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On system reliability for time-varying structure

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ABSTRACT

In reliability theory, the aging of a multi-state system is dominated by both the components and the corresponding structure functions. In previous studies, structures are usually assumed to be static, and thus the time-independent structure functions are utilized. However, due to the complex nature of aging, the structure could also vary with time, which may lead to unsatisfactory assessment reliability with the static structure-based analysis. The current investigation provides a universal approach to assessing the reliability of complex systems with time-varying structures. An open-system model is introduced to broaden the logic method of the system reliability. The analysis of open-system model implies that structure functions are probabilistically described by the time-varying structure distributions, which are the fine graining of the conditional probabilistic tables (CPTs) of the Bayesian networks. The aging of components and the time-varying structures are integrated into a probabilistic graphical model together, which is put forth to assess the time-varying reliability of complex systems. A general algorithm based on expectation–maximization (EM) for various dynamic processes for components and system structures is obtained. Two specific processes, e.g., Markov and Weibull, are studied in detail. Three examples are presented to illustrate the proposed approach and give a deeper understanding of time-varying structures.

1. Introduction

Reliability engineering plays an important role in handling the failures of the systems and equipment [1,2]. Numerous methods have been developed to assess the failure probability of systems, such as Monte Carlo simulation [3–5], first-order and second-order methods [6–8], fault trees [9–11] and etc. As a system becomes complex, the overall performance might be too complicated to be modeled by binary states, and thus the multi-state modeling should be made for such a complex system [12–14]. That is a system involving numerous components, and there are more than two states for the considered system that depends on the components' states. The dependence is described by the so-called structure function.

Various approaches to system reliability are based on mathematics of the structure functions, e.g., [15–18]. Zaitseva and Levashenko compute the Dynamic Reliability Indices base on the structure function and Logical Differential Calculus [19], and they also construct structure functions with fuzzy decision trees [18]; Merle et al. present an algebraic framework of determining the structure function of any Dynamic Fault Tree [20,21], Yuan et al. combine the structure function and the active learning kriging model [22], and the structure function with continuous states has been studied [23,24]. In these previous studies, the structure function is proposed to be static. However, due to the complex nature of aging, the structure could also vary with time and even randomly, and the assumption of static structure may lead to unsatisfactory assessment results. To the best of the authors' knowledge, this problem has not been studied systematically.

One of the most significant challenges in reliability assessment is the uncertainty of aging which includes aleatory and epistemic uncertainties [25–27]. The uncertainty usually causes a stochastic aging process. Many efforts on modeling stochastic aging processes are made, including but not limited to, Bayesian networks [28–36], survival signature [37,38], kriging model [22,39–41] and etc., to handle the reliability problems under the influence of the uncertainties. Coolen et al. generalize classical probability theory to the imprecise probability to deal with the fuzzy problems brought by the uncertainties [42,43]. In detail, incomplete data is one of the reasons that brings uncertainty. Observation data are indispensable for modeling a system. The system's structure, the components' information, and the state sequences of both components and system, etc., are needed. However, in most practical cases, such information is not easy to obtain, which leads to the study of incomplete data in reliability assessment [44–47].

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Another result of the aleatory and epistemic uncertainty is the uncertainty in structure, which may be caused by, for example, the problems of updatable and multi-function systems [48], the insufficient knowledge of the system structure, and the rough division of continuous states. In fact, the healthy conditions of components and systems, determined by continuous physical parameters, are approximated by several discrete states, which would bring uncertainty. To depict the time-varying structure and deal with the uncertainty in structure, we introduce the structure distribution in this paper, which describes the probabilities that the system takes different structure functions and their variation over time. The aging of components and the time-varying structures are integrated into a probabilistic graphical model to assess the time-varying reliability of complex systems. The expectation–maximization (EM) algorithm is used to determine the model's parameters in both types of incomplete data.

In order to give a better description of the above uncertainties, we propose an open system framework, or say open-system model. In the open-system model, the environment is composed of unknown factors. The interaction between the environment and the system makes the system uncertain, which means that a structure function cannot describe the system structure accurately anymore. A probabilistic structure function is thus needed. The structure distribution is a tool for depicting the probabilistic structure function. Further discussion and the examples of unknown factors is presented in Section 2.2.

The proposed approach in the current study mainly aims at dealing with systems with non-stationary structures. It is a rational extension of the traditional Bayesian network based models, such as those in Refs. [29,33,36], and is also used to deal with incomplete data. The structure distribution builds a connection between structure functions and Bayesian networks, where the conditional probability table (CPT) is further interpreted as the coarse graining of the structure distribution. In dynamic Bayesian networks (DBN) the graphical structure of the system at each time should not change over time, and the parameters in the CPT of each node at different time should be the same [36,49]. This assumption corresponds to the time-independent structure distribution in our proposed approach. In addition, the proposed approach can also deal with the situations with time-dependent structure distributions. To note that, CPTs, which can be seen as the marginal distributions of the structure distribution at different time slices, can be generated from different structure-varying processes, e.g., a time-invariant CPT can be margined from either a time-varying structure which is statistical independent at each time slices or a static structure. For an updated system, this could bring observable differences related to the time correlation of structures. Additionally, this study proposes two examples where estimations based on static structure assumption bring obvious error. This error is shown to be reduced to varying degrees by using different patterns of the time-varying structure.

The remainder of this paper is organized as follows. Section 2 introduces the uncertainty of the structure and gives the definition of the structure distribution. The idea of open-system model is discussed mainly in Section 2.2. Section 3 develops a graphical probabilistic model to assess reliability with the structure distribution. Meanwhile, the relationship between the Bayesian model and the structure distribution is discussed. In Section 4, EM algorithms are adopted to predict the components and system states and infer the time-varying structure distribution. Two illustrative examples are presented in Section 5 to give a demonstration of the method and a deeper understanding of uncertainty in structures. Finally, in Section 6, discussions and conclusions are presented.

2. Open system and structure uncertainty

In this section, we first review the definition of structure function. To illustrate a general picture of the probabilistic structure, we introduce a concept — open systems, which are the systems interacting with several uncertain sources called environments. Three kinds of uncertain sources are introduced to clarify the meaning of open systems in reliability assessment: unknown external factors, hidden components, and the imprecise health states division. Finally, to seek the mathematical description of open systems, we introduce the probabilistic structure function via a set of possible structures. The definition of structure distribution is described, and the relevant conceptions are clarified. The examples in Section 5 will help comprehension.

2.1. Structure function

The structure function is a discrete function that describes the influences of the reliability of components on the reliability of the system [15,16]. In most recent literature, the structure function is deterministic for a given system. A general definition of the structure function is given as follows.

Definition 2.1 (*Structure Function*). For a multi-state system with N components, the state of ith component is denoted by u_i, $u_i \in \{0, 1, \ldots, M-1\}$, where M is the total number of component states. The state S of the system is represented as

$$S = \phi(u_1, u_2, \ldots, u_N) \equiv \phi(\boldsymbol{u}) \in \{0, 1, \ldots, L-1\}, \tag{1}$$

where L is the total number of system states. The function ϕ : $\{0, 1, \ldots, M-1\}^N \to \{0, 1, \ldots, L-1\}$ is defined as the **structure function** of the system.

Commonly, the structure function reflects the logical relationship for the system's composition.

2.2. Open system

The MSS is widely applied in recent reliability assessments, where, in the MSS method, structure function is defined as the logical relationship between the components and the system. These structure functions are usually assumed to be deterministic in traditional MSS methods. This is based on a fundamental assumption that all factors directly influencing a system's functioning state are clearly listed and can be efficiently modeled by the components. To emphasize this assumption, we shall sometime use the term *closed-system model*. However, this is usually difficult, in general, to achieve in practice due to the complexity of the physical mechanics of degradation and the environment surrounding the system. Many factors are prone to be neglected or difficult to be modeled as components in practice.

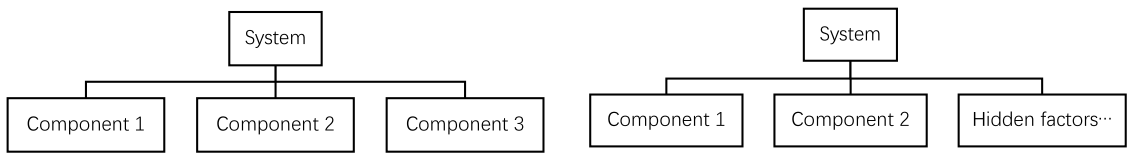
Three further examples are discussed in the following subsections.

With these arguments, we shall use the term *the open-system model* if the model only contains a part of factors, and there exist some factors that are not included in the model, which are called hidden factors. These hidden factors are called *environment* in the open-system model. The structures are shown in Fig. 1: 1(a) the closed-system model structure; 1(b) the open-system model structure.

