Nuclear Physics B299 (1988) 471–483
North-Holland, Amsterdam

A PATH INTEGRAL TO QUANTIZE SPIN

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Received 16 September 1987

We present a model for a classical spinning particle, characterized by spin magnitude, arbitrary but fixed, and continuously varying direction. A gauge freedom of the model reflects the choice of canonical coordinates in the phase space, which is spherical. We formulate the path integral for the model and find, unexpectedly, that the phase space must be punctured at the poles. It then follows that both the total spin and spin projection along any axis are quantized. The model has rotational invariance and yields the usual quantum mechanics of spin, including commutation relations, in a simple way.

1. Introduction

The Feynman path integral [1] is an intuitive bridge between classical and quantum mechanics. Quantum-mechanical spin, however, seems to be without a satisfactory derivation. Schulman [2] derived quantized spin from a path integral over the group SO(3); the configuration space is three-dimensional, and the resulting quantum top contains all spins (both fermions and bosons). A quantum-mechanical particle of definite spin is essentially one-dimensional (since it is completely specified by the eigenstates of one coordinate) so Schulman's formulation seems overly complicated. A classical spinning particle can also be described via Grassmann numbers [3], but then only a formal analogy with the continuous path integral is preserved.

The model presented here originates in the following idea: Consider a mechanical system which can only point out directions in space. For a motion with the same initial and final direction, one can define an action to be the (oriented) solid angle swept out by the motion, multiplied by a constant, λ. However, there is an ambiguity: this action is defined only up to a term $4\pi\lambda$, since a path divides a sphere into two regions. In the context of a path integral, this ambiguity would force λ to be a multiple of $\hbar/2$, and quantized spin would seem quite natural. (Other quantization schemes may also lead to this conclusion, but we do not discuss them here.)

0550-3213/88/$03.50
(North-Holland Physics Publishing Division)

A similar notation might arise if we consider Berry's phase [4] for a particle with spin: a spin eigenstate, taken in an adiabatic excursion back to its initial direction, acquires a phase $e^{is_z\Omega/\hbar}$ where s_z is the spin component and Ω is the solid angle traced out by the path. This phase is not ambiguous, since s_z is a multiple of $\hbar/2$ [5].

In order to justify such a notation of spin, we must make good on two points: We must show that it defines an action for a classical system with spin, and that this action, via path integration, leads to quantum-mechanical spin. To define a classical action, let us fix the magnitude of the spin, leaving its direction free. Since the solid angle can be written in terms of a line integral on the sphere, an appropriate lagrangian for our model is

$$L = \lambda \cos\theta\, \dot{\varphi} + \lambda\mu H \cos\theta\,. \tag{1}$$

Here μH is the product of magnetic moment and magnetic field strength, and θ, φ are the usual spherical coordinates (θ is the angle from the positive z-axis, which, for simplicity, points along the magnetic field). The equations of motion are obtained from L:

$$\begin{aligned} \lambda\dot{\theta}\sin\theta &= 0\,, \\ \lambda(\dot{\varphi} + \mu H)\sin\theta &= 0\,. \end{aligned} \tag{2}$$

These equations describe a classical particle which precesses in a magnetic field. If φ is taken to be the canonical coordinate, the conjugate momentum is

$$P_\varphi = \frac{\delta L}{\delta\dot{\varphi}} = \lambda\cos\theta\,, \tag{3}$$

so that both φ and P_φ have restricted range. (The phase space is compact.)

2. Gauge invariance of the model

By adding a total time derivative to L,

$$L' = \left(\lambda\cos\theta + \frac{\partial\Gamma}{\partial\varphi}\right)\dot{\varphi} + \frac{\partial\Gamma}{\partial\theta}\dot{\theta} + \lambda\mu H\cos\theta + \frac{\partial\Gamma}{\partial t}\,, \tag{4}$$

we seem to get two canonical coordinates, φ and θ, with conjugate momenta $\lambda\cos\theta + \partial\Gamma/\partial\varphi$ and $\partial\Gamma/\partial\theta$, but this is not the case. A general way to compute Poisson brackets when $L = \Sigma_i f_i(x)\, dx^i/dt$ is given in ref. [6]. If $F_{ij} \equiv \partial_i f_j - \partial_j f_i$ and

F^{ij} is its inverse, $\sum_j F_{ij}F^{jk} = \delta_i^k$, then

$$[Q, P] = \sum_{ij} F^{ij} \frac{\partial Q}{\partial x^i} \frac{\partial P}{\partial x^j} . \tag{5}$$

In the present case $[\varphi, \cos\theta] = 1$, as expected, and $[\theta, \varphi] = 1/\sin\theta$, so that θ and φ cannot simultaneously be canonical coordinates.

