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Dynamic asset allocation with stochastic income and interest rates ☆

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ABSTRACT

We solve for optimal portfolios when interest rates and labor income are stochastic with the expected income growth being affine in the short-term interest rate in order to encompass business cycle variations in wages. Our calibration based on the Panel Study of Income Dynamics (PSID) data supports this relation with substantial variation across individuals in the slope of this affine function. The slope is crucial for the valuation and riskiness of human capital and for the optimal stock/bond/cash allocation both in an unconstrained complete market and in an incomplete market with liquidity and short-sales constraints.

1. Introduction

It is well-documented in the theoretical asset allocation literature that the inclusion of labor income has dramatic effects on the optimal long-term portfolio choice of individual investors. Several studies, e.g. Heaton and Lucas (1997) and Cocco, Gomes, and Maenhout (2005), conclude that for an empirically reasonable insignificant correlation between labor income shocks and stock market shocks, the labor income stream is a substitute for an investment in the risk-free asset so that the financial wealth should be directed to stocks (typically, significantly levered, if possible). However, as these studies are cast in a setting where interest rates are assumed constant, they cannot distinguish short-term risk-free assets (cash deposits) from long-term risk-free assets (Treasury bonds). In order to investigate when human capital resembles a long-term bond and when it resembles cash and to assess the implications for the optimal stock/bond/cash portfolio choice, we set up, calibrate, and solve a specific model with stochastic interest rates and with a stochastic labor income that can be instantaneously correlated with interest rates, bond prices, and stock prices.

A special and important feature of our model is that the expected labor income growth rate is an affine function of the real short-term interest rate in order to

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encompass business cycle variations in wages, bonuses, and layoffs. Our calibration of the model based on PSID income data supports such a relation with a substantial variation across individuals in the business cycle sensitivity of income, i.e., the slope of the relation between expected income growth and the short-term interest rate. We demonstrate that this slope is crucial for the valuation and riskiness of the human capital and, consequently, for the optimal stock/bond/cash allocation. If the expected labor income growth is non-cyclical (zero slope), the human capital substitutes a long-term coupon bond. In that case, the optimal unconstrained investment of the financial wealth involves a large long position in stocks and significant borrowing, and will typically still involve a long position in long-term bonds for speculation and intertemporal hedging purposes. If the income is counter-cyclical (negative slope), the human capital is equivalent to a levered position in a long-term bond, and a smaller (larger) share of the financial wealth should be allocated to bonds (cash). If the income is pro-cyclical (positive slope) and the slope is exactly equal to one, the human capital will substitute for cash only. If the slope is higher than one, the human capital is like having a short position in a long-term bond and more than 100% in cash. If the slope is between zero and one, the human capital is equivalent to a moderate long position in cash and in a long-term bond. The optimal weights of the long-term bond and cash in the financial portfolio are thus highly dependent on the business cycle variations of labor income.

Throughout the paper we consider investors with time-additive power utility of consumption and terminal wealth. The dynamics of labor income, interest rates, bond prices, and stock prices are modeled by diffusion processes. First we derive a closed-form solution for the optimal consumption and investment decisions under the simplifying assumptions of no unspanned labor income risk and no portfolio constraints. While these assumptions are clearly questionable, the closed-form solution allows us to develop intuition of the economic forces at play and to understand the effect of the business cycle variations in income growth in an idealized setting. Next we allow for unspanned labor income risk and impose borrowing constraints and short-sales constraints, in which case we solve the utility maximization problem by a numerical dynamic programming technique. Our extensive numerical analysis based on the calibrated model shows that the intuition from the unconstrained, complete market version of the problem carries over to the constrained, incomplete market setting. Although the quantitative effects of the business cycle variations in income growth are dampened, the slope of the relation between expected income growth and the short-term interest rate remains an important parameter for the optimal consumption and investment decisions and, in particular, for the relative allocation between cash and long-term bonds. We illustrate the impact of this slope on the investment behavior of individuals with various levels of education using the life-cycle income profiles estimated from PSID income data, thereby generalizing the model and insights of Cocco, Gomes, and Maenhout (2005).

Let us briefly review the relevant literature for this study. As first noted by Merton (1971), long-term investors will generally hedge stochastic variations in the investment opportunity set. Stochastic interest rates are an important source of shifts in investment opportunities, and the effect of interest rate uncertainty on the optimal strategies of an investor without labor income is by now relatively well-studied. Sørensen (1999) and Brennan and Xia (2000) consider interest rate dynamics as in the Vasicek (1977) model and assume complete financial markets and constant market prices of both interest rate risk and stock market risk. They find that the optimal investment strategy of an investor with power utility of terminal wealth only is a simple combination of the mean-variance optimal portfolio, i.e., the optimal portfolio assuming investment opportunities do not change, and the zero-coupon bond maturing at the end of the investment horizon. Other studies of dynamic portfolio choice with uncertain interest rates include Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2001), Deelstra, Grasselli, and Koehl (2000), Munk and Sørensen (2004), Sangvinatsos and Wachter (2005), and Liu (2007). None of these papers take into account a labor income stream of the investor, although labor income is the main source of funds for most individuals.

On the other hand, several papers discuss how the presence of a labor income process affects the consumption and investment decisions of individual investors in an environment of constant investment opportunities. A deterministic income stream is equivalent to an implicit investment in the risk-free asset and, hence, it is optimal to invest a higher fraction of financial wealth in the risky assets than in the no-income case; cf., e.g., Hakansson (1970) and Merton (1971). With stochastic income, but fully hedgeable income risk, the optimal unconstrained strategies can be deduced from the optimal strategies without labor income, cf. Bodie, Merton, and Samuelson (1992): given the risk structure of human capital, the financial investment is determined in order to obtain the desired overall risk exposure. Since the human capital of long-term investors is often very large compared to financial wealth, labor income has dramatic effects on their optimal portfolios. Duffie, Fleming, Soner, and Zariphopoulou (1997), Koo (1998), and Munk (2000) study (mostly by use of numerical methods) the valuation of income and the optimal consumption and investment strategies of an infinite-horizon, liquidity constrained power utility investor with non-spanned income risk. The presence of liquidity constraints can significantly decrease the individual's implicit valuation of the future income stream and, hence, dampen the quantitative effects of income on portfolio choice. Other recent papers on consumption and portfolio choice with stochastic income include He and Pagès (1993), Heaton and Lucas (1997), Viceira (2001), Constantinides, Donaldson, and Mehra (2002), and Cocco, Gomes, and Maenhout (2005). Besides working with constant investment opportunities, the concrete models with stochastic income in these papers assume a single risky asset, interpreted as the stock market index. Since different risky assets will have different correlations with the labor income of a given individual, this assumption is not without loss of generality. We allow for multiple risky assets (the stock market index and bonds) and a link between labor income and investment opportunities.

Lynch and Tan (2009) consider a model where the stock market dividend yield predicts both stock market returns and the expected growth (and potentially also the volatility) of labor income leading to a negative hedging demand for stocks that partially offsets the high speculative stock demand. We model the business cycle sensitivity of income growth through the interest rate level instead of the dividend yield. While their model apparently includes time-varying interest rates, they do not allow for investments in bonds, only in cash and the stock market. Benzoni, Collin-Dufresne, and Goldstein (2007) postulate a long-run cointegration between labor income and stock market dividends and show that such a relation can substantially reduce optimal stock holdings for sufficiently risk-averse long-term investors. In contrast to these two papers, we focus on the joint implications of stochastic interest rates and labor income for the valuation of human capital and for the stock/bond/cash allocation. The model of Koijen, Nijman, and Werker (2009) includes both stochastic interest rates and labor income, but the income process is assumed to have constant expected growth and constant volatility so it cannot capture business cycle variations in income. On the other hand, they allow for time-variation in bond risk premiums and inflation. Finally, Van Hemert (2009) studies a rich life-cycle model with both stochastic income and interest rates, but he also disregards business cycle variations in income and he focuses on the optimal mortgage choice for homeowners.

The rest of the paper is organized as follows. In Section 2 we set up the general model of the financial market, specify the preferences and income of the individual, and calibrate the model to data. Section 3 focuses on the special case with unconstrained investment strategies and either locally risk-free income or spanned income risk so that we obtain closed-form solutions allowing us to understand the economic forces at play. Section 4 explains the numerical solution technique applied to the case with unspanned income uncertainty and relevant portfolio constraints and shows and discusses the results for the calibrated model. Section 5 gives some concluding remarks. The appendices contain proofs of propositions and detailed descriptions of the calibration procedure and the numerical solution technique.

2. Description of the model

We model the intertemporal consumption and investment choice of a price-taking individual who can trade in stocks, bonds, and an instantaneously risk-free asset, and receives a stochastic stream of income from non-financial sources, say labor income. We assume that the economy has a single perishable consumption good which serves as a numeraire so that all asset prices, interest rates, and income rates are specified in units of this good, i.e., in *real* terms.

2.1. Financial assets

We assume that the short-term interest rate, r_t, follows the Vasicek (1977) model,

$$dr_t = \kappa(\overline{r}-r_t)\,dt-\sigma_r\,dz_{rt}, \tag{1}$$

where κ, $\overline{r}$, and σ_r are positive constants, and $z_r=(z_{rt})_{t\geq 0}$ is a standard Brownian motion. The market price of interest rate risk, λ_r, is assumed constant. The price of a zero-coupon bond paying one unit of account at some time $\overline{T}$ is then given by

$$B_t^{\overline{T}} \equiv B^{\overline{T}}(r_t,t) = e^{-a(\overline{T}-t)-b(\overline{T}-t)r_t}, \tag{2}$$

where

$$b(\tau)=\frac{1}{\kappa}(1-e^{-\kappa\tau}),$$

$$a(\tau)=R_\infty[\tau-b(\tau)]+\frac{\sigma_r^2}{4\kappa}b(\tau)^2,$$

$$R_\infty=\overline{r}+\frac{\sigma_r\lambda_r}{\kappa}-\frac{\sigma_r^2}{2\kappa^2}. \tag{3}$$

Here R_∞ is the limit of the yield of a zero-coupon bond as maturity goes to infinity, i.e., the asymptotic long rate.

