Earthquake Quantization

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In this homage to Einstein's 144th birthday we propose a novel quantization prescription, where the paths of a path-integral are not random, but rather solutions of a geodesic equation in a random background. We show that this change of perspective can be made mathematically equivalent to the usual formulations of non-relativistic quantum mechanics. To conclude, we comment on conceptual issues, such as quantum gravity coupled to matter and the quantum equivalence principle.

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I. PROLOGUE

Given the occasion, it seems appropriate to start with an analogy in terms of a thought experiment:
Imagine that you are sitting on a balloon and observing the motion of few people walking on a square. Surprisingly, you do not see a smooth flow but rather a random zigzag. When trying to make sense of this crazy motion you might conclude that the people are

a) actually drunk and are thus moving strangely on their own cause;

b) sober, but they have a hard time trying to move steadily since they are suffering from a massive earthquake, which you in your safe balloon, can not perceive directly.

The first alternative a) of this analogy corresponds to intrinsic random motion of the path integral (PI) quantization [6]. The second alternative b) corresponds to random motion caused by a random background.

The idea of this short comment is that the paths of the PI could actually not be random by themselves, instead they are the result of a law of motion. Different paths are imposed by different causes just like the poor people in the above analogy. We will explore whether it is possible to cast the sum over random paths in the form of a sum over "classical" paths driven by random causes. The most familiar cause for deflection of paths is space-time curvature arising from deviations from the flat classical background metric. We will show that non-relativistic quantum mechanics can be expressed as weighted sum over "quantum-geodesics" resulting from fluctuations of the background metric. Because of the above analogy we call this approach earthquake quantization (EQQ) [21].

II. INTRODUCTION

A. What is a trajectory?

A classical trajectory (TJ) of a point particle associates to every given time a position. Throughout the history of physics, this simple concept did suffer several serious contraventions such as: Particles are not point-like, time and position are related themselves, and worst of all, the information one can get about positions and momenta is limited by the laws of quantum mechanics (QM). It is therefore intriguing to re-examine the concept of trajectories within the framework of quantum mechanics. As it turns out, different formulations of QM have a very different interpretation of "trajectories". Nevertheless, independent of the formulation, they all have to make sense of the fact that particle detectors "click" at certain times and certain positions, which is the essence of the above definition of trajectory. Typical interpretations of trajectories in different formulations of QM are the

- Standard [1, 2] (e.g. Kopenhagen): A TJ has NO intrinsic meaning, it is only a result of a macroscopic statistical interpretation of the evolution of the wave-function $\psi(x,t)$. Whether and how this wave-function behaves under measurement (collapse, many worlds, ...) is subject of an additional discussion beyond our concern.
- Path integrals [3] (PI): TJ do "exist", but they are neither determined by dynamical equations nor reduced to a single path between the initial and final position of the particle. Instead, the weighted sum over all these random paths dictates the evolution of the wave-function $\psi(x,t)$.
- De Broglie-Bohm theory [4, 5] (dBB): TJs do have intrinsic meaning. There exists a single path between the initial and final position, which is determined by an equation of motion. This equation of motion, however, is modified by the QM wave function ψ, called the pilot wave. The characteristic uncertainty of QM arises from the condition that the probability density is related to the statistical density of possible paths.

In this note we extend this list by a new item:

- Earthquake quantization: The TJs in the proposed EQQ share properties of both PI and dBB. On the one side a weighted sum over paths dictates the evolution of a wave-function, like in PI. On the other side these paths are determined by an equation of motion, similar to the dBB. This hybrid nature of the EQQ is shown in the schematic figure 1.

