24. Stochastic Processes

24.1 Introduction

A **stochastic process** $\{X_t : t \in T\}$ is a collection of random variables. We shall sometimes write $X(t)$ instead of X_t. The variables X_t take values in some set $\mathcal{X}$ called the **state space**. The set T is called the **index set** and for our purposes can be thought of as time. The index set can be discrete, $T = \{0, 1, 2, \ldots\}$ or continuous $T = [0, \infty)$ depending on the application.

Recall that if $X_1, \ldots, X_n$ are random variables then we can write the joint density as

$$f(x_1, \ldots, x_n) = f(x_1) f(x_2 | x_1) \ldots f(x_n | x_1, \ldots, x_{n-1}) = \prod_{i=1}^n f(x_i | \text{past}_i)$$

where past_i refers to all variables before X_i.

24.2 Markov Chains

The process $\{X_n : n \in T\}$ is a **Markov Chain** if

$$\mathbb{P}(X_n = x | X_0, \ldots, X_{n-1}) = \mathbb{P}(X_n = x | X_{n-1})$$

for all n and for all $x \in \mathcal{X}$.

For a Markov chain, the joint density function can be written as

$$f(x_1, \ldots, x_n) = f(x_1) f(x_2 | x_1) f(x_3 | x_2) \ldots f(x_n | x_{n-1})$$

A Markov chain can be represented by the following DAG:

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots$$

Each variable has a single parent, namely, the previous observation.

The theory of Markov chains is very rich and complex. Our goal is to answer the following questions:

1. When does a Markov chain "settle down" into some sort of equilibrium?

2. How do we estimate the parameters of a Markov chain?

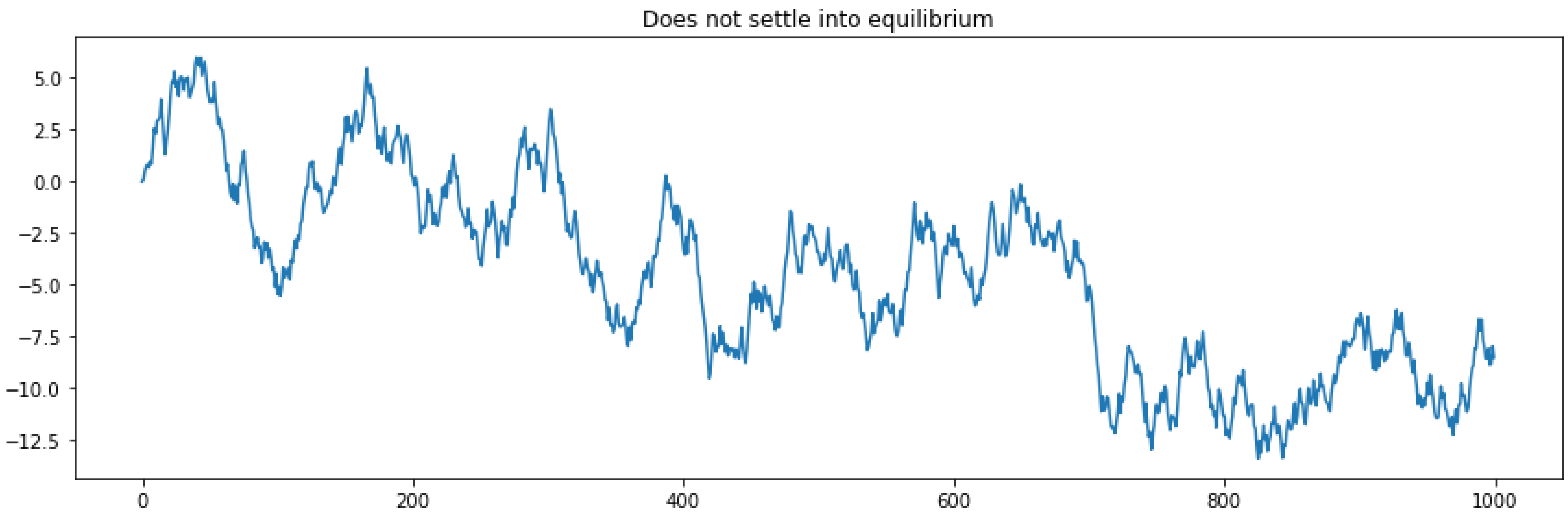
3. How can we construct Markov chains that converge to a given equilibrium and why would we want to do that?

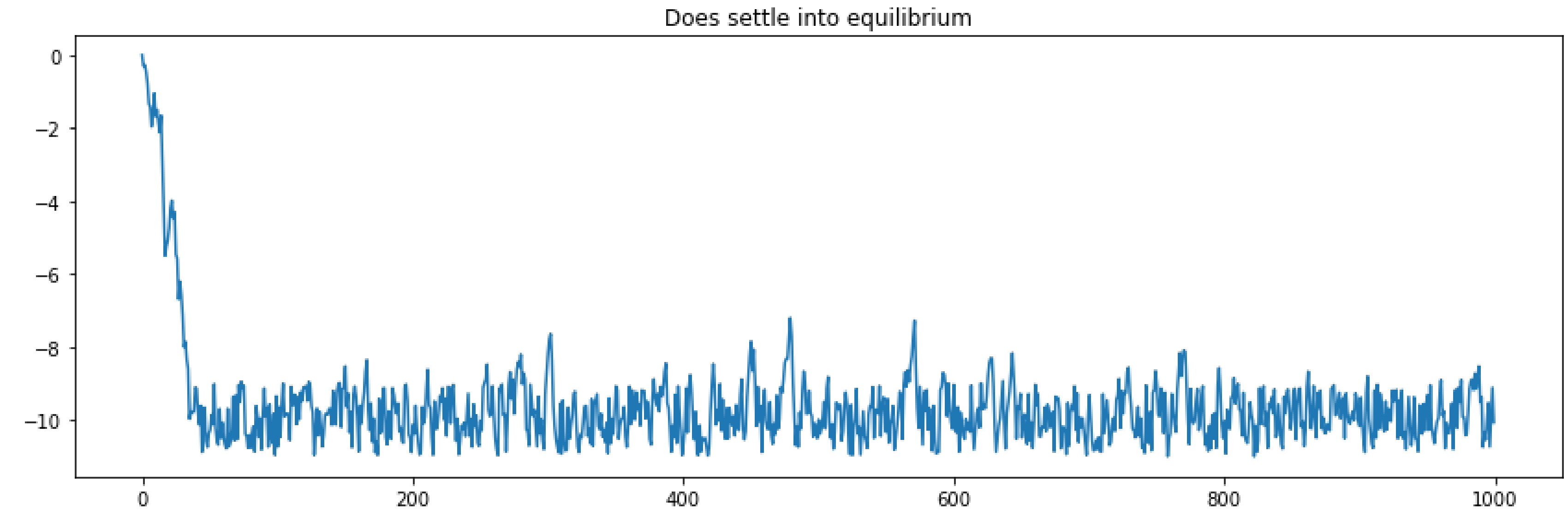
Questions 1 and 2 will be approached this chapter, question 3 in the next chapter.

In [1]:
```
import numpy as np

def generate_random_walk(n, seed=None):
    if seed is not None:
        np.random.seed(seed)

    X = np.empty(n)
    X[0] = 0
    for i in range(1, n):
        X[i] = X[i - 1] + np.random.uniform(low=-1, high=1)

    return X

def generate_random_walk_bound(n, drift=-0.4, min_value=-10, seed=None):
    if seed is not None:
        np.random.seed(seed)

    X = np.empty(n)
    X[0] = 0
    for i in range(1, n):
        X[i] = max(X[i - 1] + drift, min_value) + np.random.uniform(low=-1, high=1)

    return X
```

In [2]:
```
import matplotlib.pyplot as plt
%matplotlib inline

plt.figure(figsize=(12, 8))

A = generate_random_walk(1000, seed=0)
B = generate_random_walk_bound(1000, seed=0)

ax = plt.subplot(2, 1, 1)
ax.plot(np.arange(0, len(A)), A)
ax.set_title('Does not settle into equilibrium')

ax = plt.subplot(2, 1, 2)
ax.plot(np.arange(0, len(B)), B)
ax.set_title('Does settle into equilibrium')
```

```
plt.tight_layout()
plt.show()
```





Transition Probabilities

The key quantities of a Markov chain are the probabilities of jumping from one state into another state.