Any desired interest rate exposure can be obtained by combining deposits/loans at the short-term interest rate (interpreted as cash or the bank account) and a single default-free bond.[1] The dynamics of the price B_t of such a bond is given by

$$dB_t = B_t[(r_t+\sigma_B(r_t,t)\lambda_r)\,dt+\sigma_B(r_t,t)\,dz_{rt}], \tag{4}$$

where $\sigma_B(r_t,t)>0$ is the bond price volatility, which generally depends on both the interest rate level and the time-to-maturity and hence, on time. However, for a zero-coupon bond the volatility is $\sigma_r b(\overline{T}-t)$, which depends on the time-to-maturity $\overline{T}-t$, but not on the interest rate level. The bond price has a perfectly negative instantaneous correlation with the interest rate, $\rho_{Br}=-1$.

In addition to the bond, we assume that agents can invest in a single non-dividend paying stock, representing the stock market index, with price dynamics

$$dS_t = S_t[(r_t+\psi)\,dt+\sigma_S(\rho_{SB}\,dz_{rt}+\sqrt{1-\rho_{SB}^2}\,dz_{St})], \tag{5}$$

where z_S=(z_{St}) is a standard Brownian motion independent of z_r, ψ is the constant expected excess return, σ_S is the constant volatility, and $\rho_{SB}=-\rho_{Sr}$ is the constant correlation between the stock and the bond.

To simplify some of the following expressions, we introduce the vector $P_t=(B_t,S_t)^\top$ of prices of both risky assets. By combining the dynamics of B_t and S_t, we get

$$dP_t = \mathrm{diag}(P_t)[(r_t\mathbf{1}+\Sigma(r_t,t)\lambda)\,dt+\Sigma(r_t,t)\,dz_t], \tag{6}$$

[1] We assume that real, i.e., inflation-indexed, bonds are traded. Brennan and Xia (2002), Sangvinatsos and Wachter (2005), and Koijen, Nijman, and Werker (2009) study how real interest rate risk affects optimal investments when the traded bonds are nominal.

where $z=(z_r,z_S)^\top$ and

$$\Sigma(r_t,t)=\begin{pmatrix}\sigma_B(r_t,t) & 0\\ \sigma_S\rho_{SB} & \sigma_S\sqrt{1-\rho_{SB}^2}\end{pmatrix}.$$

Furthermore, $\lambda=(\lambda_r,\lambda_S)^\top$ is the market price of risk vector, where

$$\lambda_S=\frac{1}{\sqrt{1-\rho_{SB}^2}}\left(\frac{\psi}{\sigma_S}-\rho_{SB}\lambda_r\right).$$

2.2. The preferences and labor income of the individual

We assume throughout the paper that the individual has a time-additive utility function of consumption c_t and possibly terminal wealth W_T and seeks to maximize

$$\mathrm{E}\left[\int_0^T e^{-\delta t}U(c_t)\,dt+\varepsilon e^{-\delta T}U(W_T)\right],$$

where T is the time of death, assumed non-random, and $\varepsilon\geq 0$ defines the relative utility weighting of terminal wealth and intermediate consumption. Throughout the paper we use a power utility function

$$U(c)=\frac{1}{1-\gamma}c^{1-\gamma},$$

where $\gamma>0$ is the constant relative risk aversion.

We set up a model of labor income which is tractable and allows us to focus on the interaction between stochastic income and stochastic interest rates. We assume that the individual receives a continuous stream of non-negative income from non-financial sources until retirement at time $\tilde{T}\leq T$. The income rate at time t is denoted by y_t. We assume that y_t evolves as[2]

$$dy_t=y_t[(\xi_0(t)+\xi_1 r_t)\,dt+\sigma_y(t)\{\rho_{yP}^\top\,dz_t+\sqrt{1-\|\rho_{yP}\|^2}\,dz_{yt}\}],\tag{7}$$

where z_y=(z_{yt}) is a one-dimensional standard Brownian motion independent of z_r and z_S. The expected income growth rate is allowed to depend on the level of interest rates, reflecting the intuition that for most individuals wage increases are more frequent and larger in booming periods (high interest rates) than in recessions (low interest rates); the opposite relation may hold for individuals employed in specific industries or with specific skills. In our main analysis we assume a zero labor income in retirement, i.e., $y_t\equiv 0$ for $t\in[\tilde{T},T]$, but in a later section we will allow for a retirement income proportional to the income just before retirement.

The constant vector ρ_{yP} is defined as $(\rho_{yB},\hat{\rho}_{yS})^\top$, where $\rho_{yB}=-\rho_{yr}$ is the instantaneous correlation between the income rate and the bond price, and $\hat{\rho}_{yS}=(\rho_{yS}-\rho_{SB}\rho_{yB})/\sqrt{1-\rho_{SB}^2}$ where ρ_{yS} is the correlation between the income rate and the stock. If $\|\rho_{yP}\|^2\equiv\rho_{yB}^2+\hat{\rho}_{yS}^2=1$, the income rate is spanned, i.e., only sensitive to the traded risks represented by z. If that is the case, and there are no portfolio constraints, the income process can be replicated by some dynamic trading strategy of the traded assets and hence, valued as a traded asset.

Note that the percentage drift $\xi_0(t)+\xi_1 r_t$ and volatility $\sigma_y(t)$ are allowed to depend on time in order to reflect the empirically relevant variations in expected income growth and uncertainty over the life of an individual, cf., e.g., Hubbard, Skinner, and Zeldes (1995) and Cocco, Gomes, and Maenhout (2005). In contrast to other studies, we allow the drift to depend on the interest rate in order to incorporate the plausible link between the expected growth in income and the overall well-being of the economy.[3] Our model does not allow for jumps in income, although the risk of layoffs resulting in significantly lower income may affect consumption and portfolio choice, at least if the unemployment state is relatively persistent; see the discussions in Cocco, Gomes, and Maenhout (2005) and Lynch and Tan (2009). However, for many individuals the possible unemployment periods are likely to be rather short and in many countries individuals can partly insure against temporary income losses caused by unemployment.

2.3. Model calibration

We calibrate the model of asset prices and income to U.S. data. In this section we describe the data and the calibration results. Details of the calibration exercise can be found in Appendix A. We use the Panel Study of Income Dynamics (PSID) survey of the annual income of U.S. individuals in the period 1970–1992. PSID is the largest longitudinal U.S. data set with careful information on individual labor income and individual characteristics, and has also been applied in other studies of asset allocation with labor income, e.g., Cocco, Gomes, and Maenhout (2005). Since PSID offers only 23 time-series observations on a yearly basis, we also calibrate the model to quarterly U.S. aggregate income data and capital market data which span the period 1951–2003 with a total of 208 observation time points. The quarterly income data are obtained from the Personal Income and Its Disposition table from the National Income and Product Accounts (NIPA) published by the Bureau of Economic Analysis of the U.S. Department of Commerce, and the applied data are the per capita disposable personal

[2] As most authors, we have modeled the income stream as an exogenously given process. Of course, in real life the individual can affect her labor income to some extent by choice of education and effort. To avoid further complications of the model we do not endogenize the labor supply decision. We refer the reader to Bodie, Merton, and Samuelson (1992) and Chan and Viceira (2000).

[3] Storesletten, Telmer, and Yaron (2004) report evidence that the volatility of income shocks is significantly higher in recessions than in economic peaks. In our calibration we also tried an income volatility of the form $\sigma_y(t)\exp\{\zeta(r_t-\bar{r})\}$. Using all households in the PSID data, the estimated ζ was -5.2, but with a standard error of 7.0 the estimate is clearly insignificant. In addition, our numerical results show that the optimal consumption and investment strategies are almost the same for $\zeta=-5.2$ as for our standard case corresponding to $\zeta=0$. Furthermore, with the interest rate-dependent income volatility, it would not be possible to explicitly compute the human wealth and the optimal strategies in the unconstrained case studied in Section 3. Therefore, we stick to the specification where income volatility does not vary with interest rates, but acknowledge that the income volatility of some individuals may have a dependence on the interest rate level large enough to significantly affect their optimal decisions.

income after personal current taxes and adjusted for personal income from financial assets. The cum-dividend stock returns are constructed using quarter-end values of the Standard & Poor's (S&P) 500 index over the period, while the S&P 500 dividends and the Consumer Price Index (CPI) data are adopted from Shiller (2000); the updated data were downloaded from Robert Shiller's homepage (www.econ.yale.edu/~shiller). All income and stock prices are in real terms (using the CPI-index as deflator). Real interest rates are constructed by subtracting an estimate of the inflation rate from the three-month nominal interest rate. The subtracted inflation rate is obtained as the average realized inflation rate in the last four quarters relative to the same quarters one year earlier. The interest rate data are adopted from the estimated zero-coupon bond yields in McCulloch (1990) and McCulloch and Kwon (1993), and the yields for the period 1991 to 2003 are bootstrapped from constant maturity yields from the Federal Reserve H.15 Statistical Releases.

The applied data are illustrated in Fig. 1 where the NIPA income index and the stock index are scaled so that they start out in one in 1951 (and zero for the logarithmic value). The figure also displays the yearly PSID aggregated income time-series (which is scaled so that it equals the NIPA index in the second quarter of 1970). The correlation coefficient between changes in the two income time-series is as high as 0.9321.

