# Motion of Spin 1/2 Massless Particle in a Curved Spacetime.

### I. Lagrangian Approach

A.T. Muminov

Ulug-Beg Astronomy Institute, Astronomicheskaya 33, Tashkent 100052, Uzbekistan amuminov@astrin.uzsci.net

#### Abstract

Quasi-classical picture of motion of spin 1/2 massless particle in a curved spacetime is built on base of simple Lagrangian model. The one is constructed due to analogy with Lagrangian of massive spin 1/2 particle [9]. Equations of motion and spin propagation coincide with Papapetrou equations describing dynamic of photon in a curved spacetime [8]

**Keywords:** Spin 1/2 massless particle; Papapetrou equation.

#### 1 Introduction

Lagrangian describing motion of non scalar particles in an external field includes additional terms containing spin variables. These terms change equation of motion of the particle due to interaction of spin with the external field [1]. Motion of particles with spin in curved spacetime was subject of studies of numerous authors [2, 3, 4], see more in review [5]. It was shown in the mentioned works that equation of motion differs from geodesic equation due to a term which is contraction the curvature with spin and velocity. However, there is still no general consensus on the behavior of particle with spin in external gravitation field. Certain progress on the description of massive particles with spin 1 and 1/2 was achieved in our recent works [6, 9]. At the same time, as it was pointed in our work [8] description of massless particles cannot be derived straightforwardly from the Lagrange description of massive particles just by equating the mass to zero: m = 0. Thus, development Lagrange description of motion of spin 1/2 massless particle in a curved spacetime needs especial consideration which is given in the present work.

We consider the particle as quasi-classical. In other words this means that motion of the particle is described by its coordinates  $\{x^i\}$  and internal spin variables specifying spin degrees of freedom of the particle. Spin variables are elements of suitable spinor spaces. Determination of the spaces demands specifying an orthonormal frame in considered domain of spacetime. It is seen that procedures of constructing a field of comoving

frames along null-curve and timelike curve sufficiently differ from each other. In case of massive particle we could use length parameter s along the worldline which specified timelike vector of unit length  $\vec{n}_0 = \dot{x}^i \partial_i = \dot{\vec{x}}$ ,  $\dot{x}^i = dx^i/ds$ . The vector defined 1+3 splitting of tangent space along the worldline used for specifying spinor variables [9]. In case of massless particle vector  $\vec{n}_- = \dot{\vec{x}}$  tangent to worldline of the particle has zero length and can not be normed by the same way as timelike vector. We need second null-vector, say  $\vec{n}_+$ , which will norm the vector  $\vec{n}_-$  by the condition  $\langle \vec{n}_-, \vec{n}_+ \rangle = 1$ . This determines 2+2 splitting of tangent space along the worldline. Let  $\{\vec{n}_\alpha\}$ ,  $\alpha=1,2$  two spacelike vectors which supplements coframe  $\{\vec{n}_\pm\}$  up to frame of tangent space along the worldline. It was shown in our work [8] that vector  $\vec{n}_+$  specifies frequency of oscillations  $\omega$  of wave function of the particle which defines energy of the particle  $E=\hbar\omega$  referred to the frame. Vectors  $\vec{n}_\alpha$  specify polarization subspace of tangent space. Both pairs of vectors  $\vec{n}_\pm$  and  $\vec{n}_{1,2}$  are defined with accuracy up to arbitrary rotations which are elements of groups SO(1,1) and SO(2) accordingly.

Rotating vectors  $\vec{n}_{\pm}$  such a way that value of energy E in the frame becomes a constant along the worldline as it was done under consideration of electromagnetic field [8], we define canonical velocity  $\dot{\vec{x}} = \vec{n}_{-}$  and canonical parameter s for this null-worldline. Form of the Lagrangian and equations of motion of the particle take their simplest form under choice of the canonical parameter as time parameter along the worldline. Procedure of variation the action  $S = \int \mathcal{L} ds$  assumes independent variation of  $\dot{\vec{x}}$  and spinor variables under constraint E = const. Hence, spinor variables should be referred to coframe  $\{\vec{n}_{\alpha}\}$  of vectors of polarization. In other words, spinor variables are elements of linear spaces of representation of group SO(2).