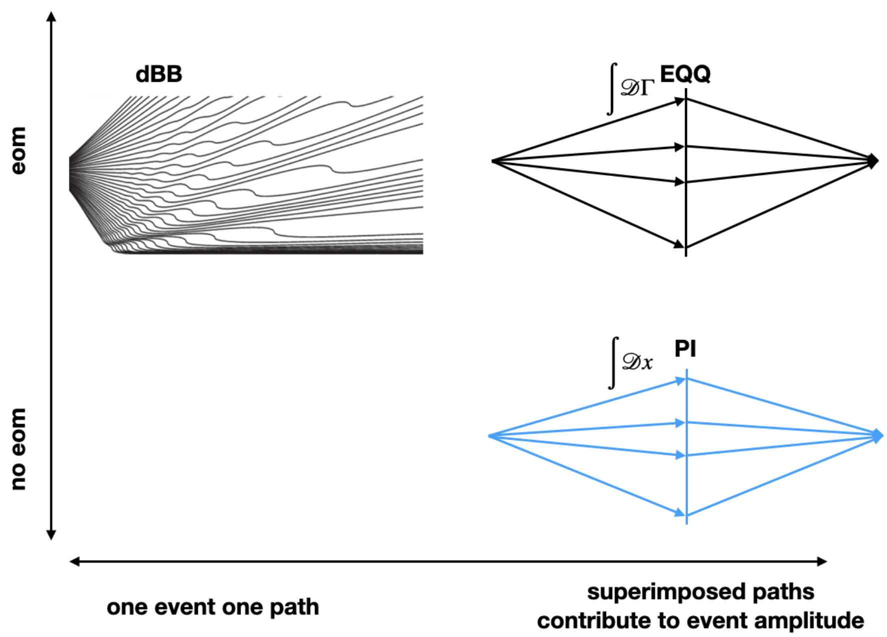

FIG. 1. Conceptual map, showing the similarities and differences between dBB, PI, and EQQ. Blue paths are "drunken" (no equation of motion (eom)), black paths are "sober" (with an eom).

B. Construction of the non-relativistic path integral

The PI of the non-relativistic point particle can be built from the infinitesimal propagator

$$\begin{aligned} K_{(0)}(\vec{x}_i,0;\vec{x}_f,\Delta t) &= \left(\frac{m}{2i\pi\hbar\Delta t}\right)^{3/2} \exp\left[\frac{im}{\hbar}\int_0^{\Delta t} dt' \frac{\dot{\vec{x}}^2}{2}\right] \\ &= \left(\frac{m}{2i\pi\hbar\Delta t}\right)^{3/2} \exp\left[\frac{i}{\hbar}S^{\ell}_{if}(\Delta t)\right], \end{aligned} \tag{1}$$

where the integral in the exponential is trivial, because the particle in this infinitesimal construction travels on a straight line

$$S^{\ell}_{if}(\Delta t) = \frac{m(\vec{x}_f-\vec{x}_i)^2}{2\Delta t}. \tag{2}$$

The transition to finite time intervals is obtained by the recursive use of the infinitesimal Kolmogorov relation

$$K_{(0)}(\vec{x}_i, 0; \vec{x}_f, 2\Delta t) = \int d^3x_1 K_{(0)}(\vec{x}_i, 0; \vec{x}_1, \Delta t) K_{(0)}(\vec{x}_1, \Delta t; \vec{x}_f, 2\Delta t). \tag{3}$$

This gives

$$\left(\frac{m}{2i\pi\hbar t}\right)^{3/2} \exp\left[\frac{i}{\hbar} S^{\ell}_{if}(t)\right] = K(x_i, 0; x_f, t) = \int \tilde{\mathcal{D}}x \exp\left[\frac{i}{\hbar} S\right], \tag{4}$$

where the action on the right hand side results from an integration along the "drunken" quantum path

$$S = \int_0^t dt' \frac{m\dot{\vec{x}}^2(t')}{2}. \tag{5}$$

The measure in (4) is defined as

$$\int \tilde{\mathcal{D}}x \equiv \left(\frac{m}{2i\hbar\pi\Delta t}\right)^{3/2} \Pi_{j=1}^N \left(\int d^3x_j \left(\frac{m}{2i\pi\hbar\Delta t}\right)^{3/2}\right), \tag{6}$$

with $t_f - t_i = N \cdot \Delta t$. This measure differs from the more common definition with only d^3x due to the normalization factor

$$\tilde{\mathcal{D}}x = \left(\frac{m}{2i\pi\hbar\Delta t}\right)^{\frac{3(N+1)}{2}} \mathcal{D}x. \tag{7}$$

This definition is useful, since we want to express the right hand side of (4) in terms of an exponential of the action (5) without further normalizations factors.

C. Metric and geodesics in the non-relativistic limit

For an arbitrary metric field $g_{\mu\nu}(x)$ we can define deviations $\delta g_{\mu\nu}(x)$ from the flat Minkowski metric $\eta_{\mu\nu}$ as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \delta g_{\mu\nu}(x). \tag{8}$$

The motion of a point particle in such a background is given by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu_{\ \alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}. \tag{9}$$

In the non-relativistic Newton limit this law of motion simplifies to

$$\ddot{x}^j = -\Gamma^j_{\ 00} \equiv -\Gamma^j. \tag{10}$$

For shorter notation in our purely non-relativistic analysis we defined the relevant terms of the affine connection as a tri-dimensional vector $\vec{\Gamma} \equiv \hat{\mathbf{e}}_j \Gamma^j_{\ 00}$ that plays the role of local and instantaneous force in the equation of motion (10). This connection is given in terms of derivatives of the metric (8) as

$$\vec{\Gamma} = \vec{\partial}\frac{c^2}{2}\delta g_{00}(x). \tag{11}$$

D. Quantum and gravity

A lot has been said, but little is yet understood about the interplay between gravity and the laws of quantum mechanics. Even more, in quantum-gravity, which is by itself a very hard topic [7], things get worse when one tries to include matter into the picture [11]. There are numerous candidates for providing a valid theory of quantum gravity, these formulations work with different fundamental degrees of freedom, such as the metric field $g_{\mu\nu}$ in the 2nd order formalism, the connection $\Gamma^\mu_{\ \alpha\beta}$ in the 1st order formalism [8], or a one-dimensional string (for a review see e.g. [9]).