The terms $\partial\Gamma/\partial\varphi$ and $\partial\Gamma/\partial\theta$ are related to a gauge invariance of the model, however. Suppose we choose new variables θ' and φ', and measure the area with reference to the axis $\theta' = 0$ located at $(\theta, \varphi) = (\theta_0, \varphi_0)$ in the original coordinate system. The new area element is $(\lambda\cos\theta' - \lambda)\,\mathrm{d}\varphi'$ which when translated into the original variables,

$$\cos\theta' = \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi - \varphi_0),$$

$$\tan(\varphi' - \varphi_0) = \frac{\sin\theta\sin(\varphi - \varphi_0)}{-\sin\theta_0\cos\theta + \cos\theta_0\sin\theta\cos(\varphi - \varphi_0)}, \tag{6}$$

looks rather complicated:

$$(\lambda\cos\theta' - \lambda)\,\mathrm{d}\varphi'$$
$$= \lambda\frac{[\sin\theta_0\sin(\varphi - \varphi_0)]\,\mathrm{d}\theta - [\cos\theta_0\sin^2\theta - \sin\theta_0\cos\theta\sin\theta\cos(\varphi - \varphi_0)]\,\mathrm{d}\varphi}{1 + \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi - \varphi_0)} .$$

But it can be written

$$(\lambda\cos\theta' - \lambda)\,\mathrm{d}\varphi' = \left(\lambda\cos\theta - \lambda + \frac{\partial\Gamma}{\partial\varphi}\right)\mathrm{d}\varphi + \frac{\partial\Gamma}{\partial\theta}\,\mathrm{d}\theta, \tag{7}$$

provided we set

$$\lambda\cos\theta - \lambda + \frac{\partial\Gamma}{\partial\varphi} = \lambda\frac{-\cos\theta_0\sin^2\theta + \sin\theta_0\cos\theta\sin\theta\cos(\varphi - \varphi_0)}{1 + \cos\theta_0\sin\theta + \sin\theta_0\sin\theta\cos(\varphi - \varphi_0)}, \tag{8a}$$

$$\frac{\partial\Gamma}{\partial\theta} = \frac{\lambda\sin\theta_0\sin(\varphi - \varphi_0)}{1 + \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi - \varphi_0)} . \tag{8b}$$

The differential equations have a solution:

$$\Gamma = -2\lambda\arctan\frac{(1 + \cos\theta_0\cos\theta - \sin\theta_0\sin\theta)\tan\left(\frac{1}{2}(\varphi - \varphi_0)\right)}{\cos\theta_0\cos\theta} + \lambda(\varphi - \varphi_0) . \tag{9}$$

Thus we can choose coordinates in phase space, φ' and $P_{\varphi'}$, with arbitrary orientation, by fixing Γ appropriately.

We can therefore identify $\lambda \sin\theta \cos\varphi$, $\lambda \sin\theta \sin\varphi$, and $\lambda \cos\theta$ as the components of spin angular momentum S_x, S_y, and S_z. Taken in pairs, they yield the appropriate Poisson brackets for angular momentum:

$$[S_x, S_y] = S_z, \qquad [S_y, S_z] = S_x, \qquad [S_z, S_x] = S_y, \tag{10a}$$

with

$$S_x^2 + S_y^2 + S_z^2 = \lambda^2. \tag{10b}$$

The equations of motion follow also from Poisson brackets between spin components and the hamiltonian, $-\lambda\mu H \cos\theta$. This hamiltonian form of the model is similar to the model given by Balachandran et al. [7] for a classical spinning particle. They define S_x, S_y, and S_z in terms of SU(2) matrices.

Adding the gradient of Γ to the lagrangian will not change the action, which is gauge invariant. The term $\lambda \cos\theta\, \dot{\varphi}$ in the action, integrated around a closed path, is just the area of phase space bounded by the path, $\oint P_\varphi \mathrm{d}\varphi$. However, gauge invariance fails when the path encloses one of the poles, $\theta = 0$ or $\theta = \pi$. Any choice of variables P_φ, φ will show a singularity for at least one point on the sphere. The same mathematical problem arises in connection with the Dirac monopole. In the present case, the "string" is due to the fact that a small closed loop around one pole picks up an "area" $\oint P_\varphi \mathrm{d}\varphi = \pm 2\pi\lambda$. This ambiguity must be resolved: the essential quantity in path integration is the relative phase between two paths, which depends on the area in phase space enclosed by the paths. (The interaction term $\lambda\mu H\cos\theta$ is not ambiguous.) There are two possible resolutions:

(i) We can demand that the area in a closed loop be independent of the choice of coordinates. Then the term $\lambda\cos\theta\dot{\varphi}$ must be replaced by $(\lambda\cos\theta - \lambda)\dot{\varphi}$, and λ must be quantized in units of $\hbar/2$. The singularly at $\theta = 0$ disappears, and the one at $\theta = \pi$ only leads to an overall multiple of 2π in the phase.