The spaces are constructed as follows. Pauli matrices  $\{\hat{\sigma}^{\alpha}\}$ ,  $\alpha=1,2$  referred to frame of polarization are introduced. The matrices generate local Clifford algebras referred to the frame. Besides, local Clifford algebra introduced this way specifies two local spinor spaces attached to the worldline. These spaces are two spaces of spinor representations of the group SO(2) which are isomorphic to each other under Hermitian conjugation. In our approach elements of the spaces  $\psi^{\dagger}$ ,  $\psi$  play role of generalized coordinates which describe spin degrees of freedom of the particle.

The desired Lagrangian should depend on generalized coordinates  $\{x^i\}$ ,  $\{\psi^{\dagger}, \psi\}$  and their derivatives over s. At the same time the Lagrangian should contain covariant derivatives of spinor variables  $\psi^{\dagger}$  and  $\psi$ . Moreover, the Lagrangian must contain term with  $\langle \vec{x}, \vec{x} \rangle$  which yields left hand side (LHS) of geodesic equation. Euler-Lagrange equations for spinor variables are expected to yield reduced form of Dirac equation for wave propagating along the worldline of the particle. In the limiting case of zero gravitation the equation becomes a consequence of Weyl equation formulated for massless spinor field. All these requirements determine the form of the Lagrangian describing motion of massless spin 1/2 particle in curved spacetime. Thus, Euler-Lagrange equations are reduced to equations describing motion of the spin along the particle worldline and the worldline shape. The equations obtained this way are similar to Papapetrou equations for photon [8].

#### 2 Lagrange formalism for massless particle of spin 1/2

First of all, we have to specify a field of orthonormal frames along the worldline. For this end we add one more null-vector  $\vec{n}_+$  transversal to null-vector  $\vec{n}_- = \dot{\vec{x}}$ . The vectors  $\vec{n}_\pm$  norm each other such a way that  $\langle \vec{n}_-, \vec{n}_+ \rangle = 1$ . This defines 2 + 2 splitting of tangent spaces  $T = (R\vec{n}_- \oplus R\vec{n}_+) \oplus (R\vec{n}_1 \oplus R\vec{n}_2)$  on the worldline. Vectors  $\{\vec{n}_a\}$ ,  $a = \pm, 1, 2$  constitute comoving orthonormal frame along the worldline. Spacelike vectors  $\{\vec{n}_a\}$ ,  $\alpha = 1, 2$  constitute frame of polarization subspace of tangent space. The vector  $\vec{n}_+$  defines the value of energy of the particle  $E = \hbar \omega$  for frame in question [8]. Vectors  $\vec{n}_+$  are defined with accuracy of arbitrary Lorentzian rotations in subspace  $R\vec{n}_- \oplus R\vec{n}_+$ . We fix the vectors by condition E = const along the worldline which, in turn, defines canonical parameter along the worldline [8]. Let  $\{\nu^a\}$  be covector comoving frame dual to  $\{\vec{n}_a\}$ . It is seen that  $\{\nu^\alpha\}$  is covector polarization frame dual to vector frame  $\{\vec{n}_\alpha\}$  in tangent polarization subspace.

In order to describe spin variables of the Lagrangian we introduce Pauli matrices  $\{\hat{\sigma}^1, \hat{\sigma}^2\}$  referred to the covector polarization frame  $\{\nu^{\alpha}\}$ . The matrices are constant in chosen frame and obey anticommutation rules as follows:

$$\{\hat{\sigma}^{\alpha}, \hat{\sigma}^{\beta}\} = -2\eta^{\alpha\beta},\tag{1}$$

where  $\eta^{\alpha\beta}$  is Minkowski tensor in subspace  $R\vec{n}_1 \oplus R\vec{n}_2$ . Algebraic span of Pauli matrices yields local sample of Clifford algebra in each point of the worldline. Union of the local Clifford algebras constitute fibre bundle of Clifford algebra along the worldline.