To the many open questions in this context we add another one: *"Are geodesics in curved space-time truly classical, or do they only appear classical because we measure them in the classical limit?"* In the latter case, fluctuations of gravity could be seen and used as the actual cause of the path history in (4), in this note on the EQQ we will consider an integral over connections Γ.

III. EARTHQUAKE QUANTIZATION

Formally one can define the EQQ from the functional integral (4) and introducing an identity in terms of a functional integral over the gravitational connection $\vec{\Gamma}$ and a functional delta function.

$$\mathbb{1} = \int \mathcal{D}\Gamma \delta\delta \left(\vec{\Gamma} - \mathrm{eom}(\vec{x}) \right). \tag{12}$$

The expression $\mathrm{eom}(\vec{x})$ stands for $-\ddot{\vec{x}}$. The functional delta $\delta\delta$ enforces the condition that at each time-step $\vec{x} = \vec{x}(t)$ is also a solution of the equation of motion (10) for every given value of the connection $\vec{\Gamma}$, including appropriate boundary conditions. Thus, a given $\vec{x} = \vec{x}_s(t)$ determines the connection $\vec{\Gamma}(t)$, or inversely a given connection determines a solution $\vec{x}_s$. With this, (4) can be rewritten as

$$\begin{aligned} K(\vec{x}_i, 0; \vec{x}_f, t) &= \int \tilde{\mathcal{D}}x \cdot \mathbb{1} \cdot \exp\left[\frac{i}{\hbar} S(x, \eta)\right] \\ &= \int \tilde{\mathcal{D}}x \int \mathcal{D}\Gamma \cdot \delta\delta \left(\vec{\Gamma} - \mathrm{eom}(\vec{x}) \right) \exp\left[\frac{i}{\hbar} S(x, \eta)\right] \\ &= \int \mathcal{D}\Gamma \frac{1}{\frac{\mathcal{D}\mathrm{eom}}{\tilde{\mathcal{D}}x}} \exp\left[\frac{i}{\hbar} S(x, \eta)\right], \end{aligned} \tag{13}$$

where in the last step we exchanged the order of integrations, to explicitly perform the functional integral over the measure $\tilde{\mathcal{D}}x$ to obtain a Jacobian factor. Note that the exponentiated action is a function of the trajectory $\vec{x}(t)$ in flat Minkowski space-time $S = S(x, \eta)$. The connections of curved space-time $\vec{\Gamma}(t')$ only come into play when defining the trajectory.

A. Free particle propagator

To learn how the Jacobian with $\frac{\mathcal{D}\mathrm{eom}}{\tilde{\mathcal{D}}x}$ is calculated explicitly let's go back to the infinitesimal integral for two steps $(0 \to \Delta t \to 2\Delta t)$, like in the Kolmogorov relation (3).

To generate the zig-zag motion of the path-integral construction from a potential, one needs to define a connection that is acting instantaneously at the intermediate time-steps. For the two-step example the intermediate time is at $t = \Delta t$, and hence we define

$$\vec{\Gamma}(x, t) \equiv \vec{\gamma}(x) \cdot \delta(t - \Delta t). \tag{14}$$

If the equations of motion (10) are fulfilled, we have

$$\ddot{\vec{x}} = -\vec{\gamma}(x) \cdot \delta(t - \Delta t). \tag{15}$$

To average this equation over one step size, we integrate from $\Delta t/2$ to $3\Delta t/2$ and divide by Δt

$$\bar{\ddot{\vec{x}}} = -\vec{\gamma}(t) \frac{1}{\Delta t}. \tag{16}$$

If the particle follows a PI trajectory with $\vec{x}_1$ as intermediate point, the averaged acceleration is kinematically defined as

$$\bar{\ddot{\vec{x}}} = \frac{1}{\Delta t} \left(\frac{\vec{x}_f - \vec{x}_1}{\Delta t} - \frac{\vec{x}_1 - \vec{x}_i}{\Delta t} \right). \tag{17}$$

Identifying (16) with (17) allows to read-off the relation between $\vec{x}_1$ and the connection at $t = \Delta t$

$$\vec{\gamma}_1 = \frac{2}{\Delta t} \left(\vec{x}_1 - \vec{x}^{\ell}_{\Delta t} \right). \tag{18}$$