We have carried out estimations both with aggregate income data (NIPA and PSID data) and with individual income data (PSID data). In line with Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002), we decompose the logarithm of an individual's income into a personal idiosyncratic component and a common component; details are given in Appendix A. The parameter estimates based on the NIPA aggregate income data are shown in Table 1, both for the full period 1951–2003 as well as for the sub-period 1970–1992 which is relevant for the PSID income data. Table 2 lists parameter estimates based on the PSID income data, both for aggregated income data within three educational groups as well as for aggregation over all households. The parameter σ_u reported in the table is volatility of the common labor income component and ρ_{uS} (ρ_{ur}) is the correlation of the common income component and the stock return (the real interest rate). For the PSID estimations we have also tabulated in Panel B of Table 2 the results for the individual household income volatility and correlations derived as explained in Appendix A.

In all estimations based on PSID income data, we have fixed the parameters describing the stock dynamics and the interest rate dynamics at the values estimated for the full sample period 1951 to 2003 with quarterly NIPA income data as shown in Table 1. These parameter estimates thus reflect all the information about dynamics of stock and interest rates which is available in our capital

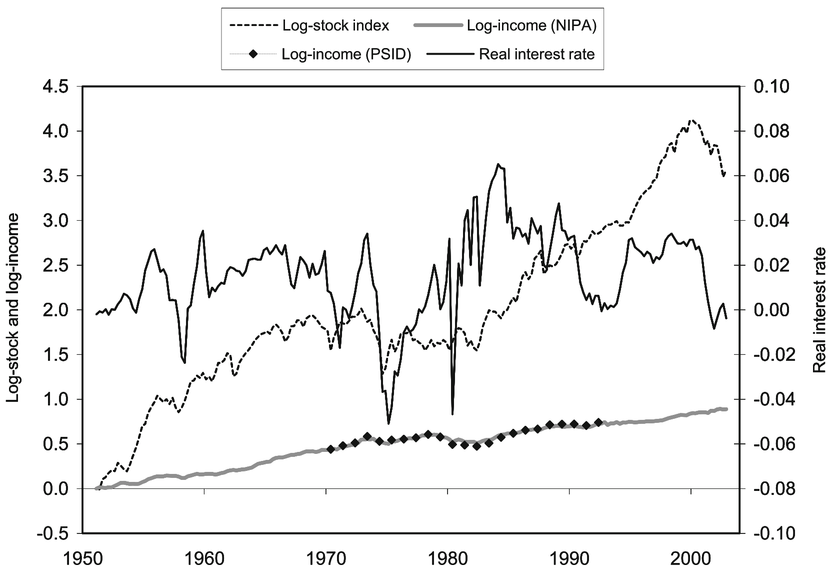

Fig. 1. Data series on income, stock index, and real interest rates applied in the calibration of our model. The thick gray line shows the log of the income index obtained from the Personal Income and Its Disposition table from the National Income and Product Accounts (NIPA) published by the Bureau of Economic Analysis. The index reflects the per capita disposable personal income after personal current taxes and adjusted for personal income from financial assets. The index value is normalized to one in 1951. The black diamonds indicate the yearly aggregated income time-series from the Panel Study of Income Dynamics (PSID) survey of the annual income of U.S. individuals in 1970–1992. The PSID aggregate income time-series is scaled so that it equals the NIPA index in the second quarter of 1970. The dashed curve reflects the log of the Standard & Poors 500 stock market index, adjusted to include dividends and initialized at one in 1951. Income numbers and stock prices have been deflated using the Consumer Price Index (CPI). The stock market and CPI data are taken from (Shiller, 2000) and the homepage of Robert Shiller (www.econ.yale.edu/~shiller). The black curve shows the real interest rate with values to be read off the right-hand side axis. The real interest rate is constructed as the three-month nominal interest rate minus the average realized inflation rate in the last four quarters relative to the same quarters one year earlier. The interest rate data are adopted from McCulloch (1990), McCulloch and Kwon (1993), and the Federal Reserve H.15 Statistical Releases.

Table 1
Estimates and benchmark values of return parameters and income parameters using NIPA aggregate income data.

The individual labor income process is decomposed into a personal idiosyncratic component and a common component as explained in detail in Appendix A. The upper part presents estimates of the constant ξ_0 and the interest rate sensitivity ξ_1 of the expected income growth, the volatility σ_u of the common labor income component, and the correlation ρ_{uS} (ρ_{ur}) between the common income component and the stock return (the real interest rate). The estimates are based on the quarterly aggregate income data from the National Income and Product Accounts (NIPA) published by the Bureau of Economic Analysis, either for the full period 1951–2003 or the sub-period 1970–1992 for which we have the alternative income data from the Panel Study of Income Dynamics. The lower part of the table shows estimates of the Sharpe ratio ψ and the volatility σ_S of the stock index, the mean reversion coefficient κ, the long-run level $\bar{r}$, and the volatility σ_r of the short-term real interest rate, as well as the correlation ρ_{Sr} between the stock return and the interest rate. In the right-most column labeled 1970–1992, the return parameters have been fixed to the estimates from the full sample period 1951–2003 as indicated by the italicized numbers. Standard errors are shown in parentheses. The right-most column shows our choice of benchmark parameter values.

Parameter	Time period 1951–2003	Time period 1970–1992	Time period 1970–1992	Benchmark value
ξ_0	0.0181	0.0123	0.0130	Varied
	(0.0043)	(0.0055)	(0.0057)	
ξ_1	−0.0434	0.1246	0.1088	Varied
	(0.1996)	(0.1059)	(0.2166)	
σ_u	0.0208	0.0235	0.0231	0.02
	(0.0010)	(0.0017)	(0.0017)	
ρ_{uS}	0.1673	0.2098	0.1997	–
	(0.0663)	(0.1027)	(0.0883)	
ρ_{ur}	0.2683	0.2343	0.1806	–
	(0.0727)	(0.1009)	(0.0753)	
ψ	0.0665	0.0497	*0.0665*	0.04
	(0.0226)	(0.0373)		
κ	0.6407	0.7139	*0.6407*	0.50
	(0.1942)	(0.2703)		
$\bar{r}$	0.0155	0.0142	*0.0155*	0.02
	(0.0047)	(0.0089)		
σ_S	0.1613	0.1790	*0.1613*	0.20
	(0.0079)	(0.0132)		
σ_r	0.0218	0.0302	*0.0218*	0.02
	(0.0012)	(0.0024)		
ρ_{Sr}	0.0052	−0.0297	*0.0052*	0
	(0.0658)	(0.1175)		

market data. The estimations based on NIPA income data for the sub-period 1970 and 1992 indicate that the other parameters are not affected much by fixing these stock and interest rate parameters. Except for the interest rate volatility parameter, the stock and interest rate parameters are not significantly different for this sub-period than the case estimated for the full sample period.[4] The sub-period 1970 to 1992, which is relevant for the PSID data, contains the period of the so-called monetary experiment in 1978 to 1982, and the interest rate volatility is estimated significantly higher at 3.02% for the PSID sub-period compared to the full sample period estimate of 2.18%.

Tables 1 and 2 also show our benchmark values of the parameters to be used in the rest of the paper. Our benchmark parameters are chosen close to the estimated values. Some parameters are chosen to be similar to those applied by, e.g., Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002) who use a stock market risk premium of $\psi = 4\%$ and a real interest rate level of 2%. The volatility in the aggregate income processes is in all cases estimated around 2%, but our benchmark parameter values are set significantly higher at 20%. This reflects the calibrated values in Panel B of Table 2, which indicate that individual household income volatility might be as high as 33–36%. These calibrated estimates are consistent with results reported by Cocco, Gomes, and Maenhout (2005) who show, however, that some of this income volatility can be attributed to transitory income effects, and therefore, we have adjusted the income volatility down to 20% in our benchmark parameter set. Also consistent with the analysis in Cocco, Gomes, and Maenhout (2005), the correlation coefficients between individual household income and the stock index are all calibrated close to zero due to the large idiosyncratic component in individual household income. The correlations between household income and the real interest rate are likewise calibrated close to zero.

It may be noted that estimations based on NIPA and PSID income, respectively, provide different estimates of the income drift parameters ξ_0 and ξ_1 although the income series are highly correlated for the sub-period 1970 to 1992. The NIPA income estimations, e.g., indicate a significant real growth in income of 1.81% per year for the full period. On the other hand, the PSID estimations indicate a clear positive relation between income growth and the level of the real interest rate. This positive relation tends to increase with the educational level and is especially significant for households where the household head has a college education.

We will focus on the effects of the interest rate sensitivity of income growth for consumption and investment decisions, and in order to disentangle those effects from age-related variations in income, we will first fix the expected income growth parameter ξ_0 and the income volatility σ_y to be time-independent. When we vary the slope parameter ξ_1 in the income growth rate, we will also vary ξ_0 so that the total expected growth rate when the risk-free rate is at its long-term average, $\xi_0 + \xi_1\bar{r}$, remains unchanged at a level of 2–3%. This level reflects a young individual's expected increase in real income of about 1–2% due to common real income growth (as reflected in the estimated value of ξ_0) and about a 1–2% expected increase in salary due to gaining working experience (as reflected in the slope of the life-cycle income profiles in Fig. 15 for younger investors). We will later allow for the full life-cycle variations in labor income that can be read off the PSID income data. Appendix A discusses how to take these life-cycle variations into account in the calibration, and the implications for portfolio choice are studied in Section 4.2.

Our estimation approach does not involve the risk premium on real interest rate risk, and our benchmark parameter value for the numerical experiments is set at

[4] A likelihood ratio test for the hypothesis that the five parameters ψ, κ, $\bar{r}$, σ_S, and ρ_{Sr} are equal to the estimates for the full period has a p-value of 75%. Including the interest rate volatility as a sixth parameter in the test results in a p-value of only 0.1%.