Invertible elements R of Clifford algebra such that

$$R^{-1} = R^{\dagger},$$

where  $R^{\dagger}$  stands for Hermitian conjugated matrix, constitute Spin(2) group [10]. There is an endomorphism  $R: SO(2) \to Spin(2)$  defined by formula:

$$R_L \hat{\sigma}^{\alpha} R_L^{-1} = L_{\beta}^{\alpha} \hat{\sigma}^{\beta}, \quad (L_{\beta}^{\alpha}) \in SO(2).$$
 (2)

Elements of local Clifford algebra are operators on two local spinor spaces referred to the frame on the worldline. The spaces are local linear spaces of representation of group Spin(2) and SO(2). Elements of the local spaces  $\psi \in S$  and  $\psi^{\dagger} \in S^{\dagger}$  are  $2 \times 1$  and  $1 \times 2$  complex matrices accordingly. This way element L of group of spatial rotations SO(2) acts on spaces of representation of the group as follows:

$$'\psi = R_L \psi, \quad '\psi^{\dagger} = \psi^{\dagger} R_L^{-1}, \quad \psi \in S, \, \psi^{\dagger} \in S^{\dagger}.$$
 (3)

Union of the local spinor spaces constitute spinor fibre bundle on the worldline.

Image of an infinitesimal rotation  $L = \mathbf{1} - \varepsilon \in SO(2)$  is:

$$R_{1-\varepsilon} = \hat{1} + \delta Q = \hat{1} + 1/4 \,\varepsilon_{\alpha\beta} \,\hat{\sigma}^{\alpha} \hat{\sigma}^{\beta}. \tag{4}$$

The infinitesimal transformation rotates elements of the rest frame:

$$\delta \nu^{\alpha} = -\varepsilon_{\beta}^{\ \alpha} \nu^{\beta}. \tag{5}$$

Accordingly (3) the rotation initiates a transformation of spinors:

$$\delta\psi = 1/8\,\varepsilon_{\beta\gamma}\,[\hat{\sigma}^{\beta},\hat{\sigma}^{\gamma}]\,\psi, \quad \delta\psi^{\dagger} = -1/8\,\varepsilon_{\beta\gamma}\,\psi^{\dagger}\,[\hat{\sigma}^{\beta},\hat{\sigma}^{\gamma}],\tag{6}$$

under which Pauli matrices rotates as follows:

$$\begin{split} '\hat{\sigma}^{\alpha} &= R\hat{\sigma}^{\alpha}R^{-1}, \quad \delta\hat{\sigma}^{\alpha} = [\delta Q, \hat{\sigma}^{\alpha}]; \\ \delta\hat{\sigma}^{\alpha} &= -\varepsilon_{\beta}^{\ \alpha}\hat{\sigma}^{\beta} = 1/4\varepsilon_{\beta\gamma} \left[\hat{\sigma}^{\beta}\hat{\sigma}^{\gamma}, \hat{\sigma}^{\alpha}\right]. \end{split}$$

It is seen that the rotation coincides with rotation of components of contravariant vector with accuracy up to opposite sign. Thus, if we take into account both of the transformations Pauli matrices become invariant as it is accepted in field theory [11].

