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The Relativistic Spherical Top

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The classical theory of the free relativistic spherical top is first developed from a Lagrangian viewpoint. Our method allows the invariant mass to be an arbitrary function of the intrinsic spin. A canonical formalism is established following the approach suggested by Dirac for constrained Hamiltonian systems. There is a second arbitrary function in the theory, in addition to the usual one due to reparametrization invariance. The usual Newton–Wigner variables are supplemented by the Euler angles. The quantum theory of the free top is discussed. The classical theory is generalized to include charged tops with magnetic moments.

INTRODUCTION

In this paper, we attempt to treat the relativistic spherical top and its electromagnetic interactions from as general a viewpoint as possible. We begin with the free classical system and progress systematically from the Lagrangian and Hamiltonian descriptions to a possible relativistic wave equation for the free top. We then introduce electromagnetic interactions and give a Lagrangian derivation of the Bargmann–Michel–Telegdi equation [1] for a spinning particle with a magnetic moment. The Dirac formalism [2–5] for constrained Hamiltonian systems is then applied to find the interacting Poisson brackets for a spherical top with no magnetic moment. Much of our presentation consists of the restatement and

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synthesis of well-known results into a unified conceptual framework. Some of our results appear to be new.

Our description of a spherical top consists of a point on a world-line in Minkowski space and a rotating frame attached to that point [6, 7]. More precisely, we begin by considering a full element of the Poincaré group and then restrict the degrees of freedom so that the remaining independent variables accurately describe a relativistic spherical top. The canonical variables in the resulting system include the spatial four-momentum, its conjugate coordinate variable, the spin momentum, and the angular coordinates conjugate to the spin momentum; these may be combined to give an extended Poincaré group algebra, including the algebraic properties of the often-omitted angular coordinates.

In our treatment, the classical and quantum mechanical mass spectra are given by a trajectory function of the angular momentum. The quantum-mechanical spectrum may be interpreted either as a family of related particles, or as a family of bound states for which the binding mechanism need not be specified. Our quantum-mechanical states all have the extra $(2j + 1)$-fold degeneracy of the nonrelativistic spherical top.

The first section of the paper is devoted to the development of a classical Lagrangian formalism for the relativistic free spherical top. In order to have the correct nonrelativistic limit, we impose constraints upon the Lagrangian in the form of a set of differential equations. We argue that the invariant mass of the top is a function of the spin determined by the form of the Lagrangian. However, we find one unusual solution of the differential equations the Lagrangian must satisfy in which the mass and spin are *separately* fixed.

The next section outlines a general procedure which we adopt from Dirac for setting up consistent Poisson brackets in a constrained Hamiltonian system; these Dirac brackets are potentially the basis of a quantum theory. The relativistic spinless particle is treated in detail as an example. Section 3 deals with setting up the Dirac brackets and the Hamiltonian formalism for the free classical top. After fixing all possible constraints, we postulate a quantum theory of the relativistic spherical top which includes the angles as canonical variables.

Section 4 introduces electromagnetic interactions using Lagrangian equations of motion. We are able to derive the Bargmann–Michel–Telegdi equations from the action principle by introducing the magnetic moment in an unusual nonlinear way. In Section 5, we use the fact that there is a certain freedom in the canonical system to fix the constraint equations of the interacting top in a convenient way. We then derive the Dirac brackets and the Hamiltonian system for the charged spherical top (with no magnetic moment) interacting with an electromagnetic field.

The final section is devoted to a discussion of various questions and problems suggested by our treatment.

Appendices give useful identities, summarize the different quantum-mechanical operators which have been used to describe spinning particles, and show how to find a classical Lagrangian corresponding to a given trajectory function relating the invariant mass and the spin.

1. Lagrangian Approach to the Free Classical Relativistic Spherical Top

There are many treatments of the classical relativistic top in the literature. One of the earliest Lagrangian approaches seems to be that of Frenkel [8]. Related approaches are given by Thomas [9], Mathisson [10], Bhabha and Corben [11], Weyssenhoff and Raabe [12], and Shanmugadhasan [13]. Recent treatments include those of Hughes [7], Itzykson and Voros [6], and Rafanelli [14]. Additional references may be found in Corben [15]. We have not seen a treatment exactly like the one we give here, although equivalent approaches must surely exist. In any event, we shall attempt here to clarify and generalize the Lagrangian approach to the relativistic spherical top.

We will represent a spinning top as a point on a world-line to which a rotating frame has been attached [6, 7].

Our first step is to denote by τ an arbitrary parameter specifying the position of the particle along its world line. Then at each value of τ, we give an element $g = (x^\mu(\tau), \Lambda^\mu{}_\nu(\tau))$ of the Poincaré group. The Lagrangian coordinates are taken tentatively to be the spacetime position $x^\mu(\tau)$ of the particle and the Lorentz matrix $\Lambda^\mu{}_\nu(\tau)$ obeying

$$\Lambda_\lambda{}^\mu \Lambda^{\lambda\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{1.1a}$$

$$\Lambda^\mu{}_\lambda \Lambda^{\nu\lambda} = g^{\mu\nu}. \tag{1.1b}$$

The velocities appearing in the Lagrangian would then be

$$\begin{aligned} \dot{x}^\mu(\tau) &= dx^\mu(\tau)/d\tau \equiv u^\mu(\tau), \\ \dot{\Lambda}^\mu{}_\nu(\tau) &= d\Lambda^\mu{}_\nu/d\tau. \end{aligned} \tag{1.2}$$

Now we must restrict the way in which $\Lambda^\mu{}_\nu$ and $\dot{\Lambda}^\mu{}_\nu$ appear in the Lagrangian so that only independent variables consistent with Eq. (1.1) appear in the Lagrangian. (Later, we will impose additional conditions so that only the three rotational degrees of freedom enter the dynamics.) Observing that the derivative of Eq. (1.1a) is

$$\dot{\Lambda}_\lambda{}^\mu \Lambda^{\lambda\nu} + \Lambda_\lambda{}^\mu \dot{\Lambda}^{\lambda\nu} = 0,$$

we deduce that the quantity

$$\sigma^{\mu\nu} = \Lambda_\lambda{}^\mu \dot{\Lambda}^{\lambda\nu} = -\sigma^{\nu\mu} \tag{1.3}$$

has only six independent components. $\sigma^{\mu\nu}$ has just the right properties to be the analog of the usual angular velocity ω^{ij} of a rotating frame, which gives the velocity of a point relative to the origin as

$$dx^i/dt = x^j \omega^{ji} \tag{1.4}$$

We will take our free Lagrangian to be a function of $\sigma^{\mu\nu}$ and u^μ alone.

Next we must indicate how the variation of the Lagrangian is to be performed. We have found it convenient to express the variation of $\sigma^{\mu\nu}$ in terms of a composite variable defined as

$$\delta\theta^{\mu\nu} = \Lambda_\lambda{}^\mu \, \delta\Lambda^{\lambda\nu} = -\delta\theta^{\nu\mu} \tag{1.5}$$

so that

$$\delta\sigma^{\mu\nu} = \delta\dot{\theta}^{\mu\nu} + \sigma^{\mu\lambda} \, \delta\theta_\lambda{}^\nu - \delta\theta^{\mu\lambda}\sigma_\lambda{}^\nu. \tag{1.6}$$

Equations (1.5) and (1.6) will be seen to give a sound Lagrangian basis for the spin equations of motion and the Poisson brackets.

A. *Poincaré-Invariant Lagrangian*

To construct the most general possible Lagrangian, we wish to consider all the independent Poincaré-invariant quantities which we can form from our variables. It is therefore essential to define Poincaré-invariance precisely. There are in fact two types of invariance under a group: right invariance and left invariance. We define an element (a, M) of the Poincaré group to act as follows on a generic group element $g = (x, \Lambda)$:

$$\text{Right transformation:} \quad (x', \Lambda') = (x, \Lambda) \cdot (a, M) = (M^{-1}x + a, \Lambda M), \tag{1.7a}$$

$$\text{Left transformation:} \quad (x', \Lambda') = (a, M) \cdot (x, \Lambda) = (\Lambda^{-1}a + x, M\Lambda). \tag{1.7b}$$

A convenient way to represent this action is to write (a, M) as the 5×5 matrix

$$(a, M) = \begin{pmatrix} M & 0 \\ a & 1 \end{pmatrix} \tag{1.8a}$$

with

$$(a, M)^{-1} = \begin{pmatrix} M^{-1} & 0 \\ -aM^{-1} & 1 \end{pmatrix}. \tag{1.8b}$$

Since $\sigma\, d\tau \equiv \Lambda^{-1}\, d\Lambda$ and $u\, d\tau \equiv dx$, the right-invariant form is

$$dg \cdot g^{-1} = (u\Lambda^{-1}, \Lambda\sigma\Lambda^{-1})\, d\tau \tag{1.9a}$$

and the left-invariant form is

$$g^{-1} \cdot dg = (u + \sigma x, \sigma)\, d\tau, \tag{1.9b}$$

where our notation for dg may be represented as

$$dg = \begin{pmatrix} d\Lambda & 0 \\ dx & 0 \end{pmatrix} = (u, \Lambda\sigma)\, d\tau. \tag{1.10}$$

We now examine the quantities $u\Lambda^{-1} = \Lambda u$, $\Lambda\sigma\Lambda^{-1}$, $u + \sigma x$, and σ. Under *right* transformation by a group-element (a, M), we have

$$\begin{aligned} \Lambda u &\xrightarrow{R} \Lambda u \\ \Lambda\sigma\Lambda^{-1} &\xrightarrow{R} \Lambda\sigma\Lambda^{-1} \\ u + \sigma x &\xrightarrow{R} M^{-1}(u + \sigma x) + M^{-1}\sigma M a \\ \sigma &\xrightarrow{R} M^{-1}\sigma M. \end{aligned} \tag{1.11}$$

Under *left* transformation, we find

$$\begin{aligned} \Lambda u &\xrightarrow{L} M(\Lambda u - \Lambda\sigma\Lambda^{-1}a), \\ \Lambda\sigma\Lambda^{-1} &\xrightarrow{L} M\Lambda\sigma\Lambda^{-1}M^{-1}, \\ u + \sigma x &\xrightarrow{L} u + \sigma x, \\ \sigma &\xrightarrow{L} \sigma. \end{aligned} \tag{1.12}$$

Thus, as we could deduce directly from Eq. (1.9)

$$\left.\begin{matrix} \Lambda u \\ \Lambda\sigma\Lambda^{-1} \end{matrix}\right\} \text{ are right-invariant} \tag{1.13a}$$

$$\left.\begin{matrix} u + \sigma x \\ \sigma \end{matrix}\right\} \text{ are left-invariant.} \tag{1.13b}$$

Physically, right transformations rotate and translate the reference system describing the top, while left transformations alter the body-fixed frame of the top. The usual meaning of Poincaré-invariance for physical systems is that the physics should not change under rotations and translations of the reference system. We now choose to allow our Lagrangian to depend only on left Lorentz scalars formed from the *right-invariant quantities* given in Eq. (1.13a) and not containing other explicit $\Lambda^{\mu}{}_{\nu}$'s. This is our definition of the relativistic spherical top. There are therefore four invariants which may be used in the Lagrangian:

$$\begin{aligned} a_1 &= u_\mu u^\mu \equiv u^2, \\ a_2 &= \sigma^{\mu\nu}\sigma_{\mu\nu} \equiv \sigma \cdot \sigma, \\ a_3 &= u_\mu \sigma^{\mu\nu}\sigma_{\nu\lambda}u^\lambda \equiv u\sigma\sigma u, \\ a_4 &= \operatorname{Det} \sigma = (1/16)(\sigma^{\mu\nu}\sigma^*_{\mu\nu})^2 \equiv (1/16)(\sigma \cdot \sigma^*)^2, \end{aligned} \tag{1.14}$$

where we define

$$\sigma^*_{\mu\nu} = (1/2)\, \epsilon_{\mu\nu\lambda\rho}\sigma^{\lambda\rho} \tag{1.15}$$

with $\epsilon^{0123} = +1$. Our free Lagrangian is therefore some function

$$L_0 = L_0(a_1, a_2, a_3, a_4)$$

of the invariants.

B. *Canonical Momenta*

The canonical momenta are now defined as the following variations of the Lagrangian,

$$P^\mu = -\partial L_0/\partial u_\mu = -2u^\mu L_1 - 2\sigma^{\mu\nu}\sigma_{\nu\lambda}u^\lambda L_3 \tag{1.16}$$

$$S^{\mu\nu} = -\partial L_0/\partial \sigma_{\mu\nu} = -4\sigma^{\mu\nu}L_2 - 2(u^\mu\sigma^{\nu\lambda}u_\lambda - u^\nu\sigma^{\mu\lambda}u_\lambda)\, L_3 - \tfrac{1}{2}\sigma^{*\mu\nu}(\sigma \cdot \sigma^*)\, L_4, \tag{1.17}$$

where

$$L_i(a_1, a_2, a_3, a_4) = (\partial/\partial a_i)\, L_0(a_1, a_2, a_3, a_4). \tag{1.18}$$

($S^{\mu\nu}$ is actually a combination of coordinates and momenta: see Section 3.A.) The minus signs in the definitions are chosen to give the correct nonrelativistic limit. For example, in the spinless case, we might have a Lagrangian like

$$L = -m(1 - v^2)^{1/2} \approx \tfrac{1}{2}mv^2 - m$$

so the momenta would be

$$P^i = mv^i/(1 - v^2)^{1/2} = \partial L/\partial v^i = -\partial L/\partial v_i\,.$$

Using $\delta u^\mu = \delta \dot{x}^\mu$ and the variation $\delta\sigma^{\mu\nu}$ given by Eq. (1.6), we find the following Euler equations as the coefficients of the arbitrary variations δx_μ and $\delta\theta_{\mu\nu}$, respectively:

$$\dot{P}^\mu = 0, \tag{1.19}$$

$$\dot{S}^{\mu\nu} + \sigma^{\mu\lambda}S_\lambda{}^\nu - S^{\mu\lambda}\sigma_\lambda{}^\nu = 0. \tag{1.20a}$$

We now use Eqs. (1.16) and (1.17) with the identities in Appendix A to show that Eq. (1.20a) may also be written

$$\dot{S}^{\mu\nu} + u^\mu P^\nu - u^\nu P^\mu = 0. \tag{1.20b}$$

Thus the quantities

$$\begin{aligned} &P^\mu, \\ &M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + S^{\mu\nu}, \end{aligned} \tag{1.21}$$

are constants of the motion which will be identified as the canonical generators of translations and of Lorentz transformations, respectively. Taken as a whole, the set of quantities (1.21) are the canonical generators of the Poincaré-group transformations.

We note that Eq. (1.20a) can be integrated directly using the definition (1.3) of $\sigma^{\mu\nu}$ to cast it in the form

$$(d/d\tau)(\Lambda_{\alpha\mu} S^{\mu\nu} \Lambda_{\beta\nu}) = 0.$$

Taking the integration constant to be $\bar{S}_{\alpha\beta}$, we have

$$S^{\mu\nu}(\tau) = \Lambda^{\alpha\mu} \bar{S}_{\alpha\beta} \Lambda^{\beta\nu} \tag{1.22}$$

so that

$$S^{\mu\nu}(\tau)\, S_{\mu\nu}(\tau) = \bar{S}^{\mu\nu} \bar{S}_{\mu\nu} = \text{const.} \tag{1.23}$$

This also follows directly from multiplying Eq. (1.20a) by $S_{\mu\nu}$. Since Eq. (1.20a) and identity (A.5) imply that

$$\dot{S}^{*\mu\nu} + \sigma^\mu{}_\lambda S^{*\lambda\nu} - S^{*\mu\lambda} \sigma_\lambda{}^\nu = 0, \tag{1.24}$$

we also have

$$S^*_{\mu\nu}(\tau) = \Lambda^\alpha{}_\mu \bar{S}^*_{\alpha\beta} \Lambda^\beta{}_\nu \tag{1.25}$$

where $\bar{S}^*_{\alpha\beta}$ is a constant. Thus

$$S^{\mu\nu}(\tau)\, S^*_{\mu\nu}(\tau) = \bar{S}^{\mu\nu} \bar{S}^*_{\mu\nu} = \text{const}, \tag{1.26}$$

as we could confirm by multiplying (1.24) by $S_{\mu\nu}$ and using identity (A.7). In the sequel $\bar{S}^{\mu\nu}$ will be related to the body-fixed components of the angular momentum of the top.

For later use, we exhibit the form of the useful four-vectors and invariants which can be made from the momenta (1.16) and (1.17). One is the Pauli–Lubanski tensor [16]

$$\begin{aligned} W^\mu &= S^{*\mu\nu} P_\nu = (1/2)\, \epsilon^{\mu\nu\lambda\sigma} S_{\nu\lambda} P_\sigma \\ &= 4\sigma^{*\mu\nu} u_\nu (2L_1 L_2 + L_3[a_3 L_3 + a_4 L_4]) \\ &\quad + \sigma^{\mu\nu} u_\nu (\sigma \cdot \sigma^*)(-L_1 L_4 - 2L_2 L_3 + a_1 (L_3)^2 + (1/2)\, a_2 L_3 L_4). \end{aligned} \tag{1.27}$$

Another is the vector

$$
\begin{aligned}
V^{\mu} &= S^{\mu\nu}P_{\nu} \\
&= 4\sigma^{\mu\nu}u_{\nu}(2L_1L_2 - L_3[a_1L_1 + a_2L_2 + a_3L_3 + a_4L_4]) \\
&\quad + \sigma^{*\mu\nu}u_{\nu}(\sigma \cdot \sigma^{*})(L_1L_4 - 2L_2L_3).
\end{aligned} \tag{1.28}
$$

We define the symbols M^2 and J^2 as

$$
M^2 \equiv P^{\mu}P_{\mu} = 4a_1(L_1)^2 + 8a_3L_1L_3 - 2a_2a_3(L_3)^2 + 4a_1a_4(L_3)^2 \tag{1.29}
$$

$$
\begin{aligned}
(1/2)\, J^2 \equiv (1/4)\, S^{\mu\nu}S_{\mu\nu} &= 4a_2(L_2)^2 + 8a_3L_2L_3 - 2a_1a_3(L_3)^2 \\
&\quad - a_4L_4(a_2L_4 + 4a_1L_3 - 16L_2).
\end{aligned} \tag{1.30}
$$

Also of use are

$$
(1/4)\, S^{\mu\nu}S^{*}_{\mu\nu} = (\sigma \cdot \sigma^{*})(2L_2[2L_2 - a_1L_3] - L_4[a_2L_2 + a_3L_3 + a_4L_4]) \tag{1.31}
$$

and

$$
\begin{aligned}
W^2 = W^{\mu}W_{\mu} &= -(1/2)\, P^{\mu}P_{\mu}S^{\alpha\beta}S_{\alpha\beta} - P^{\mu}S_{\mu\nu}S^{\nu\lambda}P_{\lambda} \\
&= -M^2J^2 + V^{\mu}V_{\mu}
\end{aligned} \tag{1.32}
$$

M^2 and W^2 are the analogs of the two Casimir invariants of the abstract Poincaré group [17].

C. *Constraints*

As our theory stands, the canonical momenta contain too many degrees of freedom. In order for our spinning particle to have only the three physical rotational degrees of freedom in its rest frame, we must eliminate three of the six components of $S^{\mu\nu}$. The classic work of Pryce [18] singles out three sensible ways of achieving this, namely

$$
\begin{aligned}
&\text{(a)} \quad V^{\mu} = S^{\mu\nu}P_{\nu} = 0, \\
&\text{(b)} \quad S^{0\mu} = 0, \\
&\text{(c)} \quad MS^{0\mu} - V^{\mu} = 0.
\end{aligned} \tag{1.33}
$$

In each case, only three of the four components of the equation are independent. Depending on the condition chosen, the particle position variables have different physical meanings. A summary of Pryce's results concerning the classical and quantum-mechanical properties of these alternative position variables is given in Appendix B. Since, in the end, the set of variables corresponding to the choice of any one of the conditions (1.33) can be reexpressed in terms of any other set, it will suffice to treat only one.

The significant convenience of having a Lorentz-covariant subsidiary condition motivates us to choose to restrict the Lagrangian so that condition (a) is satisfied [8, 10, 12]:

$$V^{\mu} = S^{\mu\nu} P_{\nu} = 0. \tag{1.34}$$

An immediate consequence of (1.34) and identity (A.7) is that

$$S^{\mu\nu} S^{*}_{\mu\nu} = 0. \tag{1.35}$$

How shall we now force the Lagrangian to obey the constraint (1.34)? Equation (1.28) tells us that a sufficient condition is to require the Lagrangian to satisfy simultaneously the two differential equations

$$2L_1 L_2 = L_3(a_1 L_1 + a_2 L_2 + a_3 L_3 + a_4 L_4), \tag{1.36}$$

$$2L_2 L_3 = L_1 L_4 . \tag{1.37}$$

However, contraction of Eq. (1.28) with $\sigma^{\mu\nu} u_{\nu}$ and $\sigma^{*\mu\nu} u_{\nu}$ reveals that (1.36) and (1.37) are equivalent if

$$a_1 a_2 a_3 + 2a_3{}^2 - 2a_1{}^2 a_4 = 0.$$

We shall see shortly that this indeed is true for the free top when the Euler equations are satisfied. In the interacting case, this expression no longer vanishes and Eqs. (1.36) and (1.37) must hold *separately* to maintain $V^{\mu} = 0$. We actually want to be able to give properties of the Lagrangian which are independent of the Euler equations, so that Eqs. (1.36) and (1.37) will generally be treated as independent conditions even in the free case. We therefore will describe the relativistic spherical top by means of a Poincaré-invariant Lagrangian satisfying Eqs. (1.36) and (1.37) in order to guarantee that our relativistic system has the right nonrelativistic limit.

There is another constraint to be imposed on the Lagrangian to ensure the arbitrary nature of the parameter τ. In order to have the analog of the τ-reparametrization-invariant spinless particle action

$$S = -m \int d\tau\, (u^{\mu} u_{\mu})^{1/2},$$

we require that the Lagrangian be homogeneous of degree one in the velocities:

$$(1/2)\, L_0(a_1, a_2, a_3, a_4) = a_1 L_1 + a_2 L_2 + 2a_3 L_3 + 2a_4 L_4 . \tag{1.38}$$

Let us now analyze the constraints on the velocities which follow from $V^{\mu} = 0$. Differentiating (1.34), we find

$$S^{\mu\nu} \dot{P}_{\nu} + \dot{S}^{\mu\nu} P_{\nu} = 0. \tag{1.39}$$

It is easy to use the Euler equations (1.19) and (1.20) to show that

$$P^\mu = M^2 u^\mu/(P^\nu u_\nu) = C^{-1} u^\mu,$$
$$\dot{S}^{\mu\nu} = 0. \tag{1.40}$$

Thus we have the following constraint among the velocities:

$$u^\mu(2a_3L_1 - a_2a_3L_3 + 2a_1a_4L_3) = 2\sigma^{\mu\nu}\sigma_{\nu\lambda}u^\lambda(a_1L_1 + a_3L_3). \tag{1.41}$$

Contracting (1.41) with u_μ, we find that if L_3 is not zero, then

$$a_1a_2a_3 + 2a_3^2 - 2a_1^2a_4 = 0. \tag{1.42}$$

Substituting Eq. (1.42) back into (1.41), we find that if $a_3(a_1L_1 + a_3L_3)$ is nonzero, the velocity constraint may be compactly expressed as

$$u^\mu = (a_1/a_3)\,\sigma^{\mu\nu}\sigma_{\nu\lambda}u^\lambda. \tag{1.43}$$

It is important to remember that Eqs. (1.40)–(1.43) are valid *after* the application of the Euler equations of motion, while $V^\mu = 0$ followed from the form of the Lagrangian alone. Bearing this in mind, we observe that the differential equations (1.36), (1.37) and (1.38) alone imply that

$$\begin{aligned}(M^2/L_0(L_3)^2)&[2a_3L_2 - a_1a_3L_3 - a_1a_4L_4]\\ &= -(M^2L_1/L_0(L_3)^3)[2a_1L_2 - a_1^2L_3 - L_4(a_3 + \tfrac{1}{2}a_1a_2)]\\ &= (a_1a_2a_3 + 2a_3^2 - 2a_1^2a_4).\end{aligned} \tag{1.44}$$

Applying the Euler equations, we find that Eq. (1.42) makes the square brackets in Eq. (1.44) vanish. Equation (1.42) may be applied again to show the equivalence of the two vanishing square brackets when the Euler equations hold. It is then a simple exercise to show the equivalence of (1.36) and (1.37) directly. Equation (1.44) and a number of other useful relations following from the differential equations alone are listed in Appendix A.