Following the fundamental assumption in MSS, the structure containing all the factors is written as a deterministic structure function as Eq. (1). All the factors that influence the system are denoted by a list of variables $u_1, u_2, \ldots, u_N$, which can be interpreted as the states of components. The question is that if we do not know the formula of the structure function ϕ and the variables $u_{N-m}, u_{N-m+1}, \ldots, u_N$ in practice, how the MSS method could be applied. To overcome this problem, one could choose an open-system model alternatively. The effective structure function of the variables $u_1, u_2, \ldots, u_{N-m-1}$ defined as

$$\begin{aligned}\bar{\phi}(u_1, u_2, \ldots, u_{N-m-1}) \equiv & \sum_{x_{N-m},\ldots,x_N} \phi(u_1, \ldots, u_{N-m-1}, x_{N-m}, \ldots, x_N) \\ & \times \Pr(u_{N-m} = x_{N-m}, \ldots, u_N = x_N),\end{aligned} \tag{2}$$

where $\Pr(u_{N-m} = x_{N-m}, u_{N-m+1} = x_{N-m+1}, \ldots, u_N = x_N)$ is the marginal distribution of the hidden factors $u_{N-m}, u_{N-m+1}, \ldots, u_N$ with certain values $x_{N-m}, x_{N-m+1}, \ldots, x_N$, and $\phi(u_1, \ldots, u_{N-m-1}, x_{N-m}, \ldots, x_N)$ could



(a) deterministic structure, often used to describe a closed system.

(b) open system structure with hidden factors

Fig. 1. Structures of open and closed system.

Table 1
The 16 structure functions in the primary set of a binary system.

u_1	0	0	1	1
u_2	0	1	0	1
$\phi_1(u_1,u_2)$	0	0	0	0
$\phi_2(u_1,u_2)$	0	0	0	1
$\phi_3(u_1,u_2)$	0	0	1	0
$\phi_4(u_1,u_2)$	0	0	1	1
$\phi_5(u_1,u_2)$	0	1	0	0
$\phi_6(u_1,u_2)$	0	1	0	1
$\phi_7(u_1,u_2)$	0	1	1	0
$\phi_8(u_1,u_2)$	0	1	1	1
$\phi_9(u_1,u_2)$	1	0	0	0
$\phi_{10}(u_1,u_2)$	1	0	0	1
$\phi_{11}(u_1,u_2)$	1	0	1	0
$\phi_{12}(u_1,u_2)$	1	0	1	1
$\phi_{13}(u_1,u_2)$	1	1	0	0
$\phi_{14}(u_1,u_2)$	1	1	0	1
$\phi_{15}(u_1,u_2)$	1	1	1	0
$\phi_{16}(u_1,u_2)$	1	1	1	1

be also regarded as a particular structure function of $u_1,u_2,\dots,u_{N-m-1}$ that depends on the values of $x_{N-m},x_{N-m+1},\dots,x_N$. One can label them by

$$\phi_{x_{N-m},\dots,x_N}(u_1,\dots,u_{N-m-1}) \equiv \phi(u_1,\dots,u_{N-m-1},x_{N-m},\dots,x_N). \tag{3}$$

It is clear that any one of these functions $\phi_{x_{N-m},\dots,x_N}(\cdot)$ is a deterministic structure function. For example, if $N-m-1=2$, $\phi_{x_3,\dots,x_N}(\cdot)$ is one of the functions in Table 1.

It follows that

$$\bar{\phi}(u_1,u_2,\dots,u_{N-m-1}) = \sum_k \phi_i(u_1,\dots,u_{N-m-1})\alpha_k, \tag{4}$$

where the label k runs over the primary set (the set consists of all the possible structure functions, defined in Definition 2.2), and α_k is the probability of the kth $(N-m)$-component structure function. With this formula, it is clear that an open-system model requires a probabilistic structure function. Furthermore, if the hidden factors vary with time, then the probabilistic structure function is also time-varying. While calculating the probability of the system state, the statistical formula in Definition 2.2 is implemented.

The uncertainty brought by the hidden factors(environment), which is written as the probabilistic structure function as Eq. (4), are called *the structure uncertainty*.

The above statement implies that all the structure distribution could be generated by an original structure function that includes both components and hidden factors. Thus, the open-system model is an extension of traditional MSS. As long as we get all the information of the system, including all the components and hidden factors, the original structure function ϕ can be precisely written, which means the system is completely deterministic so that the structure uncertainty disappears. This uncertainty caused by ignorance of the original structure is consistent with the view of subject probability [50]. To treat this problem, the open-system model could be applied.

Ref. [51] proposes a general framework to model multiple deterioration processes and their interplays. In that framework, the time-dependent capacities, demands, and external conditions imply the varying-structure nature. Compared to this framework, our study focuses on the logical structure, and the proposal is a generalization of the traditional logical method in reliability theory.

In the following, we introduce three kinds of sources that cause the structure uncertainty: unknown external factors, hidden components, and the imprecise health states division.

Unknown external factors

As pointed out by Ref. [48], one should consider the probabilistic mixture of different structure functions to deal with the updatable and multi-function systems. Updating and function switching of the system is affected by the human operation, which is a kind of unknown external factor that cannot be depicted precisely. To deal with this uncertainty, an open-system model is put into implementation.

Let $\phi_1,\dots,\phi_K$ be the potential structure functions of the system (for different functions or updating), and $\Pr(\phi_k)$ be the probability of choosing ϕ_k. The probability of the system state $S=y$ is then be

$$\Pr(S=y) = \sum_{k=1}^{K} \Pr(\phi_k)\Pr(S=y|\phi_k). \tag{5}$$

It is worth emphasizing that human operation is only one of the external factors; there are still many external unknown factors, like the weather, the location of the equipment, etc.

Hidden components

As a system is getting increasingly complex, it is challenging to launch accurate observations on every part of the system. There exist some parts of the system that cannot be accurately described. This not only means that one do not have the data of these components, but the original structure function is also unknown. We call these components *the hidden components*. Through the open-system model discussion above, these hidden components can be regarded as a part of the environment. The uncertainty enters the model through the system-environment interaction, represented by the probabilistic structure function.

Let $u_1,\dots,u_N$ be all the components of the system, and $u_1,\dots,u_M$ be the components observed($M \le N$). The components $u_{M+1},\dots,u_N$ are the hidden components. The system structure function of the full closed system is a function of variables $u_1,\dots,u_N$, i.e., $\phi(u_1,\dots,u_M,u_{M+1},\dots,u_N)$. The structure function we can write based on the observation is a function of variables $u_1,\dots,u_M$, read $\phi'(u_1,\dots,u_M)$. It is obvious that ϕ' cannot describe all the information of ϕ. In fact, ϕ' cannot exactly conform to the observation data. To deal with this uncertainty brought by hidden components, the probabilistic structure function like Eq. (4) is put into use.

For example, one can only observe two components u_1,u_2. There will be parallel-like data: u_1 survives, u_2 fails and the system survives. There will also be some series-like data: u_1 survives, u_2 fails and the system fails. A determinate ϕ' cannot describe a system with both parallel and series structures. Only the probabilistic structure function can correctly describe the system, as mentioned in Eq. (5). In this example, let $\phi_{\mathrm{para}}(u_1,u_2)$ and $\phi_{\mathrm{seri}}(u_1,u_2)$ be the parallel and series structure function, respectively. The probability of the system state $S=y$ then be

$$\Pr(S=y) = P(\phi_{\mathrm{para}})P(S=y|\phi_{\mathrm{para}}) + P(\phi_{\mathrm{seri}})P(S=y|\phi_{\mathrm{seri}}). \tag{6}$$

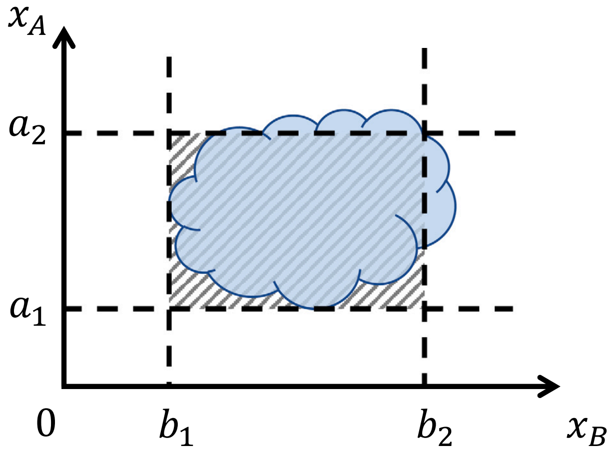


Fig. 2. Example of imprecise health states division. One can never describe the blue cloud region (determined by the physical variables of the system) by the dashed region (determined by the reliability variables and the structure function) precisely.