Here

$$\vec{x}^{\ell}_{\Delta t} = \frac{\vec{x}_i + \vec{x}_f}{2}, \tag{19}$$

is the position at Δt that corresponds to a straight undeflected line between initial $\vec{x}_i$ and final $\vec{x}_f$. The infinitesimal Kolmogorov relation with the explicit Jacobian in (13) reads then

$$K(x_i,0;x_f,2\Delta t)=\int_{-\infty}^{+\infty}d^3\vec{\gamma}_1\left(\frac{\Delta t}{2}\right)^3\left(\frac{m}{2i\pi\hbar\Delta t}\right)^{3(1+1)/2}\exp\left[\frac{i}{\hbar}\left(S^\ell_{i1}(\Delta t)+S^\ell_{1f}(\Delta t)\right)\right],\tag{20}$$

where $\vec{x}_1$ is the solution of (18)

$$\vec{x}_1=\vec{x}^\ell_{\Delta t}+\vec{\gamma}_1\frac{\Delta t}{2}.\tag{21}$$

Note that due to (21), the exponentiated action on the right hand side of (20) is a function of $\vec{x}_i\,\vec{x}_f$, and $\vec{\gamma}_1$

$$\left(S^\ell_{i1}(\Delta t)+S^\ell_{1f}(\Delta t)\right)=S(x_i\,\vec{x}_f,\vec{\gamma}_1,\Delta t)=m\frac{(\vec{x}_f-\vec{x}_i)^2+\vec{\gamma}_1^2\Delta t^2}{4\Delta t},\tag{22}$$

which has its minimum at $\vec{\gamma}_1=0$. For finite time slicing $\Delta t\to t/2$, this relation allows to factorize the exponential of the action of a straight line connecting $\vec{x}_i$ and $\vec{x}_f$

$$\exp\left[\frac{i}{\hbar}S^\ell_{if}(t)\right]=\left(\frac{tm}{2i\pi\hbar}\right)^{3/2}\int_{-\infty}^{+\infty}d^3\vec{\gamma}_1\exp\left[\frac{i}{\hbar}S^\ell_{i1}(t/2)\right]\cdot\exp\left[\frac{i}{\hbar}S^\ell_{1f}(t/2)\right].\tag{23}$$

Now, one can subdivide all steps into two parts and iteratively insert the prescription (23) into itself. Repeating this procedure n-times allows to construct a path integral with an even number of steps 2^n and an odd number of intermediate integrals $N=2^n-1$. One finds

$$\exp\left[\frac{i}{\hbar}S^\ell_{if}(t)\right]=\int\tilde{\mathcal{D}}\gamma\exp\left(\frac{i}{\hbar}S\right),\tag{24}$$

where the action integral on the right hand side is only a function of the initial and final position, the total time and the intermediate connections $S=S(\vec{x}_i,\vec{x}_f,\vec{\gamma}_k,t)$, resulting from the recursive insertion of

$$\vec{x}_k=\frac{\vec{x}_{k-1}+\vec{x}_{k+1}}{2}+\vec{\gamma}_k\frac{\Delta t}{2}\tag{25}$$

into the discretized action. Further, the normalized integral measure is defined as

$$\int\tilde{\mathcal{D}}\gamma\equiv(N+1)^{-(1+\ln(N-1))}\left(\frac{tm}{2i\pi\hbar}\right)^{\frac{3}{2}N}\Pi_{k=1}^N\left(\int d^3\vec{\gamma}_k\right).\tag{26}$$

The propagator corresponding to the path integral (24) is obtained by a multiplication with the normalization defined in (1).

B. Propagator for generic potentials

A direct solution of the path integral with external potentials is typically only possible in special cases, such as the harmonic oscillator. For the other cases, one has to recur to perturbation theory. Here, we will follow the notation of the classic book [10].

To show that EQQ is compatible with generic (weak) potentials we start of with the standard definition of the Kernel

$$K_V(\vec{x}_i,t_i;\vec{x}_f,t_f)=\int\mathcal{D}x\exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f}dt'\left\{m\frac{\dot{\vec{x}}^2}{2}-V(\vec{x},t')\right\}\right].\tag{27}$$

Note that we take the potential as a result of an external non-gravitational interaction, but it could also arise if one expands (8) around a non-flat background metric $g^{bg}_{\mu\nu}$ instead of the flat $\eta_{\mu\nu}$. For sufficiently small potentials, we expand the exponential containing the potential

$$\exp\left[-\frac{i}{\hbar}\int_{t_i}^{t_f}dsV(\vec{x},s)\right]=1-\frac{i}{\hbar}\int_{t_i}^{t_f}dsV(\vec{x},s)+\frac{1}{2\hbar^2}\left(\int_{t_i}^{t_f}dsV(\vec{x},s)\right)^2+\dots\tag{28}$$