The solution (1.40) for u^μ in terms of P^μ is completely fixed once a scale, say $u^2 = 1$, is chosen for u^μ. There is a similar ambiguity in the expression of $\sigma^{\mu\nu}$ in terms $S^{\mu\nu}$ which will turn out to be important when we introduce interactions. We first observe that if

$$\sigma^{\mu\nu} = AS^{\mu\nu} + BS^{*\mu\nu}, \tag{1.45}$$

then

$$\begin{aligned} a_1 &= C^2P^2, & a_2 &= (A^2 - B^2)(S\cdot S),\\ a_3 &= \tfrac{1}{2}B^2C^2P^2(S\cdot S), & a_4 &= \tfrac{1}{4}A^2B^2(S\cdot S)^2\end{aligned} \tag{1.46}$$

and Eqs. (1.42), (1.43), and (1.17) can be shown to hold identically. There is a relation among A, B, C, P^μ, and $S^{\mu\nu}$ which may be found as follows. From Eqs. (1.29) and (1.46) we have

$$1/C^2 = (2L_1 + L_3 B^2 S_{\alpha\beta} S^{\alpha\beta})^2. \tag{1.47}$$

But L_0 is of the form

$$L_0 = (a_1)^{1/2}\, \mathscr{L}(a_2/a_1\, , a_3/a_1^2, a_4/a_1^2),$$

so that L_1 and L_3 can be calculated in terms of derivatives of $\mathscr{L}$. When we substitute these expressions in Eq. (1.47) and replace the a_i's by the expressions (1.46), we find a relation of the form

$$F\left(\frac{A^2 - B^2}{C^2}\frac{S \cdot S}{P^2}, \frac{1}{2}\frac{B^2}{C^2}\frac{S \cdot S}{P^2}, \frac{1}{4}\frac{A^2B^2}{C^4}\frac{(S \cdot S)^2}{P^4}\right) = P^2. \tag{1.48}$$

From this equation, we see that if P^μ and $S^{\mu\nu}$ are given, only two of the functions A, B, C are independent; we can thus fix all three by choosing, say, A and B/C. The scaling of A (and hence of B and C) corresponds to the reparametrization invariance of the system in τ. A/C is then related to the angular velocity of the top. The arbitrariness in B/C will be clarified by later discussions using the Dirac formalism for constrained Hamiltonian systems. Note that the conventional nonrelativistic description of the top corresponds to setting $B = 0$ and $C = 1/M$.

D. *Trajectories*

We now argue that the constraints of the previous subsection show that M^2 is a function of J^2 which can be regarded as the relativistic analog of the relation between the total energy and the angular momentum of a classical nonrelativistic top. We first observe that Eqs. (1.36)–(1.38) combined with expressions (1.29), (1.30) for M^2, J^2 imply that we may eliminate the derivatives L_i to find consistency conditions among L_0, M^2, J^2 and the a_i. The resulting condition is a rather lengthy algebraic expression of fourth order in L_0^2. (See Appendix A.) There seems to be no evident way of simplifying this relation. As $a_4 \to 0$, however, one root of the equation for L_0^2 is given by

$$L_0^2 = a_1 M^2 + (1/2)\, a_2 J^2 + 2MJ((1/2)\, a_1 a_2 + a_3)^{1/2}. \tag{1.49}$$

If we set $M = A$, $J = B$ where A, B are constants, the resulting L_0 satisfies the differential equations (1.36) and (1.37). The resulting motion corresponds to a system with *separately fixed* M and J, whereas in general one can only infer the existence of a trajectory function

$$M^2 - f(J^2) = 0. \tag{1.50}$$

relating M^2 to J^2. Details are given in Appendix C. The basic idea is that if we define

$$\xi = a_2/a_1 , \qquad \eta = a_3/a_1^2, \qquad \theta = a_4/a_1^2,$$

then

$$L_0 = (a_1)^{1/2} \mathscr{L}(\xi, \eta, \theta)$$

and an explicit computation yields the vanishing of the Jacobians:

$$\frac{\partial(M^2, J^2)}{\partial(\xi, \eta)} = \frac{\partial(M^2, J^2)}{\partial(\xi, \theta)} = \frac{\partial(M^2, J^2)}{\partial(\eta, \theta)} = 0.$$

From this result it follows that (at least locally) there exists a function $f(J^2)$ such that (1.50) is true. On the submanifold $a_3 = a_4 = 0$, Eqs. (1.29) and (1.30) tell us that we may choose

$$\begin{aligned} M &= 2\xi\mathscr{L}' - \mathscr{L} \\ J &= 2(2\xi)^{1/2} \mathscr{L}' \end{aligned} \tag{1.51}$$

where $\mathscr{L}' = \partial\mathscr{L}(\xi, 0, 0)/\partial\xi$. We now compute the derivatives of M and J with respect to ξ, finding

$$\frac{dM}{dJ} = \frac{J}{M} f'(J^2) = \frac{\partial M}{\partial \xi} \left(\frac{\partial J}{\partial \xi}\right)^{-1} \Bigg|_{a_3=a_4=0} = \left(\frac{a_2}{2a_1}\right)^{1/2} \Bigg|_{a_3=a_4=0} . \tag{1.52}$$

This equation tells us that there is a simple intuitive relation between the trajectory slope and the angular velocity which may be extended beyond $a_3 = a_4 = 0$ by the techniques of Appendix C. Finally, Eq. (1.51) implies

$$\mathscr{L}(\xi) = -M + (\xi/2)^{1/2} J. \tag{1.53}$$

Equation (1.53) can be viewed as a family of straight lines in the M–J plane parametrized by ξ. The envelope of these lines (see Appendix C) is just the trajectory function (1.50).

The next obvious task is to find explicit solutions of the differential equations (1.36) and (1.37) corresponding to a given trajectory. The general method is to reconstruct L_0 from the trajectory using Eqs. (1.51)–(1.53) on the boundary submanifold $a_3 = a_4 = 0$ and to use (1.36) and (1.37) in a generalized eikonal procedure to continue L_0 for any a_3, a_4. The resulting calculations although straightforward are exceedingly complicated even for the simplest trajectories. A nontrivial example of an explicit solution is

$$L_0^2 = (1/2)(Aa_1 - Ba_2 + [(Aa_1 - Ba_2)^2 - 8B(Aa_3 - 2Ba_4)]^{1/2}), \tag{1.54}$$

for which the trajectory has the form

$$BM^2 - (1/2)\,AJ^2 = AB. \tag{1.55}$$

E. *Poincaré Group Generators*

For completeness, we give a simple argument to deduce the generators of the Poincaré group from the action principle. In terms of the canonical momenta (1.16) and (1.17), the variation of the action is

$$\begin{aligned}
&\delta \int_a^b L_0(a_1, a_2, a_3, a_4)\, d\tau \\
&\quad = - \int_a^b (P^\mu\, \delta u_\mu + (1/2)\, S^{\mu\nu}\, \delta\sigma_{\mu\nu})\, d\tau \\
&\quad = \int_a^b \left(\delta x_\mu \frac{dP^\mu}{d\tau} + (1/2)\, \delta\theta_{\mu\nu}\left(\frac{dS^{\mu\nu}}{d\tau} + \sigma^{\mu\lambda} S_\lambda{}^\nu - S^{\mu\lambda}\sigma_\lambda{}^\nu\right)\right) d\tau - P^\mu(b)\, \delta x_\mu(b) \\
&\qquad - (1/2)\, S^{\mu\nu}(b)\, \delta\theta_{\mu\nu}(b) + P^\mu(a)\, \delta x_\mu(a) + (1/2)\, S^{\mu\nu}(a)\, \delta\theta_{\mu\nu}(a).
\end{aligned} \tag{1.56}$$

Now if we make an infinitesimal translation of the end points, we have

$$\delta x^\mu(a) = \epsilon^\mu,$$
$$\delta\theta^{\mu\nu}(a) = 0.$$

The coefficient of ϵ_μ is

$$P^\mu = \text{Translation Generator}. \tag{1.57}$$

If we make an infinitesimal right Lorentz transformation of the endpoints,

$$\delta x^\mu(a) = \omega^{\alpha\beta}(x_\alpha g_\beta{}^\mu - x_\beta g^\mu{}_\alpha),$$
$$\delta \Lambda^{\lambda\nu}(a) = \omega^{\alpha\beta}(\Lambda^\lambda{}_\alpha g^\nu{}_\beta - \Lambda^\lambda{}_\beta g^\nu{}_\alpha),$$

so that

$$\delta\theta^{\mu\nu}(a) = \Lambda_\lambda{}^\mu(a)\, \delta\Lambda^{\lambda\nu}(a) = \omega^{\alpha\beta}(g^\mu{}_\alpha g^\nu{}_\beta - g^\mu{}_\beta g^\nu{}_\alpha),$$

the coefficient of $\omega_{\mu\nu}$ is

$$M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + S^{\mu\nu} = \text{Right Lorentz Generator}. \tag{1.58}$$

If the action is invariant under translations and Lorentz transformations, P^μ and $M^{\mu\nu}$ are conserved and generate the canonical transformations of the Poincaré group.

2. HAMILTONIAN DESCRIPTION OF A CONSTRAINED CLASSICAL SYSTEM

In order to find a consistent quantum-mechanical operator system corresponding to the classical Lagrangian description of the free spherical top given in Section 1, we must first understand how to set up the Poisson brackets and the Hamiltonian description for a constrained system. If we are fortunate, the commutation relations of the operators may be deduced directly from the Poisson brackets of the independent canonical variables.

A. *Dirac Brackets*

We now give a brief outline of Dirac's procedure [2–5] for constructing a consistent set of Poisson brackets corresponding to a Lagrangian system with constraints

$$\Phi_i(q, p) \approx 0, \qquad i = 1,..., n. \tag{2.1}$$

The symbol "$\approx$" is read "weakly equals" and is used to indicate a constraint which vanishes formally but may have nonvanishing canonical Poisson brackets.

Dirac [2, 3] originally considered only constraints which followed directly from the form of the Lagrangian ("primary constraints") or which followed from the time derivatives of the primary constraints ("secondary constraints"). The distinction between primary and secondary constraints does not seem to be important. Dirac then divides the constraints into two subsets, the "first-class constraints" whose Poisson brackets with all other constraints vanish weakly, and the "second-class constraints" which have nonvanishing Poisson brackets with at least one other constraint. (Remember that one must compute the canonical Poisson bracket first, and set the constraints weakly zero only after the computation.) The first-class constraints imply the existence of arbitrary functions in the equations of motion.

In his work on gravitation [4], Dirac introduces another kind of constraint ("gauge constraints"), *not implied* by the Lagrangian or the Euler equations, and which have nonvanishing Poisson brackets with the first-class constraints. These gauge constraints have the effect of eliminating the arbitrary functions in the Hamiltonian corresponding to some or all of the first-class constraints (which have now actually become *second-class*). As we shall see, the gauge constraints may depend explicitly on the time. This will introduce changes in the Hamiltonian, which we can treat using Hamilton's action principle.

If all arbitrary functions are not eliminated from the Hamiltonian equations of motion by the obvious gauge constraints, there is yet another type of constraint which may be used to accomplish this elimination, the "invariant relation" [5]. A set of invariant relations.

$$\psi_i(q, p) \approx 0, \qquad i = 1,..., k \tag{2.2}$$

is a set of relations among the canonical variables obeying

$$\frac{d\psi_i}{dt} \approx \sum_{i=1}^{k} M_{ij}\psi_j \tag{2.3}$$

so that if $\psi_i \approx 0$ at $t = 0$, Eq. (2.2) holds for all times. (The symbol $\approx$ is used in Eqs. (2.2) and (2.3) to indicate that all other constraints besides the ψ_i have already been set to zero.) Invariant relations are part of the definition of a canonical system in that they restrict the phase space of admissible motions.

Invariant relations can actually be treated as secondary constraints if we introduce a Lagrange multiplier λ_i corresponding to each invariant relation $\psi_i \approx 0$, $i = 1,..., k$. We now express the momenta in terms of the velocities $\dot{q}$, so $\psi_i(q, p) \to \psi_i(q, \dot{q})$, and define a new Lagrangian

$$L' = L(q, \dot{q}) + \tfrac{1}{2} \sum_{i=1}^{k} \lambda_i[\psi_i(q, \dot{q})]^2.$$

The equations of motion are now

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \sum_{i=1}^{k} \left[\lambda_i\psi_i \left(\frac{d}{dt}\frac{\partial \psi_i}{\partial \dot{q}_j} - \frac{\partial \psi_i}{\partial q_j}\right) + \lambda_i \frac{d\psi_i}{dt}\frac{\partial \psi_i}{\partial \dot{q}_j}\right] = 0,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\lambda}_i} - \frac{\partial L}{\partial \lambda_i} = -(\psi_i)^2 = 0.$$

But then the new equations of motion are exactly the old ones, except now the constraints $\psi_i \approx 0$ appear as secondary constraints following from the primary constraints $P_\lambda{}^i \equiv \partial L/\partial \dot{\lambda}_i \approx 0$. The net effect is to implement the ψ_i as ordinary constraints while the λ_i disappear completely from the dynamics.

Now we consider a second-class subset $\{\phi_\alpha(q, p); \alpha = 1,..., m\}$ of the available constraints. By definition, each ϕ_α has nonzero Poisson brackets with at least one other ϕ_β, and the matrix of Poisson brackets

$$C_{\alpha\beta} = \{\phi_\alpha, \phi_\beta\} \tag{2.4}$$

is nonsingular. We denote by $C^{-1}_{\alpha\beta}$ the matrix with the property

$$C^{-1}_{\alpha\gamma}C_{\gamma\beta} = \delta_{\alpha\beta}. \tag{2.5}$$

We have found it useful to define the prime variable

$$\xi' = \xi - \{\xi, \phi_\alpha\}\, C^{-1}_{\alpha\beta}\phi_\beta \tag{2.6}$$

corresponding to each ordinary canonical variable; note that $\xi' \approx \xi$. The Dirac bracket can then be defined as

$$\begin{aligned}\{\xi, \eta\}^* &= \{\xi, \eta\} - \{\xi, \phi_\alpha\} C_{\alpha\beta}^{-1}\{\phi_\beta, \eta\} \\ &\approx \{\xi', \eta'\} \approx \{\xi, \eta'\} \approx \{\xi', \eta\}\end{aligned} \tag{2.7}$$

and is seen from its definition in terms of ξ' and η' to satisfy the Jacobi identity. The point, of course, is that while $\xi' \approx \xi$ and $\eta' \approx \eta$, the Dirac bracket is *not* weakly equal to the Poisson bracket. The Dirac bracket is constructed in just such a way that the brackets of each constraint with any canonical variable are *strongly* zero.

To make the transition to quantum mechanics, aside from ordering problems, we take

$$\{\xi, \eta\}^* \to i[\xi, \eta] = i(\xi\eta - \eta\xi). \tag{2.8}$$

The operators ϕ_α can then be taken effectively zero on the Hilbert space of physical states because all their commutators should vanish. The question of whether or not there actually exist quantum mechanical operators realizing the algebra of the Dirac brackets apparently must be dealt with on a case-by-case basis.

One useful property of the Dirac brackets is that they are *iterative*; if you have computed the brackets for a subset of the constraints and wish to add others, you may use Eq. (2.7) with the first set of Dirac brackets replacing the ordinary Poisson brackets. The final Dirac bracket is the same regardless of the order in which the constrained brackets were computed. (See the next subsection for a proof.) This technique can be used, for example, to display the covariant nature of a set of relativistic brackets before setting $P^0 = (\mathbf{P}^2 + M^2)^{1/2}$, $x^0 = t$ and destroying the manifest covariance of the system of brackets.

In the case when only constraints following from the Lagrangian and their derivatives are considered, Dirac singles out the first-class constraints χ_i for special treatment. Since the Poisson brackets of the χ_i with all other contraints are weakly zero, they may be multiplied by arbitrary functions v_i and added to the fiirst-class Hamiltonian H_0' without changing the motion of any of the constraints. Thus Dirac takes as the total Hamiltonian

$$H = H_0' + v_i\chi_i \tag{2.9}$$

where $H \approx H_0$, but the Poisson brackets of H are not equal to those of H_0. The function of the v_i's is to specify a complete solution of the Euler equations for the motion of the system. We will give examples of the use of Eq. (2.9), but in the end we adopt a procedure in which all gauges and invariant relations are specified so that there are *no* first-class constraints, and hence no arbitrary functions in the equations of motion.

B. *Hamilton Variational Principle with Constraints*

We now develop the ideas of the previous subsection from a slightly different viewpoint which clarifies the meanings of some of the concepts. We begin by considering the Hamilton variational principle in phase space, so the p's and q's are considered as independent variables [19]:

$$\begin{aligned} 0 = \delta S &= \delta \int \left(\sum_{i=1}^{n} p_i \, dq_i - H \, d\tau \right) \\ &= \delta \int \left(\tfrac{1}{2} \sum_{i=1}^{n} (p_i \, dq_i - q_i \, dp_i) - H \, d\tau \right). \end{aligned} \tag{2.10}$$

As usual, we find Hamilton's equations,

$$\frac{\partial H}{\partial p_i} = + \frac{dq_i}{d\tau}, \qquad \frac{\partial H}{\partial q_i} = - \frac{dp_i}{d\tau}. \tag{2.11}$$

Suppose now that we know the entire set of constraints, gauges and invariant relations which restrict the phase space truly available for the particle motion. We then reexpress the $2n$ canonical variables p_i, q_i in terms of the $2m$ truly independent variables

$$z_i, \qquad i = 1,..., 2m$$

and the $2n - 2m$ constraints

$$z_i = \phi_{i-2m} \approx 0, \qquad i = 2m + 1,..., 2n.$$

Thus we may express p_i and q_i as functions of the z's and of τ, which we write

$$\begin{aligned} p_i &= p_i(z, \tau), \\ q_i &= q_i(z, \tau). \end{aligned} \tag{2.12}$$

Note that while p_i and q_i are by definition not explicit functions of τ, the z_i may be explicitly τ-dependent; the explicit τ's in Eq. (2.12) are necessary to compensate for any τ-dependence of the z_i.

Now we consider the z_i's and $\tau \equiv z_0$ as a set of independent variables, so that

$$\tfrac{1}{2} \sum_{i=1}^{n} (p_i \, dq_i - q_i \, dp_i) = \sum_{\alpha=0}^{2n} C_\alpha \, dz_\alpha \tag{2.13}$$

where

$$C_\alpha = \tfrac{1}{2} \sum_{i=1}^{n} \left(p_i \frac{\partial q_i}{\partial z_\alpha} - q_i \frac{\partial p_i}{\partial z_\alpha} \right). \tag{2.14}$$

We immediately see that

$$\frac{\partial C_\alpha}{\partial z_\beta} - \frac{\partial C_\beta}{\partial z_\alpha} = \sum_{i=1}^{n} \left(\frac{\partial q_i}{\partial z_\alpha} \frac{\partial p_i}{\partial z_\beta} - \frac{\partial q_i}{\partial z_\beta} \frac{\partial p_i}{\partial z_\alpha} \right) = (z_\alpha , z_\beta) \tag{2.15}$$

is just the *Lagrange bracket* of the new set of variables z_α, including constraints, with respect to the old set of canonical variables q_i and p_i. If we now define the *Poisson bracket* as

$$\{z_j , z_k\} = \sum_{i=1}^{n} \left(\frac{\partial z_j}{\partial q_i} \frac{\partial z_k}{\partial p_i} - \frac{\partial z_j}{\partial p_i} \frac{\partial z_k}{\partial q_i} \right), \tag{2.16}$$

we find the following properties.

$$\sum_{k=1}^{2n} \{z_k , z_i\}(z_k , z_j) = \delta_{ij} \tag{2.17}$$

$$\sum_{k=1}^{2n} \{z_k , z_i\}(z_k , z_0 = \tau) = - \frac{\partial z_i}{\partial \tau}. \tag{2.18}$$

Equation (2.18) follows from the fact that

$$\left.\frac{\partial z_i}{\partial \tau}\right|_z = 0 = \left.\frac{\partial z_i}{\partial \tau}\right|_{q,p} + \sum_{j=1}^{n} \left(\left.\frac{\partial z_i}{\partial p_j}\right|_{q,\tau} \left.\frac{\partial p_j}{\partial \tau}\right|_z + \left.\frac{\partial z_i}{\partial q_j}\right|_{p,\tau} \left.\frac{\partial q_j}{\partial \tau}\right|_z \right). \tag{2.19}$$

It is now easy to show that the equations of motion for z_i are

$$dz_i/d\tau = \{z_i , H\} + \partial z_i/\partial \tau \tag{2.20}$$

where (q_i , p_i , τ) are treated as the independent variables when computing the right-hand side of the equation.

The action principle now becomes

$$0 = \delta S = \delta \int \left(\sum_{k=1}^{2n} C_k \, dz_k + (C_0 - H) \, d\tau \right). \tag{2.21}$$

However, we require that the constraints $\phi_i \approx 0$ hold throughout the variation, so

$$\delta \phi_{k-2m} = \delta z_k = 0, \qquad k = 2m + 1, \ldots, 2n. \tag{2.22}$$

Thus the restricted action principle is

$$0 = \delta S = \delta \int \left(\sum_{i=1}^{2m} C_i \, dz_i + (C_0 - H) \, d\tau \right). \tag{2.23}$$

The variables z_i, $i = 1,..., 2m$ are independent variables whose Lagrange brackets are given by Eq. (2.15).

We now demonstrate that the Poisson brackets of the independent z_i are just the *Dirac brackets.* If we define the matrix

$$C_{ab} = \{z_a, z_b\}, \qquad a, b = 2m + 1,..., 2n,$$

we find that the Dirac brackets are

$$\{z_i, z_j\}^* = \{z_i, z_j\} - \sum_{a,b=2m+1}^{2n} \{z_i, z_a\} C_{ab}^{-1} \{z_b, z_j\}. \tag{2.24}$$

Now we multiply by the Lagrange bracket, so

$$\sum_{i=1}^{2m} (z_i, z_k)\{z_i, z_j\}^* = \sum_{i=1}^{2n} (z_i, z_k)\{z_i, z_j\}^* = \delta_{jk}, \tag{2.25}$$

where $k, j = 1,..., 2m$ and the sum can be extended from $2m$ to $2n$ because, by Eq. (2.24), $\{z_i, z_j\}^* = 0$ when $i = 2m + 1,..., 2n$. Thus the Dirac brackets are the inverse of the *restricted Lagrange brackets* following from Eq. (2.23), with only $2m$ variables; by definition, the Dirac brackets must therefore be thePoisson brackets of the restricted system. In other words, a simple restriction in the number of variables appearing in the Lagrange brackets causes drastic changes in the inverse of the Lagrange bracket matrix; the canonical Poisson brackets are changed to Dirac brackets, which can be expressed in terms of the canonical Poisson brackets only by using Eq. (2.24).

It is now trivial to prove the iterative property of the Dirac brackets mentioned earlier. Indeed, successive restrictions on the range of variables of the Lagrange brackets give the same final restricted Lagrange brackets, and hence the same inverse.

Note also that if all gauge constraints and invariant relations are not imposed, the Lagrange bracket matrix will be singular unless enough variables are removed to give a nonsingular matrix and well-defined Poisson brackets. Dirac's treatment without gauge constriants amounts to grouping the leftover variables with the Hamiltonian.