Imprecise health states division

In addition to the above two sources that bring uncertainty, there is also a significant factor: the imprecise health states division. In MSS theory, the system is assumed to have several discrete states. As mentioned in a lot of literature, this is an approximation. It means that the physical variables of the system are discretized into a finite number of reliability states. In other words, reliability states are used to approximate continuous variables (physical variables). This will inevitably result in the loss of the system's information. This division accurately reduces the region of each variable in the structure function, which will also bring a closed system to an open system.

For example, a bipartite system $S = (A, B)$, let $x_A, x_B \in (0, \infty)$ be the physical variables describing the two parts, and $u_A, u_B \in \{0, 1\}$ be the reliability states(take binary system for simplicity), respectively. To define the reliability region of each part, the division of the physical variables will be made. Let $u_A = 1$ if $a_1 < x_A \le a_2$, and $u_A = 0$ for the rest of x_A(the same division goes for u_B and x_B). From the reliability states of the components, one can easily define the survival region of the system from the structure function $\phi(u_A, u_B)$. It is noticed that continuous variables can never be precisely described by a finite number of discrete variables. These imprecise descriptions lead to uncertainty in the structure.

A simple division is shown in Fig. 2. The dashed region is the survival region of the system, which is determined by the structure function $\phi(u_A, u_B)$. The blue cloud region is the actual survival region of the system, which is determined by the physical variables x_A and x_B. One can easily see that, there is no perfect matching between the dashed and the cloud area, no matter how the division is chosen (a_1, a_2, b_1, b_2 and the structure function $\phi(u_A, u_B)$). In order to compensate for the uncertainty caused by imprecise division, the probability structure function is put into operation. We cannot describe the system by a specific structure function, so probabilities over a set of potential structure functions are used, i.e., Eq. (5).

2.3. Structure distribution

As the discussion of the open system, it is necessary to consider structure uncertainty while implementing reliability assessment. The structure uncertainty is described by the probabilities on a set of potential structure functions. We introduce the structure distribution to describe these probabilities. The related definition is given as follows.

Definition 2.2 (*Structure Distribution*)**.** For a multi-state system with N components, the system state is denoted by a random variable S. The possible structure functions of the system $\phi_1, \phi_2, \dots, \phi_K$ form **primary set** $\Phi = \{\phi_1, \phi_2, \dots, \phi_K\}$. The elements ϕ_k of Φ are called **primary structure functions**. Accordingly, the probability that the system at state is y is determined by

$$\Pr(S = y) = \sum_{k=1}^{K} \alpha_k \Pr(S = y|\phi_k), \tag{7}$$

where $\alpha_k \equiv \Pr(\phi_k)$ represents the probability of the kth primary structure function ϕ_k, which satisfies $\sum_{k=1}^{K} \alpha_k = 1$, and $\Pr(S = y|\phi_k)$ is the conditional probability of the system state S being y under the condition of the structure function ϕ_k. In definition, call $\{\alpha_1, \alpha_2, \dots, \alpha_K\}$ **the structure distribution** of the corresponding primary structure function set Φ.

For fixed component sequence $u_1 = x_1, u_2 = x_2, \dots, u_N = x_N$, the system stat y only depends on the structure functions. For a structure function ϕ_k, the state of the system is $\phi_k(x_1, x_2, \dots, x_N)$. Hence, the conditional probability of the system state $S = y$ under the component sequence $x_1, x_2, \dots, x_N$ and the structure function ϕ_k reduce to a Kronecker delta function $\Pr(S = y|u_1 = x_1, \dots, u_N = x_N, \phi_k) = \delta_{\phi(\boldsymbol{x}),y}$, which equal to 1 if $\phi(\boldsymbol{x}) = y$ and 0 for $\phi(\boldsymbol{x}) \neq y$. The quantify

$$\begin{aligned}
&\Pr(S = y|u_1 = x_1, \dots, u_N = x_N) \\
&= \sum_{k=1}^{K} \alpha_k \Pr(S = y|u_1 = x_1, \dots, u_N = x_N, \phi_k) \\
&= \sum_{k=1}^{K} \alpha_k \delta_{\phi(\boldsymbol{x}),y}
\end{aligned} \tag{8}$$

is the conditional probability of the system state $S = y$ under the given sequence $\boldsymbol{x}$. When $\alpha_m = 1$, then $\alpha_k = 0$ $(k \neq m)$, the uncertainty disappears and the structure distribution is reduced to a deterministic structure function ϕ_m.

It is natural to consider the time-dependent structure distribution, i.e., the time-varying structure:

$$\Pr(S = y) = \sum_{k=1}^{K} \alpha_k(t) \Pr(S = y|\phi_k). \tag{9}$$

Here, the time-dependence of reliability is reflected by the time-varying probabilities $\alpha_k(t)$, which must keep the normalization $\sum_{k=1}^{K} \alpha_k(t) = 1$ in any instance.

The time-varying structure would cause a big difference in reliability assessment. At the same time, this time dependence brings more difficulties in predicting the aging of such systems. The following section presents an approach to model the systems with time-varying structures and predict their aging.

3. Probabilistic graphical model

A general approach to analyzing the reliability problems for time-varying structure systems is proposed in this section. Here, we consider the statistical independence components to illustrate the method more pellucidly.

3.1. Markov aging process of components

Markov chain is assumed as a suitable mathematical model to describe the components that initially work well and finally break down [1,52]. The Markov process is applied to model the independent degradation of the components.

For a component with M states, the component state at time t is $u(t) \in \{0, 1, \dots, M-1\}$. In Markov chains, the component state at time t only depends on its state at instance $t-1$. The transition probabilities of different states are given by

$$\Pr(u(t)|u(t-1), \dots, u(0)) = \Pr(u(t)|u(t-1)). \tag{10}$$

The Markov transition matrix is $\mathcal{T}$, $\Pr(u(t+1) = b|u(t) = a) = \mathcal{T}_{ab}$.

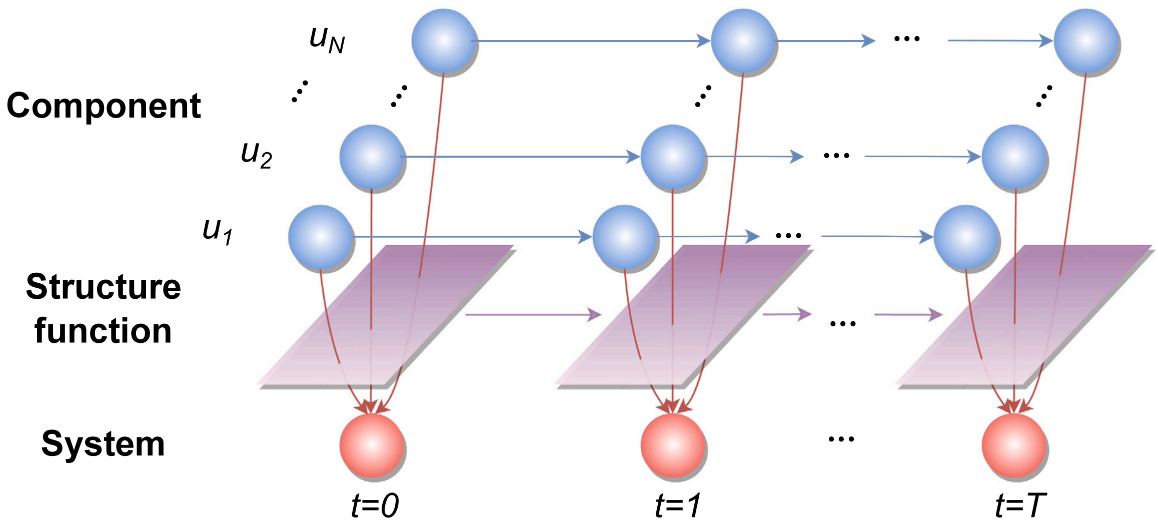


Fig. 3. The graphical representation of the approach. In the graph, blue balls, purple slabs, and red balls represent the component states, structure functions, and system states, respectively. Blue balls and blue arrows compose the Markov degradation of components. Purple slabs and purple arrows represent the time-varying structure. The system state, i.e., the red shadowed node, depends on the component states and the structure function at each instance, represented by the red arrows.

We next consider non-repairable systems [15] that the system and components perform well at the beginning and can only transition to a state with worse performance. Consequently, the transition matrix $\mathcal{T}$ becomes triangular, namely

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}_{00} & & & \\ \mathcal{T}_{10} & \mathcal{T}_{11} & & \\ \vdots & \vdots & \ddots & \\ \mathcal{T}_{M-1,0} & \mathcal{T}_{M-1,1} & \cdots & \mathcal{T}_{M-1,M-1} \end{pmatrix}. \tag{11}$$

Here, we let 0 represent the failure state, and $M-1$ represent the perfect-performed state. The smaller, the worse.