Table 2
Estimates and benchmark values of income parameters using PSID income data.

The individual labor income is decomposed into a personal idiosyncratic component and a common component as explained in detail in Appendix A. Panel A presents estimates based on various data sets of the constant ξ_0 and the interest rate coefficient ξ_1 of the expected income growth, the volatility σ_u of the common labor income component, and the correlation ρ_{uS} (ρ_{ur}) between the common income component and the stock return (the real interest rate). The estimates are based on the aggregate income data from the Panel Study of Income Dynamics (PSID) survey, using either all individuals or individuals in specific educational groups as indicated in the column headers. Panel B shows estimates of the individual income volatility σ_y and the correlation ρ_{yS} (ρ_{yr}) between individual income and the stock return (the real interest rate) based on the individual income time-series in PSID. Standard errors are shown in parentheses. In these estimations of income parameters, we fix the return parameters to the estimates from the full sample period 1951–2003, estimated together with quarterly NIPA income data, and listed in Table 1. The right-most column shows our choice of benchmark parameter values.

Parameter	PSID income All individuals	PSID income No high school	PSID income High school	PSID income College	Benchmark value
Panel A					
ξ_0	−0.0035	0.0026	−0.0046	−0.0058	Varied
	(0.0040)	(0.0055)	(0.0041)	(0.0045)	
ξ_1	0.3179	0.0322	0.3589	0.4861	Varied
	(0.1546)	(0.2124)	(0.1584)	(0.1727)	
σ_u	0.0164	0.0211	0.0163	0.0204	0.02
	(0.0025)	(0.0031)	(0.0025)	(0.0030)	
ρ_{uS}	0.3755	0.1288	0.3111	0.5677	–
	(0.1775)	(0.2054)	(0.1899)	(0.1346)	
ρ_{ur}	0.1466	0.2242	0.1064	0.0810	–
	(0.1535)	(0.1580)	(0.1591)	(0.1388)	
Panel B					
σ_y	0.3431	0.3612	0.3303	0.3296	0.20
ρ_{yS}	0.0179	0.0075	0.0154	0.0351	0
ρ_{yr}	0.0070	0.0131	0.0053	0.0050	0

$\lambda_r = 0$. We do not have reliable data on the return on real bonds to empirically justify this choice (since real bonds were first introduced in the U.S. in 1998). But the smaller risk premium on bonds relative to stocks is at least consistent with the similar results for *nominal* bonds. For example, the Brealey, Myers, and Allen (2006) textbook reports a 1.2% (7.6%) average excess return on government bonds (stocks) with a historical standard deviation on returns of 8.2% (20.1%) over the period 1900–2003 [updates of the estimated values in Dimson, Marsh, and Staunton, 2002]. In our interest data the average excess return on a five-year zero-coupon bond is likewise 1.32% (2.35%) with a volatility of 9.40% (12.52%) for the full period 1951 to 2003 (the sub-period 1970 to 1992). Together with a zero interest rate risk premium, our benchmark parameters imply a long-term real yield of 1.92%, and for an initial real short rate identical to the long-term average of 2%, the real yield curve is almost flat.

In Table 3 we have tabulated estimation results obtained for individual household time-series in the PSID data base. Here, we only consider households with at least 10 consecutive income observations for an age below 65 years of the household head and with a household income in all years (all non-consecutive observations are excluded) of at least \$1,000. These requirements reduce the number of observations from about 80,000 for 7,300 households to a total of 40,497 observations for 2,302 different households. In all estimations the stock and interest rate parameters have been fixed at the values also applied in Table 1. The results in Table 3 aim at illustrating the degree of dispersion of the obtained estimates for individual households by reporting the 10% fractile, the median, and the 90% fractile for the obtained estimates for each of the income parameters. The table also reports the median standard deviation for the different parameter estimates.

Note that especially the drift parameters ξ_0 and ξ_1 display a high degree of dispersion across individuals, and these parameters likewise seem to be estimated with much less precision than in the case for aggregate income in Tables 1 and 2. This is due to the much higher volatility in the individual household time-series. In all cases in Table 3, however, the median estimates are quite close to our applied benchmark parameter values.

2.4. Optimal strategies

The individual has to choose a consumption strategy c=(c_t) and an investment strategy $\theta = (\theta_t)$. Here, c_t is the rate at which goods are consumed at time t with the natural requirement that $c_t \geq 0$ at all times and in all states of the economy. Furthermore, θ_t is a vector $(\theta_{Bt}, \theta_{St})^\top$ of the amounts (i.e., units of the consumption good) invested at time t in the bond and the stock. With W_t denoting the financial wealth of the investor at time t, the amount invested in the bank account (held in "cash") is residually determined as $\theta_{0t} = W_t - \theta_{Bt} - \theta_{St}$. Given a consumption strategy c and an investment strategy θ, the financial wealth of the individual W_t evolves as

$$dW_t = (r_t W_t + \theta_t^\top \Sigma(r_t, t)\lambda - c_t + y_t)\, dt + \theta_t^\top \Sigma(r_t, t)\, dz_t. \tag{8}$$

The consumption and investment strategies must satisfy some technical conditions for the wealth process to be

Table 3
Parameter estimates based on individual household income data.

Individual labor income processes have been estimated by maximum likelihood (based on Eq. (38)) for each of 2,302 individuals covered by the Panel Study of Income Dynamics (PSID) survey over the period 1970–1992. Within each of the three educational groups, the table reports the 10% fractile, median, and 90% fractile of the estimates of the constant ξ_0 and the interest rate coefficient ξ_1 of the expected income growth, the income volatility σ_y, and the correlation ρ_{yS} (ρ_{yr}) between the income and the stock return (the real interest rate). In the estimations, the stock and interest rate parameters were fixed at the same values as in Table 1. The median standard errors are shown in parentheses.

Parameter	No high school			High school			College		
	90%	Median	10%	90%	Median	10%	90%	Median	10%
ξ_0	0.1744	0.0390	−0.0324	0.1829	0.0271	−0.0602	0.1833	0.0277	−0.0709
		(0.0831)			(0.0797)			(0.0796)	
ξ_1	3.6789	−0.2429	−5.8429	3.7704	−0.0933	−6.2123	5.4803	0.4912	−4.2618
		(3.3421)			(2.9230)			(2.8484)	
σ_y	0.5281	0.2857	0.1268	0.5037	0.2442	0.1148	0.4960	0.2443	0.1025
		(0.0500)			(0.0435)			(0.0429)	
ρ_{yS}	0.3554	0.0073	−0.3824	0.3715	0.0286	−0.3534	0.3658	0.0440	−0.3208
		(0.2127)			(0.2169)			(0.2133)	
ρ_{yr}	0.2869	0.0097	−0.2431	0.2868	0.0152	−0.2821	0.3418	0.0148	−0.2734
		(0.1659)			(0.1675)			(0.1660)	
No. individuals		486			1273			543	
No. observations		8530			22,198			9769	

well-defined. If there are no other restrictions on the strategies (except $c_t \geq 0$), we denote by $\mathcal{A}_t^{\text{unc}}$ the set of admissible consumption and investment strategies (c, θ) over the time interval $[t,T]$. We will also consider the case where it is not possible for the individual to borrow funds using future income as collateral so that her financial wealth W_t must stay non-negative at all times and in all states of the world.[5] In a continuous-time setting this is implemented by requiring that whenever the financial wealth hits zero, the investor must eliminate her positions in bonds and stocks. After she has received labor income, she may again enter the markets for risky securities. We denote the set of admissible strategies with this constraint by $\mathcal{A}_t^{\text{con}}$.

The indirect utility function of the individual is defined as

$$J(W,r,y,t) = \sup_{(c,\theta)\in\mathcal{A}_t} \mathrm{E}_t\left[\int_t^T e^{-\delta(s-t)}U(c_s)\,ds + \varepsilon e^{-\delta(T-t)}U(W_T)\right], \tag{9}$$

where the expectation is computed given the values of W, r, y at time t and given the strategy (c,θ). The set $\mathcal{A}_t$ is either equal to $\mathcal{A}_t^{\text{unc}}$ or $\mathcal{A}_t^{\text{con}}$. With the assumed Constant Relative Risk Aversion (CRRA) utility function the marginal utility is infinite at zero consumption so that the non-negativity constraint on consumption is not binding. The Hamilton–Jacobi–Bellman (HJB) equation associated with this dynamic optimization problem is

$$\begin{aligned}\delta J = \sup_{c,\theta}\Big\{ & U(c)+J_t+J_W(rW+\theta^\top\Sigma\lambda - c + y)+\tfrac{1}{2}J_{WW}\theta^\top\Sigma\Sigma^\top\theta \\ & +J_r\kappa[\overline{r}-r]+\tfrac{1}{2}J_{rr}\sigma_r^2+J_y y(\xi_0+\xi_1 r)+\tfrac{1}{2}J_{yy}y^2\sigma_y^2 \\ & -J_{Wr}\theta^\top\Sigma\mathbf{e}_1\sigma_r+J_{Wy}y\sigma_y\theta^\top\Sigma\rho_{yP}+J_{ry}y\rho_{yr}\sigma_y\sigma_r\Big\},\end{aligned} \tag{10}$$

where $\mathbf{e}_1 = (1,0)^\top$, subscripts on J denote partial derivatives, and we have suppressed the arguments of the functions for notational simplicity. The terminal condition is $J(W,r,y,T) = \varepsilon U(W) = \varepsilon W^{1-\gamma}/(1-\gamma)$.