(ii) We can alter the topology of the sphere, for example by puncturing it at the poles. In this case we are justified in keeping the singularity, since paths that enclose a pole are not deformable into paths that do not. There is nothing to prevent a path from acquiring an additional factor $\mathrm{e}^{i2\pi\Delta/\hbar}$ when it surrounds a pole; however, as shown by Schulman [8], the factor for a loop circling the pole n times must then be $\mathrm{e}^{i2\pi n\Delta/\hbar}$. So for this case the term $\lambda\cos\theta\,\dot{\varphi}$ in L can be replaced by $(\lambda\cos\theta + \Delta)\dot{\varphi}$. (If symmetry between the two poles is maintained, then Δ will be a multiple of $\hbar/2$.)

The first resolution is simplest, and follows the logic of other instances of topological quantization [9]; it is how the Dirac monopole is quantized. The second resolution may seem unreasonable. Since the action for a closed path is no longer rotationally invariant, the final results (propagators, transition amplitudes) can be expected to depend on the location of the punctures. But it is a logical possibility, which we shall check explicitly.

3. Path integral formulation

The sum over paths will now be formulated.

The path integral is the quantum-mechanical analogue of the canonical transformations of classical mechanics; it connects a canonical variable from a conjugate pair to a variable from a second conjugate pair. We can calculate the amplitude to go from φ_i to φ_f in a time T, since $\varphi_i(t)$ and $\varphi_f(T+t)$ are both canonical coordinates; but then we cannot specify the initial or final P_φ. A suitable discrete version of the action is

$$S = \sum_{j=1}^{N} \left[\varepsilon(\lambda \cos\theta_j + \Delta) \frac{(\varphi_j - \varphi_{j-1})}{\varepsilon} + \varepsilon\lambda\mu H \cos\theta_j \right], \tag{11}$$

where $\varepsilon \equiv T/N$ and $\varphi_0 \equiv \varphi_i$. (We take $\cos\theta_j$ to be intermediate in time between φ_{j-1} and φ_j.) We integrate over all θ_j and φ_j except φ_f and φ_i. We must also sum over paths which include whole multiples of 2π between φ_f and φ_i, in both cases (i) and (ii). The integration measure $[D\varphi][D\theta]$ includes a factor $\prod_{j=1}^{N}\cos\theta_j$ since the space is curved.

So far, the calculations for the two topologies are the same (with $\Delta = -\lambda$ for the perfect sphere). A difference arises in connection with the path measure. For the sphere, we want a sampling of paths which is independent of the orientation of coordinates; since any path on the sphere may be continuously deformed into any other, we can choose, for each time interval, the shortest path connecting two points on the sphere. Thus, the integration of θ_j is over $(0, \pi)$ and of φ_j, over $(\varphi_{j-1} - \pi, \varphi_{j-1} + \pi)$. On the sphere with two punctures, however, paths in one homotopy class cannot be deformed into paths in another; thus the integration of φ_j would extend to $(-\infty, \infty)$.

We let $x \equiv \lambda \cos\theta$, and the propagator is defined as follows

$$K(\varphi_f, \varphi_i; T) = \sum_{n=-\infty}^{\infty} \int \left[\prod_{j=1}^{N} dx_j \, d\varphi_j \exp\left(\frac{i}{\hbar} \left[\varepsilon(x_j - \lambda) \frac{\varphi_j - \varphi_{j-1}}{\varepsilon} + \mu H x_j \varepsilon \right] \right) \right]$$
$$\times \delta(\varphi_f - \varphi_N - 2\pi n) \tag{12}$$

up to a normalization factor; also the limit $N \to \infty$ should be understood. We are dealing with a phase-space path integral.

In eq. (12) let $\delta(\varphi_f - \varphi_N - 2\pi n)$ be replaced by $\int_{-\infty}^{\infty}(\mathrm{d}k/2\pi)\mathrm{e}^{ik(\varphi_f-\varphi_N-2\pi n)}$ and define $\omega_j \equiv \varphi_j - \varphi_{j-1}, \mathrm{d}\omega_j \equiv \mathrm{d}\varphi_j$ for $j = 1, \dots, N$. Then