It is emphasized that structure distributions defined above are essentially independent of the degradation of components. Thus the Markov degradation process for components can be replaced with any other rational independent degradations. Moreover, the proposed structure-distribution approach could be, in principle, extended to deal with correlated-component problems.

3.2. Graphical model

For modeling the components' degradations and the time-varying structures (i.e., structure distribution), a probabilistic graphical model is proposed to give the whole picture of the approach, see Fig. 3.

The graphical model depicts the conditional correlations between random variables [31]. In the graph, blue balls, red balls, and purple slabs represent the component states, the system states, and the structure functions, respectively. Blue arrows in the graph represent the Markov process for each component. Purple slabs and purple arrows represent the time-varying structure. As defined in Definition 2.2, the system state depends on the component states and the primary structure function at each instance, which is represented by the red arrows.

Consider an N-component system, each component possesses M states. The component state of the ith component at time t is denoted by a random variable $u_i(t)$, and the value of $u_i(t)$ is $x_t^{(i)} \in \{0, 1, \dots, M-1\}$. The system state at time t is represented by a random variable S_t, and its value is $y_t \in \{0, 1, \dots, L-1\}$, where L is the state number of the system. The structure function at time t is described by a random variable Z_t, and its value is $z_t \in \{1, 2, \dots, K\}$, K is the size of the primary set $\boldsymbol{\Phi}$.

During the working stage, as time changes from 0 to T, the observation sequence of the ith component is $\boldsymbol{x}^{(i)} = \{x_0^{(i)}, x_1^{(i)}, \dots, x_T^{(i)}\}$, the observation sequence of the system is $\boldsymbol{y} = \{y_0, y_1, \dots, y_T\}$, and the chosen structure function sequence is $\boldsymbol{z} = \{z_0, z_1, \dots, z_T\}$.

At time t, we denoted the observation data of all the components by $\boldsymbol{x}_t = \{x_t^{(1)}, x_t^{(2)} \cdots, x_t^{(N)}\}$. All the observation data of all the components are denoted as $\boldsymbol{x} = \{\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_T\} = \{\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}, \dots, \boldsymbol{x}^{(N)}\}$.

All the parameters in the model are written as $\Theta = \{\boldsymbol{\alpha}, \lambda\}$, where $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_K\}$ is the structure distribution, and λ includes the parameters in the aging process of components. In this paper, we have assumed the Markov degradation process, such that $\lambda = \mathcal{T} = \{\mathcal{T}^{(1)}, \mathcal{T}^{(2)}, \dots, \mathcal{T}^{(N)}\}$, where $\mathcal{T}^{(i)}$ is the ith component's Markov transition matrix.

3.3. Structure distribution vs. Bayesian network

The similarity and differences between the proposed model (in Fig. 3) and the Bayesian network are discussed as follows.

A structure distribution with more parameters can be coarse-grained into a Bayesian network CPT with a few parameters. As an illustration, we consider a 2-state system of two components. The total number for primary structure functions is 16, as shown in Table 1, which correspond to a structure distribution with 15 independent parameters $\alpha = \{\alpha_1, \dots, \alpha_{16}\}$, where $\sum_{i=1}^{16} \alpha_i = 1$. For example, ϕ_2 denotes the series structure function and ϕ_8 denotes the parallel structure function. In Bayesian model, there are 4 parameters in the CPT of the same system, as shown in Table 2. For example, q_1 denotes the probability of the system bing in state 0 while both components are in state 0. We can always use 15 parameters to represent 4 parameters. However, the two different structure distributions can reduce to the same CPT. This implies that structure distributions contain more information than CPTs. For example, the conditional probability table in Table 3 reduces to two different systems with structure uncertainty: (1) the system with structure distribution $\alpha_2 = \alpha_8 = 0.5, \alpha_k = 0\ (k \neq 2, 8)$, which means the system has the probability of 0.5 to take series and parallel, i.e. the primary structure functions ϕ_2 and ϕ_8, respectively; (2) the system with structure distribution $\alpha_4 = \alpha_6 = 0.5, \alpha_k = 0\ (k \neq 4, 6)$, which means the system states are only determined by the states of component 1 or component 2 with the probability of 0.5, i.e. the primary structure functions ϕ_4 and ϕ_6, respectively. In fact, the CPTs are the coarse graining of the structure distributions as the summations presented in Table 2. The structure distributions contain more detailed information than the CPTs. The information loss during the coarse graining could not be recovered from CPTs.

Apart from components, the system's structure also varies over time, which renders a time-varying structure distribution. This could be an extension and explanation of the CPTs in Bayesian networks. With the

Table 2
The conditional probability table of 2-state system.

u_1	u_2	$S=0$	$S=1$	representation by α
0	0	q_1	$1-q_1$	$q_1=\sum_{i=1,2,3,4,5,6,7,8}\alpha_i$
0	1	q_2	$1-q_2$	$q_2=\sum_{i=1,2,3,4,9,10,11,12}\alpha_i$
1	0	q_3	$1-q_3$	$q_3=\sum_{i=1,2,5,6,9,10,13,14}\alpha_i$
1	1	q_4	$1-q_4$	$q_4=\sum_{i=1,3,5,7,9,11,13,15}\alpha_i$

Table 3
An example of conditional probability table.

u_1	u_2	$S=0$	$S=1$
0	0	1	0
0	1	0.5	0.5
1	0	0.5	0.5
1	1	0	1

proposed model, one can treat an N-component system as an $(N+1)$-node Bayesian network. The extra node in the network represents the structure of the system. The states of the extra node correspond to the structure function and the conditional probability is determined by the structure distribution. The structure function can be noticed from the state of this node, whose dynamic represents the time dependence of the structure distribution. For non-repairable systems, the component degrades irreversibly, but the time-varying of the structure is not necessarily irreversible.

4. Structure distribution learning

In this section, the algorithm for obtaining the parameters of the introduced model for non-repairable systems is proposed. This algorithm is suitable for both complete and incomplete observation data. A general process for the algorithm is given as follows. With the general process, the exact algorithm under the Markov assumption is presented.

The incomplete observation data are denoted as $\widetilde{D}=\{(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}})\}$, where $\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}}$ represent the incomplete observation data of components and system, respectively. Let $\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp}$ denote the missing part of the corresponding $\widetilde{\boldsymbol{x}}$ and $\widetilde{\boldsymbol{y}}$. The primary set is written as $\Phi=\{\phi_1,\phi_2,\ldots,\phi_K\}$. Here, K is not necessarily the total number of the structure functions, which rely on prior knowledge. In practice, this prior knowledge could be obtained through expert advice, physical structure analysis, etc. The algorithm is to obtain the structure distribution of the system from the incomplete observation data $\widetilde{D}$, which is to calculate the parameters $\Theta=\{\boldsymbol{\alpha},\lambda\}$ that maximize the probability of $\widetilde{D}$, i.e., to maximize the likelihood $L(\widetilde{D})$,

$$\begin{aligned} L(\widetilde{D}) &= \prod_{(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}})\in\widetilde{D}}\Pr(\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}})=\prod_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\sum_{\boldsymbol{y}^{\perp}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \\ &= \prod_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\sum_{\boldsymbol{y}^{\perp}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{z})\Pr(\boldsymbol{x}|\boldsymbol{z})\Pr(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z}) \\ &= \prod_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\sum_{\boldsymbol{y}^{\perp}}\Pr(\boldsymbol{x})\sum_{\boldsymbol{z}}\left[\Pr(\boldsymbol{z})\prod_{t=0}^{T}\delta_{\phi_{z_t}(\boldsymbol{x}_t),y_t}\right]. \end{aligned} \tag{12}$$

The EM algorithm [53–55] is used to deal with this problem. The optimization problem is

$$\begin{aligned} &\Theta^{[l+1]} \\ &= \arg\max_{\Theta}\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp},\boldsymbol{z}}\Pr(\boldsymbol{z},\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}|\Theta). \end{aligned} \tag{13}$$

We can write

$$\begin{aligned} &\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp},\boldsymbol{z}}\Pr(\boldsymbol{z},\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}|\Theta) \\ &= \sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\Pr(\boldsymbol{x}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{x}|\Theta) \\ &\quad+\sum_{\widetilde{D}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{z}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{z}|\Theta) \\ &\quad+\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp},\boldsymbol{z}}\Pr(\boldsymbol{z},\boldsymbol{x}^{\perp},\boldsymbol{y}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z},\Theta). \end{aligned} \tag{14}$$

Notice that

$$\Pr(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z},\Theta)=\prod_{t=0}^{T}\delta_{\phi_{z_t}(\boldsymbol{x}_t),y_t}\equiv\prod_{t=0}^{T}\Pr(y_t|\boldsymbol{x}_t,z_t,\Theta), \tag{15}$$

which is not related to the optimized parameters Θ. Thus, the optimization can be simplified to

$$\begin{aligned} \Theta^{[l+1]} &= \arg\max_{\lambda}\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\Pr(\boldsymbol{x}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{x}|\Theta) \\ &\quad+\arg\max_{\alpha}\sum_{\widetilde{D}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{z}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\ln\Pr(\boldsymbol{z}|\Theta). \end{aligned} \tag{16}$$

It is worth emphasizing that the structure function has been encoded into the nodes of the Bayesian network. Thus the equation $\Pr(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{z},\Theta)=\prod_{t=0}^{T}\Pr(y_t|\boldsymbol{x}_t,z_t,\Theta)$ does not rely on any further assumptions (for example, the instantaneous CPTs are assumed to be independent of each other in the previous study [36]).