State of the particle is described by its coordinates  $\{x^i\}$  in spacetime, spinor variables  $\{\psi,\psi^{\dagger}\}$  which are elements of spinor fibre bundles on the worldline and their derivatives  $\dot{x}^i = \frac{dx^i}{ds}$  and  $\frac{d\psi}{ds}$ ,  $\frac{d\psi^{\dagger}}{ds}$  over canonical parameter s along the worldline. Polarization frame rotates along the worldline:

$$\dot{\nu}^{\alpha} = -\omega_{\beta}^{\ \alpha}(\dot{\vec{x}})\nu^{\beta},$$

where angular velocities are given by values of Cartan' rotation 1-forms  $\omega_{\beta}^{\ \alpha} = \gamma_{c\beta}^{\ \alpha} \nu^c$  on vector  $\dot{\vec{x}}$ . So do spinor variables referred to the frame. Their transformations are given by equations (6) where  $\varepsilon_{\beta}^{\ \alpha} = \gamma_{-\beta}^{\ \alpha}$ . The transformations are taken into account by covariant derivatives of spinor variables along the worldline:

$$\dot{\psi} = \frac{d\psi}{ds} + \frac{1}{4} \gamma_{b\delta\varepsilon} \dot{x}^b \, \hat{\sigma}^\delta \hat{\sigma}^\varepsilon \psi,$$

$$\dot{\psi^{\dagger}} = \frac{d\psi^{\dagger}}{ds} - \frac{1}{4} \gamma_{b\delta\varepsilon} \dot{x}^b \, \psi^{\dagger} \, \hat{\sigma}^\delta \hat{\sigma}^\varepsilon.$$

$$(7)$$

Besides, total covariant derivatives (with account of spinor transformation and rotation of vector indexes) of Pauli matrices are zero.

Since Lagrangian of the particle is covariant under internal transformations of the polarization frames, it includes derivatives (7). The Lagrangian contains term  $E/2 \psi^{\dagger} \psi < \vec{x}, \vec{x} >$  which yields geodesic equation. Euler-Lagrange equations for spin variables should lead to reduced form of Dirac equation. Analysis shows that to satisfy such the requirement we may accept the following form of the Lagrangian:

$$\mathcal{L} = \frac{E}{2} \psi^{\dagger} \psi < \dot{\vec{x}}, \dot{\vec{x}} > -\frac{i\hbar}{2} \left( \psi^{\dagger} \dot{\psi} - \dot{\psi}^{\dagger} \psi \right). \tag{8}$$

We must bear in mind that the Lagrangian is function of generalized coordinates  $x^i$ ,  $\psi$ ,  $\psi^{\dagger}$  and their velocities  $\frac{dx^i}{ds} = \dot{x}^i$ ,  $\frac{d\psi}{ds}$ ,  $\frac{d\psi^{\dagger}}{ds}$ . At the same time covariant form of the Lagrangian includes derivatives represented in orthonormal frame. Due to this we recall formulae of transformations between the frames:

$$\vec{n}_a = n_a^i \partial / \partial x^i, \quad \dot{x}^a = n_i^a \dot{x}^i, \quad \dot{x}^i = n_a^i \dot{x}^a, \quad n_a^i n_j^a = \delta_j^i, \quad n_a^i n_i^b = \delta_a^b. \tag{9}$$

#### 3 Euler-Lagrange equations for spinor variables

Due to (8) generalized momenta conjugated to generalized coordinates  $\psi^{\dagger}$  and  $\psi$  are:

$$\Psi = \partial \mathcal{L}/\partial \left(\frac{d \psi^{\dagger}}{ds}\right) = \frac{i\hbar}{2} \psi, \quad \Psi^{\dagger} = \partial \mathcal{L}/\partial \left(\frac{d \psi}{ds}\right) = -\frac{i\hbar}{2} \psi^{\dagger}.$$

Euler Lagrange equations for the considered generalized coordinates read:

$$\frac{d}{ds}\Psi = \partial \mathcal{L}/\partial \psi^{\dagger}, \quad \frac{d}{ds}\Psi^{\dagger} = \partial \mathcal{L}/\partial \psi. \tag{10}$$