$$\begin{aligned}
K(\varphi_f, \varphi_i; T) &= \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}k}{2\pi} \left[\prod_{j=1}^{N} \int \mathrm{d}x_j \, \mathrm{d}\omega_j \exp\left(\frac{i}{\hbar} \left[(x_j - \lambda)\omega_j + \mu H x_j \varepsilon - \hbar k \omega_j \right] \right) \right] \\
&\quad \times \mathrm{e}^{ik(\varphi_f - \varphi_i - 2\pi n)} \\
&= \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}k}{2\pi} \left[\int_{-\lambda}^{\lambda} \int_{-\pi}^{\pi} \mathrm{d}\omega \, \mathrm{d}x \exp\left(\frac{i}{\hbar} \left[(x - \lambda)\omega + \mu H x \varepsilon - \hbar k \omega \right] \right) \right]^N \\
&\quad \times \mathrm{e}^{ik(\varphi_f - \varphi_i - 2\pi n)} \\
&= \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}k}{2\pi} \left[2\hbar \int_{-\lambda}^{\lambda} \mathrm{d}x \, \mathrm{e}^{i\mu H x \varepsilon/\hbar} \frac{\sin[\pi(x - \lambda - \hbar k)/\hbar]}{x - \lambda - \hbar k} \right]^N \\
&\quad \times \mathrm{e}^{ik(\varphi_f - \varphi_i - 2\pi n)} \\
&= \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}k}{2\pi} [f(k)]^N \mathrm{e}^{ik(\varphi_f - \varphi_i - 2\pi n)}, \qquad (13)
\end{aligned}$$

where

$$f(k) = 2\hbar \int_{-\lambda}^{\lambda} \mathrm{d}x \, \mathrm{e}^{i\varepsilon\mu H x/\hbar} \frac{\sin[\pi(x - \lambda - \hbar k)/\hbar]}{x - \lambda - \hbar k}. \qquad (14)$$

The integral $f(k)$ cannot be evaluated directly, but we can extract from it all the necessary information. First, $f(k)$ can be written

$$f(k) = I(k) \left[1 + \frac{i\varepsilon J(k)}{I(k)} + \mathrm{O}(\varepsilon^2) \right], \qquad (15)$$

where $I(k) = 2\hbar \int_{-2\lambda-\hbar k}^{-\hbar k} (\sin(\pi x/\hbar)/x) \mathrm{d}x$ and $J(k) = 2\mu H \int_{-2\lambda-\hbar k}^{-\hbar k} (x - \lambda - \hbar k)(\sin(\pi x/\hbar)/x) \mathrm{d}x$. Thus $[f(k)]^N$ is just $[I(k)]^N$ times a complex number of unit norm. The propagator $K(\varphi_f, \varphi_i; T)$ will be dominated by the maxima of $I(k)$. These are stationary points:

$$0 = \frac{\mathrm{d}I(k)}{\mathrm{d}k} = 2\left[\frac{\sin \pi k}{k} \frac{\sin(2\pi\lambda/\hbar + \pi k)}{2\lambda/\hbar + k} \right]. \qquad (16)$$

Suppose $\lambda/\hbar$ is a positive integer. Then k is a solution for all integers except $k = 0$ and $k = -2\lambda/\hbar$, and these are the only stationary points. They are not all equivalent. By looking at a graph of $(2\hbar/x)\sin(\pi x/\hbar)$, which is being integrated

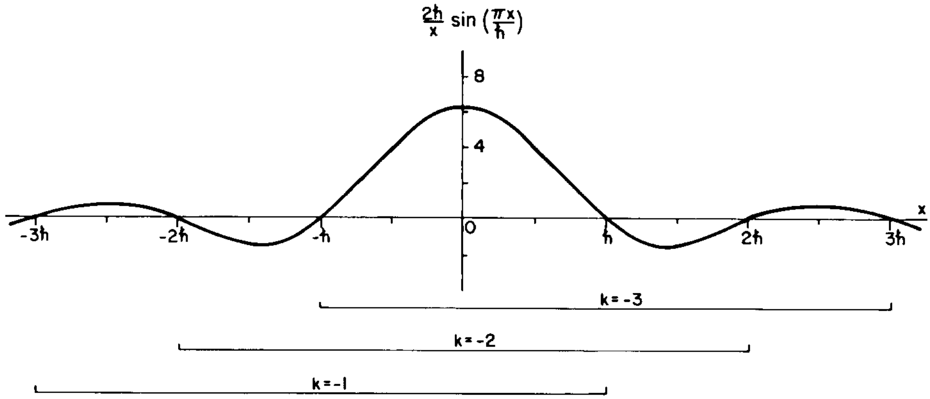

Fig. 1. Graph of $(2\hbar/x)\sin(\pi x/\hbar)$ versus x, showing the ranges of integration for the integral $I(k)$, when $\lambda = 2\hbar$ and $k = -1, -2$, and -3.