Inserting this back into (27) we can write the propagator as

$$K_V(\vec{x}_i,t_i;\vec{x}_f,t_f) = K_0(\vec{x}_i,t_i;\vec{x}_f,t_f) + K^{(1)}(\vec{x}_i,t_i;\vec{x}_f,t_f) + K^{(2)}(\vec{x}_i,t_i;\vec{x}_f,t_f) + \dots \tag{29}$$

The first term in this series expansion is given by (4). Let's now illustrate how the following terms are treated by examining $K^{(1)}$. By commuting the integration order between $ds \leftrightarrow dx_i$ we can write

$$K^{(1)}(\vec{x}_i,t_i;\vec{x}_f,t_f) = -\frac{i}{\hbar}\int_{t_i}^{t_f} ds\; F_s \tag{30}$$

with

$$F_s = \int \mathcal{D}x \exp\left[\frac{im}{\hbar}\int_{t_i}^{t_f} dt' \frac{\dot{\vec{x}}^2}{2}\right] \cdot V\left(\vec{x}(s), s\right). \tag{31}$$

Without loss of generality, and for the sake of notational simplicity, from now on we set $t_i = 0$ and $t_f \equiv t$. Now we discretize the time intervals and realize that all time steps before $t' = s$ are just an integration of the exponential of the free particle, just like all integrals after $t' = s$. For these, we can thus use the EQQ path integral (20). The remaining position integral is

$$F_s = \int d^3x_s \left(\frac{-4m^2}{\pi^2\hbar^2 s(t-s)}\right)^{\frac{3}{2}} \exp\left[\frac{im}{2\hbar}\left(\frac{(\vec{x}_s-\vec{x}_i)^2}{s} + \frac{(\vec{x}_f-\vec{x}_s)^2}{t-s}\right)\right] V\left(\vec{x}_s\right). \tag{32}$$

The exponentials only contain straight lines with two different velocities before and after the transition time s. This velocity-change can, in analogy to the previous section, be calculated from a PI perspective (17) and from the perspective of the equations of motion (16). Now, the difference is that the time variable s is not right in the middle of the total time t. Thus, the PI definition of average acceleration becomes

$$\ddot{\vec{x}} = \frac{2}{t}\left(\frac{\vec{x}_f-\vec{x}_s}{t-s} - \frac{\vec{x}_s-\vec{x}_i}{s}\right), \tag{33}$$

and the definition according to the equations of motion can be obtained from integrating (15) from $s/2$ to $s+(t-s)/2$ and dividing $t/2$

$$\ddot{\vec{x}} = -\vec{\gamma}_s \frac{2}{t}. \tag{34}$$

Comparing (33) with (34) yields the needed relation between $\vec{x}_s$ and $\vec{\gamma}_s$

$$\vec{\gamma}_s = \frac{t}{s(t-s)}\left(\vec{x}_s - \vec{x}_s^{\ell}\right). \tag{35}$$

This is the generalization of the previous relation (18), now for the case of different lengths of the time steps of the foliation. Thus, with the help of (35) we can write the integrand of (32) purely in terms of ($\vec{x}_i$, $\vec{x}_f$, and $\vec{\gamma}_s$) giving

$$F_s = \int d^3\gamma_s \left(\frac{-4m^2 s(t-s)}{\pi^2\hbar^2 t^2}\right)^{\frac{3}{2}} \exp\left[\frac{im}{2\hbar}\left(\frac{(\vec{x}_s-\vec{x}_i)^2}{s} + \frac{(\vec{x}_f-\vec{x}_s)^2}{t-s}\right)\right] V\left(\vec{x}_s\right). \tag{36}$$

The above steps and definitions can now be applied to all following terms in the expansion of the propagator (29), providing an EQQ prescription for arbitrary potentials.

C. Schrödinger equation

The propagators (27) can be used to determine the evolution of a wave function

$$\begin{aligned}\psi(\vec{x}_f, t+\delta t) &= \int d^3x_i K_V(x_i,t;x_f,t+\delta t)\psi(\vec{x}_i,t) \\ &= \int d^3x_i \left(\frac{m}{2i\pi\hbar\delta t}\right)^{3/2} \exp\left(\frac{i}{\hbar} m\frac{(\vec{x}_f-\vec{x}_i)^2}{\delta t}\right) \exp\left(-\frac{i}{\hbar}V\left(\frac{\vec{x}_f+\vec{x}_i}{2}\right)\delta t\right)\psi(\vec{x}_i,t).\end{aligned} \tag{37}$$

