Quantum Tops

The Schrödinger equations for non–relativistic spinning tops are written in terms of the Euler angles. The spherical, symmetrical, and asymmetrical tops Schrödinger equations are presented both in the free case and in the presence of external interactions. The free spherical top solutions are presented. They are written in terms of both half–integer and integer spin wave functions. The integer spin wave functions are the usual spherical harmonics functions while the half–integer spin wave functions are new. All of the cases are extended to accommodate translations in addition to spin. This work is based on "The Hurwitz Map and Harmonic Wave Functions for Integer and Half–Integer Angular Momentum", Sergio A. Hojman, Eduardo Nahmad-Achar and Adolfo Sánchez–Valenzuela, arXiv:2211.10775

I. DEFINITION OF THE HOPF MAP

The Hopf map is given by

$$\begin{aligned} x_1 &= 2(u_1u_3+u_2u_4)\ , \\ x_2 &= 2(-u_1u_4+u_2u_3)\ , \\ x_3 &= {u_1}^2+{u_2}^2-{u_3}^2-{u_4}^2\ , \end{aligned} \tag{1}$$

from $\mathbb{R}^4$ with coordinates u_α, with $1 \leq \alpha \leq 4$ to $\mathbb{R}^3$ with coordinates x_i with $1 \leq i \leq 3$. The Hopf map preserves spheres, i.e., an $\mathbb{S}^3$ sphere of radius R in $\mathbb{R}^4$ is mapped on an $\mathbb{S}^2$ sphere of radius r in $\mathbb{R}^3$ in such a way that

$$r \equiv \sqrt{{x_1}^2+{x_2}^2+{x_3}^2} = {u_1}^2+{u_2}^2+{u_3}^2+{u_4}^2 \equiv R^2\ , \tag{2}$$

as it is can be easily verified using Eqs.(1) (see, for instance, [1]).

Eqs.(1) may be inverted to get

$$\begin{aligned} u_1 &= \sqrt{r}\cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}\ , \\ u_2 &= \sqrt{r}\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}\ , \\ u_3 &= \sqrt{r}\sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}\ , \\ u_4 &= \sqrt{r}\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}\ , \end{aligned} \tag{3}$$

where r, θ and ϕ are the usual spherical coordinates related to cartesian ones by

$$\begin{aligned} x_1 &= r\sin\theta\cos\phi\ , \\ x_2 &= r\sin\theta\sin\phi\ , \\ x_3 &= r\cos\theta\ , \end{aligned} \tag{4}$$

and ψ is arbitrary (because this is a gauge (fiber) in the inverse Hopf map). Strictly speaking, θ, ϕ and ψ are the Euler angles .

Define the complex variables

$$\begin{aligned} z_1 &= u_1+iu_2 \\ &= \sqrt{r}\cos\frac{\theta}{2}\ \exp\left(i\frac{\psi+\phi}{2}\right) \\ z_2 &= u_3+iu_4 \\ &= \sqrt{r}\sin\frac{\theta}{2}\ \exp\left(i\frac{\psi-\phi}{2}\right)\ . \end{aligned} \tag{5}$$

The usual spin (angular momentum) operators may be written as [1]

$$\begin{aligned} \mathbf{L}_+ &= \left(z_1\frac{\partial}{\partial z_2}-\bar{z}_2\frac{\partial}{\partial \bar{z}_1}\right)\ , \\ \mathbf{L}_- &= -\left(\bar{z}_1\frac{\partial}{\partial \bar{z}_2}-z_2\frac{\partial}{\partial z_1}\right)\ , \\ \mathbf{L}_z &= \frac{1}{2}\left(z_1\frac{\partial}{\partial z_1}-z_2\frac{\partial}{\partial z_2}-\bar{z}_1\frac{\partial}{\partial \bar{z}_1}+\bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right), \end{aligned} \tag{6}$$

in terms of the complex variables (5), and

$$\begin{aligned} \mathbf{L}_1 &= \frac{i}{2}\Big(u_4\partial_1-u_3\partial_2+u_2\partial_3-u_1\partial_4\Big)\ , \\ \mathbf{L}_2 &= \frac{i}{2}\Big(u_3\partial_1+u_4\partial_2-u_1\partial_3-u_2\partial_4\Big)\ , \\ \mathbf{L}_3 &= \frac{i}{2}\Big(u_2\partial_1-u_1\partial_2-u_4\partial_3+u_3\partial_4\Big), \end{aligned} \tag{7}$$

in terms of the u_α variables (3).