Finally, we observe that by known theorems [20], we can find local canonical coordinates, $\tilde{p}_k$ and $\tilde{q}_k$, with Dirac brackets

$$\{\tilde{p}_k, \tilde{q}_j\}^* = -\delta_{kj}. \tag{2.26}$$

In general, the $2m$ independent z_i's will be certain τ-dependent functions of these local canonical coordinates,

$$z_i = z_i(\tilde{q}_k, \tilde{p}_k, \tau).$$

In terms of $\tilde{p}_k$ and $\tilde{q}_k$, the action principle (2.23) can be written

$$0 = \delta S = \delta \int \left(\tfrac{1}{2} \sum_{k=1}^{m} (\tilde{p}_k \, d\tilde{q}_k - \tilde{q}_k \, d\tilde{p}_k) - \tilde{H} \, d\tau \right). \tag{2.27}$$

Repeating the entire argument of Eqs. (2.13)–(2.21), we also have

$$0 = \delta S = \delta \int \left(\sum_{i=1}^{2m} \tilde{C}_i \, dz_i + (\tilde{C}_0 - \tilde{H}) \, d\tau \right), \tag{2.28}$$

where now everything is expressed in terms of the new variables $\tilde{q}_k$ and $\tilde{p}_k$. Thus

$$\begin{aligned} \tilde{C}_\alpha &= C_\alpha = \tfrac{1}{2} \sum_{k=1}^{m} \left(\tilde{p}_k \frac{\partial \tilde{q}_k}{\partial z_\alpha} - \tilde{q}_k \frac{\partial \tilde{p}_k}{\partial z_\alpha} \right), \\ \tilde{H} &= \tilde{C}_0 - C_0 + H, \end{aligned} \tag{2.29}$$

where $\alpha = 0,..., 2m$ and $z_0 \equiv \tau$. The integrands of Eqs. (2.27) and (2.28) differ at most by an exact differential which can be removed by a suitable canonical transformation on $\tilde{q}_k$ and $\tilde{p}_k$.

A specific choice of the variables $\tilde{q}_k$ and $\tilde{p}_k$ determines $\tilde{C}_0$ and hence, from (2.29), $\tilde{H}$. This choice fixes the explicit τ-dependence of the z_i appearing in the equation of motion (2.20) with $(q, p, H) \to (\tilde{q}, \tilde{p}, \tilde{H})$.

Conversely, we may always perform a suitable τ-dependent canonical transformation on the variables $\tilde{q}_k$ and $\tilde{p}_k$ which changes the Hamiltonian into any desired function. If the Dirac brackets of the z_i's do not depend explicitly on τ, then the best choice for the Hamiltonian is clearly the one which assigns no explicit τ-dependence to the z_i's.

C. *Example: Spinless Point Particle*

As a simple example of how the Dirac brackets can be used to find a quantum system, we treat the spinless point particle in the manifestly Poincaré-invariant Lagrangian formalism, where the Lagrangian is

$$L = -m \left(\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right)^{1/2} \equiv -m(u^2)^{1/2}. \tag{2.30}$$

Here x^μ is a function of τ, an arbitrary parameter describing the displacement of the particle along its world line. The action is the integral over the particle world line,

$$S = -m \int_a^b ds \equiv -m \int_a^b (dx^\mu \, dx_\mu)^{1/2} = \int_a^b d\tau \, L(x, u) \tag{2.31}$$

and is invariant under a reparametrization $\tau \to \tau'(\tau)$ of the integration parameter. If we define the canonical momentum corresponding to $x^\mu(\tau)$ as

$$P^\mu(\tau) = -\frac{\partial L}{\partial u_\mu} = m\frac{u^\mu}{(u^2)^{1/2}} = m\frac{dx^\mu}{ds}, \tag{2.32}$$

the variation of the action is

$$\delta S = \int_a^b d\tau\, \delta x_\mu \frac{dP^\mu}{d\tau} - \delta x^\mu(a)\, P_\mu(a) + \delta x^\mu(b)\, P_\mu(b). \tag{2.33}$$

With fixed end-points, the vanishing of δS implies the Euler equation

$$dP^\mu/d\tau = 0 \tag{2.34a}$$

which can also be written

$$m(d^2x^\mu/ds^2) = 0 \tag{2.34b}$$

For infinitesimal translations of the endpoints, $\delta x^\mu(a) = \epsilon^\mu$, the coefficient of ϵ^μ is the generator of the contact transformation for translations:

$$\text{Translation Generator} = P^\mu. \tag{2.35}$$

For infinitesimal Lorentz transformations, $\delta x^\mu(a) = 2x_\nu\omega^{\nu\mu}$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$, we find

$$\text{Lorentz Transformation Generator} = x^\mu P^\nu - x^\nu P^\mu. \tag{2.36}$$

From Eqs. (2.32) and (2.34), we find that both generators are constants of the motion

$$dP^\mu/d\tau = dM^{\mu\nu}/d\tau = 0 \tag{2.37}$$

and are therefore expected to generate symmetry transformations of the theory.

Next, we define the canonical Poisson brackets of the system to be

$$\{A, B\} = \frac{\partial A}{\partial x_\mu}\frac{\partial B}{\partial P^\mu} - \frac{\partial A}{\partial P^\mu}\frac{\partial B}{\partial x_\mu}. \tag{2.38}$$

(Our signs are chosen to give the correspondence of Eq. (2.8), in contrast to the conventions with no Lorentz metric in Section 2.B.) Thus

$$\{P^\mu, P^\nu\} = \{x^\mu, x^\nu\} = 0, \tag{2.39a}$$

$$\{P^\mu, x^\nu\} = -g^{\mu\nu}, \tag{2.39b}$$

$$\{P^\mu, M^{\alpha\beta}\} = g^{\mu\beta}P^\alpha - g^{\mu\alpha}P^\beta, \tag{2.39c}$$

$$\{x^\mu, M^{\alpha\beta}\} = g^{\mu\beta}x^\alpha - g^{\mu\alpha}x^\beta, \tag{2.39d}$$

$$\{M^{\mu\nu}, M^{\alpha\beta}\} = g^{\mu\alpha}M^{\nu\beta} - g^{\nu\alpha}M^{\mu\beta} + g^{\mu\beta}M^{\alpha\nu} - g^{\nu\beta}M^{\alpha\mu}, \tag{2.39e}$$

are the canonical Poisson brackets for the Poincaré group augmented by the canonical coordinate x^μ.

The canonical Hamiltonian is

$$H_0 = -P^\mu u_\mu - L \tag{2.40a}$$

and is conserved,

$$dH_0/d\tau = 0. \tag{2.41}$$

The canonical Poisson brackets of H_0 would normally generate translations in τ. However, a direct evaluation of (2.40a) gives

$$H_0 = 0. \tag{2.40b}$$

To resolve the problem of the vanishing Hamiltonian, we first observe that there is one constraint which follows from (2.32), and hence is a direct consequence of the form of the Lagrangian:

$$\chi(x, P) = P^2 - m^2 \approx 0. \tag{2.42}$$

We use the symbol ≈ 0 because this constraint has nonvanishing Poisson brackets:

$$\{P^2 - m^2, x^\mu\} = -2P^\mu \neq 0. \tag{2.43}$$

If we consider no other constraints, we may follow Dirac's prescription and add to the canonical Hamiltonian an arbitrary multiple of (2.42). The new Hamiltonian

$$H = H_0 + v(\tau)(P^2 - m^2) \approx 0 \tag{2.44}$$

has a chance of giving the right equations of motion by virtue of its nonzero Poisson brackets like (2.43). In fact, $v(\tau)$ may be related to u^μ since

$$\{H, x^\mu\} \approx -2v(\tau)\, P^\mu \tag{2.45}$$

while we define

$$\{H, x^\mu\} \equiv u^\mu. \tag{2.46}$$

Hence

$$v(\tau) = -(u^2)^{1/2}/(2m) \tag{2.47}$$

and the Hamiltonian

$$H = -[(u^2)^{1/2}/(2m)](P^2 - m^2) \tag{2.48}$$

generates the equations of motion in τ. We need not know the Poisson brackets of $(u^2)^{1/2}$, as they are always multiplied by $(P^2 - m^2) \approx 0$. Since $u^\mu = dx^\mu/d\tau$, the arbitrariness present in the definition of the integration parameter τ remains in the Hamiltonian (2.48).

Now we will eliminate all arbitrariness from the Hamiltonian system by imposing a gauge constraint which fixes the definition of τ and makes $(P^2 - m^2) \approx 0$ into a second-class constraint. Our choice is

$$\begin{aligned} \phi_1 &\equiv P^2(\tau) - m^2 \approx 0, \\ \phi_2 &\equiv x^0(\tau) - \tau \approx 0. \end{aligned} \tag{2.49}$$

Note that we are free to make other convenient choices for ϕ_2 if we desire [21]. The matrices needed to compute the Dirac bracket are

$$\begin{aligned} C_{\alpha\beta} &= \{\phi_\alpha , \phi_\beta\} = \begin{bmatrix} 0 & -2P^0 \\ 2P^0 & 0 \end{bmatrix}, \\ C^{-1}_{\alpha\beta} &= \frac{1}{2P^0} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{2.50}$$

According to Eqs. (2.6) and (2.7), we find

$$\begin{aligned} P'^\mu &= P^\mu - g^{\mu 0}(P^2 - m^2)/(2P^0), \\ x'^\mu &= x^\mu - (P^\mu/P^0)(x^0 - \tau), \end{aligned} \tag{2.51}$$

and the Dirac bracket

$$\{\xi, \eta\}^* = \{\xi, \eta\} - (1/2P^0)[\{\xi, P^2\}\{x^0, \eta\} - \{\xi, x^0\}\{P^2, \eta\}]. \tag{2.52}$$

Equation (2.39) is then replaced by

$$\{P^\mu, P^\nu\}^* = \{x^\mu, x^\nu\}^* = 0, \tag{2.53a}$$

$$\{P^\mu, x^\nu\}^* = -g^{\mu\nu} + g^{\mu 0} P^\nu/P^0, \tag{2.53b}$$

$$\{P^\mu, M^{\alpha\beta}\}^* = \{P^\mu, M^{\alpha\beta}\}, \tag{2.53c}$$

$$\{x^\mu, M^{\alpha\beta}\}^* = g^{\mu\beta}x^\alpha - g^{\mu\alpha}x^\beta - (P^\mu/P^0)(x^\alpha g^{0\beta} - x^\beta g^{0\alpha}), \tag{2.53d}$$

$$\{M^{\mu\nu}, M^{\alpha\beta}\}^* = \{M^{\mu\nu}, M^{\alpha\beta}\}. \tag{2.53e}$$

The Poincaré algebra of P^μ and $M^{\alpha\beta}$ is preserved, while P^2 and x^0 have vanishing Dirac brackets with all variables. x^0 now plays the role of the time parameter describing the evolution of the system, while the Hamiltonian is

$$H = (\mathbf{P}^2 + m^2)^{1/2}. \tag{2.54}$$

Hamilton's equations of motion become

$$\begin{aligned} \dot{A} = dA/dx^0 &= (\partial A/\partial x^0) + \{P^0, A\}^* \\ &= (\partial A/\partial x^0) + \{H, A\} \end{aligned} \tag{2.55}$$

where the second expression may be used after eliminating P^0 in terms of $\mathbf{P}$ and interpreting x^0 as a parameter instead of a canonical variable. As a check, we verify the equations of motion:

$$\begin{aligned}
\dot{P}^0 &= \dot{H} = 0, \\
\dot{\mathbf{P}} &= -\,\partial H/\partial \mathbf{x} = 0, \\
\dot{x}^0 &= 1, \\
\dot{\mathbf{x}} &= \partial H/\partial \mathbf{P} = \mathbf{P}/(\mathbf{P}^2 + m^2)^{1/2}, \\
\dot{M}^{\alpha\beta} &= 0.
\end{aligned} \tag{2.56}$$

Note that since x^0 is now a parameter, M^{0i} does *not* have vanishing Poisson brackets with H; M^{0i} is nevertheless a constant of the motion due to the compensating partial derivative with respect to x^0 in Eq. (2.55).

D. *Quantum Mechanics of Spinless Point Particle*

The traditional abstract system of quantum operators for the spinless point particle can be deduced almost directly from the Dirac brackets (2.53). With the definitions

$$\begin{aligned}
H &= P^0 = (\mathbf{P}^2 + m^2)^{1/2}, \\
x^0 &= t = \text{parameter},
\end{aligned} \tag{2.57}$$

we postulate the commutation relations

$$\begin{aligned}
i[P^i, P^j] &= i[H, P^j] = i[x^i, x^j] = i[t, x^j] = 0, \\
i[P^i, t] &= i[H, t] = 0, \\
i[P^i, x^j] &= \delta^{ij}, \\
i[H, x^i] &= P^i/(\mathbf{P}^2 + m^2)^{1/2} = dx^i/dt,
\end{aligned} \tag{2.58}$$

corresponding to Eq. (2.53). The only ordering problem occurs in the realization of $M^{\alpha\beta}$ in terms of x^μ and P^μ. This is resolved by simply requiring hermiticity, giving

$$\begin{aligned}
M^{ij} &= x^i P^j - x^j P^i, \\
M^{0i} &= tP^i - (1/2)(Hx^i + x^i H).
\end{aligned} \tag{2.59}$$

The commutators of the $M^{\alpha\beta}$ with $P^0 = H$, P^i, and x^i are then

$$\begin{aligned}
i[P^\mu, M^{\alpha\beta}] &= g^{\mu\beta} P^\alpha - g^{\mu\alpha} P^\beta, \\
i[x^i, M^{jk}] &= -\delta^{ik} x^j + \delta^{ij} x^k, \\
i[x^i, M^{0j}] &= -\delta^{ij} t + (1/2)[(P^i/H)\, x^j + x^j (P^i/H)].
\end{aligned} \tag{2.60}$$

There is a potential ordering problem in the computation of $i[M^{0i}, M^{0j}]$, but the troublesome pieces cancel to give the usual completely Poincaré-covariant theory:

$$i[M^{\mu\nu}, M^{\alpha\beta}] = g^{\mu\alpha}M^{\nu\beta} - g^{\nu\alpha}M^{\mu\beta} + g^{\mu\beta}M^{\alpha\nu} - g^{\nu\beta}M^{\alpha\mu}, \tag{2.61}$$

The Hamiltonian equations of motion are

$$dA/dt = (\partial A/\partial t) + i[H, A] \tag{2.62}$$

so that

$$\begin{aligned} dM^{0i}/dt &= P^i - (i/2)\, H[H, x^i] - (i/2)[H, x^i]\, H = 0, \\ dM^{ij}/dt &= dP^i/dt = dH/dt = 0. \end{aligned} \tag{2.63}$$

Thus P^μ and $M^{\alpha\beta}$ are constants of the motion; this would not be true if we had omitted the explicit t-dependence in Eq. (2.59).

The wave equation in coordinate space is found by using the realization $\mathbf{P} = -i\partial/\partial\mathbf{x}$ of the algebra (2.58), so that

$$H = (-\nabla^2 + M^2)^{1/2}. \tag{2.64}$$

The Schrödinger wave equation is then

$$H\phi(x) = i(\partial\phi(x)/\partial t) \tag{2.65}$$

and has a nonlocal, positive-energy character. The Klein–Gordon equation follows by iteration of Eq. (2.65):

$$((\partial^2/\partial t^2) - \nabla^2 + m^2)\, \phi(x) = 0. \tag{2.66}$$

We should also note that the norm in our Hilbert space is chosen as

$$(\phi, \phi) = \int \phi^*(x)\, \phi(x)\, d^3x \tag{2.67}$$

so that the operators (2.59), for example, are formally Hermitian. For different choices of the inner product, Eq. (2.59) would take different forms. The presence of external fields causes the Klein–Gordon wavefunction to develop negative frequency components. The norm (2.67) is then not conserved and one finds instead the conservation of a charge which is not positive definite. Therefore, one should really think of $\phi(x)$ as a field and develop a second-quantized theory. Then one would discover that our $\phi(x)$ is nonlocally related to the conventional scalar Klein–Gordon field.

E. *Remark*

Finally, we observe that the same classical description of the spinless point particle that we have derived laboriously from the Dirac bracket formalism also follows from the action

$$S = -m \int_a^b dt(1 - \mathbf{v}^2)^{1/2} = \int_a^b L \, dt \tag{2.68}$$

where $\mathbf{v} = d\mathbf{x}/dt$. The canonical momenta are

$$\mathbf{P} = \partial L/\partial \mathbf{v} = m\mathbf{v}/(1 - \mathbf{v}^2)^{1/2} \tag{2.69}$$

and the Hamiltonian is

$$H = \mathbf{P} \cdot \mathbf{v} - L = m/(1 - \mathbf{v}^2)^{1/2} = (\mathbf{P}^2 + m^2)^{1/2}. \tag{2.70}$$

The relevant Poisson brackets are

$$\begin{aligned} \{P^i, x^j\} &= \delta^{ij} \\ \{H, x^i\} &= P^i/(\mathbf{P}^2 + m^2)^{1/2} \end{aligned} \tag{2.71}$$

so that the canonical system is just that of the Dirac brackets (2.53). The corresponding quantum system is of course identical.

Why, then, bother with Dirac brackets? The point is that in some cases it will be much harder to write down a Poincaré-invariant Lagrangian involving only the independent variables. The nature of the physical system can often be seen more clearly by writing the equations in a manifestly covariant form involving redundant variables. In order to impose constraints which eliminate the redundant variables to give a consistent Hamiltonian system, we must use Dirac brackets. The Dirac brackets then provide the starting point for the development of a consistent quantum system.

3. Hamiltonian Description of Free Spherical Top

We now apply the techniques reviewed in Section 2 to develop a Hamiltonian formulation of the free top, which was described in Section 1 from a Lagrangian viewpoint.

A. *Poisson Brackets*

The first step in constructing a Hamiltonian description of a system is to define the canonical Poisson brackets. This must be done with care since our angular coordinates $\Lambda^\mu{}_\nu$ have been defined throughout to obey $\Lambda\Lambda^T = g$, so only six of the variables are independent (before further constraints are applied). Suppose that the ϕ_i, $i = 1,..., 6$, are the six independent combinations of the $\Lambda^\mu{}_\nu$ which we take as canonical coordinates. Returning to Eq. (1.5), we define $\delta\theta^{\mu\nu}$ in terms of the ϕ_i by the equation

$$\delta\theta^{\mu\nu} = \Lambda_\lambda{}^\mu \, \delta\Lambda^{\lambda\nu} = a_i^{\mu\nu}(\phi) \, \delta\phi_i \tag{3.1}$$

where $a_i^{\mu\nu} = -a_i^{\nu\mu}$. From Eq. (3.1) we also have

$$\delta\Lambda^{\mu\nu} = \Lambda^\mu{}_\alpha a_i^{\alpha\nu}(\phi) \, \delta\phi_i \, . \tag{3.2}$$

The angular velocities $\sigma^{\mu\nu}$ can then be expressed in terms of the canonical velocities $\dot{\phi}_i$ as

$$\sigma^{\mu\nu} = a_i^{\mu\nu}(\phi) \, \dot{\phi}_i \, . \tag{3.3}$$

The form of $a_i^{\mu\nu}$ is not completely arbitrary. In order for Eqs. (3.1), (3.3), and (1.6) to be compatible, we require

$$(\partial a_i^{\mu\nu}/\partial\phi_j) - (\partial a_j^{\mu\nu}/\partial\phi_i) + a_j^{\mu\lambda} a_{i,\lambda}{}^\nu - a_i^{\mu\lambda} a_{j,\lambda}{}^\nu = 0. \tag{3.4}$$

The canonical momentum conjugate to ϕ_i is defined as

$$T_i = -\frac{\partial L_0}{\partial \dot{\phi}_i} = -\frac{1}{2} \frac{\partial \sigma^{\mu\nu}}{\partial \dot{\phi}_i} \frac{\partial L_0}{\partial \sigma^{\mu\nu}} = \frac{1}{2} a_i^{\mu\nu}(\phi) \, S_{\mu\nu} \, . \tag{3.5}$$

To express $S^{\mu\nu}$ in terms of T_i, we introduce a function $b_i^{\mu\nu}$ with the properties

$$a_i^{\mu\nu} b_i^{\alpha\beta} = g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}, \tag{3.6a}$$

$$a_i^{\mu\nu} b_{j,\mu\nu} = 2\delta_{ij} \, , \tag{3.6b}$$

so that

$$S^{\mu\nu} = b_i^{\mu\nu}(\phi) \, T_i \, . \tag{3.7}$$

The spin part of the canonical Poisson bracket is found by considering the expression

$$\frac{\partial A}{\partial \phi_i} \frac{\partial B}{\partial T_i} = \frac{1}{2} \left(\Lambda^\mu{}_\alpha \frac{\partial A}{\partial \Lambda^{\mu\beta}} - \Lambda^\mu{}_\beta \frac{\partial A}{\partial \Lambda^{\mu\alpha}} \right) \frac{\partial B}{\partial S_{\alpha\beta}} + \frac{1}{8} a_j^{\alpha\beta} \frac{\partial b_j^{\mu\nu}}{\partial \phi_i} b_i^{\lambda\sigma} S_{\alpha\beta} \frac{\partial A}{\partial S^{\mu\nu}} \frac{\partial B}{\partial S^{\lambda\sigma}} \, . \tag{3.8}$$

The derivatives with respect to $\Lambda^{\mu\nu}$ and $S^{\mu\nu}$ on the right-hand side are to be taken with $S^{\mu\nu}$ and $\Lambda^{\mu\nu}$ held constant, respectively. We next note the identity

$$b_j^{\mu\nu} \frac{\partial b_i^{\alpha\beta}}{\partial \phi_j} - b_j^{\alpha\beta} \frac{\partial b_i^{\mu\nu}}{\partial \phi_j} = (-g^{\mu\alpha} g^{\nu\sigma} g^{\beta\tau} + g^{\mu\beta} g^{\nu\sigma} g^{\alpha\tau} - g^{\nu\beta} g^{\mu\sigma} g^{\alpha\tau} + g^{\nu\alpha} g^{\mu\sigma} g^{\beta\tau}) \, b_{i,\sigma\tau} \tag{3.9}$$

which incidentally means that $b_i^{\mu\nu}$ is a realization of the Lorentz group Lie algebra. From (3.8) and (3.9) we find the following expression for the canonical Poisson bracket:

$$\begin{aligned}
\{A, B\} &= \frac{\partial A}{\partial x_\mu}\frac{\partial B}{\partial P^\mu} + \frac{\partial A}{\partial \phi_i}\frac{\partial B}{\partial T_i} - \frac{\partial A}{\partial P^\mu}\frac{\partial B}{\partial x_\mu} - \frac{\partial A}{\partial T_i}\frac{\partial B}{\partial \phi_i} \\
&= \frac{\partial A}{\partial x_\mu}\frac{\partial B}{\partial P^\mu} - \frac{\partial A}{\partial P^\mu}\frac{\partial B}{\partial x_\mu} + S^{\mu\nu}\frac{\partial A}{\partial S^\mu{}_\lambda}\frac{\partial B}{\partial S^{\nu\lambda}} \\
&\quad + \frac{1}{2}\left(\Lambda^\mu{}_\alpha \frac{\partial A}{\partial \Lambda^{\mu\beta}} - \Lambda^\mu{}_\beta \frac{\partial A}{\partial \Lambda^{\mu\alpha}}\right)\frac{\partial B}{\partial S_{\alpha\beta}} \\
&\quad - \frac{1}{2}\left(\Lambda^\mu{}_\alpha \frac{\partial B}{\partial \Lambda^{\mu\beta}} - \Lambda^\mu{}_\beta \frac{\partial B}{\partial \Lambda^{\mu\alpha}}\right)\frac{\partial A}{\partial S_{\alpha\beta}}.
\end{aligned} \tag{3.10}$$

Applying Eq. (3.10), we find the canonical Poisson brackets

$$\begin{aligned}
\{P^\mu, x^\nu\} &= -g^{\mu\nu}, \\
\{P^\mu, P^\nu\} &= \{x^\mu, x^\nu\} = 0, \\
\{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\} &= 0, \\
\{\Lambda^{\mu\nu}, S^{\alpha\beta}\} &= \Lambda^{\mu\alpha}g^{\nu\beta} - \Lambda^{\mu\beta}g^{\nu\alpha}, \\
\{S^{\mu\nu}, S^{\alpha\beta}\} &= S^{\mu\alpha}g^{\nu\beta} - S^{\mu\beta}g^{\nu\alpha} + S^{\nu\beta}g^{\mu\alpha} - S^{\nu\alpha}g^{\mu\beta}.
\end{aligned} \tag{3.11}$$

We therefore see that $S^{\mu\nu}$ obeys the Lorentz group algebra all by itself, while $\Lambda^{\mu\nu}$ transforms as a four-vector under $S^{\alpha\beta}$ only on its *second index*. This indicates that the $S^{\alpha\beta}$ are the infinitesimal generators of *right* Lorentz transformations on $\Lambda^{\mu\nu}$.