We call

$$\mathbb{P}(X_{n+1} = j | X_n = i)$$

the **transition probabilities**. If the transition probabilities do not change with time, we say the chain is **homogeneous**. In this case we define $p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i)$. The matrix P whose (i, j) element is p_{ij} is called the **transition matrix**.

We will only consider homogeneous chains. Notice how each P has two properties: (i) $p_{ij} \geq 0$ and (ii) $\sum_i p_{ij} = 1$. Each row is a probability mass function. A matrix with these properties is called a **stochastic matrix**.

Let

$$p_{ij}(n) = \mathbb{P}(X_{m+n} = j | X_m = i)$$

be the probability of going from state i to state j in n steps. Let P_n be the matrix whose (i, j) element is $p_{ij}(n)$. These are called the n**-step transition probabilities**.

Theorem 24.9 (The Chapman-Kolmogorov equations). The n-step probabilities satisfy

$$p_{ij}(m+n) = \sum_k p_{ij}(m) p_{kj}(n)$$

Proof. Recall that, in general,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)\mathbb{P}(Y = y | X = x)$$

This fact is true when conditioned in another variable,

$$\mathbb{P}(X = x, Y = y | Z = z) = \mathbb{P}(X = x | Z = z)\mathbb{P}(Y = y | X = x, Z = z)$$

Also, recall the law of total probability:

$$\mathbb{P}(X = x) = \sum_y \mathbb{P}(X = x, Y = y)$$

Using these facts and the Markov property we have:

$$\begin{aligned}
p_{ij}(m+n) &= \mathbb{P}(X_{m+n}=j|X_0=i) && (1)\\
&= \sum_k \mathbb{P}(X_{m+n}=j, X_m=k|X_0=i) && (2)\\
&= \sum_k \mathbb{P}(X_{m+n}=j|X_m=k, X_0=i)\mathbb{P}(X_m=k|X_0=i) && (3)\\
&= \sum_k \mathbb{P}(X_{m+n}=j|X_m=k)\mathbb{P}(X_m=k|X_0=i) && (4)\\
&= \sum_k p_{ik}(m)p_{kj}(n) && (5)
\end{aligned}$$

Note that this definition is equivalent to matrix multiplication; hence we have shown that

$$P_{m+n} = P_m P_n$$

By definition, $P_1 = P$. Using the above theorem, we get

$$P_n = P^n \equiv \underbrace{P \times P \times \cdots \times P}_{\text{multiply matrix } n \text{ times}}$$

Let $\mu_n = (\mu_n(1), \dots, \mu_n(N))$ be a row vector where

$$\mu_n(i) = \mathbb{P}(X_n = i)$$

is the marginal probability that the chain is in state i at time n. In particular, μ_0 is called the **initial distribution**. To simulate a Markov chain, all you need to know is μ_0 and P. The simulation would look like this:

1. Draw $X_0 \sim \mu_0$. Thus, $\mathbb{P}(X_0 = i) = \mu_0(i)$.
2. Suppose the outcome of step 1 is i. Draw $X_1 \sim P$. In other words, $\mathbb{P}(X_1 = j|X_0 = i) = p_{ij}$.
3. Suppose the outcome of step 2 is j. Draw $X_2 \sim P$. In other words, $\mathbb{P}(X_2 = k|X_1 = j) = p_{jk}$.

and so on.

It might be difficult to understand the meaning of μ_n. Imagine simulating the chain many times. Collect all of the outcomes at time n from all the chains. This histogram would look approximately like μ_n. A consequence of the previous theorem is the following:

Lemma 24.10. The marginal probabilities are given by

$$\mu_n = \mu_0 P^n$$

Proof.

$$\mu_n(j) = \mathbb{P}(X_n = j) = \sum_i \mathbb{P}(X_n = j | X_0 = i)\mathbb{P}(X_0 = i) = \sum_i \mu_0(i) p_{ij}(n) = \mu_0 P^n$$

Summary

1. Transition matrix: $P(i, j) = \mathbb{P}(X_{n+1} = j | X_n = i)$
2. n-step matrix: $P_n(i, j) = \mathbb{P}(X_{m+n} = j | X_m = i)$
3. $P_n = P^n$
4. Marginal probabilities: $\mu_n(i) = \mathbb{P}(X_n = i)$
5. $\mu_n = \mu_0 P^n$

States

The states of a Markov chain can be classified according to various properties.

We say that i **reaches** j (or j is **accessible** from i) if $p_{ij}(n) > 0$ for some n, and we write $i \rightarrow j$. If $i \rightarrow j$ and $j \rightarrow i$ then we write $i \leftrightarrow j$ and we say that i and j **communicate**.

Theorem 24.12. The communication relation satisfies the following properties:

1. $i \leftrightarrow i$.
2. If $i \leftrightarrow j$ then $j \leftrightarrow i$.
3. If $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$.
4. The set of states $\mathcal{X}$ can be written as a disjoint union of **classes** $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots$ where two states i and j communicate with each other if and only if they are in the same class.

If all states communicate with each other then the chain is called **irreducible**. A set of states is **closed** if once you enter that set states you never leave. A closet set consisting of a single state is called an **absorbing state**.

Suppose we start a chain in state i. Will the chain ever return to state i? If so, that state is called persistent or recurrent.

State i is **recurrent** or **persistent** if

$$\mathbb{P}(X_n = i \text{ for some } n \geq 1 | X_0 = i) = 1$$

Otherwise, state i is **transient**.

Theorem 24.15. A state i is recurrent if and only if

$$\sum_n p_{ii}(n) = \infty$$

A state i is transient if and only if

$$\sum_n p_{ii}(n) < \infty$$

Proof. Define

$$I_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$$

The number of times that the chain is in state i is $Y = \sum_{n=0}^{\infty} I_n$. The mean of Y, given that the chain starts in state i, is

$$\begin{aligned} \mathbb{E}(Y|X_0 = i) &= \sum_{n=0}^{\infty} \mathbb{E}(I_n|X_0 = i) && (6) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(X_n = i|X_0 = i) && (7) \\ &= \sum_{n=0}^{\infty} p_{ii}(n) && (8) \end{aligned}$$

Define $a_i = \mathbb{P}(X_n = i \text{ for some } n \geq 1|X_0 = i)$. If i is recurrent, $a_i = 1$. Thus, the chain will eventually return to i. Once it does, we argue again that since $a_i = 1$, the chain will return to state i again. By repeating this argument, we conclude that $\mathbb{E}(Y|X_0 = i) = \infty$.

If i is transient, then $a_i < 1$. When the chain is in state i, there is a probability $1 - a_i > 0$ that it will never return to state i. Thus, the probability that the chain is in state i exactly n times is $a_i^{n-1}(1 - a_i)$. This is a geometric distribution that has finite mean.

Theorem 24.16. Facts about recurrence:

1. If a state i is recurrent and $i \leftrightarrow j$ then j is recurrent.
2. If a state i is transient and $i \leftrightarrow j$ then j is transient.
3. A finite Markov chain must have at least one recurrent state.
4. The states of a finite, irreducible Markov chain are all recurrent.

Theorem 24.17 (Decomposition Theorem). The state space $\mathcal{X}$ can be written as the disjoint union

$$\mathcal{X} = \mathcal{X}_T \cup \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots$$

where the $\mathcal{X}_T$ are the transient states and each $\mathcal{X}_i$ is a closed, irreducible set of recurrent states.

Convergence of Markov Chains

Suppose that $X_0 = i$. Define the **recurrence time**

$$T_{ij} = \min\{n > 0 : X_n = j\}$$

assuming X_n ever returns to the state i, otherwise define $T_{ij} = \infty$. The **mean recurrence time** of a recurrent state i is

$$m_i = \mathbb{E}(T_{ii}) = \sum_n n f_{ii}(n)$$

where

$$f_{ij}(n) = \mathbb{P}(X_1 \neq j, X_2 \neq j, \ldots, X_{n-1} \neq j, X_n \neq j | X_0 = i)$$

A recurrent state is **null** if $m_i = \infty$, otherwise it is called **non-null** or **positive**.