The above discussion is suitable for general conditions. Different specific dynamical models involve different expressions of $\Pr(\boldsymbol{x}|\Theta),\Pr(\boldsymbol{z}|\Theta)$. Taking the Markov process as an example, we assume that the Markov transition matrixes for the dynamics of components and structures are $\mathcal{T}=\{\mathcal{T}^{(1)},\mathcal{T}^{(2)},\ldots,\mathcal{T}^{(N)}\}$ and $\mathcal{A}$, respectively. The optimization problem is further expressed as

$$\begin{aligned} \Theta^{[l+1]} &= \arg\max_{\mathcal{T}}\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\Pr(\boldsymbol{x}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\sum_{i=1}^{N}\sum_{t=0}^{T-1}\ln\mathcal{T}^{(i)}_{x_t^{(i)},x_{t+1}^{(i)}} \\ &\quad+\arg\max_{\mathcal{A}}\sum_{\widetilde{D}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{z}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})\sum_{t=0}^{T-1}\ln\mathcal{A}_{z_t,z_{t+1}}. \end{aligned} \tag{17}$$

The optimization solution of the problem is obtained by the Lagrange multiplier method, which reads

$$\begin{aligned} \mathcal{T}_{ab}^{(j)[l+1]} &= \frac{\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\Pr(\boldsymbol{x}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})n_{ab}(\boldsymbol{x}^{(j)})}{\sum_{\widetilde{D}}\sum_{\boldsymbol{x}^{\perp}}\Pr(\boldsymbol{x}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})n_{a}(\boldsymbol{x}^{(j)})}, \\ \mathcal{A}_{ab}^{[l+1]} &= \frac{\sum_{\widetilde{D}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{z}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})n_{ab}(\boldsymbol{z})}{\sum_{\widetilde{D}}\sum_{\boldsymbol{z}}\Pr(\boldsymbol{z}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})n_{a}(\boldsymbol{z})}, \end{aligned} \tag{18}$$

where $n_{ab}(\boldsymbol{z})=\sum_{t=0}^{T-1}\delta_{z_t,a}\delta_{z_{t+1},b}$ is the number of transition from state a to state b in sequence $\boldsymbol{z}$, and $n_a(\boldsymbol{z})=\sum_b n_{ab}(\boldsymbol{z})$ is number of state a in sequence $\boldsymbol{z}$.

The E-step is calculating the posterior probability $\Pr(\boldsymbol{x}^{\perp}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]},\Theta^{[l]})$ and $\Pr(\boldsymbol{z}|\widetilde{\boldsymbol{x}},\widetilde{\boldsymbol{y}},\Theta^{[l]})$, the M-step is calculating the parameters with Eq. (18). Iterating E and M steps, the algorithm will converge to the optimal solution $\mathcal{T}^*,\mathcal{A}^*$, which maximizes the likelihood $L(\widetilde{D})$. The structure distribution is computed with

$$\alpha_i(t)=\sum_{\{(z_0,z_1,\ldots,z_t)|z_t=i\}}\Pr(z_0,z_1,\ldots,z_t|\mathcal{A}^*). \tag{19}$$

In addition, if one uses Weibull distributions to describe the dynamic of components and structure, the expressions of $\Pr(\boldsymbol{x}|\Theta),\Pr(\boldsymbol{z}|\Theta)$ are as follows.

$$\begin{aligned} \Pr(\boldsymbol{x}|\Theta) &= \prod_{i=1,2}a_i b_i(t_{\boldsymbol{x}^{(i)}})^{b_i-1}e^{-a_i(t_{\boldsymbol{x}^{(i)}})^{b_i}}, \\ \Pr(\boldsymbol{z}|\Theta) &= a_s b_s(t_{\boldsymbol{z}})^{b_s-1}e^{-a_s(t_{\boldsymbol{z}})^{b_s}}, \end{aligned} \tag{20}$$

where a_i,b_i,a_s,b_s are the parameters of Weibull distributions, $t_{\boldsymbol{x}^{(i)}}$ and $t_{\boldsymbol{z}}$ are the corresponding survival time of sequences $\boldsymbol{x}^{(i)}$ and $\boldsymbol{z}$ (i.e. the number of 1 in the sequence), respectively. The survival probabilities of components and structure are represented by Weibull distributions, which write $R_i(t)=e^{-a_i t^{b_i}}, i=1,2$ and $R_s(t)=e^{-a_s t^{b_s}}$, respectively.

5. Illustrative examples

In this section, the approach and algorithms we have proposed above are demonstrated by three physical examples. The examples also give an insight into the structure uncertainty and the open-system model, as discussed in Section 2.2.

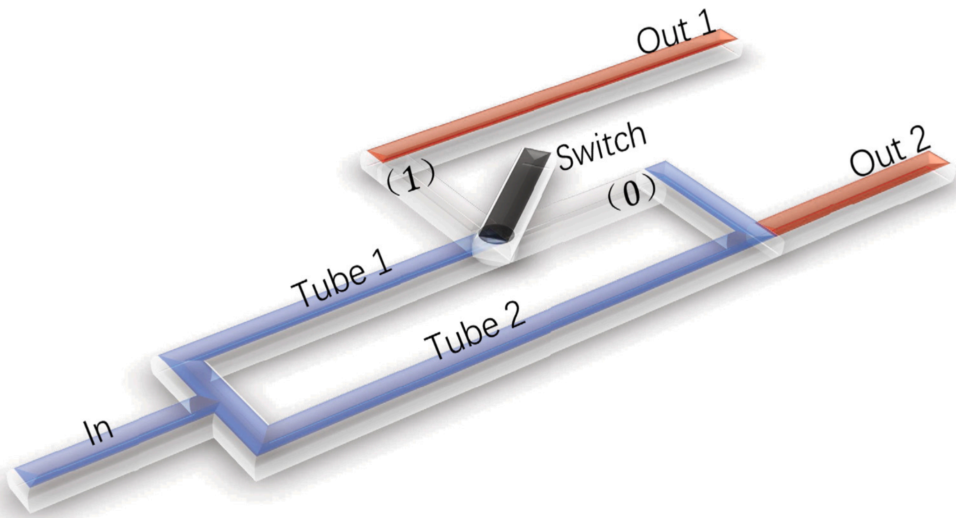


Fig. 4. The sketch of the irrigation system.

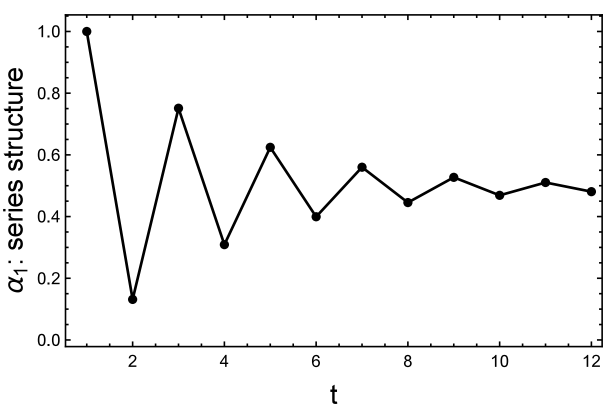


Fig. 5. The structure distribution α_1.

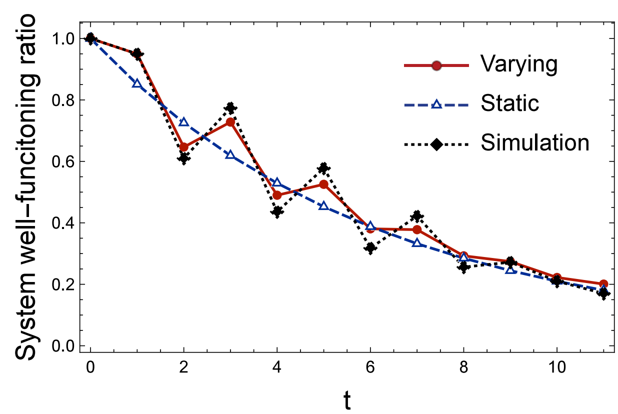


Fig. 6. The well-functioning ratios of the system with static and time-varying structure.