from $-2\lambda - \hbar k$ to $-\hbar k$ here, we see that for $|k|$ large, these local maxima and minima tend to zero; the most significant integrals include the point $x = 0$, so that $-2\lambda/\hbar < k < 0$. Nevertheless these points will also in general differ in absolute value. Consequently, this model does not produce quantum-mechanical spin. Take the case $\lambda = 2\hbar$. As fig. 1 illustrates, the maxima at $k = -1$ and $k = -3$ are equivalent, and $I(k)$ is the area integrated over the indicated range; this area exceeds the area integrated over the whole line, while for any other k, including $k = -2$, it is less. The propagator, if it exists in the limit $N \to \infty$, must be dominated by these two maxima. Indeed when $\lambda/\hbar$ is a positive integer, we have

$$K(\varphi_\mathrm{f}, \varphi_\mathrm{i}; T) \sim \int \mathrm{d}k \left[I(k)\right]^N \left[1 + i\frac{\varepsilon\mu H}{\hbar}(\lambda + \hbar k) + \mathrm{O}(\varepsilon^2)\right]^N \mathrm{e}^{ik(\varphi_\mathrm{f} - \varphi_\mathrm{i})}, \quad (17)$$

which will be, up to a factor,

$$\mathrm{e}^{-2i(\varphi_\mathrm{f} - \varphi_\mathrm{i})}\left[\mathrm{e}^{i\mu HT + i(\varphi_\mathrm{f} - \varphi_\mathrm{i})} + \mathrm{e}^{-i\mu HT - i(\varphi_\mathrm{f} - \varphi_\mathrm{i})}\right]. \quad (18)$$

This could be the propagator for a spin-1 particle, but the $s_z = 0$ term is missing. For no other value of λ does a realistic propagator emerge, either.

4. The punctured sphere

We consider the case of the sphere with punctures at its poles. The propagator is

$$K(\varphi_\mathrm{f}, \varphi_\mathrm{i}; T) = \sum_{n=-\infty}^{\infty} \int \left[\prod_{j=1}^{N} \mathrm{d}x_j\, \mathrm{d}\varphi_j \exp\left(\frac{i}{\hbar}\varepsilon\left[(x_j + \Delta)\frac{\varphi_j - \varphi_{j-1}}{\varepsilon} + \mu H x_j\right]\right)\right]$$
$$\times \delta(\varphi_f - \varphi_N - 2\pi n), \quad (19)$$

where the index n classifies paths by homotopy number. The integral over φ_i is now $(-\infty, \infty)$. With the substitutions $\omega_j \equiv \varphi_j - \varphi_{j-1}$ and $\delta(\varphi_f - \varphi_N - 2\pi n) = \int_{-\infty}^{\infty}(dk/2\pi)e^{ik(\varphi_f - \varphi_N - 2\pi n)}$ we obtain

$$
\begin{aligned}
K(\varphi_f, \varphi_i; T) &= \sum_{n=-\infty}^{\infty} \int \frac{dk}{2\pi}\left[\prod_{j=1}^{N} dx_j\, d\omega_j \exp\left(\frac{i}{\hbar}\left[(x_j + \Delta)\omega_j + \varepsilon\mu H x_j - \hbar k \omega_j\right]\right)\right] \\
&\quad \times e^{ik(\varphi_f - \varphi_i - 2\pi n)} \\
&= \sum_{n=-\infty}^{\infty} \int \frac{dk}{2\pi}\left[\prod_{j=1}^{N} dx_j\, e^{i\varepsilon\mu H x_j/\hbar} 2\pi\delta\left(\frac{x_j + \Delta}{\hbar} - k\right)\right] e^{ik(\varphi_f - \varphi_i - 2\pi n)} \\
&= \lim_{M\to\infty} (2\pi\hbar)^{N-1} \int_{-\lambda}^{\lambda} dx\, e^{i\mu H T x/\hbar} e^{i(x+\Delta)(\varphi_f - \varphi_i)/\hbar} \sum_{n=-M}^{M} e^{i2\pi n(x+\Delta)/\hbar}
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&= \lim_{M\to\infty} (2\pi\hbar)^{N-1} \int_{-\lambda}^{\lambda} dx\, e^{i\mu H T x/\hbar} e^{i(x+\Delta)(\varphi_f - \varphi_i)/\hbar} \\
&\quad \times \frac{e^{i\pi(2M+1)(x+\Delta)/\hbar} - e^{-i\pi(2M+1)(x+\Delta)/\hbar}}{e^{i\pi(x+\Delta)/\hbar} - e^{-i\pi(x+\Delta)/\hbar}}\,.
\end{aligned}
\tag{21}
$$

The integral may be done by contour integration; the two terms take contours in the upper and lower half-planes, respectively. (It does not matter how the part of the contour from $-\lambda$ to λ is drawn with respect to pole terms, as long as it is the same for both contours.) The parameter M no longer appears:

$$
K(\varphi_f, \varphi_i; T) = \sum \frac{(2\pi\hbar)^N}{2\pi} e^{i\mu H T x/\hbar} e^{i(x+\Delta)(\varphi_f - \varphi_i)/\hbar}, \tag{22}
$$

where the sum is over all values of x in $(-\lambda, \lambda)$ for which $(x + \Delta)/\hbar$ is an integer. Choosing the normalization factor to be $(2\pi\hbar)^{-N}$, we have the limit for $N \to \infty$. The remaining factor $1/2\pi$ in the propagator is required for the convolution property:

$$
K(\varphi_f, \varphi_i; T) = \int_0^{2\pi} d\varphi\, K(\varphi_f, \varphi; T - t') K(\varphi, \varphi_i; t')\,. \tag{23}
$$