The first-order condition for consumption is the standard envelope condition

$$U'(c_t) = J_W(W_t,r_t,y_t,t) \;\Rightarrow\; c_t = [J_W(W_t,r_t,y_t,t)]^{-1/\gamma}. \tag{11}$$

For the unrestricted investment case, the first-order condition for the portfolio θ implies that

$$\theta_t = -\frac{J_W}{J_{WW}}(\Sigma(r_t,t)^\top)^{-1}\lambda - \frac{J_{Wy}}{J_{WW}}y_t\sigma_y(t)(\Sigma(r_t,t)^\top)^{-1}\rho_{yP}+\frac{J_{Wr}}{J_{WW}}\frac{\sigma_r}{\sigma_B(r_t,t)}\mathbf{e}_1. \tag{12}$$

The first part corresponds to the standard mean-variance optimal portfolio, the second part is a hedge against changes in the income rate, while the third part is a hedge against changes in the interest rate. The income hedge term reflects a position in the portfolio with relative weights given by $(\Sigma(r_t,t)^\top)^{-1}\rho_{yP}/\mathbf{1}_{n+1}^\top(\Sigma(r_t,t)^\top)^{-1}\rho_{yP}$. This is the portfolio with the maximal absolute correlation with the income rate of the individual, cf. Ingersoll (1987, Chapter 13). This maximal correlation equals $\|\rho_{yP}\|$ so if the income rate is spanned, this correlation will equal one. Since the bond price is perfectly negatively correlated with the interest rate, the interest rate is hedged by a

[5] This "hard" borrowing constraint is standard in the literature. A recent paper by Davis, Kubler, and Willen (2006) studies the portfolio choice under a "soft" borrowing constraint that allows individuals to borrow even with a negative current wealth although at a rate higher than the risk-free interest rate. Their study assumes constant interest rates. To focus on the interaction between stochastic interest rates and stochastic labor income we stick to the "hard" borrowing constraint, which is easier to handle.

position in the bond only. In contrast, the income hedge and the mean-variance terms generally involve all risky assets. The remaining wealth, $W_t - \theta_t^\top \mathbf{1}_{n+1}$, is invested in the bank account.

3. No unspanned income risk and no investment constraints

In this section we derive a closed-form solution to the utility maximization problem formulated above for the case with no unspanned income risk and with unconstrained investment strategies. We therefore have to assume either that the income stream is locally risk-free, i.e., that $\sigma_y(t) \equiv 0$, or that the income stream is fully spanned by the traded assets, i.e., that $\|\rho_{yP}\| = 1$, which implies that $\rho_{yS} + \rho_{SB}\rho_{yr} = \pm\sqrt{(1-\rho_{yr}^2)(1-\rho_{SB}^2)}$. The correlation values satisfying this condition are far from the benchmark parameters determined from our calibrations, cf. Table 1, but the closed-form solution allows for a detailed understanding of the economic forces at play. First, we derive and discuss the general solution, then we study the case of a locally risk-free income and the case where the income is perfectly correlated with the stock market. Finally, we argue that the closed-form solution is also highly relevant even if the income risk is not perfectly spanned.

3.1. The general solution

When the income process is spanned, any unconstrained investor can replicate it, and it can be valued as the dividend stream from a traded asset. The market value at time t of the income stream over the time period $[t, T]$ is

$$H_t \equiv H(y_t, r_t, t) = \mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T y_s e^{-\int_t^s r_v\, dv}\, ds\right], \tag{13}$$

where $\mathbb{Q}$ denotes the unique risk-neutral probability measure. We can think of the individual selling the remaining income stream for the amount H_t, her human capital. As described below, the optimal strategies can in this case be derived from the optimal strategies for the case without income but with a financial wealth of $W_t + H_t$ instead of just W_t. Under our assumptions on the dynamics of the labor income rate and the short-term interest rate, we are able to derive an explicit expression of the human capital as shown in the following proposition. The proof is given in Appendix B.[6]

Proposition 1. *Under the assumptions above, the human capital is given by*

$$H(y,r,t) = yM(r,t) \equiv y\mathbf{1}_{\{t \le \tilde{T}\}} \int_t^{\tilde{T}} h(t,s)(B^s(r,t))^{1-\xi_1}\, ds, \tag{14}$$

where the indicator $\mathbf{1}_{\{t \le \tilde{T}\}}$ is one before retirement and zero in retirement and

$$h(t,s) = \exp\left\{\int_t^s (\xi_0(u) - \sigma_y(u)\rho_{yP}^\top\lambda - (\xi_1 - 1)\rho_{yB}\sigma_r\sigma_y(u)b(s-u))\, du + \xi_1(\xi_1 - 1)\frac{\sigma_r^2}{2\kappa^2}\left(s - t - b(s-t) - \frac{\kappa}{2}b(s-t)^2\right)\right\} \tag{15}$$

with the function b given by (3).

Due to the structure of the assumed income rate process, the human capital is separated as the product of the current income rate, y, and a multiplier, $M(r, t)$, depending only on the interest rate and time. The risk characteristics of human capital will be discussed below together with the consequences for optimal portfolio choice.

With time-additive CRRA utility, it is well-known that indirect utility function with initial wealth W and no income is of the form

$$V(W,r,t) = \frac{1}{1-\gamma} g(r,t)^\gamma W^{1-\gamma},$$

where $g(r, t)$ is a function that depends on the remaining investment horizon (and hence, on time) and the risk aversion parameter γ; see, e.g., Ingersoll (1987, Chapter 13). With Vasicek (1977) interest rate dynamics, the function $g(r, t)$ can be computed explicitly; see Sørensen (1999) for the case of terminal wealth only and see Wachter (2002) and Liu (2007) for how to extend such solutions to intermediate consumption. With a spanned income rate and no portfolio constraints, we can think of the individual having an initial financial wealth of $W_t + H(y_t, r_t, t)$ and no labor income instead of having initial wealth W_t and the income stream.[7] Under the assumptions of this section, we therefore have that the indirect utility function *with* labor income is given by

$$J(W,r,y,t) = V(W + H(y,r,t), r, t). \tag{16}$$

From the value function the optimal consumption and investment strategies can be derived from (11) and (12). The following proposition summarizes the solution, which can be verified by substitution of (17) into the HJB-equation (10).

Proposition 2. *Under the assumptions stated above, the indirect utility function is given by*

$$J(W,r,y,t) = \frac{1}{1-\gamma} g(r,t)^\gamma (W + H(y,r,t))^{1-\gamma}, \tag{17}$$

where the function $g(r, t)$ is defined by

$$g(r,t) = \int_t^T f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\, ds + \varepsilon f(T-t)(B^T(r,t))^{\gamma-1/\gamma} \tag{18}$$

with $f(\tau)$ defined by

$$\ln f(\tau) = -\left(\frac{\delta}{\gamma} + \frac{\gamma-1}{2\gamma^2}\|\lambda\|^2\right)\tau + \frac{\gamma-1}{\gamma^2}\left(\frac{\sigma_r^2}{4\kappa}b(\tau)^2 - (\overline{r} - R_\infty)(\tau - b(\tau))\right).$$

[6] Given (2), the human wealth expression in (14) can also be written in the exponential-affine form $H(y,r,t) = \int_t^T \exp\{A_0(t,s) + A_1(t,s)r + A_2(t,s)\ln y\}\, ds$. This is the case in any setting where the risk-neutral dynamics of r_t and $\ln y_t$ are affine; c.f., e.g., Duffie, Pan, and Singleton (2000). We focus on a non-trivial case where the functions A_0, A_1, A_2 can be stated in closed-form (involving some simple integrals).

[7] Bodie, Merton, and Samuelson (1992) apply this idea in the case of constant investment opportunities.

The optimal consumption rate is

$$c_t = \frac{W_t + H(y_t, r_t, t)}{g(r_t, t)}, \tag{19}$$

while the optimal investments in the bond and the stock are

$$\theta_{Bt} = \frac{1}{\gamma\sigma_B}(W_t + H_t)\left(\lambda_r - \rho_{SB}\frac{\lambda_S}{\sqrt{1-\rho_{SB}^2}}\right) - H_t\frac{\sigma_y(t)}{\sigma_B}\frac{\rho_{yB} - \rho_{yS}\rho_{SB}}{1-\rho_{SB}^2}$$
$$+(\xi_1 - 1)\frac{\sigma_r}{\sigma_B} y\int_t^T b(s-t)h(t,s)(B^s)^{1-\xi_1}\,ds$$
$$+\frac{\gamma-1}{\gamma}\frac{\sigma_r}{\sigma_B}(W_t + H_t)G(r_t, t), \tag{20}$$

$$\theta_{St} = \frac{1}{\gamma}(W_t + H_t)\frac{\lambda_S}{\sigma_S\sqrt{1-\rho_{SB}^2}} - H_t\sigma_y(t)\frac{\rho_{yS} - \rho_{SB}\rho_{yB}}{\sigma_S(1-\rho_{SB}^2)}, \tag{21}$$

where

$$G(r,t) \equiv -\frac{\gamma}{\gamma-1}\frac{g_r(r,t)}{g(r,t)}$$
$$= \frac{\int_t^T b(s-t)f(s-t)(B^s(r,t))^{\gamma-1/\gamma} + \varepsilon b(T-t)f(T-t)(B^T(r,t))^{\gamma-1/\gamma}}{\int_t^T f(s-t)B^s(r,t)^{\gamma-1/\gamma}\,ds + \varepsilon f(T-t)(B^T(r,t))^{\gamma-1/\gamma}}. \tag{22}$$