5.1. A structure-varying irrigation system

This example shows that a time-varying structure is needed for practice. The irrigation system is sketched in Fig. 4. Water goes in from the left-bottom "In" port, and a switch control two modes: (1) water goes to both outlets (Out 1 and Out 2) or (0) only goes to the second outlet (Out 2).

This irrigation system has two binary components: Tube 1 and Tube 2. Each component is unrepairable and degrades independently. If the switch is in mode (1), the structure of the system is series, and parallel for mode (0). The system structure depends on the switch's mode. Thus, the system's degradation depends not only on the components but also on the structure. A stationary structure function cannot sufficiently describe this system since the switch' mode could vary with time.

The simulation data is generated by the following rules: the tubes' hazard rates are the same to be 0.15. The switch flips periodically, and a white noise flips the instantaneous mode to the opposite one with a probability of 0.9 in every time step. The total time length is 12, and both tubes are in normal states at the beginning. The initial mode of the switch is set to be series, i.e., mode (1).

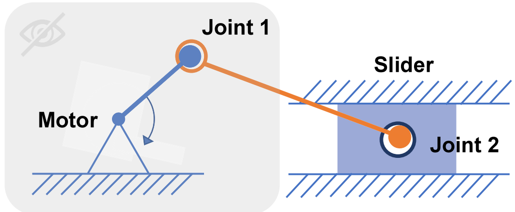
The well-functioning ratio of the system shown by the generated data is pictured in Fig. 6, the black diamond dotted line. Through the proposed approach, structure distribution is inferred and shown in Fig. 5. We can see that the structure distribution oscillates. The well-functioning ratio of the system with the distribution structure result from the proposed approach is shown in Fig. 6, the red circle line. It depicts the oscillation pattern of the data accurately. This pattern cannot be recovered by the static-structure model, which is shown in the blue triangle dashed line. This is an example of the mentioned situation of a function-switching system, for which the structure-varying approach fits better.

5.2. A mechanic slider

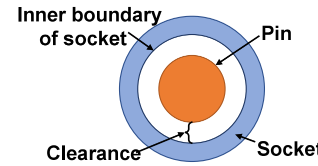
In this example, we demonstrate that structure uncertainty can be brought by hidden components. The hidden components, as claimed in Section 2.2, are the unobserved components whose existence will cause incomplete knowledge of the original structure function, thus leading to the discussion of the open-system model. A simple crank–slider mechanism considering clearance joints, as shown in Fig. 7(a), is presented here for illustration.

As the system works, the slider moves back and forth. The friction in the joints will cause the degradation of the system. A joint consists of a socket and a pin, as in Fig. 7(b). We assume that the hardness of the pin is much greater than the hardness of the socket so that the wear only happens on the inner boundary of the sockets.

The system actually consists of two joints. Suppose that, due to some reasons, one can only observe Joint 2. Thus, Joint 1 becomes the hidden component. All the information of Joint 1 as well as the structure



(a) the crank-slider mechanism



(b) the joint

Fig. 7. The sketch of the mechanic slider.

function of the entire system is inaccessible. This means that, in the view of the observer, the system contains only one component, i.e., the state of Joint 2 directly "determines" the system state. However, the hidden component Joint 1 indeed influence the system-level aging. Therefore, the mapping between the states of Joint 2 and the system is not one-to-one, so the system might be in the failure state while Joint 2 is in the normal state. Thus, the deterministic structure based function (as in Definition 2.1) is no longer valid.

Using the idea of an open-system model, as introduced in Section 2.2, the probabilistic structure function needs to be implemented. The effective structure function of this one-component system is constructed by the probability mixture of the one-variable structure functions, i.e., the primary structure functions.

We consider binary components, the normal state 1 and the failure state 0. Suppose that the primary set is $\boldsymbol{\Phi} = \{\phi_1, \phi_2\}$,

$$\phi_1(x) = \begin{cases} 1, x = 1 \\ 0, x = 0 \end{cases}, \quad \phi_2(x) = \begin{cases} 0, x = 1 \\ 0, x = 0 \end{cases}, \tag{21}$$

where x is the state of Joint 2. The probabilities of ϕ_1 and ϕ_2 are α_1 and α_2, respectively. The structure distribution is $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$. It results from Ref. [56] that the radius of clearance increases almost linearly. We assume that the degrading of joints obeys the normal distribution, i.e. the increase of the clearance $l_i(t) \sim N(\mu, \sigma)$, where $l_i(t)$ denotes the increased clearance of Joint i at time t, $i = 1, 2$. The system fails when $\sum_{i=1,2} \sum_t l_i(t) > l_c$, l_c is the failure threshold. To make the demo simple, the structure function of the two joints is assumed to be series (this structure function is only used for generating simulation data and not used in the model-testing). We consider two cases: (A) the degradation of Joint 1 (hidden component) is much slower than that of Joint 2, thus the state for Joint 2 is almost equal to that of the system and the conventional method is still valid; (B) the hazard rates of Joint 1 and Joint 2 are comparable: the conventional method breaks down due to the considerable uncertainty brought by the hidden component Joint 1. We calculate the system's failure rate in both situations, and the result is shown in Fig. 8.

It is observed from Fig. 8 that: in case (A), the failure probability of the system and Joint 2 are close to each other. Thus, one can approximately apply the deterministic-one-component structure function. In case (B), the structure uncertainty caused by the hidden component brings a big difference between the failure probabilities of the system and Joint 2. Thus, the deterministic structure function can no longer give a good description of the system. With the open-system approach, the result ("Weibull" curve) accurately reflects the failure probability of the system ("Case B" curve).

Parameters in the calculating: the increased clearance of Joint 2 l_2, Joint 1 in Case (A) l_{1A} and Joint 1 in Case (B) l_{1B} follow normal distributions, $l_2 \sim N(0.9, 0.3)$, $l_{1A} \sim N(0.6, 0.2)$, $l_{1B} \sim N(0.8, 0.3)$, respectively. The sample number is $N_{\text{sample}} = 5000$. The failure criterions of Joint 1 and Joint 2 are $l_{c2} = 30$, $l_{c1} = 20$, respectively. The units of distance and time are 10^{-5} m and hour, respectively. The selection of parameters partly refers to [56].

In this example, we demonstrate the structure uncertainty caused by hidden components, which can also be seen as the incomplete observation of the structure. Through the idea of the open-system model

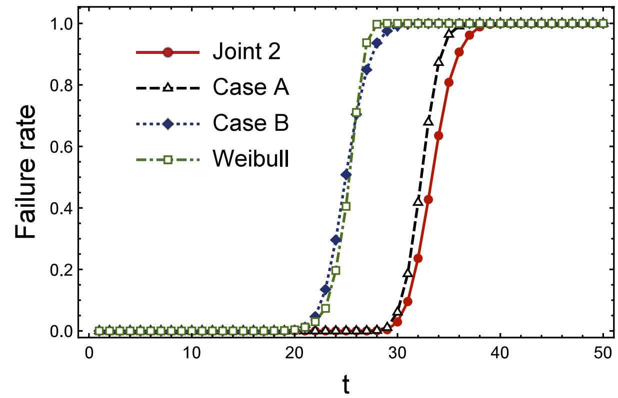


Fig. 8. The results of the probability of failure in different cases. The red circle curve, labeled "Joint 2", shows the actual Joint 2's failure probability. The dashed black triangle curve, labeled "Case A", shows the actual system failure probability in case (A): the influence brought by the degradation of Joint 1 is small enough compared to that of Joint 2. The dotted blue diamond curve, labeled "Case B", shows the actual system failure probability in case (B): the influence brought by the degradation of Joint 1 is comparable to that of Joint 2. These three curves are the direct results of the simulation data, which reflects the real failure possibility. In the conventional method, seeing the system as a one-component system, the system level aging should be the same as the Joint 2's aging. The dash-dotted green square curve, labeled "Weibull", shows the system failure probability calculated by the proposed approach. One can see that, in case (A), when the influence brought by the degradation of Joint 1 is small, using the failure probability cure of joint 2 to represent that of the system is feasible. However, in case (B), when the influence brought by the degradation of Joint 1 is comparable to that of Joint 2, the curve of joint 2 is no longer close to the system failure curve ("Case B"). Using the idea of an open-system model and the structure distribution approach, the result ("Weibull" curve) accurately reflects the failure probability of the system.

and the proposed approach, the accuracy of system aging estimation is significantly improved compared to the traditional deterministic structure function method.