B. *Constraints*

Now we must examine the constraints on the system so that we can make the transition from the canonical Poisson brackets to the self-consistent Dirac brackets of the classical system. First, let us study the consequences of requiring the Lagrangian to satisfy Eqs. (1.36) and (1.37) so that

$$V^\mu \equiv S^{\mu\nu}P_\nu \approx 0, \tag{3.12}$$

but with no gauge constraints imposed. The Poisson brackets of the three independent components of V^μ are nonzero, but there are actually only *two* independent second-class constraints among them. The constraint

$$S^{\mu\nu}S^*_{\mu\nu} \approx 0 \tag{3.13}$$

noted in Eq. (1.35) is a direct consequence of (3.12) and is first-class. The other constraint which is first-class when no gauge constraints are imposed is the trajectory relation (1.50):

$$M^2 \equiv P^\mu P_\mu \approx f(J^2 \equiv (1/2)\, S_{\mu\nu} S^{\mu\nu}). \tag{3.14}$$

The canonical Hamiltonian is

$$H_0 = -P^\mu u_\mu - (1/2)\, S^{\mu\nu}\sigma_{\mu\nu} - L_0 \equiv 0 \tag{3.15}$$

due to the homogeneity condition (1.38). The full Dirac Hamiltonian, before imposing gauge constraints, is

$$H = v_1(\tau)[P^2 - f((1/2)\, S^{\mu\nu} S_{\mu\nu})] + v_2(\tau)\, S^{\mu\nu} S^*_{\mu\nu} \tag{3.16}$$

with v_1 and v_2 arbitrary functions. Now suppose we compute the velocities

$$\{H, x^\mu\} = -2v_1 P^\mu \equiv u^\mu, \tag{3.17}$$

$$\{H, \Lambda^{\mu\nu}\} = 2v_1 f' \Lambda^{\mu\alpha} S_\alpha{}^\nu - 4v_2 \Lambda^{\mu\alpha} S_\alpha^{*\nu} \equiv \dot{\Lambda}^{\mu\nu}. \tag{3.18}$$

From (3.17) we find

$$v_1 = -u^2/2(u \cdot P) = -(u^2)^{1/2}/2M. \tag{3.19}$$

The arbitrariness in the function v_1 can be eliminated by choosing a gauge for the parameter τ, as was done in the example of Section 2.

Equation (3.18) is more interesting. Multiplying by Λ we find

$$\sigma^{\mu\nu} = 2v_1 f' S^{\mu\nu} - 4v_2 S^{*\mu\nu}. \tag{3.20}$$

The arbitrariness in v_2 means we may add to $\sigma^{\mu\nu}$ any multiple of $S^{*\mu\nu}$. This will be of more significance when we introduce interactions. Rather than pick one kind of gauge condition conjugate to $S^{\mu\nu}S^*_{\mu\nu} \approx 0$ and two others conjugate to the two second-class parts of $V^\mu \approx 0$, we shall take advantage of the fact that $S^{\mu\nu}S^*_{\mu\nu} \approx 0$ follows from $V^\mu \approx 0$ and choose a gauge which closely resembles $V^\mu \approx 0$. Since $V^\mu \approx 0$ means that our top has only the three usual components of angular momentum in its rest frame, we seek a gauge condition which constrains the six independent components of $\Lambda^{\mu\nu}$ so that only the three Euler angles appear as independent coordinates in the rest frame.

We begin by defining

$$\rho^\mu = \Lambda^{\mu\nu} P_\nu / M \tag{3.21}$$

where $\rho^2 = 1$ and ρ^μ satisfies the differential equation

$$\dot{\rho}^\mu + B\bar{S}^{*\mu\nu}\rho_\nu = 0. \tag{3.22}$$

Recall that $\bar{S}^{\mu\nu} = \Lambda^{\mu\alpha} S_{\alpha\beta} \Lambda^{\nu\beta} =$ constant of motion.

The condition (3.12) implies

$$\bar{S}^{\mu\nu}\rho_\nu \approx 0 \tag{3.23}$$

and also

$$\bar{S}^{\mu\nu}\bar{S}^*_{\mu\nu} \approx 0. \tag{3.24}$$

Since $\bar{S}^{\mu\nu}$ is a constant of the motion which is a function only of the canonical coordinates and momenta, we may choose

$$\bar{S}^{0i} \approx 0 \tag{3.25}$$

as an invariant relation. As a consequence of Eq. (3.24), only two of the conditions (3.25) are independent. The condition

$$\bar{S}'^{0i} = C^{0\mu}\bar{S}_{\mu\nu}C^{i\nu} \approx 0 \tag{3.26}$$

is of course physically equivalent to (3.25), where the role of Λ is now played by $\Lambda' = C\Lambda$ (see Section 4, Eqs. (4.49)–(4.58)).

If we define $\bar{S}^i = (1/2)\,\epsilon^{ijk}\bar{S}^{jk}$, Eqs. (3.23) and (3.25) imply that

$$\epsilon^{ijk}\bar{S}^j\rho^k \approx 0$$

so

$$\rho^i = \alpha\bar{S}^i = \alpha\bar{S}^{*0i} \tag{3.27}$$

for some α. Note that $\bar{S}^i$ is the body-fixed angular momentum.

Introducing Eq. (3.27) into Eq. (3.22), we arrive at

$$\begin{aligned} \dot{\rho}^0 - \alpha B J^2 &= 0, \\ \dot{\alpha} - B\rho^0 &= 0. \end{aligned} \tag{3.28}$$

By the same token that we can make the gauge choice $x^0 \approx \tau$, we can choose

$$\rho^0 \approx 1. \tag{3.29}$$

From Eq. (3.28), we find $B = 0$ and $\alpha =$ const. Then since

$$1 \equiv \rho^\mu\rho_\mu = 1 - \alpha^2 J^2$$

we must have $\alpha = 0$ so that

$$\rho^\mu = \Lambda^{\mu\nu}P_\nu/M \approx g^{\mu 0} \tag{3.30}$$

(Note that the $g^{\mu 0}$ appearing here is the *left* metric tensor and does not transform under right transformations.) Multiplying Eq. (3.30) by Λ, we get the most convenient form for our combined gauge and invariant relation constraints:

$$\chi^{\mu} = \Lambda^{0\mu} - P^{\mu}/M \approx 0. \tag{3.31}$$

Equation (3.31) says that $\Lambda^{\mu\nu}$ reduces to a pure rotation matrix in the rest frame of P^{μ} and thus has the correct physical properties. Just like $V^{\mu} \approx 0$, $\chi^{\mu} \approx 0$ is a four-vector constraint with only three independent components. To see this, we note that

$$\chi^{\mu}(\Lambda_{0\mu} + P_{\mu}/M) = (2/M)\, P_{\mu}\chi^{\mu} + \chi_{\mu}\chi^{\mu} \equiv 0.$$

We emphasize that by choosing (3.31), we have fixed v_2. From Eq. (3.30) and the Euler equations, we find

$$\Lambda_{\lambda}{}^{\mu}(d/d\tau)(-\Lambda^{\lambda\nu}\chi_{\nu}) = \Lambda_{\lambda}{}^{\mu}\Lambda^{\lambda\nu}P_{\nu}/M \approx 0$$

or

$$\sigma^{\mu\nu}P_{\nu} \approx 0. \tag{3.32a}$$

But this means that $\sigma^{\mu\nu}$ is proportional to $S^{\mu\nu}$, so $v_2 = 0$. Furthermore, we immediately have

$$\sigma^{\mu\nu}\sigma^{*}_{\mu\nu} \approx 0 \tag{3.32b}$$

and from Eq. (1.40)

$$\sigma^{\mu\nu}u_{\nu} \approx 0. \tag{3.32c}$$

Our constraint choice (3.31) is therefore consistent with our interpretation of it as a gauge condition which fixes the value of v_2.

C. *Preliminary Dirac Brackets*

Since we may impose constraints successively, we shall now display the properties of the covariant brackets which result from imposing only Eq. (3.12) and its conjugate gauge constraint (3.31). We first take the six independent constraints

$$V^{i} = S^{i\nu}P_{\nu} \approx 0,$$
$$\chi^{i} = \Lambda^{0i} - P^{i}/M \approx 0,$$

where $M \equiv (P^{\mu}P_{\mu})^{1/2}$, and combine them into the six-vector

$$\phi^{i} = (V^1, V^2, V^3, \chi^1, \chi^2, \chi^3), \qquad i = 1,..., 6.$$

The matrices needed to compute the Dirac brackets are

$$C^{ij} = \{\phi^i, \phi^j\} = \begin{vmatrix} S^{ij}M^2 & -(M\delta^{ij} + P^iP^j/M) \\ (M\delta^{ij} + P^iP^j/M) & 0 \end{vmatrix} \tag{3.33}$$

and its inverse

$$(C^{-1})^{ij} = \begin{vmatrix} 0 & \dfrac{1}{M}\left(\delta^{ij} - \dfrac{P^iP^j}{(P^0)^2}\right) \\ -\dfrac{1}{M}\left(\delta^{ij} - \dfrac{P^iP^j}{(P^0)^2}\right) & \left(S^{ij} + \dfrac{P^iS^{jk}P^k - P^jS^{ik}P^k}{(P^0)^2}\right) \end{vmatrix}. \tag{3.34}$$

We may cast the Dirac brackets into a very convenient manifestly covariant form by observing that

$$C_{ij} = T_{i\alpha}C^{\alpha\beta}T_{j\beta}\,, \qquad \begin{cases} i, j = 1,\dots, 6 \\ \alpha, \beta = 1,\dots, 8 \end{cases} \tag{3.35}$$

where

$$T_{i\alpha} = \begin{vmatrix} T_{k\mu} & 0 \\ 0 & T_{k\mu} \end{vmatrix}, \qquad \begin{cases} k = 1, 2, 3 \\ \mu = 0, 1, 2, 3 \end{cases} \tag{3.36a}$$

and

$$T_{k\mu} = \begin{vmatrix} P^1/P^0 & -1 & 0 & 0 \\ P^2/P^0 & 0 & -1 & 0 \\ P^3/P^0 & 0 & 0 & -1 \end{vmatrix}. \tag{3.36b}$$

The matrix $C^{\alpha\beta}$ is

$$C^{\alpha\beta} = \begin{vmatrix} M^2S^{\mu\nu} & Mg^{\mu\nu} \\ -Mg^{\mu\nu} & 0 \end{vmatrix} \tag{3.37}$$

while its inverse has the form

$$(C^{-1})^{\alpha\beta} = \begin{vmatrix} 0 & -\dfrac{g^{\mu\nu}}{M} \\ \dfrac{g^{\mu\nu}}{M} & S^{\mu\nu} \end{vmatrix}. \tag{3.38}$$

The effect of $T_{k\mu}$ is to convert V_k and χ_k into V_μ and χ_μ by virtue of the relations $P^\mu V_\mu = 0$ and $P^\mu\chi_\mu = -(1/2)\,M\chi^\mu\chi_\mu$ as follows,

$$\{A, V^k\}\,T_{k\mu} \approx \{A, V^kT_{k\mu}\} = \{A, V_\mu\}. \tag{3.39}$$

Defining the eight-vector

$$\phi^\alpha = (V^0, V^1, V^2, V^3, \chi^0, \chi^1, \chi^2, \chi^3), \qquad \alpha = 1,\ldots, 8. \tag{3.40}$$

we may write the preliminary Dirac brackets as

$$\{\xi, \eta\}' = \{\xi, \eta\} - \{\xi, \phi^\alpha\}\, C^{-1}_{\alpha\beta}\{\phi^\beta, \eta\}, \tag{3.41}$$

where we use a prime to distinguish the present preliminary brackets from the final brackets. From Eq. (3.38), we have the explicit form

$$\{\xi, \eta\}' = \{\xi, \eta\} + (1/M)\{\xi, V^\mu\}\{\chi_\mu, \eta\} - (1/M)\{\xi, \chi^\mu\}\{V_\mu, \eta\} - \{\xi, \chi_\mu\}\, S^{\mu\nu}\{\chi_\nu, \eta\}. \tag{3.42}$$

A straightforward computation gives the following primed brackets:

$$\begin{aligned}
&1. \quad \{P^\mu, x^\nu\}' = -g^{\mu\nu},\\
&2. \quad \{x^\mu, x^\nu\}' = -S^{\mu\nu}/M^2,\\
&3. \quad \{P^\mu, P^\nu\}' = 0,\\
&4. \quad \{P^\mu, S^{\nu\lambda}\}' = 0,\\
&5. \quad \{x^\mu, S^{\nu\lambda}\}' = (S^{\mu\nu}P^\lambda - S^{\mu\lambda}P^\nu)/M^2,\\
&6. \quad \{S^{\mu\nu}, S^{\alpha\beta}\}' = S^{\mu\alpha}(g^{\nu\beta} - P^\nu P^\beta/M^2) - S^{\mu\beta}(g^{\nu\alpha} - P^\nu P^\alpha/M^2)\\
&\qquad\qquad + S^{\alpha\nu}(g^{\mu\beta} - P^\mu P^\beta/M^2) - S^{\beta\nu}(g^{\mu\alpha} - P^\mu P^\alpha/M^2),\\
&7. \quad \{\Lambda^{\mu\nu}, x^\alpha\}' = \Lambda^\mu{}_\beta(P^\nu g^{\alpha\beta} - P^\beta g^{\alpha\nu})/M^2,\\
&8. \quad \{\Lambda^{\mu\nu}, P^\alpha\}' = 0,\\
&9. \quad \{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\}' = 0,\\
&10. \quad \{\Lambda^{\mu\nu}, S^{\alpha\beta}\}' = \Lambda^{\mu\alpha}(g^{\nu\beta} - P^\nu P^\beta/M^2) - \Lambda^{\mu\beta}(g^{\nu\alpha} - P^\nu P^\alpha/M^2)\\
&\qquad\qquad + g^{\mu 0}(g^{\alpha\nu}P^\beta - g^{\beta\nu}P^\alpha)/M.
\end{aligned} \tag{3.43}$$

As opposed to the canonical Poisson brackets, these brackets are all compatible with setting the constraints V^μ and χ^μ identically zero. We furthermore see that the position variables x^μ have *nonzero* brackets. This is a familiar phenomenon, noted for example by Pryce [18].

Recalling that the generator of Lorentz transformations is

$$M^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu + S^{\mu\nu},$$

we compute the brackets

$$\begin{aligned}
\{M^{\mu\nu}, M^{\alpha\beta}\}' &= g^{\nu\beta}M^{\mu\alpha} - g^{\nu\alpha}M^{\mu\beta} + g^{\mu\beta}M^{\alpha\nu} - g^{\mu\alpha}M^{\beta\nu},\\
\{M^{\mu\nu}, P^\alpha\}' &= g^{\mu\alpha}P^\nu - g^{\nu\alpha}P^\mu.
\end{aligned} \tag{3.44}$$

The system of primed brackets is therefore Poincaré-covariant. Note that

$$\{M^{\mu\nu}, x^{\alpha}\}' = g^{\mu\alpha}x^{\nu} - g^{\nu\alpha}x^{\mu} \tag{3.45}$$

so that x^{μ} is still a four-vector.

At this point, the trajectory function remains as a first-class quantity, so we could consider the Hamiltonian to be

$$H = v(\tau)[P^{\mu}P_{\mu} - f(\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu})]. \tag{3.46}$$

We compute the velocities to be

$$\begin{aligned} \{H, x^{\mu}\} &= -2vP^{\mu} \equiv u^{\mu}, \\ \{H, \Lambda^{\mu\nu}\} &= 2vf'\Lambda^{\mu\alpha}S_{\alpha}{}^{\nu} \equiv \dot{\Lambda}^{\mu\nu}, \end{aligned} \tag{3.47}$$

so that

$$\begin{aligned} v &= -(u^2)^{1/2}/(2M), \\ \sigma^{\mu\nu} &= 2vf'S^{\mu\nu}, \end{aligned} \tag{3.48}$$

in agreement with Eq. (3.32).

D. *Dirac Brackets*

The final Dirac bracket system is found by selecting a gauge constraint conjugate to the trajectory constraint and thereby eliminating all arbitrary functions. Making use of the iterative property of the Dirac brackets, we choose the final constraints

$$\begin{aligned} \phi_1 &= P^{\mu}P_{\mu} - f((1/2)\, S_{\mu\nu}S^{\mu\nu}) \approx 0, \\ \phi_2 &= x^0 - \tau \approx 0, \end{aligned} \tag{3.49}$$

and impose them on the *primed* brackets.

The required matrices are simply

$$\begin{aligned} C_{ij} &= \{\phi_i, \phi_j\}' = \begin{bmatrix} 0 & -2P^0 \\ 2P^0 & 0 \end{bmatrix} \\ C_{ij}^{-1} &= \frac{1}{2P^0}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{3.50}$$

Hence Eq. (2.7) with all Poisson brackets replaced by the primed brackets (3.43) gives the form of the final Dirac brackets, which we now denote with stars. The results are:

$$
\begin{aligned}
1.\quad & \{P^\mu, x^\nu\}^* = -g^{\mu\nu} + g^{\mu 0}P^\nu/P^0, \\
2.\quad & \{x^\mu, x^\nu\}^* = -S^{\mu\nu}/M^2 + (P^\mu S^{0\nu} - P^\nu S^{0\mu})/(M^2P^0) \\
& \qquad = -S^{\mu\nu}/M^2 + (P^\nu S^{\mu i}P^i - P^\mu S^{\nu i}P^i)/(MP^0)^2, \\
3.\quad & \{P^\mu, P^\nu\}^* = 0, \\
4.\quad & \{P^\mu, S^{\nu\lambda}\}^* = 0, \\
5.\quad & \{x^\mu, S^{\nu\lambda}\}^* = (1/M^2)(S^{\mu\nu}P^\lambda - S^{\mu\lambda}P^\nu) - (P^\mu/P^0M^2)(S^{0\nu}P^\lambda - S^{0\lambda}P^\nu), \\
6.\quad & \{S^{\mu\nu}, S^{\alpha\beta}\}^* = S^{\mu\alpha}(g^{\nu\beta} - P^\nu P^\beta/M^2) - S^{\mu\beta}(g^{\nu\alpha} - P^\nu P^\alpha/M^2) \\
& \qquad + S^{\alpha\nu}(g^{\mu\beta} - P^\mu P^\beta/M^2) - S^{\beta\nu}(g^{\mu\alpha} - P^\mu P^\alpha/M^2), \\
7.\quad & \{\Lambda^{\mu\nu}, x^\alpha\}^* = (1/M^2)\,\Lambda^\mu{}_\lambda(P^\nu g^{\alpha\lambda} - P^\lambda g^{\alpha\nu}) \\
& \qquad - (1/P^0M^2)\,P^\alpha\Lambda^\mu{}_\lambda(P^\nu g^{0\lambda} - P^\lambda g^{0\nu}) - (f'/P^0M^2)\,S^{0\alpha}\Lambda^{\mu\lambda}S_\lambda{}^\nu, \\
8.\quad & \{\Lambda^{\mu\nu}, P^\alpha\}^* = (f'/P^0)\,g^{0\alpha}\Lambda^\mu{}_\lambda S^{\lambda\nu}, \\
9.\quad & \{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\}^* = -(f'/P^0M^2)\,\Lambda^\mu{}_\lambda(P^\lambda g^{0\nu} - P^\nu g^{0\lambda})\,\Lambda^\alpha{}_\gamma S^{\gamma\beta} \\
& \qquad + (f'/P^0M^2)\,\Lambda^\alpha{}_\gamma(P^\gamma g^{0\beta} - P^\beta g^{0\gamma})\,\Lambda^\mu{}_\lambda S^{\lambda\nu}, \\
10.\quad & \{\Lambda^{\mu\nu}, S^{\alpha\beta}\}^* = \Lambda^{\mu\alpha}g^{\nu\beta} - \Lambda^{\mu\beta}g^{\nu\alpha} \\
& \qquad + (1/M^2)\,\Lambda^\mu{}_\lambda(P^\nu P^\alpha g^{\lambda\beta} - P^\nu P^\beta g^{\lambda\alpha} + P^\lambda P^\beta g^{\nu\alpha} - P^\lambda P^\alpha g^{\nu\beta}) \\
& \qquad - (f'/P^0M^2)\,\Lambda^\mu{}_\lambda S^{\lambda\nu}(P^\alpha S^{0\beta} - P^\beta S^{0\alpha}),
\end{aligned}
\tag{3.51}
$$

where $M = (P^\mu P_\mu)^{1/2} = [f((1/2)\,S_{\mu\nu}S^{\mu\nu})]^{1/2}$ and $f' = \partial f(J^2)/\partial(J^2)$, with $J^2 = (1/2)\,S^{\mu\nu}S_{\mu\nu}$.

The Hamiltonian is

$$H = P^0 = (\mathbf{P}^2 + f((1/2)\,S^{\mu\nu}S_{\mu\nu}))^{1/2} \tag{3.52}$$

and generates time translations via

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{(\mathbf{P}^2 + f((1/2)\,S^{\mu\nu}S_{\mu\nu}))^{1/2}, A\}' = \frac{\partial A}{\partial t} + \{P^0, A\}^*. \tag{3.53}$$

Here we have used the traditional notation $x^0 \equiv t$ to emphasize that x^0 is considered as a parameter and not as a canonical variable, in accordance with the gauge constraint (3.49).

The Poincaré group generators are now written:

$$
\begin{aligned}
P^0 &= H, \\
P^i &= P^i, \\
M^{0i} &= tP^i - x^iH + S^{0i} = tP^i - x^iH - S^{ij}P^j/H, \\
M^{ij} &= x^iP^j - x^jP^i + S^{ij}.
\end{aligned}
\tag{3.54}
$$

The star brackets of these quantities obey the Poincaré algebra (3.44) and the theory is again Poincaré-covariant. Applying Eq. (3.53), we confirm that the generators (3.54) are constants of the motion with respect to $x^0 = t$.

The transformation of x^μ under the Poincaré group is altered in the star brackets. $x^0 = t$ is now just a parameter, while $\mathbf{x}$ transforms as follows:

$$\begin{aligned}
\{P^0, x^i\}^* &= \{H, x^i\}^* = P^i/H, \\
\{P^i, x^j\}^* &= \delta^{ij}, \\
\{M^{ij}, x^k\}^* &= -\delta^{ik}x^j + \delta^{jk}x^i, \\
\{M^{0i}, x^k\}^* &= t\delta^{ik} - x^i P^k/H.
\end{aligned} \tag{3.55}$$

Finally, we give several properties of the Pauli-Lubanski tensor, $W^\mu = S^{*\mu\nu}P_\nu$. Since $V^\mu = S^{\mu\nu}P_\nu$ is strongly zero with respect to the star algebra, we have

$$\begin{aligned}
\{W^\mu, W^\nu\}^* &= \epsilon^{\mu\nu\lambda\sigma}W_\lambda P_\sigma = -M^2 S^{\mu\nu}, \\
\{W^\mu, P^\nu\}^* &= 0, \\
\{W^\mu, x^\nu\}^* &= (P^\mu/M^2)(W^\nu - (P^\nu/H)\, W^0).
\end{aligned} \tag{3.56}$$

The square of W^μ is

$$W^2 = W^\mu W_\mu = -(1/2)\, P_\mu P^\mu S_{\alpha\beta}S^{\alpha\beta} = -M^2 J^2 \tag{3.57}$$

while with $S^i \equiv (1/2)\, \epsilon^{ijk}S^{jk}$, we find

$$\begin{aligned}
W^0 &= -\mathbf{P}\cdot\mathbf{S}, \\
\mathbf{W} &= -(1/H)(M^2\mathbf{S} + \mathbf{P}(\mathbf{P}\cdot\mathbf{S})),
\end{aligned} \tag{3.58}$$

and

$$\begin{aligned}
\mathbf{S} &= -(H/M^2)\,\mathbf{W} + \mathbf{P}(\mathbf{P}\cdot\mathbf{W})/HM^2, \\
S^{0i} &= (1/M^2)\,\epsilon^{ijk}P^j W^k.
\end{aligned} \tag{3.59}$$

By virtue of Eq. (3.59), we may express the Poincaré generators (3.54) in terms of $\mathbf{W}$ if we wish.