We can think of the CRRA investor first determining how sensitive her total wealth should be towards the shocks to the economy, i.e., the optimal percentage volatility vector of total wealth. This desired volatility vector is independent of how the total wealth is comprised by financial wealth and human capital. It can be shown that, following the optimal strategies stated in the proposition above, the dynamics of total wealth $W_t + H_t$ will be

$$\frac{d(W_t + H_t)}{W_t + H_t} = \left(r_t + \frac{1}{\gamma}\|\lambda\|^2 - \sigma_r\lambda_r\frac{g_r(r_t,t)}{g(r_t,t)} - \frac{1}{g(r_t,t)}\right)dt$$
$$+\frac{1}{\gamma}\lambda^\top dz_t - \sigma_r\frac{g_r(r_t,t)}{g(r_t,t)}\,dz_{rt}, \tag{23}$$

which is equivalent to the optimal wealth dynamics in a setting with Vasicek interest rates but no labor income. Note that, for $\gamma > 1$, we have $g_r/g < 0$ so that the optimally invested wealth is negatively related to the shock $-\sigma_r\,dz_{rt}$ to the interest rate. A positive (negative) shock to the interest rate leads to better (worse) investment opportunities and, due to the intertemporal hedging motive, is therefore optimally accompanied by lower (higher) wealth. By Itô's lemma and the spanning assumption, the dynamics of human capital are

$$dH_t = (r_tH_t + H_y(y_t, r_t, t)y_t\sigma_y(t)\rho_{yP}^\top\lambda - H_r(y_t, r_t, t)\sigma_r\lambda_r - y_t)\,dt$$
$$+H_y(y_t, r_t, t)y_t\sigma_y(t)\rho_{yP}^\top\,dz_t - H_r(y_t, r_t, t)\sigma_r\,dz_{rt}.$$

Subtracting dH_t from $d(W_t+H_t)$ shows how the dynamics of financial wealth should optimally be, which is obtained using the strategies stated in the proposition above. The terms in the expression for θ_t that involve H compensate exactly for the dynamics of the human capital. In other words, the consumer-investor computes her optimal investment of total wealth and then corrects the investment strategy for the implicit investment that the income stream represents. The numerical examples in the subsections below will illustrate how the human wealth, the financial wealth, and the sum of the two are expected to evolve over the life-cycle. The computation of these expectations can be found in Appendix B.

In the problem without labor income, the optimal strategies are such that wealth stays positive with probability one. By analogy, the optimal strategies for the problem with labor income ensure that *total* wealth stays positive with probability one. However, financial wealth in itself may very well go negative in some situations. For positive values of financial wealth it makes sense to talk of portfolio weights, i.e., the fractions of financial wealth invested in the different assets. Investment strategies are usually stated in such portfolio weights. We denote portfolio weights by $\pi = \theta/W$. Using (14), we get from (20) and (21) that the optimal portfolio weights can be written compactly as

$$\pi_t = \frac{1}{\gamma}\left(1 + \frac{y_t}{W_t}M(r_t,t)\right)(\Sigma(r_t,t)^\top)^{-1}(\lambda - \gamma\sigma_y(t)\rho_{yP})$$
$$+\sigma_y(t)(\Sigma(r_t,t)^\top)^{-1}\rho_{yP} + \left(\frac{y_t}{W_t}M_r(r_t,t) - \frac{g_r(r_t,t)}{g(r_t,t)}\right.$$
$$\left.\times\left(1 + \frac{y_t}{W_t}M(r_t,t)\right)\right)\frac{\sigma_r}{\sigma_B(r_t,t)}\mathbf{e}_1. \tag{24}$$

It is now clear that the optimal portfolio weights do not depend on current financial wealth and labor income separately but only through the wealth-to-income ratio.

Investment in the stock. The first term in θ_{St} is the speculative demand and the second term is the correction term for stock-like income risk. The presence of labor income magnifies the optimal investment in the stock due to a wealth effect. The sign of the correction term depends on the correlation structure. If the stock is uncorrelated with the bond, the hedge demand is positive (negative) if the income-stock correlation is negative (positive). The total effect of income on the demand of the stock depends on the sign of $\lambda_S/\gamma - \sigma_y(t)\hat{\rho}_{yS}$. Since H is increasing in T, this sign will also determine how the optimal stock demand varies with the investment horizon. For a stock in positive demand the popular advice to decrease the fraction of wealth invested in stocks over the life-cycle so that θ_{St} increases with T is true for a risk-free income stream, but with income uncertainty the validity of this advice is highly dependent on risk aversion, risk premiums, and the correlation coefficients.

Investment in the bond. The optimal bond investment in (20) is the sum of a speculative demand (first term), a correction for bond-like income risk (second and third term), and a hedge against interest rate risk (last term). Both the speculative term and the hedge term are magnified due to the presence of human capital. The first correction for bond-like income risk shows up if the contemporaneous "stock-filtered" correlation between the income rate and the short-term interest rate is non-zero. The second correction term can be rewritten as $\sigma_r(\partial H/\partial r)/\sigma_B$, where

$$\frac{\partial H}{\partial r} = (\xi_1 - 1)y_t\int_t^T h(t,s)b(s-t)(B^s(r_t,t))^{1-\xi_1}\,ds$$

is the interest rate sensitivity of the human capital. The parameter ξ_1, i.e., the slope of the relation between expected income growth and the short rate, is crucial for the risk characteristics of the human capital and, consequently, for the optimal bond investment. For the case $\xi_1 = 1$, the human capital is insensitive to variations in the interest rate. An increase in the discount rate will be exactly offset by an increase in the expected future income rates. Ignoring contemporaneous income-asset correlations, the human capital will be equivalent to a short-term risk-free asset (cash) if $\xi_1 = 1$. Since in that case there is no long-term bond investment implicit in the human capital, there is no correction in the optimal bond demand. For $\xi_1 = 0$ (non-cyclical labor income), the human capital is like a long-term bond paying a coupon of $h(t, s)$ at time s and, hence, the explicit bond investment is reduced. For ξ_1 between zero and one, the interest rate sensitivity of the human capital is equivalent to that of a portfolio of a long position in cash and a long position in the long-term coupon bond. For $\xi_1 < 0$, i.e., a counter-cyclical income, the human capital is like a levered position in the long-term bond. For $\xi_1 > 1$ (strongly pro-cyclical income), the interest rate risk of the human capital is equivalent to that of a portfolio of more than 100% in cash and a short position in the long-term bond. The explicit bond investments thus have to be corrected upwards. Recall from Table 2 that based on aggregate PSID income data the estimate of ξ_1 is 0.3179 but, as reflected by Table 3, there is substantial variation across individuals. For example, for college graduates the median ξ_1 is 0.4912 and the 10% (90%) fractile is -4.2618 (5.4803).

In the case with no income, the second and third terms on the right-hand side of (20) disappear and H=0 in the speculative term and the hedge term. In Appendix C we show that $G(r, t)$ is increasing in T. The optimal bond demand of a conservative investor with no labor income is increasing in the investment horizon. This is inconsistent with the traditional advice of investing more in bonds as the horizon shrinks. As discussed by Munk and Sørensen (2004), the hedge position in the traded bond is combined with a short-term deposit or loan to mimic an investment in a specific coupon bond reflecting the expected consumption stream of the investor. Other things equal, a longer horizon will increase the duration and volatility of this desired coupon bond, which requires a larger weight on the traded bond in the mimicking strategy. The presence of labor income can either reinforce, dampen, or reverse the horizon effect. In Appendix C we also show that $G(r, t)$ is decreasing in r if $\gamma > 1$ and increasing in r if $\gamma < 1$. Keeping the bond volatility fixed, it follows that in absence of labor income, the optimal bond allocation is a decreasing function of the interest rate level. Due to the interest rate sensitivity of human capital, the optimal bond allocation with income may respond very differently to interest rate changes.

Consumption. Using (19), we see that the propensity to consume out of wealth and the propensity to consume out of current income, respectively, are given by

$$\frac{c_t}{W_t} = \frac{1+\frac{y_t}{W_t}M(r_t,t)}{g(r_t,t)}, \quad \frac{c_t}{y_t} = \frac{\frac{W_t}{y_t}+M(r_t,t)}{g(r_t,t)}, \tag{25}$$

which also depend on the wealth-income ratio. As expected, the propensity to consume out of income, c_t/y_t, is increasing in the wealth-income ratio and the expected income growth rate and decreasing in the current income rate, while the dependence on the income volatility, the risk aversion coefficient, the investment horizon, and the interest rate level is parameter specific.

3.2. Locally risk-free income

In order to illustrate how the business cycle dependence of the income growth affects optimal decisions over the life-cycle, we assume now for simplicity that the income is locally risk-free, i.e., $\sigma_y(t) \equiv 0$. Since the interest rate enters the income growth rate (unless $\xi_1 = 0$) and evolves stochastically, future income will still be stochastic. It is clear from (21) that the fraction of total wealth optimally invested in the stock, $\theta_{St}/(W_t+H_t)$, is then constant. The fraction of total wealth optimally invested in the bond reduces to

$$\frac{\theta_{Bt}}{W_t+H_t} = \frac{1}{\gamma\sigma_B}\left(\lambda_r - \rho_{SB}\frac{\lambda_S}{\sqrt{1-\rho_{SB}^2}}\right) + (\xi_1-1)\frac{\sigma_r}{\sigma_B}\frac{H_t}{W_t+H_t}\frac{M_r(r_t,t)}{M(r_t,t)} + \frac{\gamma-1}{\gamma}\frac{\sigma_r}{\sigma_B}G(r_t,t).$$

The fraction of total wealth invested in the short-term bank account is residually determined as $1-(\theta_{St}+\theta_{Bt}+H_t)/(W_t+H_t)$.

Since G and M_r/M are relatively insensitive to the interest rate level, the variations in the fraction of total wealth invested in the bond are mainly due to the variations in the human wealth to total wealth ratio, $H_t/(W_t+H_t)$, and to the shortening of the time horizon. For relatively long horizons, both G and M_r/M are fairly stable, so the relative bond investment is mainly determined by the decomposition of wealth.