5.3. An electricity circuit

In this subsection, an electrical circuit is discussed to illustrate the structure uncertainty brought by the imprecise health states division (as claimed in Section 2.2). Consider a parallel circuit involving two components, i.e., two resistors, sketched in Fig. 9.

5.3.1. Physical analysis

The operation of the circuit raises the temperature, and the increase in temperature accelerates the aging of the resistor. There are two terms describing the change of resistance, the temperature-dependent term $R_{\mathrm{a},i}$, and the process-dependent term $R_{\mathrm{b},i}$, $i = 1, 2$. The temperature-dependent term is reversible, meaning that the resistance decreases while the temperature decreases. The process-dependent term is irreversible and relates to all the historical operations of the system. The irreversible term will not decrease if the temperature drops and always increases with working time.

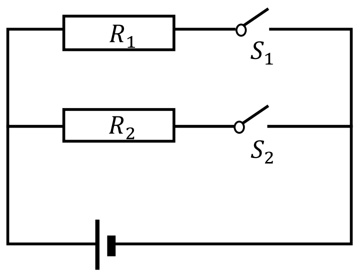


Fig. 9. The sketch of the parallel circuit.

The resistance R_i of component i is a sum of the temperature-dependent part and the process-dependent part,

$$R_i(t) = R_{\mathrm{a},i}(t) + R_{\mathrm{b},i}(t),\ i = 1, 2. \tag{22}$$

The temperature-dependent part $R_{\mathrm{a},i}$ reads

$$R_{\mathrm{a},i}(t) = c_a(T_i(t) - T_0) + R_0, \tag{23}$$

where R_0 is the initial resistance of the component, T_0 is the room temperature, and c_a is the temperature coefficient of resistance. The process-dependent part $R_{\mathrm{b},i}$ is assumed to be

$$R_{\mathrm{b},i}(t) = c_g \int_0^t T_i(t')g_i(t')\,\mathrm{d}t', \tag{24}$$

where c_g is the coefficient relates to aging, $g(t)$ is the switch function which is 1 for on and 0 for off respectively. Explicitly, let $\mathcal{O} = \cup_n \mathcal{O}_n$ denote the union of all the time intervals of switching on, and its complement $\mathbb{R}^+ - \mathcal{O}$ for switching off. The interval $\mathcal{O}_n$ is randomly picked in $\mathbb{R}^+$ and satisfy the condition $\mathcal{O}_n \cap \mathcal{O}_m = \emptyset,\ n \neq m$. It follows that $g(t)$ is given as

$$g(t) = \begin{cases} 1, & t \in \mathcal{O} \\ 0, & t \in \mathbb{R}^+ - \mathcal{O}. \end{cases} \tag{25}$$

The heat production of component i with dissipation is described by

$$\frac{\mathrm{d}Q_i}{\mathrm{d}t} = \frac{U^2 g_i(t)}{R_i(t)} - \gamma(T_i(t) - T_0), \tag{26}$$

where γ is the dissipation coefficient, and U is voltage. With specific heat relations $\mathrm{d}Q_i = c\,\mathrm{d}t$ where c is the specific heat coefficient, we obtain the differential equations

$$\begin{cases} \dfrac{\mathrm{d}T_i}{\mathrm{d}t} = \dfrac{1}{c}\left[\dfrac{U^2 g_i(t)}{c_a(T_i(t) - T_0) + R_{\mathrm{b},i}(t) + R_0} - \gamma(T_i(t) - T_0)\right], \\ \dfrac{\mathrm{d}R_{\mathrm{b},i}}{\mathrm{d}t} = c_g T_i(t) g_i(t) \end{cases} \tag{27}$$

with initial conditions $T_i(0) = T_0,\ R_{\mathrm{b},i}(0) = 0,\ (i = 1, 2)$.

The parameter to quantify the component's health condition is its resistance at room temperature. Namely, the failure of a resistor is defined as that the irreversible term $R_{\mathrm{b},i}$ become greater than a failure criterion R_c, i.e., $R_{\mathrm{b},i} > R_c$. The irreversible part of the system resistance $R_{\mathrm{b},sys}$ is defined as

$$R_{\mathrm{b},sys}(t) \equiv \frac{1}{\frac{1}{R_{\mathrm{b},1}(t)+R_0} + \frac{1}{R_{\mathrm{b},2}(t)+R_0}} - \frac{R_0}{2}, \tag{28}$$

which is the difference between the system resistance at room temperature T_0 and the initial system resistance $R_0/2$. The failure criterion of the system is that the irreversible term $R_{\mathrm{b},sys}$ becomes greater than a failure criterion R_d, i.e., $R_{\mathrm{b},sys} > R_d$. In this example, we let $R_d = R_c/2$. In the working stage, two switch functions are independently random.

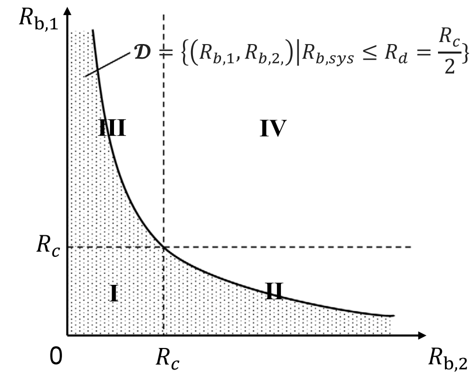


Fig. 10. The failure criteria for the system and components. The failure criterion of component 1 is $R_{\mathrm{b},1} > R_c$, which corresponds to the survival region of component 1: I + II. Similarly, the survival region of component 2 is I + III. The failure criterion of the system is determined by $R_{\mathrm{b},sys} > R_d = R_c/2$, which corresponds to the survival region of the system: the shadowed region, denoted by $\mathcal{D}$. The failure criterion of a logical series system is $R_{\mathrm{b},1} > R_c$ and $R_{\mathrm{b},2} > R_c$, which corresponds to the survival region of the system: I. The failure criterion of a logical parallel system is $R_{\mathrm{b},1} > R_c$ or $R_{\mathrm{b},2} > R_c$, which corresponds to the survival region of the system: I + II + III.

5.3.2. Structure distribution of the circuit

In this subsection, a theoretical analysis of the structure distribution is presented. The random switch of the switches will cause a time-varying probability density $p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t)$ of two irreversible terms. The normalization condition

$$\iint_{\mathbb{R}^+\times\mathbb{R}^+} \mathrm{d}R_{\mathrm{b},1}\mathrm{d}R_{\mathrm{b},2}\, p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t) = 1 \tag{29}$$

holds for every instant. The failure criteria for the system and components are sketched in Fig. 10.

The time-varying survival probabilities r_1 and r_2 of two resistances are calculated according to the criterion $R_{\mathrm{b},i} \le R_c$,

$$\begin{aligned} r_1(t) &= \int_0^{R_c} \mathrm{d}R_{\mathrm{b},1} \int_0^{\infty} \mathrm{d}R_{\mathrm{b},2}\, p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t), \\ r_2(t) &= \int_0^{R_c} \mathrm{d}R_{\mathrm{b},2} \int_0^{\infty} \mathrm{d}R_{\mathrm{b},1}\, p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t). \end{aligned} \tag{30}$$

Similarly, the survival probability r_{sys} of the system is given as

$$r_{sys}(t) = \iint_{\mathcal{D}} \mathrm{d}R_{\mathrm{b},1}\mathrm{d}R_{\mathrm{b},2}\, p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t), \tag{31}$$

where the region $\mathcal{D}$ is determined by the criterion $R_{\mathrm{b},sys} \le R_d = R_c/2$, i.e. the shadowed region in Fig. 10.