We replace x by s_z, since it is the component of spin along the z-axis, now quantized. The particle's total spin will be denoted by s. For a fermion, $\Delta = \hbar/2$, while for a boson, $\Delta = 0$. Only the integer part of $\lambda/\hbar$ is fixed:

$$
s + \Delta \leqslant \lambda < s + \Delta + \hbar\,. \tag{24}
$$

The propagator

$$K(\varphi_{\rm f}, \varphi_{\rm i}; T) = \frac{1}{2\pi} \sum_{s_z=-s}^{s} e^{i\mu H T s_z/\hbar} e^{i(s_z+\Delta)(\varphi_{\rm f}-\varphi_{\rm i})/\hbar} \tag{25}$$

operates on a wave function $\langle \varphi_{\rm i}, t | \Psi_{\rm i} \rangle$, to yield $\langle \varphi_{\rm f}, t+T | \Psi_{\rm f} \rangle$:

$$\langle \varphi_{\rm f}, t+T | \Psi_{\rm f} \rangle = \int_0^{2\pi} d\varphi_{\rm i} \, K(\varphi_{\rm f}, \varphi_{\rm i}; T) \langle \varphi_{\rm i}, t | \Psi_{\rm i} \rangle \tag{26}$$

and the only Fourier components of $\langle \varphi_{\rm i}, t | \Psi_{\rm i} \rangle$ that can propagate are those with the appropriate values of s_z. Using these states as a basis, $|\Psi_{\rm i}\rangle$ can be written

$$|\Psi_{\rm i}\rangle = \sum_{s_z=-s}^{s} a(s_z) |s_z\rangle, \tag{27}$$

where the $a(s_z)$ are complex constants forming a normalized vector. With the wave function represented as a vector, the propagator takes the form of a matrix:

$$K(\Psi_{\rm f}, \Psi_{\rm i}; T) = \langle \Psi_{\rm f} | e^{i\mu H T \hat{s}_z/\hbar} | \Psi_{\rm i} \rangle, \tag{28}$$

where $\hat{s}_z$ is the diagonal matrix operator for s_z, for a particle with total spin s.

5. Rotational invariance of the model

In order to study the effect of rotations about axes other than the z-axis, we can either rotate the coordinates or choose $\boldsymbol{H}$ to make an angle with the z-axis. Suppose $\boldsymbol{H}$ points along the y-axis; then the term $\lambda\mu H \cos\theta$ in the lagrangian is replaced by $\lambda\mu H \sin\theta \sin\varphi$. Such a term calls for care since it combines non-commuting operators. The standard prescription [8] for a term of the form $P(p)Q(q)$ is to transcribe it as $P(p_i)Q[\frac{1}{2}(q_i + q_{i-1})]$ in the path integral. Thus the interaction terms for the magnetic moment, $\varepsilon\mu H x_j$, are replaced by

$$\varepsilon\mu H \left[\lambda^2 - \left(\frac{x_j + x_{j-1}}{2} \right)^2 \right]^{1/2} \sin\varphi_j, \tag{29a}$$

or

$$\varepsilon\mu H \left[\lambda^2 - x_j^2 \right]^{1/2} \sin\frac{\varphi_j + \varphi_{j-1}}{2}. \tag{29b}$$

It is easier to start with the propagator for an infinitesimal time ε. After a calculation similar to those shown above, the result (using either version of the interaction term) is

$$K(\varphi_{\mathrm{f}}, \varphi_{\mathrm{i}}, \varepsilon) = \frac{1}{2\pi} \sum_{s_z=-s}^{s} \mathrm{e}^{i(s+\Delta)(\varphi_{\mathrm{f}}-\varphi_{\mathrm{i}})/\hbar} \left[1 + \frac{\varepsilon\mu H}{2\hbar} \left[\lambda^2 - (s_z + \hbar/2)^2 \right]^{1/2} \mathrm{e}^{i\varphi_{\mathrm{f}}} - \frac{\varepsilon\mu H}{2\hbar} \left[\lambda^2 - (s_z - \hbar/2)^2 \right]^{1/2} \mathrm{e}^{-i\varphi_{\mathrm{f}}} \right], \quad (30)$$

where, for simplicity, $\lambda \geqslant s + \hbar/2$ is assumed. For states of definite spin projection s_z and s_z', this result implies

$$K(s_z', s_z, \varepsilon) = \delta(s_z', s_z) + \frac{\varepsilon\mu H}{2\hbar} \left[\lambda^2 - (s_z + \hbar/2)^2 \right]^{1/2} \delta(s_z', s_z + \hbar) - \frac{\varepsilon\mu H}{2\hbar} \left[\lambda^2 - (s_z - \hbar/2)^2 \right]^{1/2} \delta(s_z', s_z - \hbar). \quad (31)$$