Straightforward calculation of the right hand side (RHS) of the above equations gives:

$$\partial \mathcal{L}/\partial \psi^{\dagger} = -\frac{i\hbar}{2}\dot{\psi} - \frac{i\hbar}{2} \cdot \frac{1}{4}\gamma_{b\delta\varepsilon}\dot{x}^{b}\,\hat{\sigma}^{\delta}\hat{\sigma}^{\varepsilon}\psi,$$
$$\partial \mathcal{L}/\partial \psi = \frac{i\hbar}{2}\dot{\psi}^{\dagger} - \frac{i\hbar}{2} \cdot \frac{1}{4}\gamma_{b\delta\varepsilon}\dot{x}^{b}\,\psi^{\dagger}\hat{\sigma}^{\delta}\hat{\sigma}^{\varepsilon}.$$

Now it is seen that the RHS of the equations (10) completes ordinary derivatives of spinor variables in the LHS up to covariant derivatives. This way Euler-Lagrange equations for  $\psi^{\dagger}$ ,  $\psi$  generalized coordinates become:

$$i\hbar\dot{\psi} = 0, \qquad i\hbar\dot{\psi^{\dagger}} = 0.$$
 (11)

The equations coincide with Weyl equations for massless freely propagating spinor field of definite helicity [13].

## 4 Generalized momentum conjugated with $x^i$ and conservation of spin

Generalized momentum conjugated with  $x^i$  is  $p_i = \partial \mathcal{L}/\partial \dot{x}^i$ . However it is convenient to operate with generalized momenta expressed in orthonormal frame:  $p_a = n_a^i p_i$ . Differentiating (8) over  $\dot{x}^a$  we obtain:

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{x}^a} = E \psi^{\dagger} \psi \, \eta_{ab} \dot{x}^b - \frac{i\hbar}{2} \cdot \frac{1}{2} \gamma_{a\delta\varepsilon} \, \psi^{\dagger} \hat{\sigma}^{\delta} \hat{\sigma}^{\varepsilon} \psi.$$

We define spin of the particle as:

$$S^{\delta\varepsilon} = -\frac{i\hbar}{4} \,\psi^{\dagger} \hat{\sigma}^{[\delta} \,\hat{\sigma}^{\,\varepsilon]} \psi, \tag{12}$$

where [,] stands for commutator of the Pauli matrices. By construction, the spin is element of space which is tensor product of two copies of tangent polarization space. Thus, it has no  $\pm$ -components and condition of orthogonality of the spin to velocity is

satisfied:  $\dot{x}^b S_b{}^c \equiv 0$ . It is seen that RHS of (12) can be represented as  $\hbar/2\psi^{\dagger}\hat{\sigma}^3\psi$  which is eigenvalue of operator of helicity  $\hbar/2\hat{\sigma}^3$  in state described by wave function  $\psi$  in the tangent polarization space. Moreover, it can be shown that definition (12) is in accord with formula for "–"-component of current of spin derived from Noether theorem in field theory [13].

Due to the definition of the generalized momentum  $p_a$  given above the one can be represented as follows:

$$p_a = E \,\psi^{\dagger} \psi \,\eta_{ab} \dot{x}^b + \frac{1}{2} \gamma_{a\delta\varepsilon} S^{\delta\varepsilon} = \pi_a + \frac{1}{2} \gamma_{a\delta\varepsilon} S^{\delta\varepsilon}. \tag{13}$$

It is seen that form of (13) coincides with analogous equation obtained for massive particle of spin 1/2 [9].

According to the equation (11) straightforward calculation of covariant derivative of spin (12) gives:

$$\frac{D S^{\delta \varepsilon}}{ds} = \frac{d S^{\delta \varepsilon}}{ds} + \dot{x}^a \left( \gamma_{a\gamma}^{\ \delta} S^{\gamma \varepsilon} + \gamma_{a\gamma}^{\ \varepsilon} S^{\delta \gamma} \right) \equiv 0, \tag{14}$$

where we took into account the fact that total covariant derivatives of Pauli matrices are zero.