Now, we introduce the EQQ connections at the initial time

$$\vec{\Gamma}(t') = \vec{\gamma}_i \delta(t'-t). \tag{38}$$

These produce the velocity kicks, which determine the relation between initial and final velocities and positions

$$\vec{v}_f = \vec{v}_i - \vec{\gamma}_i \tag{39}$$
$$\vec{x}_f = \vec{x}_i + \vec{v}_f \delta t. \tag{40}$$

With this, we can change the d^3x_i integration to a $d^3\gamma_i$ integration, yielding

$$\psi(\vec{x}_f, t+\delta t) = \int d^3\gamma_i \left(\frac{m\delta t}{2i\pi\hbar}\right)^{3/2} \tag{41}$$
$$\exp\left(\frac{i}{\hbar} m(\vec{v}_i - \vec{\gamma}_i)^2 \delta t\right) \exp\left(-\frac{i}{\hbar} V\left(\vec{x}_f - \frac{\vec{v}_i - \vec{\gamma}_i}{2}\delta t\right)\delta t\right) \psi\left(\vec{x}_f - (\vec{v}_i - \vec{\gamma}_i)\delta t, t\right).$$

This is the EQ version of relation (37). Interestingly, it seems to suggest that the evolved wave function at $t+\delta t$ depends on the velocity of the paths at t. This dependence is, however, fictitious. To see this, we will now use the integral relation (41) for the evolution of wave functions to derive a differential eqution with the same purpose, the Schrödinger equation. For this, we adapt the usual steps outlined in [10]. First, we replace $d^3\gamma$ by a new integration variable

$$\vec{\omega} = (\vec{v}_i - \vec{\gamma}_i)\delta t. \tag{42}$$

This simplifies the Gaussian integral. Second, we expand one of the exponentials for small potentials and small δt

$$\exp\left(-\frac{i}{\hbar} V\left(\vec{x}_f - \frac{\vec{v}_i - \vec{\gamma}_i}{2}\delta t\right)\delta t\right) \approx 1 - \frac{i}{\hbar} V(\vec{x}_f)\delta t. \tag{43}$$

Third, we expand the wave function for small δt

$$\tilde{\psi}\left(\vec{x}_f - (\vec{v}_i - \vec{\gamma}_i)\delta t, t\right) \approx \tilde{\psi}\left(\vec{x}_f, t\right) - \vec{\omega}\cdot\vec{\nabla}\tilde{\psi}\left(\vec{x}_f, t\right) + \frac{1}{2}\omega^j\omega^k \nabla_j \nabla_k \tilde{\psi}\left(\vec{x}_f, t\right). \tag{44}$$

After performing the integral in $d^3\omega = (\delta t)^3 d^3\gamma_i$, (41) reads

$$\psi(\vec{x}_f, t+\delta t) = \left(1 - iV(\vec{x}_f)\frac{\delta t}{\hbar} + i\frac{\hbar\delta t}{2m}\vec{\nabla}^2\right)\psi\left(\vec{x}_f, t\right) \tag{45}$$

Then we pull $\tilde{\psi}(\vec{x}_f, t)$ to the left, and multiply by $i\hbar/\delta t$. This gives, in the limit of $\delta t \to 0$, the familiar Schrödinger equation

$$i\hbar\partial_t \psi(\vec{x}_f, t) = \left(-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{x}_f)\right)\psi(\vec{x}_f, t). \tag{46}$$

D. Ehrenfest principle and classical equation of motion

Let us consider a generic functional of the particle trajectory, $\mathcal{F}[\vec{x}(t)]$. Its expectation value, from the path-integral formulation, is given by the formal expression

$$\langle \mathcal{F} \rangle = \int \mathcal{D}x\, e^{\frac{i}{\hbar}S[\vec{x}(t)]} \mathcal{F}[\vec{x}(t)], \tag{47}$$

where we consider the generic non-relativistic action for an external potential $V(\vec{x})$

$$S[\vec{x}(t)] = \int_{t_i}^{t_f} \left\{\frac{m}{2}\dot{\vec{x}}^2 - V(\vec{x}(t))\right\} dt. \tag{48}$$

If we now consider a small variation of the tajectory $\vec{x}(t) \to \vec{x}(t) + \vec{\eta}(t)$, the measure of the path integral $\mathcal{D}x$ remains invariant, and hence we have (expanding up to first order in $\vec{\eta}(t)$)

$$\begin{aligned}\langle \mathcal{F} \rangle &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}S[\vec{x}(t)+\vec{\eta}(t)]} \mathcal{F}[\vec{x}(t)+\vec{\eta}(t)] \\ &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}S[\vec{x}(t)]} \left\{\mathcal{F}[\vec{x}(t)] + \int ds \vec{\eta}(s)\cdot\frac{\delta\mathcal{F}}{\delta\vec{x}(s)} + \frac{i}{\hbar}\mathcal{F}[\vec{x}(t)] \int ds\vec{\eta}(s)\cdot\frac{\delta S}{\delta\vec{x}(s)}\right\}\end{aligned} \tag{49}$$