One can now consider all kinds of rearrangements of the independent variables x^i, P^i, S^{ij} and Λ^{ij} and compute their brackets using Eq. (3.51). We note here a set of variables which has particularly simple brackets and corresponds to the Pryce–Newton–Wigner variables [22, 18, 23]. We define

$$S^i = (1/2)\,\epsilon^{ijk}S^{jk}$$

and then examine

$$\mathbf{q} = \mathbf{x} - (\mathbf{S} \times \mathbf{P})/H(H + M) = x^i - S^{0i}/(H + M), \tag{3.60a}$$

$$\mathbf{J} = (M/H)\mathbf{S} + \mathbf{P}(\mathbf{P} \cdot \mathbf{S})/H(H + M), \tag{3.60b}$$

$$\begin{aligned} R^{ij} &= \Lambda^{ij} - \Lambda^{i0}P^j/(H + M) \\ &= \Lambda^{ij} - \Lambda^{ik}P^kP^j/H(H + M). \end{aligned} \tag{3.60c}$$

We retain the same variable $\mathbf{P}$ and the Hamiltonian

$$H = P^0 = (\mathbf{P}^2 + f(\mathbf{J}^2))^{1/2},$$

where $\mathbf{J}^2 = (1/2)\, S^{\mu\nu}S_{\mu\nu}$, and repeat the warning that $M^2 = f(\mathbf{J}^2)$ has non-zero brackets with $\Lambda^\mu{}_\nu$. Observe that

$$\begin{aligned} R^{ik}R^{jk} &= \delta^{ij}, \\ R^{ki}R^{kj} &= \delta^{ij}, \end{aligned} \tag{3.61}$$

so that only three components of R^{ij} are independent. From Eq. (3.51), we find the following brackets for the new variables.

$$\begin{array}{ll} 1. & \{P^i, q^j\}^* = +\delta^{ij}, \\ & \{H, q^j\}^* = P^j/H, \\ 2. & \{q^i, q^j\}^* = 0, \\ 3. & \{P^\mu, P^\nu\}^* = 0, \\ 4. & \{P^\mu, J^i\}^* = 0, \\ 5. & \{q^i, J^j\}^* = 0, \\ 6. & \{J^i, J^j\}^* = -\epsilon^{ijk}J^k, \\ 7. & \{R^{ij}, q^k\}^* = 0, \\ 8. & \{R^{ij}, P^k\}^* = 0, \\ & \{R^{ij}, H\}^* = (f'/H)\, R^{ik}\epsilon^{jkl}J^l, \\ 9. & \{R^{ij}, R^{lm}\}^* = 0, \\ 10. & \{R^{ij}, J^k\}^* = \epsilon^{kjl}R^{il}. \end{array} \tag{3.62}$$

The Poincaré group generators become H, $\mathbf{P}$, and

$$\begin{aligned} M^{ij} &= q^iP^j - q^jP^i + \epsilon^{ijk}J^k, \\ M^{0i} &= tP^i - q^iH + \epsilon^{ijk}J^jP^k/(H + M). \end{aligned} \tag{3.63}$$

Since

$$\{M^{0i}, q^j\}^* = t\delta^{ij} - \frac{q^iP^j}{H} - \frac{\epsilon^{ijk}J^k}{H + M} - \frac{P^j\epsilon^{ikl}J^kP^l}{H(H + M)^2} \tag{3.64}$$

and

$$\{M^{0i}, R^{jk}\}^* = \frac{\delta^{ik} R^{jl} P^l - P^k R^{ji}}{H + M} + \frac{f'}{H}\left(q^i R^{jl} \epsilon^{klm} J^m - \frac{\epsilon^{ikn} J^n R^{jl} J^l}{M(H + M)}\right), \quad (3.65)$$

q^j and R^{jk} do not transform under boosts as we would expect the space parts of four-vectors to transform.

E. *Quantum Mechanics*

The system of single-particle positive-energy quantum-mechanical operators describing the relativistic spherical top can be deduced almost directly from the Dirac brackets for the final constrained system. For the sake of sidestepping the ordering problems inherent in Eq. (3.51), we will define the quantum system in terms of the variables (3.60) and their Dirac brackets (3.62). Most other ordering problems can be resolved by choosing the operators to be Hermitian with respect to the norm of the Hilbert space, for which we choose the convention

$$(\phi, \phi) = \int d^3x\, \phi^*(\mathbf{x}, t)\, \phi(\mathbf{x}, t). \quad (3.66)$$

We then take

$$\begin{aligned} t &= q^0 = c\text{-number}, \\ M^2 &\equiv P^\mu P_\mu = f(\mathbf{J}^2) = q\text{-number}, \end{aligned} \quad (3.67)$$

and the Hamiltonian

$$H = (\mathbf{P}^2 + f(\mathbf{J}^2))^{1/2}, \quad (3.68)$$

where the twelve independent operators are

$$q^i,\ P^i,\ J^i \text{ and three of the } R^{ij}.$$

From the Dirac brackets (3.62), we take the commutation relations of these operators to be

$$\begin{aligned} i[P^i, q^j] &= \delta^{ij}, \\ i[P^i, J^j] &= i[P^i, P^j] = i[P^i, R^{jk}] = 0, \\ i[q^i, q^j] &= i[q^i, J^j] = i[q^i, R^{jk}] = 0, \\ i[J^i, J^j] &= -\epsilon^{ijk} J^k, \\ i[R^{ij}, J^k] &= \epsilon^{kjl} R^{il}, \\ i[R^{ij}, R^{lm}] &= 0. \end{aligned} \quad (3.69)$$

The Hamiltonian (3.68) then has the commutation relations

$$\begin{aligned} i[H, q^i] &= P^i/H, \\ i[H, P^i] &= 0, \\ i[H, J^i] &= 0. \end{aligned} \tag{3.70}$$

The explicit computation of $i[H, R^{ij}]$ poses nontrivial ordering problems and will not be reported here.

The Poincaré group algebra is generated by the operators H, $\mathbf{P}$, and

$$\begin{aligned} M^{ij} &= q^i P^j - q^j P^i + \epsilon^{ijk} J^k, \\ M^{0i} &= tP^i - (1/2)(q^i H + Hq^i) + \epsilon^{ijk} J^j P^k/(H + M). \end{aligned} \tag{3.71}$$

The Hamiltonian equations of motion for an operator A are

$$dA/dt = i[H, A] + \partial A/\partial t. \tag{3.72}$$

Thus, we see, for example, that

$$dM^{0i}/dt = -(i/2)(H[H, q^i] + [H, q^i]\, H) + P^i = 0,$$

so that the Poincaré group generators are constants of the motion.

We remark that the algebra (3.69) can be realized with the differential operators

$$\begin{aligned} P^k &= (1/i)(\partial/\partial q^k), \\ \epsilon^{ijk} J^k &= (1/i)(R^{ki}(\partial/\partial R^{kj}) - R^{kj}(\partial/\partial R^{ki})). \end{aligned} \tag{3.73}$$

The eigenfunctions of $\mathbf{J}^2$ for the spherical top are just the matrix elements

$$D^j_{m'm}(R^{ik})$$

of the unitary representations of $SU(2)$ [24]. It is well-known that these functions provide a complete basis for the Hilbert space of square-integrable functions on $SU(2)$. Ordinary rotations act on the right index m only. The left index m' is necessary to represent the Euler angles as operators on the Hilbert space of physical states. The additional label m' implies the same $(2j + 1)$-fold degeneracy of the spectrum as occurs for the nonrelativistic spherical top. It is obvious that any given top will be either a fermion (j = half-integer) or a boson (j = integer).

The Schrödinger wave equation corresponding to the Heisenberg picture equation (3.72) would be

$$H\psi(\mathbf{x}, t; R^{ij}) = (-\nabla^2 + f(\mathbf{J}^2))^{1/2}\psi = i(\partial\psi/\partial t), \tag{3.74}$$

where J^i is understood to be the operator in Eq. (3.73). One might thus conjecture a Klein–Gordon field equation of the form

$$(\Box + f(\mathbf{J}^2))\, \psi(\mathbf{x}, t; R^{ij}) = 0. \tag{3.75}$$

It would be amusing to see if this equation could be realized in a second-quantized field theory, to study the nature of the Green's functions, and to try to introduce interactions.

4. Lagrangian Description of the Relativistic Spherical Top Interacting with Electromagnetism

We will now develop an approach to the inclusion of electromagnetic interactions in our Lagrangian for the relativistic spherical top. At this point, we must simply postulate that the basic electromagnetic interaction appears in the form

$$L_I = e(dx_\mu/d\tau)\, A^\mu(x). \tag{4.1}$$

Then, under a gauge transformation, the Lagrangian changes only by a divergence

$$L_I' = e(dx_\mu/d\tau)(A^\mu(x) + \partial\Lambda(x)/\partial x_\mu) = L_I + e\, d\Lambda/d\tau \tag{4.2}$$

and the action remains the same. In addition, we assume that if any derivatives of $A^\mu(x)$ appear, they occur only in the gauge-invariant combination

$$F^{\mu\nu}(x) = A^{\mu,\nu} - A^{\nu,\mu} = (\partial A^\mu/\partial x_\nu) - (\partial A^\nu/\partial x_\mu). \tag{4.3}$$

We will defer the question of whether or not the field acts exactly at the point x^μ appearing as a Lagrangian coordinate in our treatment of the classical free top, and will simply examine the consequences of our assumptions.

A. *Example: Spinless Particles*

For comparison with the spinning particle, we give the Lagrangian formulation of a spinless relativistic particle interacting with electromagnetism. We take the Lagrangian of a spinless particle in an external vector potential A^μ to be

$$L = -m(u^2)^{1/2} - eu_\mu A^\mu(x) \tag{4.4}$$

so the canonical momentum is

$$P^\mu = -\partial L/\partial u_\mu = m(u^\mu/(u^2)^{1/2}) + eA^\mu(x). \tag{4.5}$$

We define the mechanical momentum as

$$\pi^\mu = m(u^\mu/(u^2)^{1/2}) = P^\mu - eA^\mu(x). \tag{4.6}$$

The Euler equations

$$\dot{P}^\mu = eu_\nu A^{\nu,\mu} \tag{4.7a}$$

may thus be written in terms of π^μ as

$$\dot{\pi}^\mu = -eF^{\mu\nu}u_\nu \tag{4.7b}$$

where $\dot{A}^\mu = u_\nu A^{\mu,\nu}$.

Note that

$$(d/d\tau)(\pi^2) = 0 \tag{4.8}$$

so that π^2 is a constant of the motion. In fact, the constraint

$$\pi^2 - m^2 = P^2 - 2e(P \cdot A) + e^2A^2 - m^2 = 0 \tag{4.9}$$

is a constraint among the canonical variables P^μ and x^μ following from the form of the Lagrangian.

B. *The Bargmann–Michel–Telegdi Equations*

We now turn to the relativistic spherical top. Suppose that, in addition to the interaction (4.1), our spinning particle has a magnetic moment. The Euler equations should then contain the equations of Bargmann, Michel, and Telegdi (BMT) [1].

If we make the obvious choice

$$L = L_0 - eu_\mu A^\mu - (1/2)\, g\sigma_{\mu\nu}F^{\mu\nu} \tag{4.10}$$

for the interaction Lagrangian, we find the canonical momenta

$$\begin{aligned} P^\mu &= -(\partial L_0/\partial u_\mu) + eA^\mu = \pi^\mu + eA^\mu \\ S^{\mu\nu} &= -(\partial L_0/\partial \sigma_{\mu\nu}) + gF^{\mu\nu} = \Sigma^{\mu\nu} + gF^{\mu\nu}. \end{aligned} \tag{4.11}$$

The Euler equations are

$$\begin{aligned} \dot{\pi}^\mu &= -eF^{\mu\nu}u_\nu + (1/2)\, g\sigma_{\alpha\beta}F^{\alpha\beta,\mu}, \\ &\dot{S}^{\mu\nu} + \sigma^{\mu\lambda}S_\lambda{}^\nu - S^{\mu\lambda}\sigma_\lambda{}^\nu = 0 \\ &= \dot{S}^{\mu\nu} + u^\mu\pi^\nu - u^\nu\pi^\mu + g(\sigma^{\mu\lambda}F_\lambda{}^\nu - F^{\mu\lambda}\sigma_\lambda{}^\nu) = 0. \end{aligned} \tag{4.12}$$

Now if we take L_0 to satisfy Eqs. (1.36)–(1.38), then

$$\Sigma^{\mu\nu}\pi_\nu = 0 \tag{4.13}$$

so

$$\Sigma^{\mu\nu}\Sigma^*_{\mu\nu} = 0. \tag{4.14}$$

However, from Eq. (4.12),

$$S^{\mu\nu}S^{*}_{\mu\nu} = (\Sigma^{\mu\nu} + gF^{\mu\nu})(\Sigma^{*}_{\mu\nu} + gF^{*}_{\mu\nu}) = \text{const.} \tag{4.15}$$

Since the constant vanishes when the field vanishes, but is not a function of the field, the constant is zero. Applying (4.14) to (4.15), we find

$$2F^{\mu\nu}\Sigma^{*}_{\mu\nu} + gF^{\mu\nu}F^{*}_{\mu\nu} = 0.$$

In the presence of a pure electric field, we find $(\mathbf{E} \cdot \mathbf{\Sigma}) = 0$; this is clearly an unphysical restriction, so Eqs. (4.13) and (4.14) are physically unreasonable.

We must thus find some alternative to Eq. (4.10) in order to couple the magnetic moment in such a way that we can derive the BMT equation in a consistent manner. It is possible that there are several methods of accomplishing this. The method which we found most successful was the following: make the substitution

$$\sigma^{\mu\nu} \rightarrow \tilde{\sigma}^{\mu\nu} = \sigma^{\mu\nu} + gL_0(u, \tilde{\sigma})\, F^{\mu\nu}, \tag{4.16}$$

where g is a constant and the presence of L_0 maintains τ-reparametrization invariance. Equation (4.16) is a highly nonlinear, implicit definition of $\tilde{\sigma}$ that amounts to a kind of minimal coupling principle for magnetic dipole moments.

Now we set

$$a_1 = u^2, \qquad a_2 = \tilde{\sigma} \cdot \tilde{\sigma}, \qquad a_3 = u\tilde{\sigma}\tilde{\sigma}u, \qquad a_4 = \operatorname{Det} \tilde{\sigma},$$

and take

$$L(a_i) = L_0(a_i) - eu_\mu A^\mu. \tag{4.17}$$

Let us define

$$\begin{aligned} \psi^\mu &= P^\mu_{\text{free}}(u, \tilde{\sigma}) = -\left.\frac{\partial L_0}{\partial u_\mu}\right|_{\tilde{\sigma}=\text{const}}, \\ \Gamma^{\mu\nu} &= S^{\mu\nu}_{\text{free}}(u, \tilde{\sigma}) = -\frac{\partial L_0}{\partial \tilde{\sigma}_{\mu\nu}}, \end{aligned} \tag{4.18}$$

so that ψ^μ and $\Gamma^{\mu\nu}$ have the same functional form as the free momenta, except that $\sigma^{\mu\nu}$ is replaced by $\tilde{\sigma}^{\mu\nu}$. The canonical momenta are

$$\begin{aligned} P^\mu &= -\partial L/\partial u_\mu = \pi^\mu + eA^\mu \\ S^{\mu\nu} &= -\partial L/\partial \sigma_{\mu\nu} = -\partial L_0/\partial \sigma_{\mu\nu}. \end{aligned} \tag{4.19}$$

The mechanical momenta can be expressed in terms of Eq. (4.18) as

$$\pi^\mu = -\frac{\partial L_0}{\partial u_\mu} = \psi^\mu - \frac{1}{2}\frac{\partial L_0}{\partial \tilde{\sigma}_{\alpha\beta}}\frac{\partial \tilde{\sigma}_{\alpha\beta}}{\partial u_\mu}, \tag{4.20}$$

$$S^{\mu\nu} = -\frac{\partial L_0}{\partial \sigma_{\mu\nu}} = \Gamma^{\mu\nu} - \frac{1}{2}\frac{\partial L_0}{\partial \tilde{\sigma}_{\alpha\beta}}\frac{\partial(\tilde{\sigma}_{\alpha\beta} - \sigma_{\alpha\beta})}{\partial \sigma_{\mu\nu}}. \tag{4.21}$$

We now observe that

$$\begin{aligned} \partial L_0/\partial\tilde{\sigma}_{\alpha\beta} &= -\Gamma^{\alpha\beta}, \\ \partial\tilde{\sigma}_{\alpha\beta}/\partial u_\mu &= -gF_{\alpha\beta}\pi^\mu, \\ \partial(\tilde{\sigma}_{\alpha\beta} - \sigma_{\alpha\beta})/\partial\sigma_{\mu\nu} &= -gF_{\alpha\beta}S^{\mu\nu}. \end{aligned} \tag{4.22}$$

Thus the choice of the coefficient L_0 in Eq. (4.16) results in the following remarkable simplifications in Eqs. (4.20) and (4.21):

$$\begin{aligned} \pi^\mu &= \psi^\mu - (1/2)\,\pi^\mu g\Gamma_{\alpha\beta}F^{\alpha\beta} \\ S^{\mu\nu} &= \Gamma^{\mu\nu} - (1/2)\,S^{\mu\nu}g\Gamma_{\alpha\beta}F^{\alpha\beta} \end{aligned} \tag{4.23}$$

so that

$$\begin{aligned} \pi^\mu &= (1 - (1/2)\,gF\cdot S)\,\psi^\mu \\ S^{\mu\nu} &= (1 - (1/2)\,gF\cdot S)\,\Gamma^{\mu\nu}. \end{aligned} \tag{4.24}$$

If $L_0(a_i[u, \tilde{\sigma}])$ is chosen to satisfy the same differential equations in a_i as before, Eqs. (1.36)–(1.38), then

$$\Gamma^{\mu\nu}\psi_\nu = 0 \tag{4.25}$$

and (4.24) gives

$$V^\mu = S^{\mu\nu}\pi_\nu = 0. \tag{4.26}$$

The Euler equations for $\dot{P}^\mu$ are easily found to be

$$\dot{P}^\mu = eu_\nu A^{\nu,\mu} - \partial L_0/\partial x_\mu \tag{4.27}$$

where

$$\frac{\partial L_0}{\partial x_\mu} = \frac{1}{2}\frac{\partial L_0}{\partial\tilde{\sigma}_{\alpha\beta}}\frac{\partial(\sigma^{\alpha\beta} + gL_0F^{\alpha\beta})}{\partial x_\mu}$$

so that

$$\partial L_0/\partial x_\mu = -(1/2)\,gL_0S_{\alpha\beta}F^{\alpha\beta,\mu}. \tag{4.28}$$

The mechanical Euler equations are thus

$$\dot{\pi}^\mu = -eF^{\mu\nu}u_\nu + (1/2)\,gL_0S_{\alpha\beta}F^{\alpha\beta,\mu}. \tag{4.29}$$

We now show how the BMT equation arises. The Euler equation for $\dot{S}^{\mu\nu}$, as usual, is

$$\dot{S}^{\mu\nu} + \sigma^{\mu\lambda} S_\lambda{}^\nu - S^{\mu\lambda}\sigma_\lambda{}^\nu = 0. \tag{4.30}$$

This can be rewritten, using (4.24) and the known forms (1.16)–(1.17) for ψ^μ and $\Gamma^{\mu\nu}$, as

$$\dot{S}^{\mu\nu} = \pi^\mu u^\nu - u^\mu \pi^\nu + gL_0(F^{\mu\lambda} S_\lambda{}^\nu - S^{\mu\lambda} F_\lambda{}^\nu). \tag{4.31}$$

Contracting with π_ν and using the derivative of the constraint (4.26),

$$S^{\mu\nu}\dot{\pi}_\nu + \dot{S}^{\mu\nu}\pi_\nu = 0 \tag{4.32}$$

we find

$$\pi^2 u^\mu = \pi^\mu (u \cdot \pi) + S^{\mu\nu}\dot{\pi}_\nu - gL_0 S^{\mu\lambda} F_{\lambda\nu}\pi^\nu. \tag{4.33}$$

The antisymmetric product of (4.33) with π^ν gives

$$\pi^2(\pi^\mu u^\nu - u^\mu \pi^\nu) = \pi^\mu S^{\nu\lambda}\dot{\pi}_\lambda - \pi^\nu S^{\mu\lambda}\dot{\pi}_\lambda - gL_0(\pi^\mu S^{\nu\lambda} F_{\lambda\rho}\pi^\rho - \pi^\nu S^{\mu\lambda} F_{\lambda\rho}\pi^\rho).$$

Inserting this expression in (4.31) gives one form of the BMT equations,

$$\begin{aligned} \pi^2 \dot{S}^{\mu\nu} = {} & gL_0 \pi^2 (F^{\mu\lambda} S_\lambda{}^\nu - S^{\mu\lambda} F_\lambda{}^\nu) - gL_0(\pi^\mu S^{\nu\lambda} F_{\lambda\rho}\pi^\rho - \pi^\nu S^{\mu\lambda} F_{\lambda\rho}\pi^\rho) \\ & + \pi^\mu S^{\nu\lambda}\dot{\pi}_\lambda - \pi^\nu S^{\mu\lambda}\dot{\pi}_\lambda \,. \end{aligned} \tag{4.34}$$

From Eq. (4.34), we can derive the following equation of motion for $W^\mu = S^{*\mu\nu}\pi_\nu$:

$$\pi^2 \dot{W}^\mu = -\pi^\mu W_\alpha \dot{\pi}^\alpha - gL_0(\pi^2 F^{\mu\nu} W_\nu + \pi^\mu W_\alpha F^{\alpha\beta}\pi_\beta). \tag{4.35}$$

The identification with the standard BMT equation is complete if we set

$$\begin{aligned} gL_0 \,|_{us} &= +ge/2m \,|_{\mathrm{BMT}} \\ \pi^\mu \,|_{us} &= mu^\mu \,|_{\mathrm{BMT}} \,. \end{aligned} \tag{4.36}$$

The BMT method, which does not use a Lagrangian, could not distinguish between π^μ and mu^μ, which are not necessarily proportional in our treatment. Since $(d\tau)^2 = dx_\mu\, dx^\mu$ in the standard BMT equation, direct comparison of our result to BMT requires a definite choice of the gauge relating $d\tau$ to dx and $d\Lambda$. Choosing L_0 to be a numerical constant on the equations of motion, as implied by Eq. (4.36), is precisely the condition which makes τ the ordinary proper time in the spinless case.

Clearly it would be desirable to derive the BMT equations by introducing magnetic couplings into our formalism in a simpler way.