Lemma 24.18. If a state is null and recurrent, then $p_{ii}^n \to 0$.

Lemma 24.19. In a finite state Markov chain, all recurrent states are positive.

Consider a three state chain with transition matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Suppose we start the chain in state 1. Then we will be in state 3 at times $3, 6, 9, \ldots$. This is an example of a periodic chain. Formally, the **period** of state i is d is $p_{ii}(n) = 0$ whenever n is not divisible by d and d is the largest integer with this property. Thus, $d = \gcd\{n : p_{ii}(n) = 0\}$, where gcd means "greatest common divisor". State i is **periodic** if $d(i) > 1$ and **aperiodic** if $d(i) = 1$.

Lemma 24.20. If a state i has period d and $i \leftrightarrow j$ then j has period d.

A state is **ergodic** if it is recurrent, non-null and aperiodic. A chain is ergodic if all its states are ergodic.

Let $\pi = (\pi_i : i \in \mathcal{X})$ be a vector of non-negative numbers that sum to one. Thus π can be thought of as a probability mass function.

We say that π is a **stationary** (or **invariant**) distribution if $\pi = \pi P$.

Here is the intuition. Draw X_0 from distribution π and suppose that π is a stationary distribution. Now draw X_1 according to the transition probability of the chain. The distribution of X_1 is then $\mu_1 = \mu_0 P = \pi P = \pi$. Continuing this way, the distribution of X_n is $\mu_n = \mu_0 P^n = \pi P^n = \pi$. In other words: if at any time the chain has distribution π, then it will continue to have distribution π forever.

We say that a chain has **limiting distribution** if

$$P^n \to \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}$$

for some π. In other words, $\pi_j = \lim_{n \to \infty} P^n_{ij}$ exists and is independent of i.

Theorem 24.24. An irreducible, ergodic Markov chain has a unique stationary distribution π. The limiting distribution exists and is equal to π. If g is any bounded function, then, with probabiity 1,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(X_n) = \mathbb{E}_\pi(g) \equiv \sum_j g(j)\pi_j$$

The last statement of the theorem is the law of large numbers for Markov chains. It says that sample averages converge to their expectations. Finally, there is a special condition which will be useful later. We say that π satisfies **detailed balance** if

$$\pi_i p_{ij} = p_{ji} \pi_j$$

Detailed balance guarantees that π is a stationary distribution.

Theorem 24.25. If π satisfies detailed balance then π is a stationary distribution.

Proof. We need to show that $\pi P = \pi$. The j-th element of πP is $\sum_i \pi_i p_{ij} = \sum_i p_{ji} \pi_j = \pi_j \sum_i p_{ji} = \pi_j$.

The importance of detailed balance will become clear when we discuss Markov chain Monte Carlo methods.

Warning: Just because a chain has a stationary distribution does not mean it converges.

Inference for Markov Chains

Consider a chain with finite state space $\mathcal{X} = \{1, 2, \dots, N\}$. Suppose we observe n observations $X_1, \dots, X_n$ from this chain. The unknown parameters of a Markov chain are the initial probabilities $\mu_0 = (\mu_0(1), \mu_0(2), \dots)$ and the elements of the transition matrix P. Each row of P is a multinomial distribution, so we are essentially estimating N distributions (plus the initial probabilities). Let n_{ij} be the observed number of transitions from state i to state j. The likelihood function is

$$\mathcal{L}(\mu_0, P) = \mu_0(x_0) \prod_{r=1}^{n} p_{X_{r-1}, X_r} = \mu_0(x_0) \prod_{i=1}^{N} \prod_{j=1}^{N} p_{ij}^{nij}$$

There is only one observation on μ_0 so we can't estimate that. Rather, we focus on estimating P. The MLE is obtained by maximizing the likelihood subject to the constraint that the elements are non-negative and the rows sum to 1. The solution is

$$\hat{p}_{ij} = \frac{n_{ij}}{n_i}$$

where $n_i = \sum_{j=1}^{N} n_{ij}$. Here we are assuming that $n_i > 0$. If not, we set $\hat{p}_{ij} = 0$ by convention.

Theorem 24.30 (Consistency and Asymptotic Normality of the MLE). Assume that the chain is ergodic. Let $\hat{p}_{ij}(n)$ denote the MLE after n observations. Then $\hat{p}_{ij}(n) \xrightarrow{\text{P}} p_{ij}$. Also,

$$\left[\sqrt{N_i(n)}(\hat{p}_{ij} - p_{ij})\right] \rightsquigarrow N(0, \Sigma)$$

where the left hand side is a matrix, $N_i(n) = \sum_{r=1}^{n} I(X_r = i)$ is the count of observations at state i up to time n, and the covariance matrix Σ is a $t \times t$ matrix, where t is the number of transitions from state i to j, with elements

$$\Sigma_{ij,k\ell} = \begin{cases} p_{ij}(1 - p_{ij}) & \text{if } (i,j) = (k, \ell) \\ -p_{ij} p_{i\ell} & \text{if } i = k, j \neq \ell \\ 0 & \text{otherwise} \end{cases}$$

24.3 Poisson Process

As the name suggests, the Poisson process is intimately related to the Poisson distribution. Let's first review the Poisson distribution.

Recall that X has a Poisson distribution with parameter λ, written $X \sim \text{Poisson}(\lambda)$, if

$$\mathbb{P}(X = x) \equiv p(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad = 0, 1, 2, \ldots$$

Also recall that:

- $X \sim \text{Poisson}(\lambda)$ distribution has mean $\mathbb{E}(X) = \lambda$ and variance $\mathbb{V}(X) = \lambda$.
- If $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\nu)$ and X and Y are independent, then $X + Y \sim \text{Poisson}(\lambda + \nu)$.
- If $N \sim \text{Poisson}(\lambda)$ and $Y | N = n \sim \text{Binomial}(n, p)$ then the marginal distribution of Y is $Y \sim \text{Poisson}(\lambda p)$.

Now we describe the Poisson process. Imagine that whenever an event occurs you record its timestamp. Let X_t be the number of events that occured up until time t. Then, $\{ X_t : t \in [0, \infty) \}$ is a stochastic process with state space $\mathcal{X} = \{0, 1, 2, \dots \}$. A process of this form is called a **counting process**.

In what follows, we will sometimes write $X(t)$ instead of X_t. Also, we will need little-o notation: write $f(h) = o(h)$ if $f(h) / h \rightarrow 0$ as $h \rightarrow 0$. This means that $f(h)$ is smaller than h when h is close to 0. For example, $h^2 = o(h)$.

A **Poisson process** is a stochastic process $\{ X_t : t \in [0, \infty) \}$ with state space $\mathcal{X} = \{0, 1, 2, \dots \}$ such that

1. $X(0) = 0$
2. For any increasing times $0 = t_0 < t_1 < t_2 < \dots < t_n$, the count increments

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent.

1. There is a function $\lambda(t)$ such that

$$\begin{align} \mathbb{P}\left(X(t + h) - X(t) = 1\right) &= \lambda(t) h + o(h) \tag{9} \\ \mathbb{P}\left(X(t + h) - X(t) \geq 2\right) &= o(h) \tag{10} \end{align}$$

We call $\lambda(t)$ the **intensity function**.

Theorem 24.32. If X_t is a Poisson process with intensity function $\lambda(t)$, then

$$X(s + t) - X(s) \sim \text{Poisson}\left(m(s + t) - m(s)\right)$$

where

$$m(t) = \int_0^t \lambda(s) ds$$

In particular, $X(t) \sim \text{Poisson}(m(t))$. Hence, $\mathbb{E}(X(t)) = m(t)$ and $\mathbb{V}(X(t)) = m(t)$.