The choice of λ is crucial. Up to now λ has been somewhat arbitrary, but eq. (31) shows that $\lambda = s + \hbar/2$ is required for the propagator to be invariant under rotations. For $\lambda = s + \hbar/2$, rotation symmetry is easy to establish: Recall that the matrix elements for the raising and lowering operators $\hat{J}_+$ and $\hat{J}_-$ are

$$\langle s_z' | \hat{J}_+ | s_z \rangle = \delta(s_z', s_z + \hbar) \left[(s + \hbar/2)^2 - (s_z + \hbar/2)^2 \right]^{1/2},$$
$$\langle s_z' | \hat{J}_- | s_z \rangle = \delta(s_z', s_z - \hbar) \left[(s + \hbar/2)^2 - (s_z - \hbar/2)^2 \right]^{1/2}. \quad (32)$$

So the propagator takes a familiar form:

$$K(s_z', s_z; \varepsilon) = \langle s_z' | 1 + \frac{\varepsilon\mu H}{2\hbar} (\hat{J}_+ - \hat{J}_-) | s_z \rangle = \langle s_z' | 1 + i \frac{\varepsilon\mu H}{\hbar} \hat{J}_y | s_z \rangle. \quad (33)$$

Similarly, $K(s_z', s_z, \varepsilon) = \langle s_z' | 1 + i(\varepsilon\mu H/\hbar) \hat{J}_x | s_z \rangle$ when $\boldsymbol{H}$ points along the x-axis, and for a general $\boldsymbol{H}$,

$$K(s_z', s_z, \varepsilon) = \langle s_z' | 1 + \frac{i}{\hbar} \varepsilon\mu \boldsymbol{H} \cdot \hat{\boldsymbol{J}} | s_z \rangle. \quad (34)$$

Via convolution this propagator extends to non-infinitesimal times:

$$K(s_z', s_z; T) = \langle s_z' | \mathrm{e}^{i\mu \boldsymbol{H} \cdot \hat{\boldsymbol{J}} T/\hbar} | s_z \rangle. \quad (35)$$

Eq. (35) shows explicitly that s_z transforms as a vector under rotations. Therefore the location of the punctures in the sphere is unobservable. The punctures simply reflect the choice of coordinates and quantization axis.

Eq. (35) has an additional application. The path integral is a quantum version of classical transformations between one of a pair of canonical coordinates at time t, and one of a pair of canonical coordinates at time $t + T$. As discussed earlier, $(\varphi, \lambda \cos\theta)$ and $(\varphi', \lambda \cos\theta')$ (referenced to different axes) are also pairs of canonical coordinates. So $\langle s_x|s_z\rangle$, for example, should have a path-integral expression [10]. The classical generator of infinitesimal rotations is angular momentum, here as well. (The generators may be derived explicitly from eqs. (6).) The generator which rotates the z-axis into the x-axis is $L_y = \lambda \sin\theta \sin\varphi$, so a rotation of ε in this direction corresponds to eq. (33) without the μH. Similarly, eq. (35) has the following form for a rotation:

$$K(s_z', s_z; a) = \langle s_z'| \mathrm{e}^{i\alpha \boldsymbol{n}\cdot \hat{J}/\hbar} |s_z\rangle . \tag{36}$$

Here $\boldsymbol{n}$ is a unit vector along the new quantization axis, and α is the angle it makes with the z-axis. This equation summarizes the behavior of state vectors under rotations.

6. Comparison with the harmonic oscillator

Although we have obtained a path integral for spin, it does not seem to fit the original proposal; whatever the spin s, a path enclosing one of the punctures will pick up a phase of -1. The restoration of rotational symmetry is hard to understand. Also, to derive the propagator in a field of arbitrary direction, we applied the standard midpoint prescription [8, 11] to products of non-commuting operators. The prescription was essential: without it, the factors $[\lambda^2 - (s_z + \hbar/2)^2]^{1/2}$ and $[\lambda^2 - (s_z - \hbar/2)^2]^{1/2}$ in eqs. (31) and (32) could have been something different. Such extreme prescription dependence is a common feature of path integrals; however, the usual applications of the midpoint prescription are to path integrals with gaussian terms, which are not present here.