These expressions in terms of the Euler angles θ, ϕ, and ψ are

$$\begin{aligned} \mathbf{L}_+ &= e^{i\phi}\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}-i\frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\right)\ , \\ \mathbf{L}_- &= e^{-i\phi}\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}+i\frac{1}{\sin\theta}\frac{\partial}{\partial\psi}\right)\ , \\ \mathbf{L}_3 &= -i\frac{\partial}{\partial\phi}\ , \end{aligned} \tag{8}$$

are the usual ones, except for the terms in red. Note that the terms in red vanish when acting on the usual spherical harmonics spin wave functions. Nevertheless, they are important when acting on spherical harmonics half–integer spin wave functions. It is important to realize that the new spin operators satisfy the usual $SO(3)$ commutation relations.

II. NEW OPERATORS AND WAVEFUNCTIONS

In this section, we display new operators and wavefunctions which do not appear in the usual spin treatment. These include raising and lowering operators $\mathbf{j}^{++}$, $\mathbf{j}^{+-}$, $\mathbf{j}^{-+}$, $\mathbf{j}^{--}$ for angular momentum which modifies wavefunctions by raising and lowering the total spin quantum number j and third component of the spin m by $\pm 1/2$ respectively, new spin wavefunctions which carry half–integer angular momentum quantum numbers j and m and a new operator $\jmath$ with (integer and half–integer) eigenvalues j, related to the usual spin square operator by $\mathbf{L}_1{}^2 + \mathbf{L}_2{}^2 + \mathbf{L}_3{}^2 = \mathbf{j}^2 + \mathbf{j}$ when acting on the harmonic functions space. The approach also recovers all of the usual results for integer values of j and m.

A. Total angular momentum raising and lowering operators

The total angular momentum raising operators are

$$\begin{aligned} \mathbf{j}^{++} &= z_1 \ , \\ \mathbf{j}^{++} &= \bar{z}_2 \ , \\ \mathbf{j}^{+-} &= \bar{z}_1 , \\ \mathbf{j}^{+-} &= z_2 \ . \end{aligned} \tag{9}$$

The total angular momentum lowering operators are

$$\begin{aligned} \mathbf{j}^{-+} &= \frac{\partial}{\partial z_2} \ , \\ \mathbf{j}^{-+} &= \frac{\partial}{\partial \bar{z}_1} \ , \\ \mathbf{j}^{--} &= \frac{\partial}{\partial z_1} , \\ \mathbf{j}^{--} &= \frac{\partial}{\partial \bar{z}_2} \ . \end{aligned} \tag{10}$$

Let us start by defining the wavefunctions in terms of the complex variables z_1, and z_2. The dictionaries given above allow for the translation of the different kinds of variables.

Use bra–ket notation $|j, m>$ for the angular momentum vectors, define $|0, 0>= 1$ and leave the normalization questions aside, for the moment. Define the $j = 1/2$ kets by

$$|1/2, +1/2 >= \mathbf{j}^{++}|0, 0 >= z_1|0, 0 > \ , \tag{11}$$

$$|1/2, -1/2 >= \mathbf{j}^{+-}|0, 0 >= z_2|0, 0 > \ , \tag{12}$$

to get the $j = 1/2$ harmonic functions

$$f_{1/2,+1/2}(r, \theta, \phi) = \sqrt{r} \cos\frac{\theta}{2} \ \exp\left(i\frac{\psi + \phi}{2}\right) \ , \tag{13}$$

and

$$f_{1/2,-1/2}(r, \theta, \phi) = \sqrt{r} \sin\frac{\theta}{2} \ \exp\left(i\frac{\psi - \phi}{2}\right) \ . \tag{14}$$

The $j = 1/2$ angular momentum functions $f_{1/2,m}$ are "ψ gauge" dependent and we know the $j = 1$ angular momentum functions should not depend on ψ to agree with usual results.