A Poisson process with constant intensity function $\lambda(t) \equiv \lambda$ for some $\lambda > 0$ is called a **homogeneous Poisson process** with rate λ. In this case,

$$X(t) \sim \text{Poisson}(\lambda t)$$

Let $X(t)$be a homogeneous Poisson process with rate λ. Let W_n be the time at which the n-th event occurs and set $W_0 = 0$. The random variables $W_0, W_1, \ldots$ are called **waiting times**. Let $S_n = W_{n+1} - W_n$. Then $S_0, S_1, \ldots$ are called **sojourn times** or **interarrival times**.

Theorem 24.34. The sojourn times $S_0, S_1, \ldots$ are IID random variables. Their distribution is exponential with mean $1/\lambda$, that is, they have density

$$f(s) = \lambda e^{-\lambda s}, \quad s \geq 0$$

The waiting time has distribution $W_n \sim \text{Gamma}(n, 1/\lambda)$, that is, it has density

$$f(w) = \frac{1}{\Gamma(n)} \lambda^n w^{n-1} e^{-\lambda t}$$

Hence, $\mathbb{E}(W_n) = n/\lambda$ and $\mathbb{V}(W_n) = n/\lambda^2$.

Proof. First, we have

$$\mathbb{P}(S_1 > t) = \mathbb{P}(X(t) = 0) = e^{-\lambda t}$$

which shows that the CDF for S_1 is $1 - e^{-\lambda t}$. This shows the result for S_1. Now,

$$\begin{aligned}
\mathbb{P}(S_2|t|S_1 = s) &= \mathbb{P}\left(\text{no events in } (s, s+t] | S_1 = s\right) & (11) \\
&= \mathbb{P}\left(\text{no events in } (s, s+t]\right) \quad (\text{increments are independent}) & (12) \\
&= e^{-\lambda t} & (13)
\end{aligned}$$

Hence, S_2 has an exponential distribution and is independent of S_1. The result follows by repeating the argument. The result for W_n follows since a sum of exponentials has a Gamma distribution.

24.6 Exercises

Exercise 24.6.1 Let $X_0, X_1, \ldots$ be a Markov chain with states $\{0, 1, 2\}$ and transition matrix

$$P = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.9 & 0.1 & 0.0 \\ 0.1 & 0.8 & 0.1 \end{bmatrix}$$

Assume that $\mu_0 = (0.3, 0.4, 0.3)$. Find $\mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 2)$ and $\mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 1)$.

Solution.

We have:

$$\begin{aligned}
\mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 2) &= \mathbb{P}(X_0 = 0)\mathbb{P}(X_1 = 1 | X_0 = 0)\mathbb{P}(X_2 = 2 | X_1 = 1) && (14)\\
&= \mu_0(1) P_{12} P_{23} && (15)\\
&= 0.3 \cdot 0.2 \cdot 0.0 && (16)\\
&= 0 && (17)
\end{aligned}$$

which can also be seen since there is no probability for a transition from state 1 to state 2 (the corresponding value in the P matrix is zero).

We also have:

$$\begin{aligned}
\mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 2) &= \mathbb{P}(X_0 = 0)\mathbb{P}(X_1 = 1 | X_0 = 0)\mathbb{P}(X_2 = 1 | X_1 = 1) && (18)\\
&= \mu_0(1) P_{12} P_{22} && (19)\\
&= 0.3 \cdot 0.2 \cdot 0.1 && (20)\\
&= 0.006 && (21)
\end{aligned}$$

Exercise 24.6.2. Let $Y_1, Y_2, \dots$ be a sequence of iid observations such that $\mathbb{P}(Y = 0) = 0.1$, $\mathbb{P}(Y = 1) = 0.3$, $\mathbb{P}(Y = 2) = 0.2$, $\mathbb{P}(Y = 3) = 0.4$. Let $X_0 = 0$ and let

$$X_n = \max\{Y_1, \dots, Y_n\}$$

Show that $X_0, X_1, \dots$ is a Markov chain and find the transition matrix.

Solution. By definition,

$$X_{n+1} = \max\{Y_1, \dots, Y_{n+1}\} = \max\{X_n, Y_{n+1}\}$$

so X_{n+1} is defined based only on its predecessor and on a IID variable $Y_{n+1} \sim Y$. Thus:

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \begin{cases} \frac{\mathbb{P}(Y=j)}{\sum_{k \geq i} \mathbb{P}(Y=k)} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}$$

This, paired with a state space $\mathcal{X} = \{0, 1, 2, 3\}$ and initial probabilities $\mu_0 = (1, 0, 0, 0)$, defines a Markov chain for the X_i's. The explicit transition matrix is:

$$P = \begin{bmatrix} 1/10 & 3/10 & 1/5 & 2/5 \\ 0 & 1/3 & 2/9 & 4/9 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 24.6.3. Consider a two state Markov chain with states $\mathcal{X} = \{1, 2\}$ and transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

where $0 < a < 1$ and $0 < b < 1$. Prove that

$$\lim_{n \to \infty} P^n = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

Solution. Note that the Markov chain is irreducible and ergodic, given the bounds on a and b. Note also that it has a stationary distribution

$$\pi = \left(\frac{b}{a+b}, \frac{a}{a+b} \right)$$

since $\pi = \pi P$:

$$\begin{aligned}
\pi P &= \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} && (22) \\
&= \frac{1}{a+b} \begin{bmatrix} b(1-a) + ab & ab + (1-b)a \end{bmatrix} && (23) \\
&= \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix} && (24) \\
&= \pi && (25)
\end{aligned}$$

By theorem 24.24, the limit of P^n is as given,

$$\lim_{n \to \infty} P^n = \begin{bmatrix} \pi \\ \pi \end{bmatrix} = \begin{bmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{bmatrix}$$

Exercise 24.6.4. Consider the chain from question 3 and set $a = .1$ and $b = .3$. Simulate the chain. Let

$$\hat{p}_n(1) = \frac{1}{n}\sum_{i=1}^{n} I(X_i = 1) \quad \text{and} \quad \hat{p}_n(2) = \frac{1}{n}\sum_{i=1}^{n} I(X_i = 2)$$

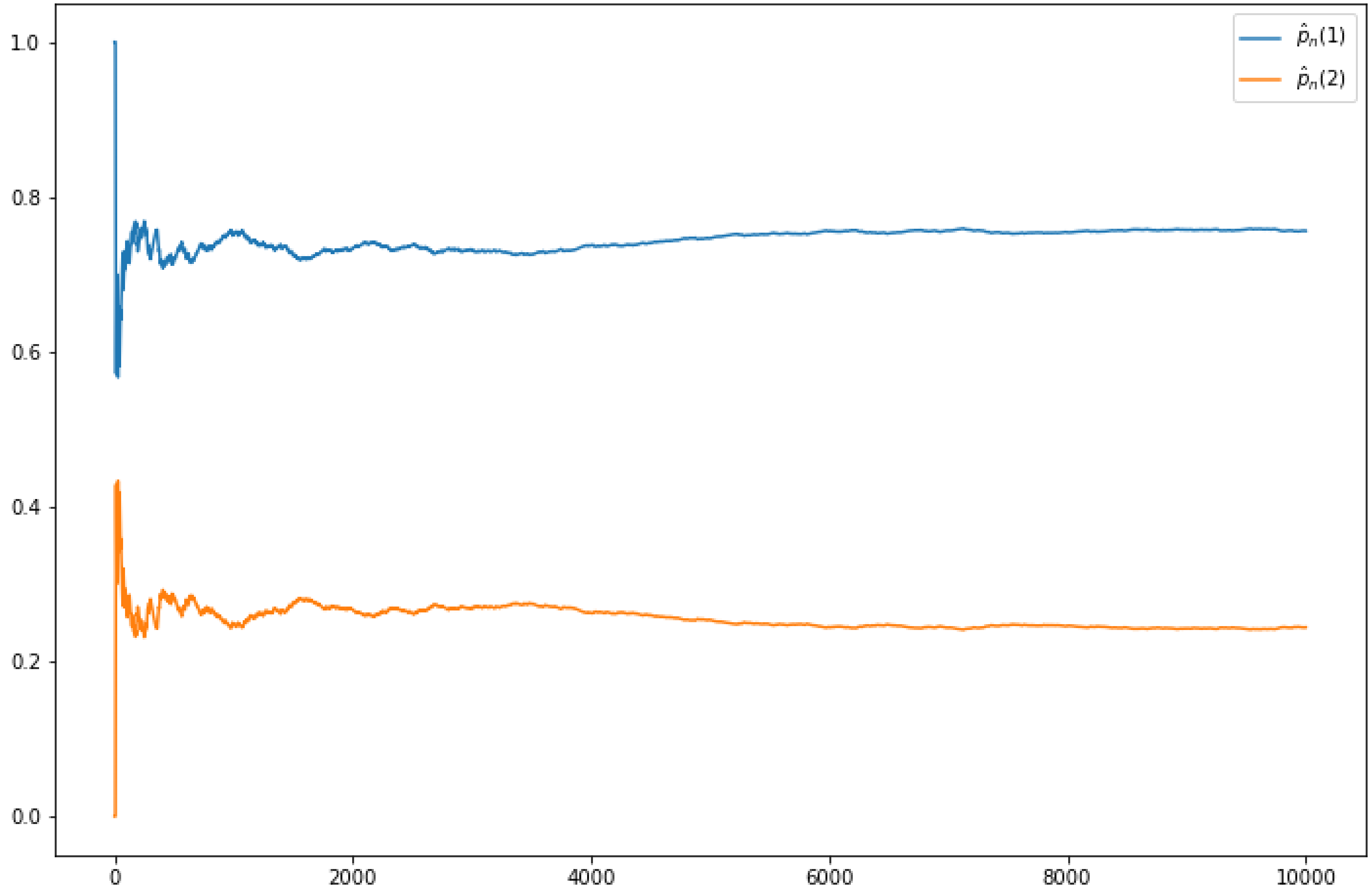
be the proportion of times the chain is in state 1 and state 2. Plot $\hat{p}_n(1)$ and $\hat{p}_n(2)$ versus n and verify that they converge to the values predicted from the answer in the previous question.