Upon separating the three terms on the right hand side, and exchanging the order of integration, we recover expression

$$\langle\mathcal{F}\rangle = \langle\mathcal{F}\rangle + \int ds\vec{\eta}(s)\cdot\left[\langle\frac{\delta\mathcal{F}}{\delta\vec{x}(s)}\rangle + \frac{i}{\hbar}\langle\frac{\delta S}{\delta\vec{x}(s)}\mathcal{F}\rangle\right] \tag{50}$$

Cancelling out the term $\langle\mathcal{F}\rangle$ on both sides, and considering that the variation $\vec{\eta}(s)$ is arbitrary, we conclude

$$\langle\frac{\delta\mathcal{F}}{\delta\vec{x}(s)}\rangle = -\frac{i}{\hbar}\langle\frac{\delta S}{\delta\vec{x}(s)}\mathcal{F}\rangle \tag{51}$$

For the action defined in Eq.(48), the variation is (for fixed initial $\vec{x}(t_i) = \vec{x}_i$ and final $\vec{x}(t_f) = \vec{x}_f$ points)

$$\frac{\delta S}{\delta\vec{x}(s)} = -m\ddot{\vec{x}}(s) - \nabla V(\vec{x}(s)) \tag{52}$$

which substituted into Eq.(51) yields

$$\langle\frac{\delta\mathcal{F}}{\delta\vec{x}(s)}\rangle = \frac{i}{\hbar}\langle\left\{m\ddot{\vec{x}}(s) + \nabla V(\vec{x}(s))\right\}\mathcal{F}\rangle \tag{53}$$

In particular, for the choice $\mathcal{F} = 1$ into Eq.(53), we recover the classical limit of the equation of motion at the level of the expectation values, a manifestation of Ehrenfest's principle

$$m\langle\ddot{\vec{x}}(s)\rangle = -\langle\nabla V(\vec{x}(s))\rangle \tag{54}$$

Even though the previous argument was formulated via the measure $\mathcal{D}\vec{x}$ of the path integral in coordinates space, the same conclusion follows if, as discussed in the previous section, we change the integration variables to incorporate the random velocity kicks between consecutive time steps, i.e. $\vec{x}_{k+1} = \vec{x}_k + \vec{\gamma}_k\delta t$, and $d^3\vec{x}_k = (\delta t)^3 d^3\vec{\gamma}_k$, which in the continuum limit accounts for a corresponding trivial scaling of the measure $\mathcal{D}x \rightarrow \mathcal{D}\Gamma$. The same argument can be applied to the relativistic covariant action for the single-particle

$$S = -mc\int_i^f \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau \tag{55}$$

Here, closely following the argument in Section C, we shall assume that the metric is a superposition of a deterministic classical background and a random local fluctuation, i.e. $g^{\alpha\beta}(x) = g^{\alpha\beta}_{BG}(x) + \delta g^{\alpha\beta}(x)$. For this purpose, it is more convenient to write Eq.(53) directly as a general relation between functional variations

$$\langle\delta\mathcal{F}\rangle = -\frac{i}{\hbar}\langle\delta S\,\mathcal{F}\rangle \tag{56}$$

Now, following the usual procedure to variate the relativistic action in Eq. (55) that involves the variation of the background metric $g^{\mu\nu}_{BG}(x)$ due to the variation of the trajectories δx^μ (with fixed endpoints), one obtains

$$\begin{aligned}\delta S &= -2mc\int_i^f\left\{g_{BG,\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\left(\partial_\alpha g_{BG,\mu\nu} + \partial_\nu g_{BG,\mu\alpha} - \partial_\mu g_{BG,\alpha\nu}\right)\right\}\delta x^\mu d\tau\\ &= -2mc\int_i^f\left\{g_{BG,\mu\beta}\left(\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\right)\right\}\delta x^\mu d\tau,\end{aligned} \tag{57}$$

where in the last line we identified the Christoffel symbol for the background metric $g^{\mu\nu}_{BG}$

$$\Gamma^\beta_{\alpha\nu} = \frac{1}{2}g^{\mu\beta}_{BG}\left(\partial_\alpha g_{BG,\mu\nu} + \partial_\nu g_{BG,\mu\alpha} - \partial_\mu g_{BG,\alpha\nu}\right) \tag{58}$$

By inserting Eq. (57) into Eq. (56), and choosing as before $\mathcal{F} = 1$, we obtain

$$\langle\delta S\rangle = 0 = -2mc\int_i^f\left\langle g_{BG,\mu\beta}\left(\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\right)\right\rangle\delta x^\mu d\tau, \tag{59}$$

which implies the geodesic equation at the level of the expectation value

$$\left\langle g_{BG,\mu\beta}\left(\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\alpha\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}\right)\right\rangle = 0, \tag{60}$$

in agreement with Ehrenfest's principle, just as in the non-relativistic case.