The following graphs are constructed assuming a relative risk aversion of $\gamma = 4$, a time preference parameter of $\delta = 0.03$, and $\varepsilon = 3$ indicating that a terminal wealth of x provides roughly as much utility as consuming x in each of the last three years. Benchmark values of the market parameters are used, cf. Table 1, and the current short rate is assumed identical to the long-term average. We assume that the bond that the investor trades in is, at any date, a zero-coupon bond maturing ten years later. The relative bond investment, and thus the relative investment in the bank account, depend on the interest rate sensitivity of income growth. We assume for the moment that ξ_0 is independent of age and, when varying ξ_1, we determine ξ_0 so that the expected income growth rate at the long-term average interest rate equals 2% per year, i.e., $\xi_0 + \xi_1\bar{r} = 0.02$.

Fig. 2 shows how the investments relative to total wealth depend on the human wealth to total wealth ratio. We assume 30 years to retirement and 15 years in retirement, but up to a few years before retirement the picture is similar. Of course, for a typical investor the human wealth to total wealth is quite high early in life and then decreases to zero as retirement approaches.

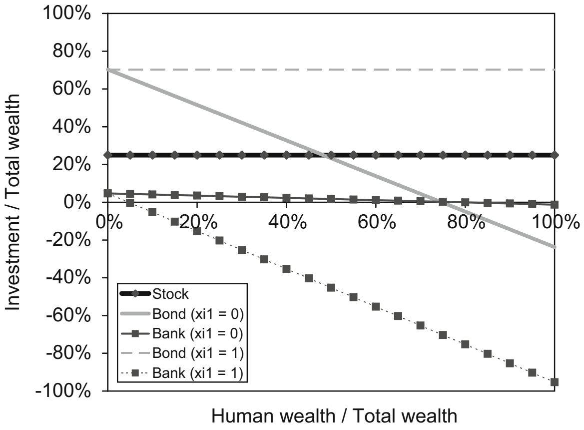

Fig. 2. Optimal unconstrained investments and the decomposition of wealth with locally risk-free income. The figure shows how the optimal fractions of total wealth invested in the different assets depend on the ratio of human wealth to total wealth. The income is assumed locally risk-free, $\sigma_y(t)\equiv 0$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The individual faces no portfolio constraints and has 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The flat line with the diamonds reflects the investment in the stock, which is independent of the value of the parameter ξ_1. The gray solid (dashed) line depict the investment in the bond, when $\xi_1=0$ ($\xi_1=1$). The solid (dashed) line with the boxes show the investment in the bank account, when $\xi_1=0$ ($\xi_1=1$).

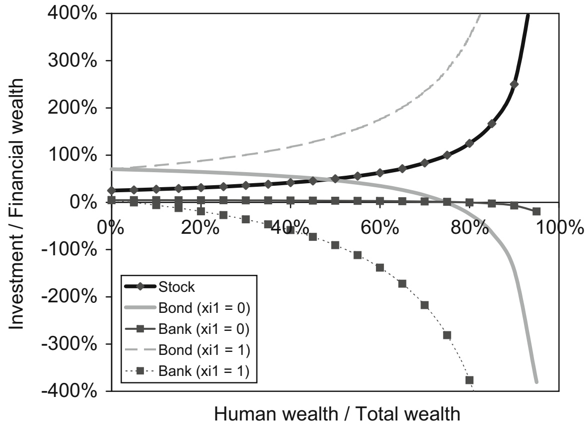

Fig. 3. Optimal unconstrained investments and the decomposition of wealth with locally risk-free income. The figure shows how the optimal fractions of financial wealth invested in the different assets depend on the ratio of human wealth to total wealth. The income is assumed locally risk-free, $\sigma_y(t)\equiv 0$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The individual faces no portfolio constraints and has 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The curve with the diamonds reflects the investment in the stock, which is independent of the value of the parameter ξ_1. The gray solid (dashed) curve depict the investment in the bond, when $\xi_1=0$ ($\xi_1=1$). The solid (dashed) curve with the boxes show the investment in the bank account, when $\xi_1=0$ ($\xi_1=1$).

Of total wealth, 25% is optimally invested in the stock, independently of the decomposition of both wealth and income growth. For zero human wealth, 70.3% is invested in the bond (solely to hedge against interest rate risk) and 4.7% in the bank account. For high values of the human wealth to total wealth ratio (early in life), the investments in the bond and the bank account vary significantly with the parameter ξ_1. The solid curves represent the case $\xi_1=0$, where the income growth rate is constant and human wealth constitutes an implicit investment in the bond. The direct investment in the bond is therefore very small and even negative when human wealth is very high. The dashed curves are for $\xi_1=1$, where the human wealth is insensitive to the interest rate and therefore, constitutes an implicit investment in the bank account. The direct investment in the bond is therefore constant at the 70.3%, while significant borrowing is optimal when human wealth dominates. Note that while the magnitude of the fractions of total wealth invested in the different assets may seem reasonable, the fractions of financial wealth invested in the assets, π_t, can be quite extreme, when the human wealth dominates financial wealth, since $\pi_t=\theta_t/(W_t+H_t)/(1-H_t/W_t+H_t)$. This is clear from Fig. 3, which is similar to Fig. 2 except that the investments are now expressed in fractions of financial wealth. Many investors face liquidity and short-selling constraints that do not allow such strategies to be implemented, which motivates our analysis in Section 4.

Fig. 4 illustrates how the portfolio changes over the life-cycle. The individual has an initial financial wealth equal to the initial annual income, 30 years to retirement, and

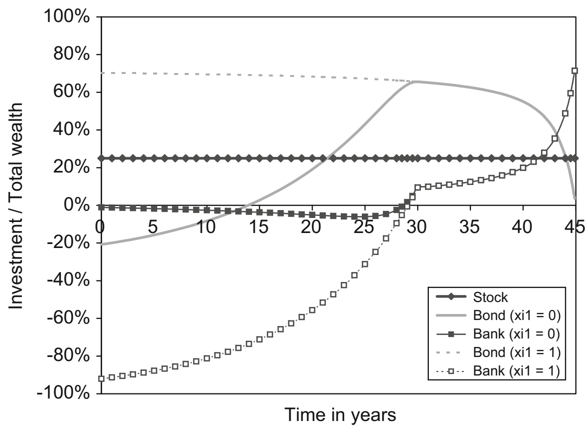

Fig. 4. Optimal unconstrained investments over the life-cycle with locally risk-free income. The figure shows how the optimal fractions of total wealth invested in the different assets vary over the life-cycle. The income is assumed locally risk-free, $\sigma_y(t)\equiv 0$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The initial annual income y_0 and the initial financial wealth W_0 are assumed identical. The individual faces no portfolio constraints and has initially 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The line with the diamonds reflects the investment in the stock, which is independent of the value of the parameter ξ_1. The gray solid (dashed) curve depict the investment in the bond, when $\xi_1=0$ ($\xi_1=1$). The solid (dashed) curve with the boxes show the investment in the bank account, when $\xi_1=0$ ($\xi_1=1$). The average interest rate $\bar{r}$ is used in G and M_r/M, and the future ratio of human wealth to total wealth is approximated by the ratio of time zero expectations.

15 years in retirement. The future optimal portfolios will depend on the future interest rate and the future value of the ratio of human-to-total wealth. In the graph we use the average interest rate $\bar{r}$ in G and M_r/M, and we use the ratio of expected human wealth to expected total wealth, $\mathrm{E}_0[H_t]/\mathrm{E}_0[W_t+H_t]$. The expected human and total wealth levels are computed in Appendix B. Again, the optimal fraction of total wealth invested in the stock is flat at 25%, while the optimal fractions invested in the bond and the bank account vary both before and in retirement, and before retirement these fractions are highly dependent on the interest rate coefficient ξ_1 in the income growth rate. In the retirement phase, the bond demand is purely determined by the intertemporal hedging motive. The optimal interest rate hedge can be seen as a bond paying a coupon equal to the expected future consumption (under the forward measure), cf. Munk and Sørensen (2004), which becomes a shorter and shorter bond as the individual is running out of time. The way to replicate this hedge is to put an increasingly smaller weight on the ten-year bond and higher weight on the bank account. Again, note that the value of ξ_1 has a big impact on the allocation between bonds and cash, especially early in life.

Fig. 5 shows how the expected decomposition of wealth changes over life. Initial financial wealth and initial annual income are both assumed equal to \$20,000. Human wealth dominates early in life and, with the parameter values given, is expected to decrease monotonically to zero at retirement. The expected total wealth is also monotonically decreasing over life and ends up well above zero due to the utility weight on terminal wealth. Of course, total wealth and financial wealth are identical in retirement. The expected financial wealth has a hump shape so that financial wealth is accumulated when working in order to finance consumption in retirement. Comparing the graphs for $\xi_1=1$ (pro-cyclical income growth; solid curves) and $\xi_1=-1$ (counter-cyclical income growth; dashed curves), we see that the value of ξ_1 has only minor effects on the life-cycle pattern of wealth, so ξ_1 is mainly important for the allocation of wealth to long-term bonds and short-term deposits or loans.

3.3. An example with spanned income risk

Next, we consider the case where labor income is perfectly positively correlated with the stock market so that $\rho_{yS}=1$, while $\rho_{yB}=\rho_{SB}=0$. This will be relevant for stock traders, although the perfect correlation is surely extreme. Note that the present value of the future income is decreasing in the income-stock correlation, just as the value of a pro-cyclical dividend stream is smaller than a counter-cyclical dividend stream, other things equal. Other parameter values are as in the previous subsection.

Fig. 6 shows the investments relative to total wealth as a function of the ratio of human wealth to total wealth.