#### 5 Euler-Lagrange equations for $x^i$ variables

Euler-Lagrange equations for  $x^i$  variables read:

$$dp_i/ds = \partial \mathcal{L}/\partial x^i. \tag{15}$$

Under comparison (8) with the Lagrangian of spin 1/2 massive particle [9] we see that parts of the Lagrangians contributing to  $\partial \mathcal{L}/\partial x^i$  formally coincide. At the same time, as it was pointed above expressions for the generalized momenta both massless and massive particles of spin 1/2 are formally identical. Noting that (14) has the same form as equation for spin of massive particle [9] we expect that Euler-Lagrange equations (15) will be reduced to the same form as in case of massive particle [6, 9]:

$$D\pi_a/ds = 1/2 R_{\delta \varepsilon ab} \dot{x}^b S^{\delta \varepsilon},$$

where

$$\Omega_{c.}^{\phantom{c}d}=d\omega_{c.}^{\phantom{c}d}+\omega_{e.}^{\phantom{e}d}\wedge\omega_{c.}^{\phantom{c}e}=1/2R_{c\cdot ab}^{\phantom{c}d}\,\nu^a\wedge\nu^b$$

is 2-form of curvature. Now substituting  $\pi_a = E\psi^{\dagger}\psi \,\eta_{ab}\dot{x}^b$  into the above equation and keeping in mind fact that the coefficient at  $\eta_{ab}\dot{x}^b$  is constant we rewrite the equation as follows:

$$E\psi^{\dagger}\psi \, \frac{D\,\dot{x}^a}{ds} = 1/2\,R_{\delta\varepsilon \cdot b}^{\quad a}\,\dot{x}^b S^{\delta\varepsilon}. \tag{16}$$

It is seen that (14) and (16) constitute set of equations of motion of massless particle of spin s = 1/2 which is similar to system of Papapetrou equations for motion of photon in curved space-time as they presented in the work [8].

#### Acknowledgment

The author express his gratitude to Z. Ya. Turakulov who motivated the author to carry out these studies and whose critical remarks provided significant improve of present article. This research was supported by project FA-F2-F061 of Uzbekistan Academy of Sciences.

### References

- [1] Frenkel J.Z. fur Physik **37**, 243 (1926)
- [2] Papapetrou A, Proc. R. Soc. **A209**, p248 (1951)
- [3] W.G. Dixon *Proc. Roy. Soc. A* vl. 314, no. 1519 pp. 499-527 (1970)
- [4] Turakulov Z Ya, Classical Mechanics of Spinning Particle in a Curved Space. arXive:dg-ga/9703008 v.1, 14 March (1997)
- [5] A. Frydryszak, Lagrangian Models of Particles with Spin: the First Seventy Years. arXive:hep-th/9601020 v.1, 6 Jan (1996)
- [6] Turakulov Z Ya, Safonova M, Motion of a Vector Particle in a Curved Space-Time. I. Lagrangian Approach. Mod Phys Lett A18 (2003) 579
- [7] Turakulov Z Ya, Safonova M, Motion of a Vector Particle in a Curved Space-Time. II. First-Order Correction to a Geodesic in a Schwarzschild background. Mod Phys Lett A 20 (2005) 2785
- [8] Turakulov Z. Ya, Muminov A. T, Electromagnetic field with constraints and Papapetrou equation. Zeitschrift fur Naturforschung 61a, 146 (2006)
- [9] Muminov A T, Motion of Spin 1/2 Massive Particle in a Curved Space-Time. arXiv:0709.4607v1 [gr-qc] (2007)
- [10] M. Berg, C. DeWitt-Morette, Sh. Gwo and E. Kramer, *The Pin Groups in Physics:* C,P and T, arXive:math-ph/0012006 (2000).
- [11] Seminaire Arthur Besse 1978/79 Geometrie Riemannienne en Dimension 4 CEDIC/FERNAND NATHAN Paris (1981)
- [12] Messia A. Quantum Mechanics. vol 1,2 New York: J. Wiley & Sons (1958)
- [13] Bogoliubov N N and Shirkov D V, Introduction to the Theory of Quantized Fields. New York: Wiley-Interscience (1959)