Recall that

$$\mathbf{L}_{\pm} f_{j,m} = \sqrt{(j \mp m)(j \pm m + 1)} \ f_{j,m\pm 1} \ . \tag{15}$$

It is important to realize that

$$\mathbf{L}_{\mp}|1/2, \pm 1/2 >= |1/2, \mp 1/2 > \ , \tag{16}$$

and

$$\mathbf{L}_3|1/2, \pm 1/2 >= \pm 1/2 \ |1/2, \pm 1/2 > \ , \tag{17}$$

where $\mathbf{L}_{\pm}$ and $\mathbf{L}_3$ are defined in (6).

Define the spin operator $\mathbf{j}$ by

$$\mathbf{j} \equiv r\frac{\partial}{\partial r} \ , \tag{18}$$

and get

$$\mathbf{j} f_{1/2,m} = 1/2 f_{1/2,m} = j f_{1/2,m} \ . \tag{19}$$

The general way to prove that $\mathbf{L}_1{}^2 + \mathbf{L}_2{}^2 + \mathbf{L}_3{}^2 = \mathbf{j}^2 + \mathbf{j}$ when acting on harmonic functions is by writing the Laplace operator in terms of the square of the angular momentum operator and use the fact that harmonic functions satisfy Laplace's equation.

The usual $j = 1$ ("ψ gauge" invariant) harmonic functions may be written in terms of the complex variables z as

$$f_{1,+1}(r, \theta, \phi) = -r \exp(i\phi)\frac{\sin\theta}{2} = -\bar{z}_2 z_1|0, 0 >, \tag{20}$$

$$f_{1,0}(r, \theta, \phi) = -\frac{r}{\sqrt{2}} \cos\theta = z_1\bar{z}_1 - \bar{z}_2 z_1|0, 0 >, \tag{21}$$

and

$$f_{1,-1}(r, \theta, \phi) = r \exp(-i\phi)\frac{\sin\theta}{2} = \bar{z}_1 z_2|0, 0 >, \ . \tag{22}$$

Note that $f_{1,0}(r, \theta, \phi)$ and $f_{1,-1}(r, \theta, \phi)$ may be obtained starting from $f_{1,+1}(r, \theta, \phi)$ by applying $\mathbf{L}_-$ successively to $f_{1,+1}(r, \theta, \phi)$ (or starting from $f_{1,-1}(r, \theta, \phi)$) by applying $\mathbf{L}_+$ successively to $f_{1,-1}(r, \theta, \phi)$).

In general, to get higher (half–integer or integer) j spin harmonic functions it is enough to start from $f_{j,j}(r,\theta,\phi,\psi)$ and apply $\mathbf{j}^{++}$, i.e.,

$$f_{j+1,j+1}(r,\theta,\phi,\psi) = z_1 f_{j,j}(r,\theta,\phi,\psi) \qquad (23)$$

or

$$f_{j+1,j+1}(r,\theta,\phi,\psi) = \bar{z}_2 f_{j,j}(r,\theta,\phi,\psi) \qquad (24)$$

depending on whether j is integer or half–integer and use $\mathbf{L}_-$ repeatedly, to lower the value of m.

Similarly, one may apply $\mathbf{j}^{+-}$ to $f_{j,-j}(r,\theta,\phi,\psi)$ to get

$$f_{j+1,-j-1}(r,\theta,\phi,\psi) = z_2 f_{j,-j}(r,\theta,\phi,\psi) \qquad (25)$$

or

$$f_{j+1,-j-1}(r,\theta,\phi,\psi) = \bar{z}_1 f_{j,j}(r,\theta,\phi,\psi) \qquad (26)$$

depending on whether j is integer or half–integer and use $\mathbf{L}_+$ repeatedly, to raise the value of m.