To better judge the results, we examine the harmonic oscillator in the action-angle representation. We have

$$x = \left(\frac{2J}{m\omega}\right)^{1/2}, \qquad p = -(2Jm\omega)^{1/2} \tag{37}$$

and the hamiltonian is $J\omega$. The propagator is

$$K(\theta_{\mathrm{f}}, \theta_{\mathrm{i}}; T) = \sum_n \int \prod_{j=1}^{N} \left[\mathrm{d}J_j\, \mathrm{d}\theta_j \exp\left(\frac{i}{\hbar}\left[(J_j + \Delta)(\theta_j - \theta_{j-1}) - J_j\omega\varepsilon\right]\right)\right]$$
$$\times \delta(\theta_{\mathrm{f}} - \theta_N - 2\pi n) . \tag{38}$$

We would expect Δ to be zero, and the integrations over $\mathrm{d}\theta_j$ to have measure 2π. However, in this case the propagator does not have any consistent limit. On the other hand, if $\Delta = \hbar/2$ and the integrations over $\mathrm{d}\theta_j$ have infinite range, we obtain the correct energy levels; furthermore, the standard mid-point prescription gives the correct matrix elements for the raising and lowering operators, $a^+ = \sqrt{J/\hbar}\, \mathrm{e}^{i\theta}$ and $a = \sqrt{J/\hbar}\, \mathrm{e}^{-i\theta}$.

Yet the usual position-space path integral for the harmonic oscillator is completely equivalent to the Schrödinger formulation. The problem must be with the use of the new canonical coordinates. The path integral is written correctly in configuration space, and translation into other coordinates is a problem for which, at present, there is no comprehensive solution. (The transformation to action-angle variables is not among those treated in ref. [10].) It is may be that the correct transformation of the harmonic oscillator action into action-angle variables includes a factor -1 for each cycle of 2π in θ. We note that the usual propagator contains a factor $-i$ for each classical turning point, thus -1 for each cycle, and it is these factors that lead to the zero-point term in the energy levels [8, 12].

7. Discussion

Quantized spin appears here as the result of a path integral over continuous variables. The calcualtions require little work. A particle of definite spin s, either bosonic or fermionic, is described by the parameters $\lambda = s + \hbar/2$ with $\Delta = 0$ for a boson, $\Delta = \hbar/2$ for a fermion. Equivalently, we can eliminate Δ by replacing $(\lambda\cos\theta + \Delta)\dot{\varphi}$ in the action by $(\lambda\cos\theta - \lambda + \hbar/2)\dot{\varphi}$. The SU(2) algebra emerges in the appropriate representation. Expectation values of operators and of their commutators can be derived by standard path-integral techniques.

The essential ingredient in the two models analyzed is an action which is an area in phase space. Such a term is present in any action as $\int p\,\mathrm{d}q$, since

$$\text{action} = \int (p\dot{q} - H)\mathrm{d}t\,. \tag{39}$$

When the phase space is spherical, however, this term is ambiguous. The ambiguity can be removed by quantizing the parameters of the theory. In the first model, the parameters were necessarily quantized before the paths were summed, otherwise the relative phases among paths would not have been well-defined. Nevertheless, the sum did not result in quantum-mechanical spin. In the second model, there was no need to quantize until the path-integral limit had been taken. Then it could be seen that the "punctures" which had been introduced to define the sum, but which mutilate the rotational invariance of the classical theory, disappear for particular values of λ. A normal limit exists for the path integral, which yields quantum spin.

It is not clear, though, how faithful the second model is to the original conception of spin as a Wess-Zumino [9] term. By changing the action from $(\lambda\cos\theta - \lambda)\dot{\varphi}$ to $(\lambda\cos\theta - \lambda + \hbar/2)\dot{\varphi}$, we have brought back the singularities that we wanted to eliminate. We conjecture, however, that the extra term $\hbar\dot{\varphi}/2$ is dictated by the particular canonical coordinate system used, as it appears to be for the harmonic oscillator. We hope our work will inspire further research on canonical transformations in the path integral.

The model introduced here may have applications. Possibly the Dirac equation could be obtained from a path integral over continuous variables. What has been done for the spinning particle could be extended to fields, making possible new formulations of quantum field theory without Grassmann variables [13].

A description of fields carrying spin, using only continuous variables, might be useful for treating fermions in lattice gauge theory. But the area term in the exponent of our path integral is purely imaginary, and would remain so after a rotation to imaginary time. That this term remains imaginary is essential, for without the periodicity coming from the imaginary exponent, the spin would not be quantized. Complex actions are ill-suited to Monte Carlo simulation, although there has been some success with numerical solution via the Langevin equation [14].

For now, the main attraction of this model for spin is the prospect that, like the Feynman path integral itself, it can add an intuitive picture to quantum calculations.

We thank E. Gozzi and B. Durhuus for help with references, and J. Ambjørn for encouragement.

Note added in proof

A related spin path integral has been formulated by L.D. Faddeev, together with a student, but not published [15].

K. Johnson [16] has shown how to obtain the condition $\lambda = s + \hbar/2$ (which we discuss below eq. (31)) independently of the path integral. Namely, he extends the usual spin operator algebra to include $\hat{\varphi}$ (or at least $e^{i\hat{\varphi}}$) along the lines of sect. 2, and shows that the extension has unitary representations only for $\lambda = s + \hbar/2$. With this algebra and conventional quantum mechanics as the starting point, he obtains a new derivation of the path integral of sect. 4.

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