C. *The Evolution Equation*

Finally, we show that all the equations of motion can be written in a simple general form. First, we multiply (4.30) by π_ν and use (4.32) to get

$$S^{\mu\nu}(\dot{\pi}_\nu + \sigma_{\mu\nu}\pi^\nu) = 0.$$

Since $W^\mu = S^{*\mu\nu}\pi_\nu$ and π^μ are the only null vectors of $S^{\mu\nu}$ we must have

$$\dot{\pi}^\mu + (Ag^{\mu\nu} + BS^{*\mu\nu} + \sigma^{\mu\nu})\,\pi_\nu = 0. \tag{4.39}$$

Contracting with π^μ and W^μ we find

$$\begin{aligned} A &= -\pi_\mu \dot{\pi}^\mu/\pi^2, \\ B &= -[W_\mu \dot{\pi}^\mu + W_\mu \sigma^{\mu\nu}\pi_\nu]/W^2, \\ &= [\dot{W}_\mu \pi^\mu + \pi_\mu \sigma^{\mu\nu} W_\nu]/W^2. \end{aligned} \tag{4.40}$$

Thus we may write

$$\frac{d}{d\tau}\left[\frac{\pi^\mu}{(\pi^2)^{1/2}}\right] + \Omega^{\mu\nu}\left[\frac{\pi_\nu}{(\pi^2)^{1/2}}\right] = 0, \tag{4.41}$$

where

$$\Omega^{\mu\nu} = \sigma^{\mu\nu} + BS^{*\mu\nu}. \tag{4.42}$$

Similarly, if we multiply $\dot{S}^{*\mu\nu}$ by π_ν and use Eq. (4.41), we may show

$$\frac{d}{d\tau}\left[\frac{W^\mu}{(W^2)^{1/2}}\right] + \Omega^{\mu\nu}\left[\frac{W_\nu}{(W^2)^{1/2}}\right] = 0. \tag{4.43}$$

It is also obvious that Eq. (4.30) can be written

$$\frac{d}{d\tau}\left[\frac{S^{\mu\nu}}{(S\cdot S)^{1/2}}\right] + \Omega^{\mu\lambda}\left[\frac{S_\lambda{}^\nu}{(S\cdot S)^{1/2}}\right] - \left[\frac{S^{\mu\lambda}}{(S\cdot S)^{1/2}}\right]\Omega_\lambda{}^\nu = 0. \tag{4.44}$$

Thus $\Omega^{\mu\nu}$ looks like a generalized angular velocity, giving the change in the momenta as a function of the momenta.

We observe that we may use Eqs. (1.22)–(1.26) to rewrite the coefficient B. If we define

$$\rho^\mu = \Lambda^{\mu\nu}\pi_\nu/(\pi^2)^{1/2}, \tag{4.45}$$

where $\rho^2 = 1$, then Eq. (4.39) can be written

$$\dot{\rho}^\mu + B\bar{S}^{*\mu\nu}\rho_\nu = 0. \tag{4.46}$$

Thus

$$B = -\frac{\rho^\mu \bar{S}^*_{\mu\nu}\dot{\rho}^\nu}{\rho^\mu \bar{S}^*_{\mu\nu}\bar{S}^{*\nu\lambda}\rho_\lambda} = -\frac{\rho^\mu \bar{S}^*_{\mu\nu}\dot{\rho}^\nu}{J^2}. \tag{4.47}$$

At this point, we notice that the arguments developed in Eqs. (3.21)–(3.31) for the free case hold for the interacting case as well when P^μ is replaced by π^μ. In particular, we are free to imitate Eqs. (3.25) and (3.29) and impose the invariant and gauge relations $\bar{S}^{0i} \approx 0$ and $\rho^0 \approx 1$. This corresponds to

$$\begin{aligned} \rho^\mu &\approx g^{\mu 0}, \\ B &= 0, \\ \Omega^{\mu\nu} &= \sigma^{\mu\nu}. \end{aligned} \tag{4.48}$$

Now, however, $\sigma^{\mu\nu}$ is not proportional to $S^{\mu\nu}$ and in general $a_3 \neq 0$, $a_4 \neq 0$.

When we fix B, we naturally fix the time evolution of Λ, since $\Lambda^{-1}\dot{\Lambda} = \sigma$ and σ depends on B. There is a convenient factorization which may be used to exhibit the B-dependence of Λ and to show that the same physical motion follows from different choices of B. (In the language of Dirac's formalism described in Section 2, B depends on the arbitrary function multiplying the first-class variable $S^{\mu\nu}S^*_{\mu\nu}$ in the Hamiltonian, so different B's should be physically equivalent. Compare Eq. (3.20) and (4.42).) Now let Λ_1 and Λ_2 be two Lorentz matrices resulting from different choices of B, with

$$\Lambda_2 = M\Lambda_1, \tag{4.49}$$

where M is a Lorentz matrix to be determined. Then, since $\sigma_1 = \Lambda_1^{-1}\dot{\Lambda}_1$ and $\sigma_2 = \Lambda_2^{-1}\dot{\Lambda}_2$, we have

$$\begin{aligned} \sigma_2 &= \Lambda_1^{-1}M^{-1}\dot{M}\Lambda_1 + \Lambda_1^{-1}\dot{\Lambda}_1 \\ &= \sigma_1 + \Lambda_1^{-1}M^{-1}\dot{M}\Lambda_1. \end{aligned} \tag{4.50}$$

But from Eq. (3.20), we can deduce that the difference between σ_1 and σ_2 is given even in the interacting case by

$$\sigma_2 = \sigma_1 + S^*\, db/d\tau \tag{4.51}$$

for some function $b(\tau)$. But then

$$M^{-1}\dot{M} = (db/d\tau)\, \Lambda_1 S^* \Lambda_1^{-1} = (db/d\tau)\, \bar{S}^* \tag{4.52}$$

where $\bar{S}^*$ is the constant matrix given in Eq. (1.25). We may integrate (4.52) directly to get

$$M^{\mu\nu} = C^{\mu\lambda}[\exp(\bar{S}^* b(\tau))]_\lambda{}^\nu \tag{4.53}$$

where C is a constant matrix. Thus

$$\Lambda_2 = M\Lambda_1 = Ce^{S*b}\Lambda_1 . \tag{4.54}$$

Now we examine the expressions

$$\begin{aligned} \rho_1{}^{\mu} &= \Lambda_1^{\mu\nu}\pi_\nu/(\pi^2)^{1/2}, \\ \rho_2{}^{\mu} &= \Lambda_2^{\mu\nu}\pi_\nu/(\pi^2)^{1/2} = M^{\mu}{}_{\lambda}\rho_1{}^{\lambda}, \end{aligned} \tag{4.55}$$

where

$$\begin{aligned} \dot\rho_1 + B_1\bar S^*\rho_1 &= 0, \\ \dot\rho_2 + B_2\Lambda_2 S^*\Lambda_2^{-1}\rho_2 &= 0. \end{aligned} \tag{4.56}$$

But from Eqs. (4.54) and (4.55),

$$\begin{aligned} \dot\rho_2 &= \dot M\rho_1 + M\dot\rho_1 = (db/d\tau)\, M\bar S^*\rho_1 - MB_1\bar S^*\rho_1 \\ &= ((db/d\tau) - B_1)\, \Lambda_2 S^*\Lambda_2^{-1}\rho_2 . \end{aligned}$$

Thus

$$B_2 = B_1 - db/d\tau. \tag{4.57}$$

We see that by choosing $db/d\tau = B_1$, so that B_2 vanishes, we may express any Lorentz matrix Λ_1 as

$$\Lambda_1(\tau) = M(\tau)^{-1}\Lambda_2(\tau)$$

where

$$\Lambda_2^{\mu\nu}\pi_\nu/(\pi^2)^{1/2} = \text{const.} \tag{4.58}$$

$M^{\mu\nu}(\tau)$ is then physically irrelevant, while Λ_2 is physically identifiable as a pure rotation in the rest frame of π^μ, so Λ_2 defines the Euler angles of the top.

D. *Constraints*

The velocity constraints treated in Eqs. (1.39)–(1.44) for the free case have more complicated analogs in the interacting case. We only show a brief sketch of how one would proceed to study the constraints in the $g = 0$ case. From the equations of motion, (4.29) and (4.30), together with Eq. (4.33), we find

$$\pi^\mu(\pi \cdot u) = u^\mu M^2 - eS^{\mu\nu}F_{\nu\lambda}u^\lambda. \tag{4.59}$$

Substituting the explicit forms of the momenta, we find

$$\begin{aligned} 0 = u^\mu L_3&[-2e(u\sigma Fu) - 4a_3L_1 + 2a_2a_3L_3 - 4a_1a_4L_3] \\ &+ 4\sigma^{\mu\nu}\sigma_{\nu\lambda}u^\lambda L_3(a_1L_1 + a_3L_3) \\ &+ 4eL_2\sigma^{\mu\nu}F_{\nu\lambda}u^\lambda + (1/2)\, eL_4(\sigma \cdot \sigma^*)\, \sigma^{*\mu\nu}F_{\nu\lambda}u^\lambda. \end{aligned} \tag{4.60}$$

The contractions of (4.60) with $\sigma^{\mu\nu}u_\nu$ and u^μ give two scalar constraint equations

$$(u\sigma\sigma Fu) \equiv u^\mu\sigma_{\mu\nu}\sigma^{\nu\lambda}F_{\lambda\rho}u^\rho = 0 \tag{4.61}$$

and

$$\begin{aligned}&2L_3{}^2(a_1a_2a_3 + 2a_3{}^2 - 2a_1{}^2a_4) + 2e(u\sigma Fu)(2L_2 - a_1L_3)\\ &\quad + (1/2)\, eL_4(\sigma\cdot\sigma^*)(u\sigma^*Fu) = 0.\end{aligned} \tag{4.62}$$

All other contractions give equivalent expressions. The corresponding expressions for $g \neq 0$ are easy to derive.

We mention at this point a similar topic, the trajectory constraints for $g \neq 0$. Starting with Eq. (4.25) and proceeding by analogy to the free case, we deduce the existence of a trajectory constraint

$$\psi^\mu\psi_\mu - f((1/2)\,\Gamma^{\mu\nu}\Gamma_{\mu\nu}) = 0 \tag{4.63}$$

or

$$(1 - (1/2)\, g(F\cdot S))^{-2}\pi^2 - f((1/2)\, S\cdot S[1 - (1/2)\, g(F\cdot S)]^{-2}) = 0. \tag{4.64}$$

5. Hamiltonian Description of Interacting Top with No Magnetic Moment

We now examine the Hamiltonian formalism for the electromagnetically interacting relativistic spherical top of Section 4. For simplicity, we will set the magnetic moment coefficient g *equal to zero.*

A. *Spinless Particle*

For comparison to the spinning case, we again outline the treatment of a spinless particle. With the Lagrangian (4.4) and momenta (4.5) and (4.6), we get the canonical Hamiltonian

$$H_0 = -P^\mu u_\mu - L = 0. \tag{5.1}$$

The constraint of Eq. (4.9) is now interpreted as the solitary first-class weak constraint

$$\pi^2 - m^2 \approx 0 \tag{5.2}$$

of the system, so that the Hamiltonian may be chosen as

$$H = v(\tau)(\pi^2 - m^2). \tag{5.3}$$

The canonical Poisson brackets are

$$\begin{aligned} \{P^\mu, P^\nu\} &= \{x^\mu, x^\nu\} = 0, \\ \{P^\mu, x^\nu\} &= -g^{\mu\nu}. \end{aligned} \tag{5.4}$$

The Poisson brackets of the mechanical momenta, $\pi^\mu = P^\mu - eA^\mu(x)$, are then

$$\begin{aligned} \{\pi^\mu, \pi^\nu\} &= -e((\partial A^\mu/\partial x_\nu) - (\partial A^\nu/\partial x_\mu)) \equiv -eF^{\mu\nu}, \\ \{\pi^\mu, x^\nu\} &= -g^{\mu\nu}. \end{aligned} \tag{5.5}$$

The Hamiltonian (5.3) generates the following motion for x:

$$\{H, x^\mu\} = -2v\pi^\mu \equiv u^\mu \tag{5.6}$$

so that

$$\begin{aligned} \pi^\mu &= -u^\mu/2v \\ v &= -(u^2)^{1/2}/2m. \end{aligned} \tag{5.7}$$

The equation of motion for π^μ is

$$\dot{\pi}^\mu = \{H, \pi^\mu\} = -2ev\pi_\nu F^{\nu\mu} = -eF^{\mu\nu}u_\nu \tag{5.8}$$

in agreement with Eq. (4.7).

Now we choose a gauge constraint to fix the parameter and make the constraint (5.2) second-class. Then, as usual, the Dirac brackets following from

$$\begin{aligned} \phi_1 &= \pi^2 - m^2 \approx 0, \\ \phi_2 &= x^0 - \tau \approx 0, \end{aligned} \tag{5.9}$$

are

$$\{\xi, \eta\}^* = \{\xi, \eta\} - (1/2\pi^0)[\{\xi, \pi^2\}\{x^0, \eta\} - \{\xi, x^0\}\{\pi^2, \eta\}]. \tag{5.10}$$

We therefore find

$$\begin{aligned} \{x^\mu, x^\nu\}^* &= 0, \\ \{\pi^\mu, x^\nu\}^* &= -g^{\mu\nu} + g^{\mu 0}\pi^\nu/\pi^0, \\ \{\pi^\mu, \pi^\nu\}^* &= -eF^{\mu\nu} + (e/\pi^0)(F^{\mu\lambda}\pi_\lambda g^{0\nu} - F^{\nu\lambda}\pi_\lambda g^{0\mu}). \end{aligned} \tag{5.11}$$

The Hamiltonian is now

$$\begin{aligned} H &= \pi^0 + eA^0(x) = (\boldsymbol{\pi}^2 + m^2)^{1/2} + eA^0(x) \\ &= ([\mathbf{P} - e\mathbf{A}]^2 + m^2)^{1/2} + eA^0(x). \end{aligned} \tag{5.12}$$

B. *Preliminary Spinning Dirac Brackets*

We take the Lagrangian and the canonical momenta for the interacting $g = 0$ spherical top from Section 4, Eqs. (4.17) and (4.19). The canonical Hamiltonian vanishes as usual,

$$H_0 = -P^\mu u_\mu - (1/2)\, S^{\mu\nu}\sigma_{\mu\nu} - L = 0. \tag{5.13}$$

The canonical Poisson brackets are taken from Section 3, Eqs. (3.10).

Given our assumption that the argument of the external potential A^μ is the canonical coordinate x^μ, we find that the mechanical momentum, $\pi^\mu = P^\mu - eA^\mu(x)$, has the canonical Poisson brackets

$$\{\pi^\mu, x^\nu\} = -g^{\mu\nu} \tag{5.14}$$

$$\{\pi^\mu, \pi^\nu\} = -eF^{\mu\nu}. \tag{5.15}$$

The Lagrangian is chosen as before to obey the constraint

$$V^\mu = S^{\mu\nu}\pi_\nu \approx 0 \tag{5.16}$$

among the canonical variables. As noted before, there is a combination of the three independent constraints which is first-class and which can be taken to be

$$S^{\mu\nu}S^*_{\mu\nu} \approx 0. \tag{5.17}$$

By imitating our previous arguments, we find also in the interacting case the first-class trajectory constraint

$$\pi^2 - f(\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}) + \frac{2eV^\mu F_{\mu\nu}\pi^\nu}{\pi^2 - e(S\cdot F)/2} \approx 0. \tag{5.18}$$

Before fixing any gauge constraints, we may thus take the Hamiltonian to be

$$H = v_1(\tau)\left(\pi^2 - f(\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}) + \frac{2eV^\mu F_{\mu\nu}\pi^\nu}{\pi^2 - e(S\cdot F)/2}\right) + v_2(\tau)\, S^{\mu\nu}S^*_{\mu\nu}\,. \tag{5.19}$$

Thus

$$u^\mu \equiv \{H, x^\mu\} = -2v_1\left(\pi^\mu - \frac{eS^{\mu\nu}F_{\nu\lambda}\pi^\lambda}{\pi^2 - e(S\cdot F)/2}\right). \tag{5.20}$$

We also find that Eq. (3.20) for $\sigma^{\mu\nu}$ is now replaced by

$$\sigma^{\mu\nu} = v_1\left(2f'S^{\mu\nu} + \frac{2e(\pi^\mu F^{\nu\lambda}\pi_\lambda - \pi^\nu F^{\mu\lambda}\pi_\lambda)}{\pi^2 - e(S\cdot F)/2}\right) - 4v_2 S^{*\mu\nu}. \tag{5.21}$$

The equations of motion are thus

$$\begin{aligned}\dot{\pi}^\mu &= \{H, \pi^\mu\} = -eF^{\mu\nu}u_\nu\,,\\ \dot{S}^{\alpha\beta} &= \{H, S^{\alpha\beta}\}\\ &= S^{\alpha\gamma}\sigma_\gamma{}^\beta - \sigma^\alpha{}_\gamma S^{\gamma\beta},\end{aligned} \tag{5.22}$$

where we use Eq. (5.20) for u^μ and Eq. (5.21) for $\sigma_{\mu\nu}$. Equations (5.22) thus agree with Eqs. (4.29) and (4.30). (Remember that $g = 0$.)

Now we could compute Dirac brackets using the two independent second-class constraints in $V^\mu \approx 0$. We shall instead proceed directly to impose constraints conjugate to $V^\mu \approx 0$, as we did in Section 3. By the arguments of Eqs. (4.45)–(4.58), we have in our system a gauge freedom which corresponds to $S^{\mu\nu}S^*_{\mu\nu} \approx 0$ and two integral relations corresponding to the second-class parts of $V^\mu \approx 0$. We showed in Section 4 that a consistent constraint was

$$\rho^\mu = \Lambda^{\mu\nu}\pi_\nu/(\pi^2)^{1/2} \approx g^{\mu 0}$$

which we choose to write as

$$\chi^\mu = \Lambda^{0\mu} - \pi^\mu/(\pi^2)^{1/2} \approx 0. \tag{5.23}$$

Since $\rho^2 = 1$ (or $\pi^\mu\chi_\mu = -(1/2)\,M\chi^2$), only three of these constraints are independent, just as only three of the V^μ are independent.

We now compute the Dirac brackets consistent with setting $V^\mu \approx 0$ and with eliminating the corresponding arbitrary functions by setting $\chi^\mu \approx 0$. The set of constraints is summarized in the eight-vector

$$\phi^\alpha = (V^0, V^1, V^2, V^3, \chi^0, \chi^1, \chi^2, \chi^3), \qquad \alpha = 1,\ldots, 8. \tag{5.24}$$

For convenience, we define

$$\begin{aligned}M &= (\pi^\mu\pi_\mu)^{1/2},\\ N &= M - (e/2M)\,S_{\alpha\beta}F^{\alpha\beta},\\ \tilde{F}^{\mu\nu} &= F^{\mu\nu} + M^{-2}(\pi^\mu F^{\nu\lambda}\pi_\lambda - \pi^\nu F^{\mu\lambda}\pi_\lambda).\end{aligned} \tag{5.25}$$

The arguments of Eqs. (3.33)–(3.39) are then repeated to give the Dirac-bracket matrices

$$\begin{aligned}C^{\alpha\beta} &= \begin{vmatrix} MNS^{\mu\nu} & Mg^{\mu\nu} + \dfrac{e}{M}S^{\mu\lambda}\tilde{F}_\lambda{}^\nu \\ -Mg^{\mu\nu} - \dfrac{e}{M}\tilde{F}^{\mu\lambda}S_\lambda{}^\nu & -\dfrac{e}{M^2}\tilde{F}^{\mu\nu}\end{vmatrix},\\ (C^{-1})^{\alpha\beta} &= \begin{vmatrix} -\dfrac{e}{NM^3}\tilde{F}^{\mu\nu} & -\dfrac{g^{\mu\nu}}{M} \\ \dfrac{g^{\mu\nu}}{M} & S^{\mu\nu}\end{vmatrix}.\end{aligned} \tag{5.26}$$

The preliminary Dirac brackets may then be written explicitly as

$$\{\xi, \eta\}' = \{\xi, \eta\} + \{\xi, V_\mu\} \frac{e}{NM^3} \tilde{F}^{\mu\nu}\{V_\nu, \eta\} - \{\xi, \chi_\mu\} S^{\mu\nu}\{\chi_\nu, \eta\}$$
$$+ \{\xi, V_\mu\} \frac{1}{M}\{\chi^\mu, \eta\} - \{\xi, \chi_\mu\} \frac{1}{M}\{V^\mu, \eta\}. \tag{5.27}$$

The Dirac brackets can now be computed. We first define the following frequently occurring quantity,

$$G^{\mu\nu} = g^{\mu\nu} - (e/NM) S^{\mu\alpha}F_\alpha{}^\nu, \tag{5.28}$$

where $G^{\mu\nu}$ is *not* symmetric and $G_\mu{}^\mu = 2(M+N)/N$. The resulting brackets are:

1. $\{\pi^\mu, x^\nu\}' = -g^{\mu\nu} + (e/NM) S^{\nu\alpha}F_\alpha{}^\mu = -G^{\nu\mu}$,
2. $\{x^\mu, x^\nu\}' = -S^{\mu\nu}/MN = -(1/M^2)\, G^{\mu\lambda}S_\lambda{}^\nu = (1/M^2)\, G^{\nu\lambda}S_\lambda{}^\mu$,
3. $\{\pi^\mu, \pi^\nu\}' = -eF^{\mu\nu} + (e^2/NM) F^{\mu\alpha}S_{\alpha\beta}F^{\beta\nu} = -eF^{\mu\alpha}G_\alpha{}^\nu = eF^{\nu\alpha}G_\alpha{}^\mu$,
4. $\{\pi^\mu, S^{\nu\lambda}\}' = (e/NM)(F^{\mu\alpha}S_\alpha{}^\lambda\pi^\nu - F^{\mu\alpha}S_\alpha{}^\nu\pi^\lambda)$,
5. $\{x^\mu, S^{\nu\lambda}\}' = (1/NM)(S^{\mu\nu}\pi^\lambda - S^{\mu\lambda}\pi^\nu)$,
6. $\{S^{\mu\nu}, S^{\alpha\beta}\}' = S^{\mu\alpha}\left(g^{\nu\beta} - \dfrac{\pi^\nu\pi^\beta}{NM}\right) - S^{\mu\beta}\left(g^{\nu\alpha} - \dfrac{\pi^\nu\pi^\alpha}{NM}\right) + S^{\alpha\nu}\left(g^{\mu\beta} - \dfrac{\pi^\mu\pi^\beta}{NM}\right) - S^{\beta\nu}\left(g^{\mu\alpha} - \dfrac{\pi^\mu\pi^\alpha}{NM}\right)$,
7. $\{\Lambda^{\mu\nu}, x^\alpha\}' = (1/M^2)(g^{\nu\lambda}\Lambda^{\mu\rho} - \Lambda^{\mu\lambda}g^{\nu\rho})\, G^\alpha{}_\rho\pi_\lambda$,
8. $\{\Lambda^{\mu\nu}, \pi^\alpha\}' = (e/M^2)(\Lambda^{\mu\lambda}g^{\nu\rho} - \Lambda^{\mu\rho}g^{\nu\lambda})\, \pi_\lambda F^{\alpha\beta}G_{\beta\rho}$
9. $\{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\}' = -(e/NM^3)(\Lambda^{\mu\sigma}g^{\nu\rho} - \Lambda^{\mu\rho}g^{\nu\sigma})(\Lambda^{\alpha\tau}g^{\beta\gamma} - \Lambda^{\alpha\gamma}g^{\beta\tau})\, \pi_\rho\pi_\gamma F_{\sigma\tau}$
10. $\{\Lambda^{\mu\nu}, S^{\alpha\beta}\}' = \Lambda^{\mu\alpha}g^{\nu\beta} - \Lambda^{\mu\beta}g^{\nu\alpha} + (1/M^2)(\Lambda^{\mu\rho}g^{\nu\lambda} - \Lambda^{\mu\lambda}g^{\nu\rho})\, \pi_\lambda(\pi^\alpha G^\beta{}_\rho - \pi^\beta G^\alpha{}_\rho)$. (5.29)

Since the constraints (5.16) and (5.23) are strongly zero in the prime system of brackets, they may be used wherever convenient to re-express the right-hand sides of Eqs. (5.29). The symbol $M = (\pi^\mu\pi_\mu)^{1/2}$ is *not* a constant in Eqs. (5.29), but has nontrivial prime brackets. Among the prime brackets of the Pauli–Lubanski tensor are

$$\{W^\mu, W^\nu\}' = -M^2S^{\mu\nu} - \frac{e}{2N} S^{*\mu\nu}\left[M(S^*\cdot F) - \frac{e}{4M}(S\cdot S)(F^*\cdot F)\right], \tag{5.30}$$
$$\{W^\mu, \pi^\alpha\}' = -eS^{*\mu\nu}F_{\nu\lambda}G^{\lambda\alpha}.$$

The Hamiltonian for our new, partially constrained, prime bracket theory is just the trajectory constraint, which is first-class because there are no other constraints:

$$H = v(\tau)(\pi^2 - f((1/2)\, S_{\mu\nu}S^{\mu\nu})). \tag{5.31}$$

Now we must compute the equations of motion using the prime brackets (5.29). We find

$$u^\mu \equiv \{H, x^\mu\}' = -2v(\pi^\mu - (e/NM)\, S^{\mu\nu}F_{\nu\lambda}\pi^\lambda) \tag{5.32}$$

so that now

$$v = -(1/2)(u^2)^{1/2}NM(N^2M^4 + e^2\pi^\alpha F_{\alpha\beta}S^{\beta\gamma}S_{\gamma\delta}F^{\delta\epsilon}\pi_\epsilon)^{-1/2}. \tag{5.33}$$

Also

$$\begin{aligned}\sigma^{\mu\nu} &\equiv \Lambda_\lambda{}^\mu\{H, \Lambda^{\lambda\nu}\}' \\ &= 2vf'S^{\mu\nu} - (2ev/M^2)(\pi^\mu\pi^\alpha F_{\alpha\beta}G^{\beta\nu} - \pi^\nu\pi^\alpha F_{\alpha\beta}G^{\beta\mu}).\end{aligned} \tag{5.34}$$

After using the identities in Appendix A, the above equation can be rewritten in the form (5.21) with

$$\begin{aligned}v_1(\tau) &= v(\tau), \\ v_2(\tau) &= +(e^2v(\tau)/8NM)\, F^{\alpha\beta}F^*_{\alpha\beta}\,.\end{aligned} \tag{5.35}$$

Our gauge choice thus implies that Eqs. (5.22) continue to hold in the new system and that

$$(dK^\mu/d\tau) + \sigma^{\mu\nu}K_\nu = 0,$$

where $K^\mu = \pi^\mu/M$ or $K^\mu = W^\mu/(-W^2)^{1/2}$. Thus the evolution equations (4.41)–(4.44) hold with $\Omega^{\mu\nu} = \sigma^{\mu\nu}$.