However, it is found that the survival probability r_{sys} of the system is different from that evaluated by either the logical series and logical parallel, i.e., the two primary structure functions. The survival probability of the system evaluated by logical series is written as

$$\begin{aligned} r_{\mathrm{series}}(t) &= \iint_{\mathrm{I}} p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t)\,\mathrm{d}R_{\mathrm{b},1}\mathrm{d}R_{\mathrm{b},2} \\ &= r_1(t) \times r_2(t), \end{aligned} \tag{32}$$

where the region I means $R_{\mathrm{b},1} \le R_c$ and $R_{\mathrm{b},2} \le R_c$. Similarly, survival probability for logical parallel is written as

$$\begin{aligned} r_{\mathrm{parallel}}(t) &= \iint_{\mathrm{I+II+III}} p(R_{\mathrm{b},1}, R_{\mathrm{b},2}; t)\,\mathrm{d}R_{\mathrm{b},1}\mathrm{d}R_{\mathrm{b},2} \\ &= r_1(t) + r_2(t) - r_1(t) \times r_2(t), \end{aligned} \tag{33}$$

where the integration region I + II + III means $R_{\mathrm{b},1} \le R_c$ or $R_{\mathrm{b},2} \le R_c$. Here, one can see that, $r_{\mathrm{series}} \le r_{sys} \le r_{\mathrm{parallel}}$. Thus, there exist an unique $\alpha(t)$ such that

$$\alpha(t) \times r_{\mathrm{parallel}}(t) + (1 - \alpha(t)) \times r_{\mathrm{series}}(t) = r_{sys}(t). \tag{34}$$

Here, $\alpha(t)$ is the probability for the parallel structure function. It is worth mentioning that the structure function is defined by the logical

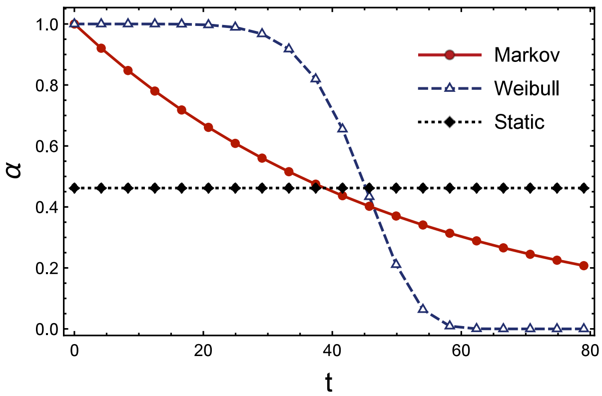


Fig. 11. Structure distributions with different processes: Markov process, Weibull distribution, and static structure. The degradations of components are described by Weibull distribution. The results are calculated under the incomplete data with drop ratio $d_p = 10\%$.

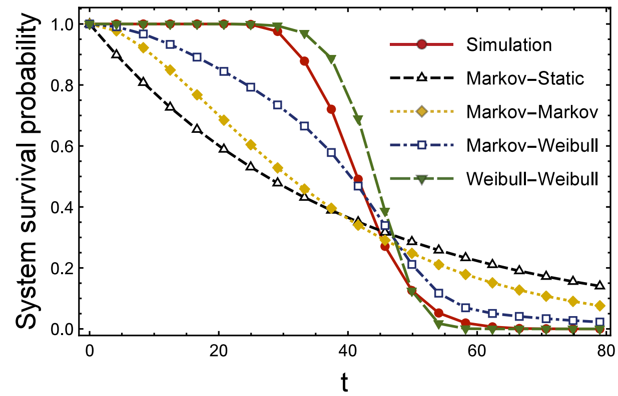


Fig. 12. System survival probabilities for different aging processes: Markov components and static structure, Markov components and Markov structure distribution, Markov components and Weibull structure distribution, Weibull components and Weibull structure distribution. The results are compared to the numerical simulation data. The results are calculated under the incomplete data with drop ratio $d_p = 10\%$.

structure which, in common, is not equivalent to the physical structure. In this example, the system is physically a parallel circuit but logically a mixture of parallel and series.

As discussed in Section 2.2, one cannot represent the shadowed region in Fig. 10 by two binary components(seem as the cloud region in Fig. 2). There exist situations that the system is in a different health state while the components' states are the same, i.e. the components are in normal and fault states respectively, and the system can be in a normal or a fault state. The structure becomes probabilistic and the structure distribution is needed.

5.3.3. Illustration of the approach

In this part, we test the proposed method with the simulation data for the circuit. The numerical simulation generates 10,000 sample data for a total of 80 time steps. The observation data are the collection of the generated data every 4 time step. We further consider two kinds of data incompletion: random missing and time truncation. For random missing data, we drop the data in the observation data randomly with the drop ratio d_p. For time truncation data, we just drop all the data after the cut-off time T_c.

To show the difference between different aging processes, we use the Markov and Weibull aging processes to model the dynamics of components and structure. The structure distributions of different processes are shown in Fig. 11. The survival probability of the system is shown in Fig. 12. The time-varying property of structure distribution can significantly increase the performance of the graphical model since the figures show that the Weibull distribution based model possesses the highest accuracy. The Markov-static curve reveals the performance of the model for the time-independent CPT presented in previous studies.

To deal with the incomplete observation data, we compare the results of two different cases: random missing and time truncation. The performance of the model with missing data is shown in Fig. 13, which presents a high tolerance for different missing ratios. However, the cut-off time T_c does affect the fitting results, as shown in Fig. 14. A short cut-off time that prevents the observation of most of the failed events leads to the failure of forecasting the dynamics.

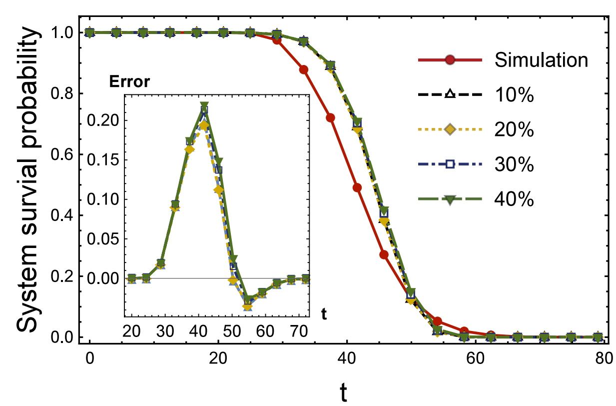


Fig. 13. System survival probabilities and error with different drop ratios. The fitting model is chosen as "Weibull-Weibull". From the error plot, we can see that the approach is robust with different ratios of incomplete data.

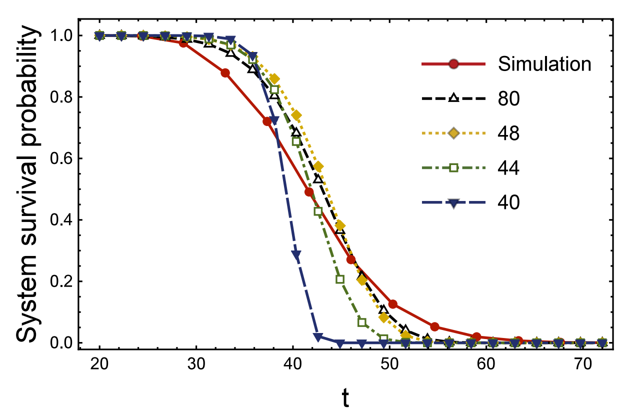


Fig. 14. System survival probabilities with different cut-off times. The fitting model is chosen as "Weibull-Weibull". The longer the observation time, the more accurate the results will be.

6. Discussion and conclusions

In this paper, we propose a systematical approach to the reliability assessment for systems with time-varying and uncertain structures. An idea of open system is introduced to generalize the description of the system reliability. Three kinds of sources that can bring uncertainty are discussed: unknown external factors, hidden components, and the imprecise health states division. By introducing the structure distribution, the structure uncertainty is well depicted and is shown as an extension of CPTs in Bayesian network based approaches. To be more detailed, the CPTs are the coarse-graining of the structure distributions. By using the structure distribution, the dynamics of components and structures are described by a novel graphical model. A general algorithm based on the EM for various dynamic processes for components and system structures is obtained. Two specific processes, e.g., Markov and Weibull, are studied in detail. Three illustrations are presented to show the necessity of considering the time-varying structure and demonstrate the proposed approach with different uncertainty sources mentioned above.

For the proposed approach, there are several advantages:

(a) The open-system model broadens the logic method in the reliability assessment. Because of the complexity of realistic systems, it is usually difficult to determine all the factors that directly influence the system reliability. Instead of original structure function that defined on the all factors, the efficient structure function of the known factors is represented as the probabilistic mixture of the primary structure functions.
(b) It is able to model both the system with time-varying and static structure, which could be regarded as a significant improvement of the previous studies [29,33,36].
(c) It can deal with incomplete observation data: (1) all information and the existence of some components are missing; (2) the information of the components is partially missing, for example, due to the observation with limited frequency or limited time domain.
(d) By clarifying the correspondence between the structure functions and CPTs in dynamic Bayesian networks, the interpretability of the model is improved, and it could be applied for updatable systems and multi-function systems [48].

In this paper, we mainly consider the interplays between components and the system, the correlation among components has not been considered in detail. This is related to the study on the interdependent components, where the model based on, for example, Markov Random Field, could be studied. The extended model for repairable systems could be further discussed. The approach can be directly extended to hierarchical multi-state systems. However, for the hierarchical structure with a large number of levels, the proposed algorithm needs to be further optimized to preserve the efficiency of simulations.

CRediT authorship contribution statement

L.X. Cui: Writing – original draft, Software, Methodology, Formal analysis, Data curation, Conceptualization. **Yi-Mu Du:** Writing – review & editing, Methodology, Formal analysis, Conceptualization. **C.P. Sun:** Writing – review & editing, Supervision, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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