Solution.

In [3]:
```
import numpy as np

a, b = 0.1, 0.3
P = np.array([[1 - a, a], [b, 1 - b]])
```

In [4]:
```
# Do a *single* simulation starting from, say, state 1

def generate_series(n, seed=None, initial_state=1):
    if seed is not None:
        np.random.seed(0)

    random_values = np.random.uniform(low=0, high=1, size=n)

    X = np.empty(n, dtype=int)
    X[0] = initial_state
    for i in range(1, n):
        X[i] = 1 if random_values[i] < P[X[i - 1] - 1, 0] else 2

    return X

n = 10000
X = generate_series(n, seed=0)
p1 = np.cumsum(X == 1) / np.arange(1, n + 1)
p2 = np.cumsum(X == 2) / np.arange(1, n + 1)
```

In [5]:
```
import matplotlib.pyplot as plt
%matplotlib inline

plt.figure(figsize=(12, 8))
plt.plot(np.arange(1, n+1), p1, label=r'$\hat{p}_n(1)$')
plt.plot(np.arange(1, n+1), p2, label=r'$\hat{p}_n(2)$')
plt.legend()
plt.show()
```



Note that the values do converge to $\pi = (0.75, 0.25)$.

Exercise 24.6.5. An important Markov chain is the **branching process** which is used in biology, genetics, nuclear physics and many other fields. Suppose that an animal has Y children. Let $p_k = \mathbb{P}(Y = k)$. Hence $p_k \geq 0$ for all k and $\sum_{k=0}^{\infty} p_k = 1$. Assume each animal has the same lifespan and that they produce offspring according to the distribution p_k. Let X_n be the number of animals in the n-th generation. Let $Y_1^{(n)}, \dots, Y_{X_n}^{(n)}$ be the offspring produced in the n-th generation. Note that

$$X_{n+1} = Y_1^{(n)} + \dots + Y_{X_n}^{(n)}$$

Let $\mu = \mathbb{E}(Y)$ and $\sigma^2 = \mathbb{V}(Y)$. Assume throughout this question that $X_0 = 1$. Let $M(n) = \mathbb{E}(X_n)$ and $V(n) = \mathbb{V}(X_n)$.

(a) Show that $M(n+1) = \mu M(n)$ and that $V(n+1) = \sigma^2 M(n) + \mu^2 V(n)$.

(b) Show that $M(n) = \mu^n$ and that $V(n) = \sigma^2 \mu^{n-1}(1 + \mu + \cdots + \mu^{n-1})$.

(c) What happens to the variance if $\mu > 1$? What happens to the variance if $\mu = 1$? What happens to the variance if $\mu < 1$?

(d) The population goes extinct if $X_n = 0$ for some n. Let us thus define the extinction time N by

$$N = \min\{n : X_n = 0\}$$

Let $F(n) = \mathbb{P}(N \leq n)$ be the CDF of the random variable N. Show that

$$F(n) = \sum_{k=0}^{\infty} p_k (F(n-1))^k, \quad n = 1, 2, \dots$$

Hint: note that the event $\{N \leq n\}$ is the same event as $\{X_n = 0\}$. Thus $\mathbb{P}(N \leq n) = \mathbb{P}(X_n = 0)$. Let k be the number of offspring of the original event. The population becomes extinct at time n if and only if each of the k sub-populations generated from the k offspring goes extinct in $n-1$ generations.

(e) Suppose that $p_0 = 1/4$, $p_1 = 1/2$, $p_2 = 1/4$. Use the formula from (d) to compute the CDF $F(n)$.

Solution.

(a) We have:

$$M(n+1) = \mathbb{E}[X_{n+1}] = \mathbb{E}\left[\sum_{i=1}^{X_n} Y_i^{(n)}\right] = \mathbb{E}\left[\sum_{i=1}^{X_n} \mathbb{E}[Y_i^{(n)}]\right] = \mathbb{E}\left[\sum_{i=1}^{X_n} \mu\right] = \mu\mathbb{E}[X_n] = \mu M(n)$$

We also have:

$$
\begin{aligned}
V(n+1) &= \mathbb{V}[X_{n+1}] = \mathbb{E}[X_{n+1}^2] - \mathbb{E}[X_{n+1}]^2 && (26)\\
&= \mathbb{E}\left[\left(\sum_{i=1}^{X_n} Y_i^{(n)}\right)^2\right] - \mu^2 M(n)^2 && (27)\\
&= \mathbb{E}\left[\sum_{i=1}^{X_n}\left(Y_i^{(n)}\right)^2 + \sum_{i=1}^{X_n}\sum_{j=1,j\neq i}^{X_n} Y_i^{(n)} Y_j^{(n)}\right] - \mu^2 M(n)^2 && (28)\\
&= \mathbb{E}\left[\sum_{i=1}^{X_n}\mathbb{E}\left[\left(Y_i^{(n)}\right)^2\right] + \sum_{i=1}^{X_n}\sum_{j=1,j\neq i}^{X_n} \mathbb{E}\left[Y_i^{(n)} Y_j^{(n)}\right]\right] - \mu^2 M(n)^2 && (29)\\
&= \mathbb{E}\left[\sum_{i=1}^{X_n}\left(\mathbb{V}[Y_i^{(n)}] + \mathbb{E}\left[Y_i^{(n)}\right]^2\right) + \sum_{i=1}^{X_n}\sum_{j=1,j\neq i}^{X_n} \mathbb{E}\left[Y_i^{(n)}\right]\mathbb{E}\left[Y_j^{(n)}\right]\right] - \mu^2 M(n)^2 && (30)\\
&= \mathbb{E}\left[\sum_{i=1}^{X_n}(\sigma^2+\mu^2) + \sum_{i=1}^{X_n}\sum_{j=1,j\neq i}^{X_n} \mu^2\right] - \mu^2 M(n)^2 && (31)\\
&= \mathbb{E}\left[X_n(\sigma^2+\mu^2) + X_n(X_n-1)\mu^2\right] - \mu^2 M(n)^2 && (32)\\
&= \mathbb{E}\left[X_n\sigma^2 + X_n^2\mu^2\right] - \mu^2 M(n)^2 && (33)\\
&= \sigma^2\mathbb{E}[X_n] + \mu^2\mathbb{E}[X_n^2] - \mu^2 M(n)^2 && (34)\\
&= \sigma^2 M(n) + \mu^2(V(n) + M(n)^2) - \mu^2 M(n)^2 && (35)\\
&= \sigma^2 M(n) + \mu^2 V(n) && (36)
\end{aligned}
$$

(b)

Since the initial population is 1, $M(0)=1$; since $M(n+1)=\mu M(n)$, it follows by induction that $M(n)=\mu^n$.

