infty}\Gamma_{0n} = 1$, whose amplitude and phase are 1 and 0, respectively. (c) If $\mathrm{R}=0$, Z_n is an purely imaginary number, which leads to $|\mathrm{Z}_n - \mathrm{Z}_0| = |\mathrm{Z}_n + \mathrm{Z}_0|$.

Answer 9.11. (a)

$$h_{\boldsymbol{\psi}}[\ell] = \underbrace{\begin{bmatrix} 1 \\ e^{\mathrm{j}\psi_1} \\ \vdots \\ e^{\mathrm{j}\psi_N}\end{bmatrix}^\mathrm{T}}_{=\boldsymbol{\psi}^\mathrm{T}} \underbrace{\begin{bmatrix} c_\mathrm{s}[\ell] \\ \sum_{j=1}^{L_\mathrm{r}}\sum_{i=1}^{L_\mathrm{t}} c_{i,j}[\ell]\mathbf{a}_{N_\mathrm{H},N_\mathrm{V}}(\varphi_{\mathrm{o},j},\theta_{\mathrm{o},j}) \odot \mathbf{a}_{N_\mathrm{H},N_\mathrm{V}}(\varphi_{\mathrm{i},i},\theta_{\mathrm{i},i})\end{bmatrix}}_{=\mathbf{h}[\ell]}.$$

(b) $\sqrt{S}\begin{bmatrix} h_{\boldsymbol{\psi}}[0] & h_{\boldsymbol{\psi}}[1] & \dots & h_{\boldsymbol{\psi}}[T]\end{bmatrix}\mathbf{F}_{S,1:T+1,:} = \boldsymbol{\psi}^\mathrm{T}\begin{bmatrix}\check{\mathbf{h}}[0] & \check{\mathbf{h}}[1] & \dots & \check{\mathbf{h}}[S-1]\end{bmatrix}$ where $\mathbf{F}_{S,1:T+1,:} \in \mathbb{C}^{(T+1)\times S}$ matrix that consists of the first $T+1$ columns of a $S\times S$ DFT matrix. (c) When transmitting $\bar{x}[\nu] = \sqrt{q}$ at the first $T+1$ subcarriers throughout $N+1$ OFDM symbols with the surface configuration $\boldsymbol{\Psi} = \begin{bmatrix}\boldsymbol{\psi}[1] & \dots & \boldsymbol{\psi}[N+1]\end{bmatrix}^\mathrm{T}$, the ML estimate of $\mathbf{H} = \begin{bmatrix}\mathbf{h}[0] & \mathbf{h}[1] & \dots & \mathbf{h}[T]\end{bmatrix}$ can be obtained as $\widehat{\mathbf{H}} = \frac{1}{\sqrt{qS}}\boldsymbol{\Psi}^{-1}\bar{\mathbf{Y}}\mathbf{F}_{S,1:T+1,1:T+1}^{-1}$ where $\bar{\mathbf{Y}} = \begin{bmatrix}\bar{y}_1[0] & \bar{y}_1[1] & \dots & \bar{y}_1[T] \\ \vdots & \vdots & \vdots & \vdots \\ \bar{y}_{N+1}[0] & \bar{y}_{N+1}[1] & \dots & \bar{y}_{N+1}[T]\end{bmatrix}$ and $\bar{y}_l[\nu]$ is the received signal at the subcarrier ν of the lth OFDM symbol. Here, $\mathbf{F}_{S,1:T+1,1:T+1} \in \mathbb{C}^{(T+1)\times(T+1)}$ is the matrix that consists of the first $T+1$ rows and the first $T+1$ columns of the $S\times S$ DFT matrix.

Algorithm A.9.1 Reconfigurable surface configuration for downlink sum capacity maximization.

Initialization: Set $\psi_1, \dots, \psi_N$ randomly and select the number of iterations L

for $i = 1, \dots, L$ **do**

    Solve the following problem using current $\psi_1, \dots, \psi_N$

$$\underset{\substack{P_1^{\mathrm{ul}}, \dots, P_K^{\mathrm{ul}} \\ P_1^{\mathrm{ul}} + \dots + P_K^{\mathrm{ul}} \leq P}}{\text{maximize}} \quad B \log_2 \left(\det \left(\mathbf{I}_M + \frac{1}{N_0} \mathbf{H} \mathrm{diag} \left(\frac{P_1^{\mathrm{ul}}}{B}, \dots, \frac{P_K^{\mathrm{ul}}}{B} \right) \mathbf{H}^{\mathrm{H}} \right) \right).$$

    Set $\mathbf{Q} = \mathrm{diag}\left(\frac{P_1^{\mathrm{ul}}}{B}, \dots, \frac{P_K^{\mathrm{ul}}}{B} \right)$ using the solution to the problem in the last line

    for $n = 1, \dots, N$ **do**

        Compute $\mathbf{A}_n$ and $\mathbf{b}_n$ using current $\psi_1, \dots, \psi_N$

        $\psi_n \leftarrow -\arg(\mathbf{b}_n^{\mathrm{H}} \mathbf{A}_n^{-1} \mathbf{h}_{\mathrm{r},n})$

    end for

end for

Output: $P_1^{\mathrm{ul}}, \dots, P_K^{\mathrm{ul}}$, $\psi_1, \dots, \psi_N$

Answer 9.12. The capacity maximizing surface configuration is $\psi_n = \arg(\mathbf{a}_{\mathrm{r}}^{\mathrm{H}} \mathbf{h}_{\mathrm{s}}) - \arg(b_{\mathrm{r},n} h_{\mathrm{t},n}) + 2\pi k_n$, where k_n is the integer that gives $\psi_n \in [-\pi, \pi)$, for $n = 1, \dots, N$.

Answer 9.13. (a)

$$C = \log_2 \left(\prod_{k=1}^{M} \left(1 + \frac{q s_k^2}{M N_0} \right) \right) \leq M \log_2 \left(1 + \frac{q \sum_{k=1}^{M} s_k^2}{M^2 N_0} \right) = M \log_2 \left(1 + \frac{q \|\mathbf{H}\|_{\mathrm{F}}^2}{M^2 N_0} \right)$$

where the equality in the above expression holds if and only if $s_1 = \ldots = s_M$. (b) The high-SNR channel capacity is maximized by configuring the surface as $\psi_n = \arg(\mathbf{a}_{\mathrm{r}}^{\mathrm{H}} \mathbf{a}_{\mathrm{s}}) + \arg(\mathbf{b}_{\mathrm{s}}^{\mathrm{T}} \mathbf{b}_{\mathrm{t}}^*) - \arg(b_{\mathrm{r},n} h_{\mathrm{t},n}) + 2\pi k_n$, where k_n is the integer that gives $\psi_n \in [-\pi, \pi)$, for $n = 1, \dots, N$

Answer 9.14. The algorithm is given in Algorithm A.9.1, which uses the notation

$$\mathbf{b}_n = \frac{1}{N_0} \mathbf{H}_n \mathbf{Q} \vec{\mathbf{h}}_{\mathrm{t},n}^*, \quad \mathbf{A}_n = \mathbf{I}_M + \frac{1}{N_0} \mathbf{H}_n \mathbf{Q} \mathbf{H}_n^{\mathrm{H}} + \frac{1}{N_0} \mathbf{h}_{\mathrm{r},n} \vec{\mathbf{h}}_{\mathrm{t},n}^{\mathrm{T}} \mathbf{Q} \vec{\mathbf{h}}_{\mathrm{t},n}^* \mathbf{h}_{\mathrm{r},n}^{\mathrm{H}}.$$