E. Discussion

When realizing our proofs, we basically just performed a series of changes of variables, re-definitions, and mathematical identities to the usual PI quantization. Further, the integrals over connections always involve the previous and subsequent position and thus, a multi-step application can make the algebra more cumbersome.

"*So what is the point*"?

The benefit and new insight is not in the mathematical steps, but in the largely different conceptual meaning: The EQQ in flat space-time can be interpreted as an integral over geodesics in a space-time with virtual deformations at intermediate times. These deformations occur at all points in space-time, but according to the definition (26) they are reduced to a functional integral over connections $\vec{\gamma}$. Relating two different functional integrals in this way, e.g. with a functional delta like in (13) will have consequences for all programs which attempt to quantize geometric degrees of freedom Γ and matter fields ϕ [11, 12], e.g. in terms of seemingly independent functional integrals

$$\int \mathcal{D}\Gamma \int \mathcal{D}\phi \, e^{\frac{i}{\hbar}S} \dots \tag{61}$$

Imposing a $\delta\delta$ restriction in (61) will reduce the functional measure and eventually help to avoid infinities. This is reminiscent of the mechanism that renders delta gravity finite [17, 18]. Further similarities of our proposal exist to a line of research which explores the possibility of understanding quantum mechanics in terms of geometric concepts [19, 20].

In addition to this, there are other intriguing and philosophical aspects of the EQQ that we'd like to touch on.

- Background resolution vs. particle resolution:
  The first comment comes from the hypothesis that the motion of a quantum particle is caused by fluctuating local curvatures. This motion only appears to be erratic, since it is perceived from the macroscopic perspective of a flat background. The perspective of a flat background is accounted for by the fact that the actions, which enter in the exponential weight of the curvature integrals (24), are actually free actions in flat space-time. If it would be possible to increase the δt resolution below the scale of the space-time fluctuations (the Planck scale?), then the measurement would be aware of the local changes $\vec{\gamma}_i$. This would imply that the local experiment would actually co-move with the point particle. In this extreme case, the exponential action should also contain the effects of the local $\vec{\gamma}_i$.

- Equivalence principle (EP):
  The second comment relates to the EP, which states that "*the effects of gravity in terms of a particular background metric and the effects of acceleration are indistinguishable*" [13]. By many, the EP is seen as the conceptual corner stone of GR. Vexatiously, the EP in this form is already in conflict with simple quantum mechanics (QM), even at "mesoscopic" distance scales (scales far bigger than the Planck length, but small enough to be sensitive to quantum effects). For example, one of the challenges in reconciling the EP with QM is the fact that quantum mechanics allows for the possibility of superpositions, which are combinations of different states that can interfere with each other. This can lead to scenarios where the two particle states, such as being at rest or being accelerated are superimposed at mesoscopic scales, while the above EP needs to relate to a metric and thus demands to pick one of the two states. The obvious way out of this dilemma would be to allow for metric superpositions at mesoscopic scales. This, however, could mean to release the beast of quantum gravity at distance scales much larger than the Planck scale. There are numerous attempts to reformulate the EP, to a quantum version (QEP) in such a way that it is at least compatible with QM at mesoscopic scales [14–16].

  The EQQ approach might add to this discussion since it can be cast into the statement: "*The random motion of PI's is indistinguishable from geodesic motion caused by random gravitational fluctuations*". This is a new candidate for a QEP since the classical limit of QM paths leads to classical paths with accelerations and the classical limit of gravitational fluctuations leads to macroscopic classical curvature. Thus, we are tempted to conjecture that the macroscopic classical limit of this QEP-candidate is the usual EP. Even more, this QEP-candidate has the advantage, that it is formulated at the level of paths in the amplitude. As such, it naturally allows for superposition and avoids the usual conflicts of the classical EP with superposition and non-locality of the wave function.

IV. CONCLUSION AND OUTLOOK

We have shown, that it is possible to formulate non-relativistic QM in terms of an integral over geodesic paths on a random background instead of an integral over random free paths on a flat background. This novel perspective introduces a wealth of new questions and opportunities for further exploration, including the generalization to systems with many particles, particles possessing spin, relativistic point particles, quantum field theory, and even the coupling of quantum gravity with matter.

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[21] Note that this name is not meant as trivialization of the devastating effects an earthquake can provoke in inhabited regions. Instead, we encourage everyone to donate to those who are helping in the aftermath of such an event.