Now, we also have that the initial population is known, so $V(0)=0$. By induction,

$$
\begin{aligned}
V(n+1) &= \sigma^2 M(n) + \mu^2 V(n) && (37)\\
&= \sigma^2\mu^n + \mu^2\left(\sigma^2\mu^{n-1}\left(1+\mu+\cdots+\mu^{n-1}\right)\right) && (38)\\
&= \sigma^2\mu^n\left(1+\mu\left(1+\mu+\cdots+\mu^{n-1}\right)\right) && (39)\\
&= \sigma^2\mu^n\left(1+\mu+\cdots+\mu^n\right) && (40)
\end{aligned}
$$

(c) If $\mu \neq 1$, we can add up the geometric sum and write

$$V(n) = \sigma^2 \mu^n \frac{1 - \mu^n}{1 - \mu}$$

- For $\mu > 1$, $V(n)$ grows exponentially.
- For $\mu < 1$, $V(n)$ converges to 0.
- For $\mu = 1$, we have $V(n) = n\sigma^2$, which grows linearly.

(d) Following the reasoning in the hint,

$$\begin{aligned}
F(n) &= \mathbb{P}(N \leq n) = \mathbb{P}(X_n = 0) && (41)\\
&= \sum_{k=0}^{\infty} \mathbb{P}(X_n = 0 | X_1 = k) \mathbb{P}(X_1 = k) && (42)\\
&= \sum_{k=0}^{\infty} \mathbb{P}(Y_i^{(n-1)} = 0 \text{ for all } i | X_1 = k) \mathbb{P}(X_1 = k) && (43)\\
&= \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} \mathbb{P}(Y_i^{(n-1)} = 0 | X_1 = k) \right) \mathbb{P}(X_1 = k) && (44)\\
&= \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} F(n-1) \right) p_k && (45)\\
&= \sum_{k=0}^{\infty} p_k (F(n-1))^k && (46)
\end{aligned}$$

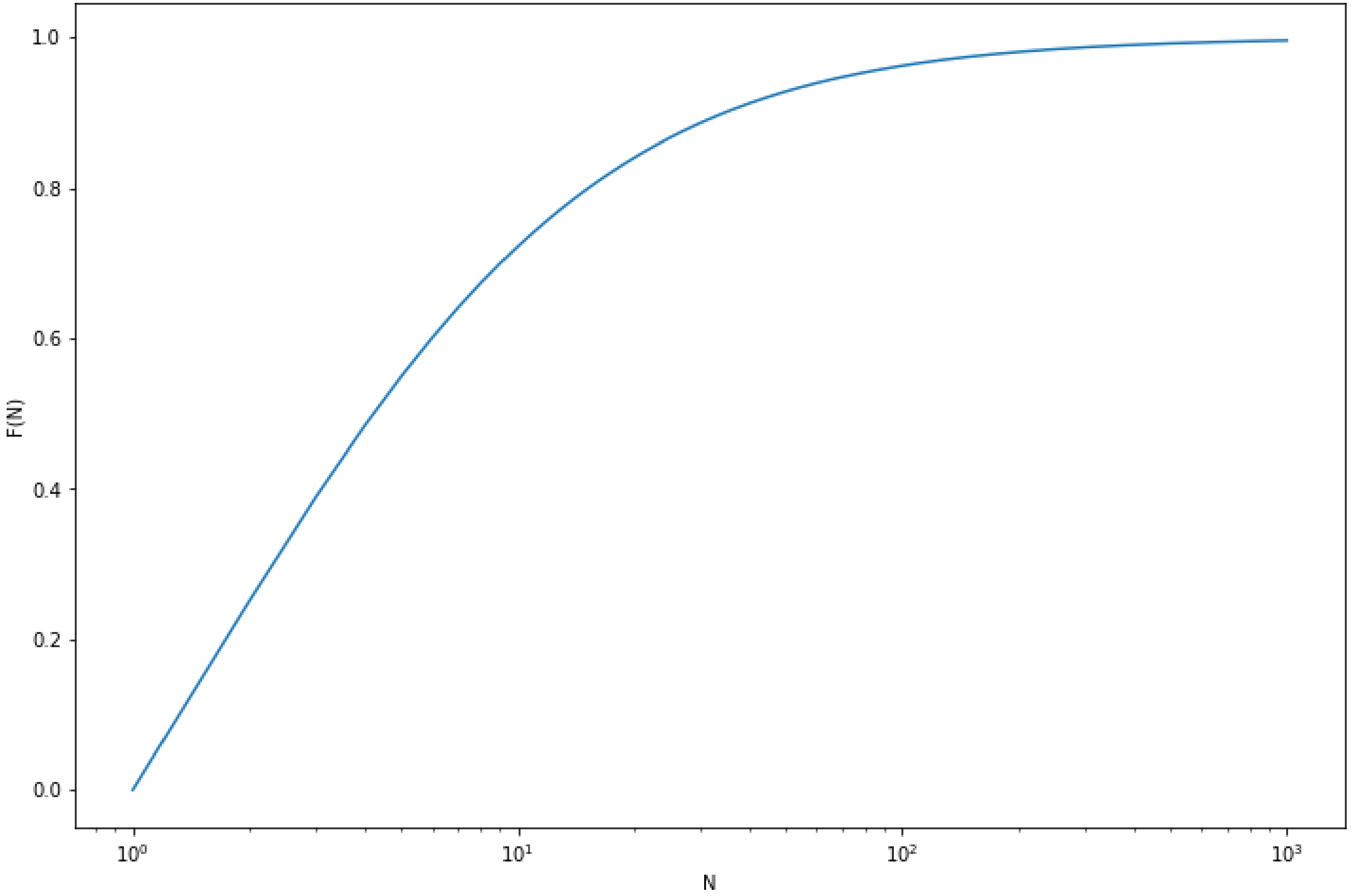
(e) The recurrence formula is:

$$F(n+1) = p_0 + p_1 F(n) + p_2 F(n)^2 = \frac{1}{4} + \frac{1}{2} F(n) + \frac{1}{4} F(n)^2 = \frac{1}{4}(F(n) + 1)^2$$

with initial value $F(0) = 0$ (since we start with a non-extinct population).

In [6]:
```
import numpy as np

N = 1000
F = np.empty(N)

F[0] = 0
for n in range(1, N):
    F[n] = (F[n - 1] + 1)**2 / 4
```

In [7]:
```
import matplotlib.pyplot as plt
%matplotlib inline

plt.figure(figsize=(12, 8))
plt.plot(np.arange(1, N + 1), F)
plt.xlabel('N')
plt.ylabel('F(N)')
plt.xscale('log')
plt.show()
```



Exercise 24.6.6. Let

$$P = \begin{bmatrix} 0.40 & 0.50 & 0.10 \\ 0.05 & 0.70 & 0.25 \\ 0.05 & 0.50 & 0.45 \end{bmatrix}$$

Find the stationary distribution π.

Solution. The stationary distribution π satisfies $\pi = \pi P$, so it is a (normalized) left eigenvector of P, or a right eigenvector of P^T.

In [8]:
```
import numpy as np
from numpy.linalg import eig

P = np.array([[0.4, 0.5, 0.1], [0.05, 0.7, 0.25], [0.05, 0.5, 0.45]])
```

In [9]:
```
w, v = eig(P.T)
pi = v[:, 0]
print(pi)
```

```
[0.11041049 0.89708523 0.42784065]
```

In [10]:
```
print(pi @ P)
```

```
[0.11041049 0.89708523 0.42784065]
```

Exercise 24.6.7. Show that if i is a recurrent state and $i \leftrightarrow j$, then j is a recurrent state.

Solution.