C. *Fixed Mass Dirac Brackets*

Now we impose our last available gauge constraint to remove all arbitrary functions from the Hamiltonian system. We take

$$\begin{aligned}\phi_1 &= \pi^2 - f((1/2)\, S_{\mu\nu}S^{\mu\nu}) \approx 0, \\ \phi_2 &= x^0 - \tau \approx 0,\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}\{\phi_1, \phi_2\}' &= -2E \equiv -2G^{0\mu}\pi_\mu \\ &= -2(\pi^0 - (e/NM)\, S^{0\nu}F_{\nu\mu}\pi^\mu).\end{aligned} \tag{5.37}$$

The final Dirac brackets are thus

$$\{\xi, \eta\}^* = \{\xi, \eta\}' - \frac{1}{2E}\{\xi, \pi^2 - f(\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu})\}'\{x^0, \eta\}' + \frac{1}{2E}\{\xi, x^0\}'\{\pi^2 - f(\tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}), \eta\}'. \tag{5.38}$$

With a little labor, we arrive at the final system of brackets for a charged spherical top consistent with the constraints (5.16), (5.23) and (5.36):

1. $\{\pi^\mu, x^\nu\}^* = -G^{\nu\mu} - \dfrac{e}{EMN} S^{0\nu}F^{\mu\lambda}G_{\lambda\rho}\pi^\rho + \dfrac{1}{E} G^{0\mu}G^{\nu\lambda}\pi_\lambda ,$

2. $\{x^\mu, x^\nu\}^* = -\dfrac{S^{\mu\nu}}{MN} + \dfrac{1}{EMN}(G^{\mu\lambda}\pi_\lambda S^{0\nu} - G^{\nu\lambda}\pi_\lambda S^{0\mu}),$

3. $\{\pi^\mu, \pi^\nu\}^* = -eF^{\mu\alpha}G_\alpha{}^\nu + \dfrac{e}{E}(F^{\mu\alpha}G_{\alpha\beta}\pi^\beta G^{0\nu} - F^{\nu\alpha}G_{\alpha\beta}\pi^\beta G^{0\mu}),$

4. $$\{\pi^\mu, S^{\nu\lambda}\}^* = \frac{e}{NM}(F^{\mu\alpha}S_\alpha{}^\lambda\pi^\nu - F^{\mu\alpha}S_\alpha{}^\nu\pi^\lambda) + \frac{e}{ENM}(F^{\mu\alpha}G_{\alpha\beta}\pi^\beta(S^{0\nu}\pi^\lambda - S^{0\lambda}\pi^\nu) - G^{0\mu}(\pi^\nu S^{\lambda\alpha}F_{\alpha\beta}\pi^\beta - \pi^\lambda S^{\nu\alpha}F_{\alpha\beta}\pi^\beta)),$$

5. $$\{x^\mu, S^{\nu\lambda}\}^* = \frac{1}{NM}(S^{\mu\nu}\pi^\lambda - S^{\mu\lambda}\pi^\nu) - \frac{1}{ENM}G^{\mu\alpha}\pi_\alpha(S^{0\nu}\pi^\lambda - S^{0\lambda}\pi^\nu) + \frac{e}{EN^2M^2}S^{0\mu}(\pi^\nu S^{\lambda\alpha}F_{\alpha\beta}\pi^\beta - \pi^\lambda S^{\nu\alpha}F_{\alpha\beta}\pi^\beta),$$

6. $$\begin{aligned}\{S^{\mu\nu}, S^{\alpha\beta}\}^* &= S^{\mu\alpha}\left(g^{\nu\beta} - \frac{\pi^\nu\pi^\beta}{NM}\right) - S^{\mu\beta}\left(g^{\nu\alpha} - \frac{\pi^\nu\pi^\alpha}{NM}\right) \\ &+ S^{\alpha\nu}\left(g^{\mu\beta} - \frac{\pi^\mu\pi^\beta}{NM}\right) - S^{\beta\nu}\left(g^{\mu\alpha} - \frac{\pi^\mu\pi^\alpha}{NM}\right) \\ &+ \frac{e}{EN^2M^2}(\pi^\mu S^{0\nu} - \pi^\nu S^{0\mu})(\pi^\alpha S^{\beta\lambda}F_{\lambda\rho}\pi^\rho - \pi^\beta S^{\alpha\lambda}F_{\lambda\rho}\pi^\rho) \\ &- \frac{e}{EN^2M^2}(\pi^\alpha S^{0\beta} - \pi^\beta S^{0\alpha})(\pi^\mu S^{\nu\lambda}F_{\lambda\rho}\pi^\rho - \pi^\nu S^{\mu\lambda}F_{\lambda\rho}\pi^\rho),\end{aligned}$$

7. $$\{\Lambda^{\mu\nu}, x^\alpha\}^* = \frac{1}{M^2}\Lambda^\mu{}_\lambda(\pi^\nu G^{\alpha\lambda} - \pi^\lambda G^{\alpha\nu}) - \frac{1}{EM^2}\Lambda^\mu{}_\lambda G^{\alpha\beta}\pi_\beta(\pi^\nu G^{0\lambda} - \pi^\lambda G^{0\nu}) + \frac{1}{EMN}\Lambda^\mu{}_\lambda S^{0\alpha}\left[\frac{e}{M^2}\pi^\rho F_{\rho\sigma}(\pi^\lambda G^{\sigma\nu} - \pi^\nu G^{\sigma\lambda}) - f'S^{\lambda\nu}\right],$$

$$
\begin{aligned}
8.\quad \{\Lambda^{\mu\nu}, \pi^{\alpha}\}^* = & -\frac{\Lambda^{\mu}{}_{\lambda}G^{0\alpha}}{E}\left[\frac{e}{M^2}\pi^{\rho}F_{\rho\sigma}(\pi^{\lambda}G^{\sigma\nu} - \pi^{\nu}G^{\sigma\lambda}) - f'S^{\lambda\nu}\right] \\
& + \frac{e}{M^2}\Lambda^{\mu}{}_{\lambda}F^{\alpha}{}_{\beta}(\pi^{\lambda}G^{\beta\nu} - \pi^{\nu}G^{\beta\lambda}) \\
& + \frac{e}{EM^2}\Lambda^{\mu}{}_{\lambda}F^{\alpha\beta}G_{\beta\gamma}\pi^{\gamma}(\pi^{\nu}G^{0\lambda} - \pi^{\lambda}G^{0\nu}), \\
9.\quad \{\Lambda^{\mu\nu}, \Lambda^{\alpha\beta}\}^* = & -\frac{e}{NM^3}(\Lambda^{\mu\sigma}g^{\nu\rho} - \Lambda^{\mu\rho}g^{\nu\sigma})(\Lambda^{\alpha\tau}g^{\beta\gamma} - \Lambda^{\alpha\gamma}g^{\beta\tau})\,\pi_{\rho}\pi_{\gamma}F_{\sigma\tau} \\
& + \frac{1}{EM^2}\Lambda^{\mu}{}_{\eta}\Lambda^{\alpha}{}_{\xi}(\pi^{\eta}G^{0\nu} - \pi^{\nu}G^{0\eta}) \\
& \times\left[\frac{e}{M^2}\pi^{\lambda}F_{\lambda\sigma}(\pi^{\xi}G^{\sigma\beta} - \pi^{\beta}G^{\sigma\xi}) - f'S^{\xi\beta}\right] \\
& - \frac{1}{EM^2}\Lambda^{\alpha}{}_{\eta}\Lambda^{\mu}{}_{\xi}(\pi^{\eta}G^{0\beta} - \pi^{\beta}G^{0\eta}) \\
& \times\left[\frac{e}{M^2}\pi^{\lambda}F_{\lambda\sigma}(\pi^{\xi}G^{\sigma\nu} - \pi^{\nu}G^{\sigma\xi}) - f'S^{\xi\nu}\right], \\
10.\quad \{\Lambda^{\mu\nu}, S^{\alpha\beta}\}^* = & \,\Lambda^{\mu\alpha}g^{\nu\beta} - \Lambda^{\mu\beta}g^{\nu\alpha} + \frac{1}{M^2}\Lambda^{\mu}{}_{\lambda}(\pi^{\nu}g^{\rho\lambda} - \pi^{\lambda}g^{\rho\nu})(\pi^{\alpha}G^{\beta}{}_{\rho} - \pi^{\beta}G^{\alpha}{}_{\rho}) \\
& + \frac{1}{ENM}\Lambda^{\mu}{}_{\lambda}(\pi^{\alpha}S^{0\beta} - \pi^{\beta}S^{0\alpha}) \\
& \times\left[\frac{e}{M^2}\pi^{\rho}F_{\rho\sigma}(\pi^{\lambda}G^{\sigma\nu} - \pi^{\nu}G^{\sigma\lambda}) - f'S^{\lambda\nu}\right] \\
& + \frac{e}{ENM^3}\Lambda^{\mu}{}_{\lambda}(\pi^{\nu}G^{0\lambda} - \pi^{\lambda}G^{0\nu})[\pi^{\alpha}S^{\beta\gamma}F_{\gamma\delta}\pi^{\delta} - \pi^{\beta}S^{\alpha\gamma}F_{\gamma\delta}\pi^{\delta}].
\end{aligned}
\tag{5.39}
$$

For the convenience of the reader, we repeat the definitions of the symbols in (5.39):

$$
\begin{aligned}
f' &= \partial f(J^2)/\partial(J^2), \qquad J^2 = \tfrac{1}{2}S_{\mu\nu}S^{\mu\nu}, \\
N &= M - \frac{e}{2M}S_{\alpha\beta}F^{\alpha\beta}, \\
G^{\mu\nu} &= g^{\mu\nu} - \frac{e}{NM}S^{\mu\alpha}F_{\alpha}{}^{\nu}, \\
E &= G^{0\mu}\pi_{\mu}\,.
\end{aligned}
$$

The constraints (5.36) are now strongly zero and have vanishing star brackets. Thus wherever π^0 appears in (5.39), we may replace it by

$$
\pi^0 = (\boldsymbol{\pi}^2 + f((1/2)\,S^{\mu\nu}S_{\mu\nu}))^{1/2}. \tag{5.40}
$$

The problem is now to find a revealing form for the Hamiltonian; we are faced with the problem alluded to in Section 2.B. If we use $P^0 = \pi^0 + eA^0$ as the Hamiltonian, the explicit time dependence of our variables is complicated. The ideal solution would be to express all variables in terms of Newton–Wigner type coordinates (3.60) as done in the free case. The time-dependence of these coordinates would be entirely implicit and this would select a Hamiltonian unique except for time-independent canonical transformations. This Hamiltonian would correspond to the Erikson and Kolsrud [25] treatment of the Dirac electron (see also Dixon [26]). This appears to be a difficult task and we have not carried it out. In the static case, however, the choice $P^0 = \pi^0 + eA^0$ assigns no explicit time dependence to all variables and the corresponding algebra has time-independent Poisson brackets. It is clear, however, that the theory contains no intrinsic magnetic moment. This conclusion is reasonable because our classical theory applies to systems of particles only, without the introduction of antiparticles. We will discuss this point further in the final section.

6. Conclusion

There are many unanswered questions and new subjects for investigation which are suggested by the treatment we have given for the charged relativistic spherical top. This final section is devoted to a discussion of a number of these topics and to suggestions for directions in which further work might proceed.

The first question to which we address ourselves is the consistency of taking the electromagnetic potentials to act at the "position" x^μ of the particle. In the one-particle quantum theory with an external field, we would presumably be led to awkward functions $A^\mu(\mathbf{x}, t)$ where the $\mathbf{x}$'s would not commute. Furthermore, we should include the energy and momentum of the electromagnetic field in the Lagrangian in order to make the entire system Poincaré-invariant, with Poincaré group generators which are constants of the motion. Then, however,we would have to develop simultaneously the constrained canonical formalism for the electromagnetic field. (This is not too difficult, and is essentially equivalent to the method of Bjorken and Drell [27]. See also Dirac [3].) But if $A^\mu(x)$ is treated as a canonical coordinate, the variables x^μ have the meaning of *labels* on the infinite degrees of freedom of $A^\mu(x)$, rather than canonical coordinates. We conclude that we cannot have local interactions with a canonical electromagnetic field unless the x^μ can be interpreted as commuting four-vectors labeling the field.

One approach to resolving this problem would be that of Jordan and Mukunda [28]. They double the space of quantum-mechanical states by introducing antiparticles. The new degrees of freedom enable them to derive Abelian four-vector coordinates which are essentially a Foldy–Wouthuysen transformation [29] of the Newton–Wigner coordinates for the particle and antiparticle. It appears, however, that the currents which result from their method are local only for spin $\frac{1}{2}$. This limits the usefulness of the Jordan–Mukunda method for the introduction of local interactions.

It therefore appears that the concept of the quantum-mechanical interacting top is not well-defined unless we enlarge the quantization scheme to a second-quantized system including tops and antitops. It is possible that the methods of Weinberg [30] or of Bhabha [31] could be used. The present theory would then be a classical limit of the quantum electrodynamics of tops for slowly varying fields and with pair-production neglected. Even so, there may be difficulties for particles of spin greater than $\frac{1}{2}$, as illustrated by the work of Velo and Zwanziger [32].

[We remark that the finding of Velo and Zwanziger, that high-spin wave equations may violate causality in regions of intense fields, has in fact a classical counterpart in our theory. If the quantity

$$E = \pi^0 - e(NM)^{-1} S^{0\mu} F_{\mu\nu} \pi^\nu$$

vanishes at any point on the path of the top, then the corresponding four-velocity can become spacelike while the mechanical momentum remains timelike due to $\pi^\mu \pi_\mu = M^2 = \text{const}$. One could then find particle trajectories bending backwards in time. The present Hamiltonian formalism would then probably be inadequate to deal with the problem, since ordinary time would no longer be an appropriate label on the particle trajectory.]

There are many things which might be clarified by a better understanding of the quantum theory of spinning particles. In addition to clarifying the problems of electromagnetically interacting high-spin fields [33], some light might be shed on the question of the intrinsic magnetic moments; is $g = 1/j$ as indicated by Hagen and Hurley [34], or does it depend, for example, on the nature of the trajectory function?

Equation (3.75) raises other intriguing questions. Can it be interpreted as a field equation for a quantum field operator representing a family of particles lying on a trajectory? What are the properties of the Green's functions? Are they related to the Gribov calculus [35] or to the dual resonance models? How do fermions and bosons differ? And how could interactions and currents be introduced?

We mention a number of topics which we have left open even in the classical theory. One is to understand better how the intrinsic magnetic moment g enters the classical theory and to treat the $g \neq 0$ case in the Dirac–Hamiltonian formalism. There is also no reason why a formalism could not be developed to include higher multipoles beyond the pure dipole case we have treated. An extension of our methods to the asymmetric top would be interesting because the known elementary particles do not possess the same degeneracy as does the quantized spherical top; the asymmetric top could provide a description of elementary particles (e.g., the electron and muon could from a basis for a spin $\frac{1}{2}$ representation of an asymmetric top). Both the asymmetric top and the top with higher multipoles will probably involve Lagrangians which depend explicitly on $\Lambda^{\mu}{}_{\nu}$.

Since the spinless particle action has a clear interpretation as a classical path length, it would be interesting to find an analogous geometrical interpretation for our spinning action function and to develop the quantum theory using Feynman's path-integral approach [36].

Finally, our treatment of electromagnetic interactions should be extended to get the "true" Hamiltonian for time-dependent fields, and to reach a point in the quantum-mechanics (or quantum field theory), where the connection between our approach and treatments based on the Dirac equation is clearly apparent. In addition to electromagnetism, one might also attempt to include other interactions such as gravitation in our formalism.

APPENDIX A: IDENTITIES

We list here a number of identities which we have used repeatedly in carrying out calculations mentioned in the text. All Greek indices run from 0 to 3 and are raised and lowered by the metric $g^{\mu\nu}$ with diagonal elements $(1, -1, -1, -1)$. The totally antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ is chosen to have $\epsilon^{0123} = +1$.

1. *Antisymmetric Tensor Identities*

Let $S^{\mu\nu} = -S^{\nu\mu}$ and $T^{\mu\nu} = -T^{\nu\mu}$ be four-tensors and define the dual tensor

$$S^{*\mu\nu} = (1/2)\,\epsilon^{\mu\nu\alpha\beta}S_{\alpha\beta}\,. \tag{A.1}$$

Then

$$S^{**\mu\nu} = -S^{\mu\nu}, \tag{A.2}$$

$$S^{\alpha\beta}T_{\alpha\beta} = -S^{*\alpha\beta}T^{*}_{\alpha\beta}\,, \tag{A.3}$$

$$S^{\alpha\lambda}T_{\lambda}{}^{\beta} - T^{*\alpha\lambda}S_{\lambda}^{*\beta} = -(1/2)\,g^{\alpha\beta}(S^{\mu\nu}T_{\mu\nu}), \tag{A.4}$$

$$S^{\alpha\mu}T_{\mu\nu}S^{\nu\beta} = -(1/2)\,S^{\alpha\beta}(S^{\mu\nu}T_{\mu\nu}) - (1/4)\,T^{*\alpha\beta}(S^{\mu\nu}S^{*}_{\mu\nu}), \tag{A.5}$$

$$(S^{\mu\lambda}T_{\lambda}{}^{\nu} - T^{\mu\lambda}S_{\lambda}{}^{\nu})^{*} = S^{*\mu\lambda}T_{\lambda}{}^{\nu} - T^{\mu\lambda}S_{\lambda}^{*\nu} = S^{\mu\lambda}T_{\lambda}^{*\nu} - T^{*\mu\lambda}S_{\lambda}{}^{\nu}, \tag{A.6}$$

$$S^{*\alpha\lambda}S_{\lambda}{}^{\beta} = S^{\alpha\lambda}S_{\lambda}^{*\beta} = -(1/4)\,g^{\alpha\beta}(S^{\mu\nu}S^{*}_{\mu\nu}). \tag{A.7}$$

2. *Vector and Antisymmetric Tensor Identities*

Let P^μ be a four-vector and $S^{\mu\nu} = -S^{\nu\mu}$ an antisymmetric tensor and define

$$V^\mu = S^{\mu\nu} P_\nu , \tag{A.8}$$

$$W^\mu = S^{*\mu\nu} P_\nu = (1/2)\, \epsilon^{\mu\nu\lambda\sigma} S_{\nu\lambda} P_\sigma . \tag{A.9}$$

Then

$$\begin{aligned}(P^\mu V^\nu - P^\nu V^\mu)^* &= \epsilon^{\mu\nu\lambda\sigma} P_\lambda V_\sigma \\ &= W^\mu P^\nu - W^\nu P^\mu - P^2 S^{*\mu\nu}, \end{aligned} \tag{A.10}$$

$$\begin{aligned}(P^\mu W^\nu - P^\nu W^\mu)^* &= \epsilon^{\mu\nu\lambda\sigma} P_\lambda W_\sigma \\ &= P^\mu V^\nu - P^\nu V^\mu + P^2 S^{\mu\nu}, \end{aligned} \tag{A.11}$$

$$W^\mu W_\mu = -(1/2)\, P^\mu P_\mu S_{\alpha\beta} S^{\alpha\beta} - P^\alpha S_{\alpha\beta} S^{\beta\gamma} P_\gamma . \tag{A.12}$$

3. *Three-Vector Identities*

In three dimensions, we have the following if $S^i = (1/2)\, \epsilon^{ijk} S^{jk}$, $S^{ij} = \epsilon^{ijk} S^k$:

$$S^{ij} P^j = (\mathbf{P} \times \mathbf{S})^i, \tag{A.13}$$

$$\begin{aligned} P^i S^{jk} P^k - P^j S^{ik} P^k &= P^i \epsilon^{jkl} P^k S^l - P^j \epsilon^{ikl} P^k S^l \\ &= \epsilon^{ijk} P^k (\mathbf{P} \cdot \mathbf{S}) - \epsilon^{ijk} S^k (\mathbf{P} \cdot \mathbf{P}). \end{aligned} \tag{A.14}$$

4. *Identities Following From the Differential Equations*

From Eqs. (1.27)–(1.32) and the constraints (1.36)–(1.38), we find the following relations

$$\begin{aligned} a_1 a_2 a_3 + 2a_3^2 - 2a_1^2 a_4 &= (M^2/L_0 L_3^2)[2a_3 L_2 - a_1 a_3 L_3 - a_1 a_4 L_4] \\ &= -(M^2 L_1/L_0 L_3^3)[2a_1 L_2 - a_1^2 L_3 - (a_3 + (1/2)\, a_1 a_2)\, L_4], \end{aligned} \tag{A.15}$$

$$\begin{aligned} (1/2)\, L_0 L_3 &= 2L_1 L_2 + L_3(a_3 L_3 + a_4 L_4) \\ &= 4L_1 L_2 - L_3(a_1 L_1 + a_2 L_2) \\ &= L_1(2L_2 - a_1 L_3) + L_2(2L_1 - a_2 L_3) \\ &= (L_1/L_3)[L_3(2L_2 - a_1 L_3) + (1/2)\, L_4(2L_1 - a_2 L_3)] \\ &= (1/L_0)[(1/2)\, J^2 L_1 + M^2 L_2], \end{aligned} \tag{A.16}$$

$$\begin{aligned} M^2 = P^\mu P_\mu &= 4a_1 L_1^2 + 8a_3 L_1 L_3 - 2a_2 a_3 L_3^2 + 4a_1 a_4 L_3^2 \\ &= L_0(2L_1 - a_2 L_3 - a_4 L_3 L_4/L_2) \\ &= (L_0/L_2)(2L_1 L_2 - L_3(a_2 L_2 + a_4 L_4)) \\ &= (L_0 L_3/L_2)(a_1 L_1 + a_3 L_3) \\ &= -(1/2)(L_0 L_3/L_2)(u_\mu P^\mu), \end{aligned} \tag{A.17}$$

$$\begin{aligned} J^2 = (1/2)\, S_{\mu\nu}S^{\mu\nu} &= 8a_2L_2^2 + 16a_3L_2L_3 - 4a_1a_3L_3^2 \\ &\quad - 2a_4L_4(a_2L_4 + 4a_1L_3 - 16L_2) \\ &= 2L_0(2L_2 - a_1L_3 + a_4L_4L_3/L_1) \\ &= (2L_0/L_1)(2L_1L_2 - a_1L_1L_3 + a_4L_4L_3) \\ &= (2L_0L_3/L_1)(a_2L_2 + a_3L_3 + 2a_4L_4) \\ &= -(1/2)(L_0L_3/L_1)(\sigma_{\mu\nu}S^{\mu\nu}). \end{aligned} \tag{A.18}$$