By the same token, to get lower (integer or half–integer) j spin harmonic functions it is enough to start from $f_{j,-j}(r,\theta,\phi,\psi)$ and apply $\mathbf{j}^{-+}$ to it

$$f_{j-1,-j+1}(r,\theta,\phi,\psi) = \frac{\partial}{\partial z_2} f_{j,-j}(r,\theta,\phi,\psi) \qquad (27)$$

or

$$f_{j-1,-j+1}(r,\theta,\phi,\psi) = \frac{\partial}{\partial \bar{z}_1} f_{j,-j}(r,\theta,\phi,\psi) \qquad (28)$$

depending on whether j is half–integer or integer and use $\mathbf{L}_+$ repeatedly, repeatedly to raise the value of m.

Similarly, one may apply $\mathbf{j}^{--}$ to $f_{j,j}(r,\theta,\phi,\psi)$ to get

$$f_{j-1,j-1}(r,\theta,\phi,\psi) = \frac{\partial}{\partial z_1} f_{j,-j}(r,\theta,\phi,\psi) \qquad (29)$$

or

$$f_{j-1,j-1}(r,\theta,\phi,\psi) = \frac{\partial}{\partial \bar{z}_2} f_{j,j}(r,\theta,\phi,\psi) \qquad (30)$$

depending on whether j is half–integer or integer and use $\mathbf{L}_-$ repeatedly, to lower the value of m.

This approach solves the problem of finding a unified way of dealing with half–integer and integer spherical harmonics (by dividing $f_{j,m}$ by r^j and normalizing).

The treatment presented here aims to construct the usual spherical harmonics. One still has the choice to use the other coordinates (r, ψ and ϕ) and give rise to totally new (equivalent) representations of the integer and half–integer spherical harmonics.

III. THE SCHRÖDINGER EQUATIONS

We start by writing the Schrödinger equation, for a spherical free top

$$\mathbf{H}\Psi(X,Y,Z,\theta,\phi,\psi,t) = -i\hbar\frac{\partial\Psi(X,Y,Z,\theta,\phi,\psi,t)}{\partial t}\ , \qquad (31)$$

where the Hamiltonian $\mathbf{H}$ is given by (I is the moment of inertia of the top)

$$\mathbf{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial Z^2}\right) - \frac{\hbar^2}{2I}{\mathbf{L}_{New}}^2\ , \qquad (32)$$

in terms of the coordinates X, Y, and Z of the "center of mass" of the top and where the (square of the "new") spin operator ${\mathbf{L}_{New}}^2$ is written in terms of the (square of the "old") spin operator ${\mathbf{L}_{Old}}^2$ and the Euler angles as

$${\mathbf{L}_{New}}^2 = {\mathbf{L}_{Old}}^2 + 2\csc(\theta)\cot(\theta)\frac{\partial}{\partial\theta}\frac{\partial}{\partial\psi} - \csc(\theta)^2\frac{\partial^2}{\partial\psi^2}\ . \qquad (33)$$

It is a straightforward matter to realize that the wave functions are given by

$$\Psi_{E,j,m}(X,Y,Z,r,\theta,\phi,\psi,t) = f_{j,m}(r,\theta,\phi,\psi)\, e^{(i(P_X X + P_Y Y + P_z Z - Et)/\hbar)}\ , \qquad (34)$$

and

$$E = \frac{1}{2m}({P_X}^2 + {P_Y}^2 + {P_Z}^2) + \frac{\hbar^2}{2I}j(j+1)\ . \qquad (35)$$

Ya me cansé por hoy. Después sigo. El r está relacionado con el tamaño del trompo. El momento de inercia I, puede estar relacionado con r y con m. También es claro cómo introducir trompos no esféricos y también las interacciones con campos magnéticos, aunque no he encontrado las soluciones todavía. Parece que las traslaciones y las rotaciones no se mezclan en la teoría no relativista.

[1] Sergio A. Hojman, Eduardo Nahmad-Achar and Adolfo Sánchez–Valenzuela, "The Hurwitz Map and Harmonic Wave Functions for Integer and Half–Integer Angular Mo-

mentum", arXiv:2211.10775