Since i is recurrent, $\sum_n p_{ii}(n) = \infty$. But

$$\sum_n p_{jj}(n) \geq \sum_{a,b,c} p_{ji}(a) p_{ii}(b) p_{ij}(c) \geq \sum_b p_{ji}(a') p_{ii}(b) p_{ij}(c')$$

for some a' such that $p_{ji}(a') > 0$ (which exists because $j \to i$) and some c' such that $p_{ij}(c') > 0$ (which exists because $i \to j$). Then this sum is lower bounded by

$$\frac{1}{p_{ji}(a') p_{ij}(c')} \sum_b p_{ii}(b)$$

which diverges because $\sum_n p_{ii}(n) = \infty$, so j must be a recurrent state.

Exercise 24.6.8. Let

$$P = \begin{bmatrix} 1/3 & 0 & 1/3 & 0 & 0 & 1/3 \\ 1/2 & 1/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

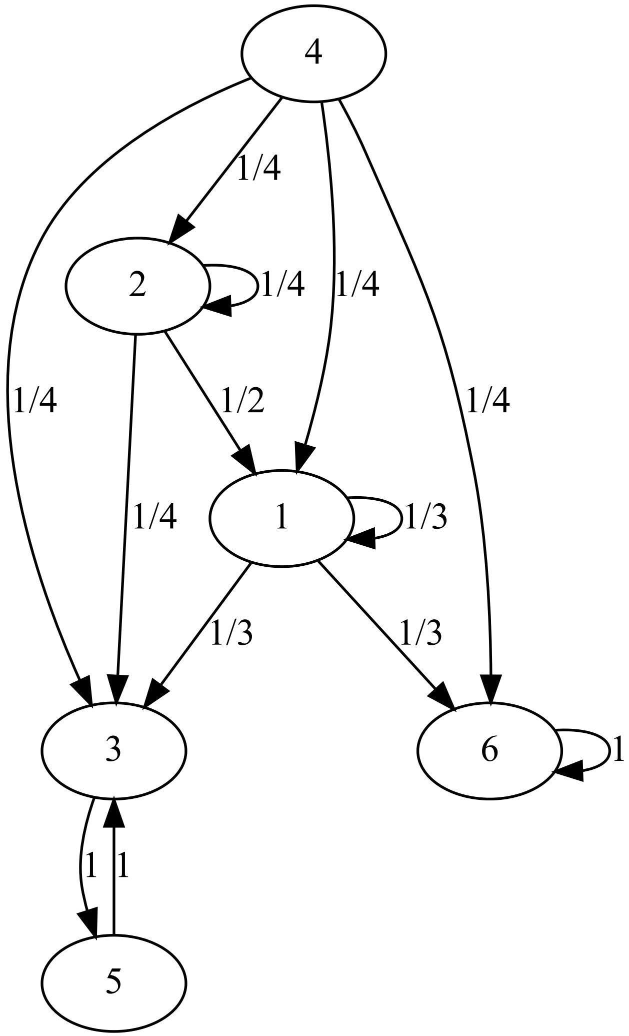
Which states are transient? Which states are recurrent?

Solution.

In [11]:

```
from graphviz import Digraph

d = Digraph()

d.edge('1', '1', label='1/3')
d.edge('1', '3', label='1/3')
d.edge('1', '6', label='1/3')

d.edge('2', '1', label='1/2')
d.edge('2', '2', label='1/4')
d.edge('2', '3', label='1/4')

d.edge('3', '5', label='1')

d.edge('4', '1', label='1/4')
d.edge('4', '2', label='1/4')
d.edge('4', '3', label='1/4')
d.edge('4', '6', label='1/4')

d.edge('5', '3', label='1')

d.edge('6', '6', label='1')

d
```

Out[11]:



States 3, 5, 6 are recurrent:

- A chain starting on state 6 will forever stay in state 6.
- A chain starting on states 3 or 5 will bounce between states 3 and 5 forever.

Other states are not recurrent, as they may eventually fall into state 6 and never reach other state after.

Exercise 24.6.9. Let

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Show that $\pi = (1/2, 1/2)$ is a stationary distribution. Does this chain converge? Why / why not?

Solution. The distribution is stationary if and only if $\pi = \pi P$, and

$$\pi P = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} = \pi$$

This chain does not converge: it will always oscillate between states 1 and 2 with probability 1.

Exercise 24.6.10. Let $0 < p < 1$ and $q = 1 - p$. Let

$$P = \begin{bmatrix} q & p & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ q & 0 & 0 & p & 0 \\ q & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the limiting distribution of the chain.

Solution.

Solving $\pi P = \pi$, we get:

$$\pi = \begin{bmatrix} \frac{1-p}{p^4(1-p^5)} & \frac{1-p}{p^3(1-p^5)} & \frac{1-p}{p^2(1-p^5)} & \frac{1-p}{p(1-p^5)} & \frac{1-p}{1-p^5} \end{bmatrix}$$

We can verify this is the limiting distribution:

$$\begin{aligned}
\pi P &= \begin{bmatrix} \frac{1-p}{p^4(1-p^5)} & \frac{1-p}{p^3(1-p^5)} & \frac{1-p}{p^2(1-p^5)} & \frac{1-p}{p(1-p^5)} & \frac{1-p}{1-p^5} \end{bmatrix} \begin{bmatrix} 1-p & p & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 1-p & 0 & 0 & p & 0 \\ 1-p & 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} && (47) \\
&= \begin{bmatrix} \frac{(1-p)^2}{p^4(1-p^5)} + \frac{(1-p)^2}{p^3(1-p^5)} + \frac{(1-p)^2}{p^2(1-p^5)} + \frac{(1-p)^2}{p(1-p^5)} + \frac{1-p}{1-p^5} & \frac{1-p}{p^3(1-p^5)} & \frac{1-p}{p^2(1-p^5)} & \frac{1-p}{p(1-p^5)} & \frac{1-p}{1-p^5} \end{bmatrix} && (48) \\
&= \begin{bmatrix} \frac{(1-p)^2(1+p+p^2+p^3)+p^4(1-p)}{p^4(1-p^5)} & \frac{1-p}{p^3(1-p^5)} & \frac{1-p}{p^2(1-p^5)} & \frac{1-p}{p(1-p^5)} & \frac{1-p}{1-p^5} \end{bmatrix} && (49) \\
&= \begin{bmatrix} \frac{1-p}{p^4(1-p^5)} & \frac{1-p}{p^3(1-p^5)} & \frac{1-p}{p^2(1-p^5)} & \frac{1-p}{p(1-p^5)} & \frac{1-p}{1-p^5} \end{bmatrix} && (50) \\
&= \pi && (51)
\end{aligned}$$

Exercise 24.6.11. Let $X(t)$ be an inhomogeneous Poisson process with intensity function $\lambda(t) > 0$. Let $\Lambda(t) = \int_0^t \lambda(u)du$. Define $Y(s) = X(t)$ where $s = \Lambda(t)$. Show that $Y(s)$ is a homogeneous Poisson process with intensity $\lambda = 1$.

Solution.

We have: $Y(s) = X(\Lambda^{-1}(s))$ and $X(t) \sim \text{Poisson}(\Lambda(t))$, so

$$Y(s) \sim \text{Poisson}(\Lambda(\Lambda^{-1}(s)) = \text{Poisson}(s) = \text{Poisson}(\lambda_Y \cdot s)$$

which is the desired result with $\lambda_Y = 1$.

Exercise 24.6.12. Let $X(t)$ be a Poisson process with intensity λ. Find the conditional distribution of $X(t)$ given that $X(t+s) = n$.