Finally, we give the algebraic equation which the Lagrangian must satisfy when the dependence on the L_i is expressed in terms of M^2 and J^2:

$$\begin{aligned} 0 = {} & L_0^8(a_2^2 + 16a_4) \\ & + L_0^6M^2(-2a_1a_2^2 + 4a_2a_3 - 48a_1a_4) \\ & + L_0^6J^2(-a_2^3 - 16a_2a_4) \\ & + L_0^4M^4(a_1^2a_2^2 + 4a_3^2 - 8a_1a_2a_3 + 48a_1^2a_4) \\ & + L_0^4M^2J^2(-a_1a_2^3 - 8a_2^2a_3 - 4a_1a_2a_4 - 80a_3a_4) \\ & + L_0^4J^4((1/4)\, a_2^4 + 2a_2^2a_4 - 32a_4^2) \\ & + L_0^2M^6(4a_1^2a_2a_3 - 8a_1a_3^2 - 16a_1^3a_4) \\ & + L_0^2M^4J^2(-4a_1a_2^2a_3 - 20a_2a_3^2 + 20a_1^2a_2a_4 + 88a_1a_3a_4) \\ & + L_0^2M^2J^4(a_2^3a_3 + 4a_2a_3a_4 - 8a_1a_2^2a_4 - 80a_1a_4^2) \\ & + L_0^2J^6(a_2^3a_4 + 16a_2a_4^2) \\ & + M^8(4a_1^2a_3^2) \\ & + M^6J^2(-4a_1a_2a_3^2 - 16a_3^3 - 8a_1^2a_3a_4) \\ & + M^4J^4(a_2^2a_3^2 + 48a_3^2a_4 + 8a_1a_2a_3a_4 + 4a_1^2a_4^2) \\ & + M^2J^6(-2a_2^2a_3a_4 - 4a_1a_2a_4^2 - 48a_3a_4^2) \\ & + J^8(a_2^2a_4^2 + 16a_4^3). \end{aligned} \tag{A.19}$$

APPENDIX B: SINGLE-PARTICLE COORDINATE OPERATORS

The classic work on the subject of single-particle coordinate operators for spinning bodies is that of Pryce [18]. As an aid to the physical interpretation of our theory of the spherical top, we give a summary of Pryce's results. The three interesting types of coordinates for spinning particles are

(a) *Rest-frame center of momentum* is the center of momentum calculated when the body as a whole is at rest, then Lorentz-transformed to an arbitrary frame. Classically, one would write

$$\mathbf{x}_a = \frac{\sum_i E_0{}^i \mathbf{x}_i}{\sum_i E_0{}^i} \tag{B.1}$$

where $E_0{}^i$ is the energy of the ith component particle computed when the total momentum is zero. $\mathbf{x}_a$ transforms as the space part of a four-vector. The corresponding spin matrix obeys the covariant constraint

$$V^\mu = S_a{}^{\mu\nu} P_\nu = 0. \tag{B.2}$$

(b) *Center of momentum* is the center of momentum of a body computed in the frame of an arbitrary observer. Classically,

$$\mathbf{x}_b = \frac{\sum_i E^i \mathbf{x}_i}{\sum_i E^i} \tag{B.3}$$

where E^i is the total energy of the ith particle in the observer's frame. This coordinate coincides with (a) only in the rest frame; in other frames, the relativistic mass-increases of diametrically opposite particles will differ, causing $\mathbf{x}_b$ to shift away from $\mathbf{x}_a$. $\mathbf{x}_b$ is defined in a frame-dependent manner, and so is not part of a four-vector. The corresponding spin matrix satisfies

$$S_b^{0\mu} = 0. \tag{B.4}$$

(c) *Newton–Wigner Coordinate* [23] is the name now usually given to the peculiar weighted average of (a) and (b) noticed earlier by Pryce [22]:

$$\mathbf{x}_c = (M\mathbf{x}_a + E\mathbf{x}_b)/(M + E). \tag{B.5}$$

M and E are the total invariant mass and energy of the system, respectively. $\mathbf{x}_c$ is the only single-particle coordinate variable which commutes with all of its own components. $\mathbf{x}_c$ is frame-dependent and is not a four-vector but its commutative properties have given it an important place in quantum theory. The corresponding spin constraint is

$$MS_c^{0\mu} - S_c^{\mu\nu} P_\nu = 0. \tag{B.6}$$

Our notation has the following relation to that of Pryce:

Here	Pryce
(a) $\mathbf{x} = \mathbf{x}_a$	(d) $\mathbf{X}$
(b) $\mathbf{x}_b$	(c) $\mathbf{q}$
(c) $\mathbf{q} = \mathbf{x}_c$	(e) $\tilde{\mathbf{q}}$
(a) $\mathbf{S} = \mathbf{S}_a$	(d) $\boldsymbol{\Sigma}$
(b) $\mathbf{S}_b$	(c) $\mathbf{S}$
(c) $\mathbf{J} = \mathbf{S}_c$	(e) $\tilde{\mathbf{S}}$

Since only three components of $S^{\mu\nu}$ are independent in each case, we will generally use the three-vector portion:

$$S^i = (1/2)\,\epsilon^{ijk}S^{jk}. \tag{B.7}$$

The simplest commutation relations are those of the variables (c), which obey

$$\begin{aligned} i[x_c^{\,i}, x_c^{\,j}] &= i[P^i, P^j] = 0,\\ i[x_c^{\,i}, S_c^{\,j}] &= i[P^i, S_c^{\,j}] = 0,\\ i[S_c^{\,i}, S_c^{\,j}] &= -\epsilon^{ijk}S_c^{\,k},\\ i[P^i, x_c^{\,j}] &= \delta^{ij}. \end{aligned} \tag{B.8}$$

With the Hamiltonian

$$P^0 = H = (\mathbf{P}^2 + m^2)^{1/2}, \tag{B.9}$$

the Poincaré algebra is generated by P^μ and the operators

$$\begin{aligned} M^{ij} &= x_c^{\,i}P^j - x_c^{\,j}P^i + \epsilon^{ijk}S_c^{\,k},\\ M^{0i} &= tP^i - (1/2)(Hx_c^{\,i} + x_c^{\,i}H) + \epsilon^{ijk}S_c^{\,j}P^k/(H+m). \end{aligned} \tag{B.10}$$

The commutation relations of the operators (a) and (b) can be deduced from their expressions as functions of the operators (c). The following equations express the relations among the operators:

$$\begin{aligned} \mathbf{x}_a &= \mathbf{x}_c - \frac{\mathbf{P}\times\mathbf{S}_c}{m(H+m)} = \mathbf{x}_b - \frac{\mathbf{P}\times\mathbf{S}_b}{m^2},\\ \mathbf{S}_a &= \frac{H}{m}\mathbf{S}_c - \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S}_c)}{m(H+m)} = \frac{H^2}{m^2}\mathbf{S}_b - \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S}_b)(H-m)}{m^2(H+m)}. \end{aligned} \tag{B.11}$$

$$\begin{aligned} \mathbf{x}_b &= \mathbf{x}_c + \frac{\mathbf{P}\times\mathbf{S}_c}{H(H+m)} = \mathbf{x}_a + \frac{\mathbf{P}\times\mathbf{S}_a}{H^2},\\ \mathbf{S}_b &= \frac{m}{H}\mathbf{S}_c + \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S}_c)}{H(H+m)} = \frac{m^2}{H^2}\mathbf{S}_a + \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S}_a)(H-m)}{H^2(H+m)}. \end{aligned} \tag{B.12}$$

$$\begin{aligned} \mathbf{x}_c &= \mathbf{x}_a - \frac{\mathbf{S}_a\times\mathbf{P}}{H(H+m)} = \mathbf{x}_b + \frac{\mathbf{S}_b\times\mathbf{P}}{m(H+m)},\\ \mathbf{S}_c &= \frac{m}{H}\mathbf{S}_a + \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S}_a)}{H(H+m)} = \frac{H}{m}\mathbf{S}_b - \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{S}_b)}{m(H+m)}. \end{aligned} \tag{B.13}$$

By making use of the constraints (B.2), (B.4), and (B.6), one may show that the Lorentz group generators have the form

$$\begin{aligned} M^{ij} &= x_\alpha{}^i P^j - x_\alpha{}^j P^i + S_\alpha^{ij}, \\ M^{0i} &= tP^i - (1/2)(Hx_\alpha{}^i + x_\alpha{}^i H) + S_\alpha^{0i}, \end{aligned} \tag{B.14}$$

where $\alpha = a, b, c$.

The interesting commutation relations are those of the x's,

$$\begin{aligned} i[x_a{}^i, x_a{}^j] &= -\frac{S_a^{ij}}{m^2} + \frac{P^i S_a^{0j} - P^j S_a^{0i}}{m^2 H} \\ &= -\frac{S_a^{ij}}{H^2} - \frac{\epsilon^{ijk} P^k (\mathbf{P} \cdot \mathbf{S}_a)}{m^2 H^2}, \end{aligned} \tag{B.15a}$$

$$i[x_b{}^i, x_b{}^j] = S_b^{ij}/H^2, \tag{B.15b}$$

$$i[x_c{}^i, x_c{}^j] = 0, \tag{B.15c}$$

and those of the x's with M^{0i}:

$$i[M^{0i}, x_a{}^j] = t\delta^{ij} - \frac{1}{2}\left(x_a{}^i \frac{P^j}{H} + \frac{P^j}{H} x_a{}^i\right), \tag{B.16a}$$

$$i[M^{0i}, x_b{}^j] = t\delta^{ij} - \frac{1}{2}\left(x_b{}^i \frac{P^j}{H} + \frac{P^j}{H} x_b{}^i\right) - \epsilon^{ijk} S_b{}^k/H, \tag{B.16b}$$

$$i[M^{0i}, x_c{}^j] = t\delta^{ij} - \frac{1}{2}\left(x_c{}^i \frac{P^j}{H} + \frac{P^j}{H} x_c{}^i\right) - \frac{\epsilon^{ijk} S_c{}^k}{H + m} - \frac{P^j \epsilon^{ikl} S_c{}^k P^l}{H(H + m)^2}. \tag{B.16c}$$

It is evident that $\mathbf{x}_a$ is a noncommuting four-vector, $\mathbf{x}_c$ is a commuting non-four-vector, and $\mathbf{x}_b$ is neither commuting nor a four-vector.

We note that the commuting four-vector coordinates of Jordan and Mukunda [28] can be written

$$\mathbf{x} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{x}_b + \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix} \frac{m}{H^2} \mathbf{S}_a . \tag{B.17}$$

In Section 3, we give a prescription for augmenting the Pryce system of operators $\mathbf{x}$, $\mathbf{S}$ and $\mathbf{P}$ by including the angle operators R^{ij} in the algebra.

APPENDIX C: Trajectories and the Differential Equations

In this Appendix, we develop a method for finding solutions of the differential equations (1.36)–(1.38) which we require the Lagrangian of the spherical top to satisfy. To solve the equations, we can begin by specifying the boundary conditions on the surface $a_3 = a_4 = 0$ corresponding to a given trajectory function. In particular, we give the function

$$L_0(a_1, a_2, 0, 0) = (a_1)^{1/2} \mathscr{L}(\xi, 0, 0).$$

This function is known once the trajectory function $M^2 = f(J^2)$ is given. In fact, from (1.51), we have

$$2/\xi = 2a_1/a_2 = (dJ/dM)^2 = \psi(J^2) \tag{C.1}$$

where $\psi(J^2)$ is a known function of J^2. Therefore by inverting this equation, we find a function $\phi(\xi) = J^2$ so that $M^2 = f(\phi(\xi))$. From Eq. (1.53), we have

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial \xi} &= \frac{J}{2(2\xi)^{1/2}} = \frac{1}{2}\left(\frac{\phi(\xi)}{2\xi}\right)^{1/2} \\ \mathscr{L}(\xi) &= -M(\xi) + (1/2)(2\xi\phi(\xi))^{1/2}. \end{aligned} \tag{C.2}$$

Therefore we assume that $L_0(a_1, a_2, 0, 0)$ is known once the desired trajectory is specified.

We now introduce new homogeneous coordinates x_i, $i = 1, \ldots, 5$ such that

$$a_i = x_i/x_5, \qquad i = 1, 2, 3, 4.$$

In terms of these variables, we find

$$\begin{aligned} \frac{\partial L_0}{\partial x_i} &= \frac{1}{x_5} L_i = \lambda_i, \qquad i = 1, 2, 3, 4 \\ \frac{\partial L_0}{\partial x_5} &= -\frac{1}{x_5^2} \sum_{i=1}^{4} x_i L_i = -\frac{1}{x_5} \sum_{i=1}^{4} a_i L_i \equiv \lambda_5 . \end{aligned} \tag{C.3}$$

Thus Eqs. (1.36) and (1.37) may be written

$$\begin{aligned} 2\lambda_1\lambda_2 + \lambda_3\lambda_5 &= 0, \\ \lambda_1\lambda_4 - 2\lambda_2\lambda_3 &= 0. \end{aligned} \tag{C.4}$$

In x-space we now consider a family Σ of surfaces $x^i(\sigma, \tau)$ determined by the differential equations

$$\partial x_i/\partial\sigma = v_i(x), \qquad \partial x_i/\partial\tau = w_i(x), \tag{C.5}$$

where

$$v_1(x) = \lambda_4, \qquad v_2(x) = -2\lambda_3, \qquad v_3(x) = -2\lambda_2,$$
$$v_4(x) = \lambda_1, \qquad v_5(x) = 0$$

and

$$w_1(x) = 2\lambda_2, \qquad w_2(x) = 2\lambda_1, \qquad w_3(x) = \lambda_5,$$
$$w_4(x) = 0, \qquad w_5(x) = \lambda_3.$$

A surface of the family lies on an $L_0 =$ constant hypersurface. If we express L_0 in terms of $x_i(\sigma, \tau)$, we find

$$\frac{\partial L_0}{\partial \sigma} = \sum_1^5 \lambda_i v_i = 0,$$

$$\frac{\partial L_0}{\partial \tau} = \sum_1^5 \lambda_i w_i = 0,$$

by virtue of (C.4). We also find

$$\begin{aligned}\frac{\partial \lambda_1[x(\sigma, \tau)]}{\partial \sigma} &= \sum_1^5 v_i \frac{\partial^2 L_0}{\partial x_i\, \partial x_1} \\ &= \lambda_4 \frac{\partial^2 L_0}{\partial {x_1}^2} + \lambda_1 \frac{\partial^2 L_0}{\partial x_1\, \partial x_4} - 2\lambda_2 \frac{\partial^2 L_0}{\partial x_3\, \partial x_1} - 2\lambda_3 \frac{\partial^2 L_0}{\partial x_2\, \partial x_1} \\ &= \frac{\partial}{\partial x_1} (\lambda_1 \lambda_4 - 2\lambda_2 \lambda_3) \equiv 0.\end{aligned}$$

Similarly,

$$\partial \lambda_i / \partial \sigma = \partial \lambda_i / \partial \tau = 0 \tag{C.6}$$

for all i. Obviously we then have

$$\partial v_i/\partial \sigma = \partial w_i/\partial \sigma = \partial v_i/\partial \tau = \partial w_i/\partial \tau = 0.$$

Notice that $\partial v_i/\partial \tau = \partial w_i/\partial \sigma$ is trivially satisfied and therefore (C.5) is completely integrable, or else (C.5) would not define a surface. Moreover,

$$\partial^2 x_i/\partial \sigma^2 = \partial^2 x_i/\partial \tau^2 = \partial^2 x_i/\partial \sigma\, \partial \tau = 0, \tag{C.7}$$

and the x_i must be linear in σ, τ so that

$$x_i = \rho_i + v_i \sigma + w_i \tau \tag{C.8}$$

where ρ_i, v_i and w_i do not depend on σ, τ. Notice that $x_i(0, 0) = \rho_i$, so that

$$v_i(x) = v_i(\rho), \qquad w_i(x) = w_i(\rho)$$

since v_i, w_i are independent of σ, τ.

Through each point in x-space, there is only one surface belonging to our family of surfaces. Consider the 3-dimensional manifold $\mathscr{M}$ defined by $x_3 = x_4 = 0$. Through each point P of $\mathscr{M}$, there is exactly one surface $\Sigma(P)$ of the family. We now choose the point ρ_i to lie on $\mathscr{M}$ so that $\rho_3 = \rho_4 = 0$. In this case, if we know $L_0(\rho_1, \rho_2, 0, 0, \rho_5)$, we know $\lambda_1, \lambda_2, \lambda_5$ and, from (C.6), we know also λ_3, λ_4. Hence we know $v_j(\rho)$, $w_j(\rho)$ and the corresponding surface through P is determined. Now suppose we are given a generic point Q, not necessarily on $\mathscr{M}$, with generic coordinates x_i. If

$$x_i = \rho_i + \sigma v_i(\rho) + \tau w_i(\rho) \tag{C.9}$$

for some σ, τ, then Q is on $\Sigma(P)$. But then, since L_0 is constant on $\Sigma(P)$, we have $L_0(Q) = L_0(P)$.

The solution procedure now runs as follows: We give x_i and solve the nonlinear system (C.9) of 5 equations in the 5 unknowns, $\rho_1, \rho_2, \rho_5, \sigma$ and τ, in terms of the x_i. Therefore (C.9) defines $\rho_1(x)$, $\rho_2(x)$, $\rho_5(x)$, $\sigma(x)$ and $\tau(x)$. Finally, we recall that

$$L_0(\rho_1, \rho_2, 0, 0, \rho_5) = (\rho_1/\rho_5)^{1/2}\, \mathscr{L}(\rho_2/\rho_1, 0, 0)$$

is explicitly given by specifying the trajectory function. Therefore

$$L_0(x_1, x_2, x_3, x_4, x_5) = L_0(x) = (\rho_1(x)/\rho_5(x))^{1/2}\, \mathscr{L}(\rho_2(x)/\rho_1(x), 0, 0)$$

is the desired solution. Since Eq. (C.9) may have more than one solution, there may be different Lagrangians corresponding to the same trajectory.

We now turn to a discussion of the implicit relation between M^2 and J^2. From (A.17) and (A.18), we have

$$\begin{aligned} M^2 &= (L_0L_3/L_2)(a_1L_1 + a_3L_3) = L_0(\lambda_3/\lambda_2)(x_1\lambda_1 + x_3\lambda_3), \\ J^2 &= 2(L_0L_3/L_1)(a_2L_2 + a_3L_3 + 2a_4L_4) = 2L_0(\lambda_3/\lambda_1)(x_2\lambda_2 + x_3\lambda_3 + 2x_4\lambda_4). \end{aligned}$$

Consider now a surface of the family Σ through the point P. Setting

$$\begin{aligned} M^2(\sigma, \tau) &= M^2(x^i[\sigma, \tau]), \\ J^2(\sigma, \tau) &= J^2(x^i[\sigma, \tau]), \end{aligned}$$

we find that

$$\begin{aligned} \frac{\partial M^2}{\partial \sigma} &= L_0 \frac{\lambda_3}{\lambda_2} \left(\frac{\partial x_1}{\partial \sigma} \lambda_1 + \frac{\partial x_3}{\partial \sigma} \lambda_3 \right) \\ &= L_0 \frac{\lambda_3}{\lambda_2} (\lambda_4\lambda_1 - 2\lambda_2\lambda_3) = 0, \end{aligned}$$

and similarly

$$\partial M^2/\partial \tau = \partial J^2/\partial \sigma = \partial J^2/\partial \tau = 0.$$

(Recall that L_0 and λ_i are constant with respect to σ, τ.) Therefore on $\Sigma(P)$, M^2 and J^2 are constants. But M^2 and J^2 are functions of the reparametrization-invariant variables

$$\xi = a_2/a_1, \qquad \eta = a_3/a_1^2, \qquad \theta = a_4/a_1^2.$$

On $\Sigma(P)$, ξ, η and θ are also functions of σ, τ. One can then let $\sigma_1 \equiv \sigma$, $\sigma_2 \equiv \tau$ and compute

$$\frac{\partial M}{\partial \sigma_i} = \frac{\partial M}{\partial \xi}\frac{\partial \xi}{\partial \sigma_i} + \frac{\partial M}{\partial \eta}\frac{\partial \eta}{\partial \sigma_i} + \frac{\partial M}{\partial \theta}\frac{\partial \theta}{\partial \sigma_i} = 0, \tag{C.10}$$

$$\frac{\partial J}{\partial \sigma_i} = \frac{\partial J}{\partial \xi}\frac{\partial \xi}{\partial \sigma_i} + \frac{\partial J}{\partial \eta}\frac{\partial \eta}{\partial \sigma_i} + \frac{\partial J}{\partial \theta}\frac{\partial \theta}{\partial \sigma_i} = 0. \tag{C.11}$$

But now Eqs. (C.10) and (C.11) can be considered as two identical linear systems in three homogeneous unknowns, $\partial M/\partial\xi$, $\partial M/\partial\eta$, $\partial M/\partial\theta$ and $\partial J/\partial\xi$, $\partial J/\partial\eta$, $\partial J/\partial\theta$ respectively. The coefficients are the same in each case: $\partial\xi/\partial\sigma_i$, $\partial\eta/\partial\sigma_i$, $\partial\theta/\partial\sigma_i$. Therefore the corresponding solutions are proportional and we have

$$\partial M/\partial\xi = \gamma\, \partial J/\partial\xi,$$

etc. We conclude that the Jacobians vanish,

$$\frac{\partial M}{\partial \xi}\frac{\partial J}{\partial \eta} - \frac{\partial M}{\partial \eta}\frac{\partial J}{\partial \xi} = 0, \qquad \text{cyclic in } \xi, \eta, \theta.$$

But this means that there exists a functional relationship

$$M^2 = f(J^2),$$

with the same function $f(J^2)$ as was given on the manifold $\mathscr{M}$, but which is valid at any point Q, insofar as there is always a surface $S(Q)$ through Q and meeting $\mathscr{M}$ at some point P.

A particular choice for $\mathscr{L}$ is

$$\mathscr{L} = (A - B\xi)^{1/2}$$

so

$$L_0 = (Aa_1 - Ba_2)^{1/2}.$$

Using the techniques given here, one can reconstruct the general L_0 with the result

$$L_0^2 = (1/2)(Aa_1 - Ba_2 + [(Aa_1 - Ba_2)^2 - 8B(Aa_3 - 2Ba_4)]^{1/2})$$

where

$$BM^2 - (1/2)\,AJ^2 = AB$$

is the trajectory. This choice of L_0 can be seen after a short calculation to satisfy the differential equations (1.36) and (1.37). The actual construction of L_0 for a given trajectory is difficult to carry out in all but the simplest cases.

The trajectory function can be viewed as the *envelope* of a family of curves in the M–J plane specified by $\mathscr{L}$ and parametrized by ξ. A simpler example of an envelope than those given so far is provided by setting $\mathscr{L}(\xi) = \xi/2$ in Eq. (C.2). We then find a family of straight lines in the M–J plane given by the equation

$$(\xi/2) + M - (\xi/2)^{1/2} J = 0.$$

The ξ-derivative is then

$$(1/2)(1 - J(2\xi)^{-1/2}) = 0.$$

When one eliminates ξ from these two equations, one finds that the envelope of the family of lines is just the parabola

$$J^2 = 4M.$$

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