Solution. The random variables $X(t) - X(0) = X(t)$ and $Y(s) = X(t+s) - X(t)$ are independent, as they are both count increments, with $X(t) \sim \text{Poisson}(\lambda t)$ and $Y(s) \sim \text{Poisson}(\lambda s)$. Then:

$$\begin{aligned}
\mathbb{P}\left(X(t)=x \mid X(t+s)=n\right) &= \mathbb{P}\left(X(t)=x \mid X(t)+Y(s)=n\right) && (52)\\
&= \frac{\mathbb{P}\left(X(t)=x, X(t)+Y(s)=n\right)}{\mathbb{P}\left(X(t)+Y(s)=n\right)} && (53)\\
&= \frac{\mathbb{P}\left(X(t)=x, Y(s)=n-x\right)}{\sum_{0\le j\le n}\mathbb{P}\left(X(t)=j, Y(s)=n-j\right)} && (54)\\
&= \frac{\mathbb{P}\left(X(t)=x\right)\mathbb{P}\left(Y(s)=n-x\right)}{\sum_{j=0}^{n}\mathbb{P}\left(X(t)=j\right)\mathbb{P}\left(Y(s)=n-j\right)} && (55)\\
&= \frac{f(x)}{\sum_{j=0}^{n} f(j)} && (56)
\end{aligned}$$

where $f(x)=\mathbb{P}\left(X(t)=x\right)\mathbb{P}\left(Y(s)=n-x\right)$. But we have:

$$f(x)=\frac{(\lambda t)^x e^{-\lambda t}}{x!}\frac{(\lambda s)^{n-x}e^{-\lambda s}}{(n-x)!}=\frac{\lambda^n e^{-\lambda(t+s)}}{n!}\binom{n}{x}t^x s^{n-x}$$

Replacing on the expression above and cancelling the factors that do not depend on x, we get

$$\mathbb{P}\left(X(t)=x \mid X(t+s)=n\right)=\frac{f(x)}{\sum_{j=0}^{n} f(j)}=\frac{\binom{n}{x}t^x s^{n-x}}{\sum_{j=0}^{n}\binom{n}{j}t^j s^{n-j}}=\frac{\binom{n}{x}t^x s^{n-x}}{(t+s)^n}=\binom{n}{x}\left(\frac{t}{t+s}\right)^x\left(\frac{s}{t+s}\right)^{n-x}$$

using the binomial expansion $(t+s)^n=\sum_{j=0}^{n}\binom{n}{j}t^j s^{n-j}$. Therefore, the conditional distribution follows a Binomial distribution,

$$X(t) \mid X(t+s)=n \sim \text{Binomial}\left(n, \frac{t}{t+s}\right)$$

Exercise 24.6.13. Let X be a Poisson process with intensity λ. Find the probability that $X(t)$ is odd, i.e. $\mathbb{P}(X(t)=1,3,5,\dots)$.

Solution. Expanding using the probability mass function,

$$\mathbb{P}(X(t)\text{ is odd})=\sum_{k=0}^{\infty}\frac{(\lambda t)^{2k+1}e^{-\lambda t}}{(2k+1)!}=e^{-\lambda t}\sinh(\lambda t)=\frac{1}{2}\left(1-e^{-2\lambda t}\right)$$

Exercise 24.6.14. Suppose that people logging in to the University computer system is described by a Poisson process $X(t)$ with intensity λ. Assume that a person stays logged in for some random time with CDF G. Assume these times are all independent. Let $Y(t)$ be the number of people on the system at time t. Find the distribution of $Y(t)$.

Solution.

The number of people arriving at time period j is $A(j) \equiv X(j) - X(j-1)$, where $X(0) = 0$. Let $W_i^{(j)}$ be the amount of time spent by the i-th person that arrived at time j. Then, the total number of people at time t is:

$$Y(t) = \sum_{j=1}^{t} \sum_{i=1}^{A(j)} I\left(W_i^{(j)} \geq t - j\right)$$

where $I(x) = 1$ if x is true and 0 otherwise, and the terms count only the people who are still logged in after an extra time $t - j$ elapsed.

Each value $I\left(W_i^{(j)} \geq x\right)$ follows a distribution given by $1 - G(x)$, measuring the probability of a person staying longer than time x, and they are all independent.

We can write the sum above as:

$$Y(t) = \sum_{j=1}^{t} B(j), \quad B(j) = \sum_{i=1}^{A(j)} I\left(W_i^{(j)} \geq t - j\right)$$

Note that this implies that $B(j) | A(j) = n$ follows a binomial distribution with parameters n and $\mathbb{P}\left(W_i^{(j)} \geq t - j\right) = 1 - G(t - j)$.

Since $A(j) \sim \text{Poisson}(\lambda)$ and $B(j) | A(j) = n \sim \text{Binomial}(n, 1 - G(t - j))$, the marginal distribution of $B(j)$ is $B(j) \sim \text{Poisson}\left(\lambda(1 - G(t - j))\right)$. Therefore, Y being the sum of independent Poisson variables $B(j)$, we get that Y is a Poisson process with inhomogeneous rate $\lambda_Y(t)$,

$$Y(t) \sim \text{Poisson}\left(\lambda_Y(t)\right), \quad \lambda_Y(t) = \lambda \left[\sum_{j=1}^{t} (1 - G(t - j)) \right]$$

Exercise 24.6.15. Let $X(t)$ be a Poisson process with intensity λ. Let $W_1, W_2, \ldots$ be the waiting times. Let f be an arbitrary function. Show that

$$\mathbb{E}\left[\sum_{i=1}^{X(t)} f(W_i) \right] = \lambda \int_0^t f(w) dw$$

Solution.

When conditioning on $X(t)=n$, the waiting times are uniformly distributed on the interval $[0,t]$, so $\sum_{i=1}^n f(W_i)$ has the same distribution as $\sum_{i=1}^n f\left(U_i^{(n)}\right)$, for $U_i^{(n)} \sim \text{Uniform}(0,t)$. Then the expectation of $f\left(U_i^{(n)}\right)$ is

$$\mathbb{E}\left[f\left(U_i^{(n)}\right)\right] = \int_0^t f(w) f_{U_i^{(n)}}(w) dw = \frac{1}{t}\int_0^t f(w)dw$$

Then, we have:

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{X(t)} f(W_i)\right] &= \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{X(t)} f(W_i) \,\middle|\, X(t)=n\right] \mathbb{P}\left[X(t)=n\right] && (57)\\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\sum_{i=1}^{n} f\left(U_i^{(n)}\right)\right] \mathbb{P}\left[X(t)=n\right] && (58)\\
&= \sum_{n=0}^{\infty} \frac{n}{t}\left(\int_0^t f(w)dw\right) \mathbb{P}\left[X(t)=n\right] && (59)\\
&= \frac{1}{t}\left(\int_0^t f(w)dw\right) \sum_{n=0}^{\infty} n\, \mathbb{P}\left[X(t)=n\right] && (60)\\
&= \frac{1}{t}\left(\int_0^t f(w)dw\right) \mathbb{E}\left[X(t)\right] && (61)\\
&= \frac{1}{t}\left(\int_0^t f(w)dw\right) \lambda t && (62)\\
&= \lambda \int_0^t f(w)dw && (63)
\end{aligned}$$

Exercise 24.6.16. A two dimensional Poisson process is a process of random points in the plane such that (i) for any set A, the number of points falling in A is Poisson with mean $\lambda\mu(A)$ where $\mu(A)$ is the area of A, (ii) the number of points in nonoverlapping regions is independent. Consider an arbitrary point x_0 in the plane. Let X denote the distance from x_0 to the nearest point. Show that

$$\mathbb{P}(X>t) = e^{-\lambda\pi t^2}$$

and

$$\mathbb{E}(X) = \frac{1}{2\sqrt{\lambda}}$$

Solution. The distance X from x_0 to the nearest point is at least t if and only if no points lie within a circle of radius t centered in x_0. This circle C has area $\mu(C) = \pi t^2$, and so

$$\mathbb{P}(X > t) = \mathbb{P}(\#\{\text{points in C}\} = 0) = f_C(0) = \frac{\lambda_C^0 e^{-\lambda C}}{0!} = e^{-\lambda C} = e^{-\lambda \pi t^2}$$

Given that X has a CDF of $F_X(x) = 1 - e^{-\lambda \pi x^2}$ for $x \geq 0$, its PDF is $f_X(x) = 2\lambda \pi x e^{-\lambda \pi x^2}$, and its mean is

$$\mathbb{E}(X) = \int_0^\infty x f_X(x)\ dx = \int_0^\infty 2\lambda \pi x^2 e^{-2\lambda \pi x^2}\ dx = \frac{1}{2\sqrt{\lambda}}$$