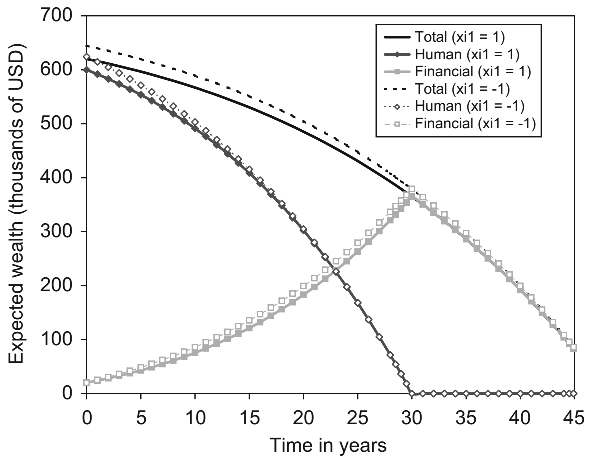

Fig. 5. Expected wealth over the life-cycle with locally risk-free income and no investment constraints. The figure shows how the expected human wealth, the expected financial wealth, and the sum of the two vary over life. The income is assumed locally risk-free, $\sigma_y(t)\equiv 0$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The initial annual income y_0 and the initial financial wealth W_0 are assumed identical and equal to \$20,000. The individual faces no portfolio constraints and has initially 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The graphs are drawn using the expressions for expected wealth derived in Appendix B. The solid (dashed) curves are for the case $\xi_1=1$ ($\xi_1=-1$). The curves with boxes (diamonds) depict financial (human) wealth, while the curves without markers depict the total wealth. In retirement, human wealth is zero so total wealth is identical to financial wealth.

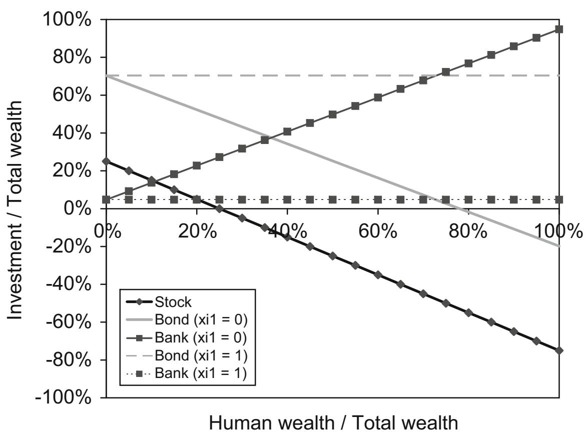

Fig. 6. Optimal unconstrained investments and the decomposition of wealth with stock-like income. The figure shows how the optimal fractions of total wealth invested in the different assets depend on the ratio of human wealth to total wealth. The income is assumed to be perfectly correlated with the stock, $\rho_{yS}=1$, and uncorrelated with the bond, $\rho_{yB}=0$. The income has a constant volatility $\sigma_y=0.20$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The individual faces no portfolio constraints and has 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The line with the diamonds reflects the investment in the stock, which is independent of the value of the parameter ξ_1 for a fixed ratio of human wealth to total wealth. The gray solid (dashed) line depict the investment in the bond, when $\xi_1=0$ ($\xi_1=1$). The solid (dashed) line with the boxes show the investment in the bank account, when $\xi_1=0$ ($\xi_1=1$).

The relative stock investment is now

$$\frac{\theta_{St}}{W_t+H_t} = \frac{1}{\gamma}\frac{\lambda_S}{\sigma_S} - \frac{\sigma_y(t)}{\sigma_S}\frac{H_t}{W_t+H_t},$$

and decreases linearly in the human-total wealth ratio because human wealth constitutes an implicit stock position. When human wealth dominates, the optimal strategy involves significant short-selling of the stock. For a fixed human-total wealth ratio, the relative stock investment is independent of ξ_1. The optimal bond investment is very similar to the case of locally risk-free income. The optimal position in short-term deposits is residually determined. Whether the negative explicit stock position for high human wealth is mainly balanced by a positive position in the long-term bond or short-term deposits depends on ξ_1.

The optimal portfolios over the life-cycle are illustrated in Fig. 7. As before, we use the average interest rate $\bar{r}$ in G and M_r/M and the ratio of expected human wealth to expected total wealth, $\mathrm{E}_0[H_t]/\mathrm{E}_0[W_t+H_t]$, when computing the future portfolio weights. Again, we note the significant short-selling of stocks early in life, and we can again see the importance of the parameter ξ_1 for the allocation between long-term bonds and short-term deposits.

The initial expectation of future wealth is depicted in Fig. 8. The expected total wealth is qualitatively very similar to the case of locally risk-free income and decreases monotonically over the life. Human wealth is now expected to increase slightly in the early years, since the income multiplier $M(r_t,t)$ with the assumed correlations decreases relatively slowly in the beginning and this is more than outweighed by the expected increase in the income rate. Consequently, the financial wealth is expected to decrease early in life and will even go negative for some parameter constellations. After some years, we have a hump-shaped curve as in the previous subsection. Again, the value of ξ_1 is relatively unimportant for the life-cycle pattern of wealth.

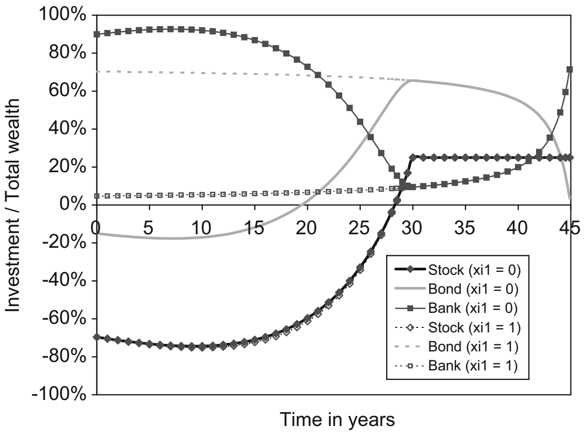

Fig. 7. Optimal unconstrained investments over the life-cycle with stock-like income. The figure shows how the optimal fractions of total wealth invested in the different assets vary over the life-cycle. The income is assumed to be perfectly correlated with the stock, $\rho_{yS}=1$, and uncorrelated with the bond, $\rho_{yB}=0$. The income has a constant volatility $\sigma_y=0.20$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The initial annual income and the initial financial wealth are assumed identical. The individual faces no portfolio constraints and has initially 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The solid (dashed) curve with the diamonds reflect the investment in the stock; these curves are almost indistinguishable. The gray solid (dashed) curve depict the investment in the bond, when $\xi_1=0$ ($\xi_1=1$). The solid (dashed) curve with the boxes show the investment in the bank account, when $\xi_1=0$ ($\xi_1=1$). The average interest rate $\bar{r}$ is used in G and M_r/M, and the future ratio of human wealth to total wealth is approximated by the ratio of time zero expectations.

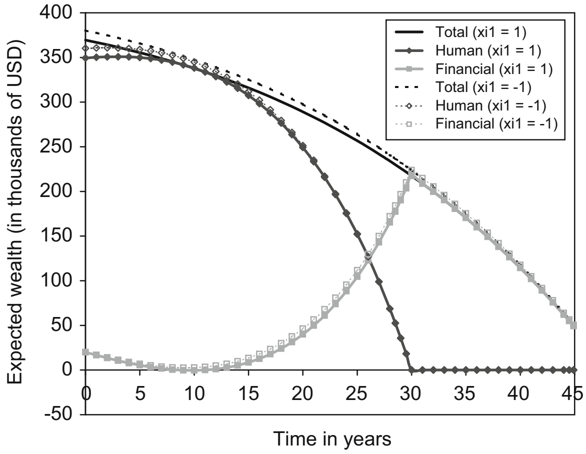

Fig. 8. Expected wealth over the life-cycle with stock-like income and no investment constraints. The figure shows how the expected human wealth, the expected financial wealth, and the sum of the two vary over life. The income is assumed to be perfectly correlated with the stock, $\rho_{yS}=1$, and uncorrelated with the bond, $\rho_{yB}=0$. The income has a constant volatility $\sigma_y=0.20$, and the average income growth rate $\xi_0+\xi_1\bar{r}$ is fixed at 2%. The initial annual income y_0 and the initial financial wealth W_0 are assumed identical and equal to \$20,000. The individual faces no portfolio constraints and has initially 30 years to retirement plus 15 years in retirement. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For market parameters, the benchmark values from Table 1 are used. The graphs are drawn using the expressions for expected wealth derived in Appendix B. The solid (dashed) curves are for the case $\xi_1=1$ ($\xi_1=-1$). The curves with boxes (diamonds) depict financial (human) wealth, while the curves without markers depict the total wealth. In retirement, human wealth is zero so total wealth is identical to financial wealth.

3.4. Unspanned income risk

The strategies in Proposition 2 are optimal if there are no portfolio restrictions and either (i) labor income is instantaneously risk-free or (ii) labor income risk is spanned by traded assets. Neither of those two cases are realistic for most individuals. Of course, you can compute a value of the income stream using Proposition 1 and apply the consumption and investment strategies of Proposition 2 whether or not any of the two assumptions are satisfied, but the strategies are then not necessarily optimal. In a slightly simpler setting (interest rates assumed constant), Bick, Kraft, and Munk (2009) quantify the loss incurred when following this sub-optimal strategy instead of the optimal strategy, which is unknown when income risk is unspanned. They find that the utility loss corresponds to at most 14% of initial total wealth even when income is uncorrelated with asset returns. The loss gets much smaller the higher the income-asset correlation and, of course, approaches zero as the income-asset correlation goes to 1.[8] The results in Propositions 1 and 2 are therefore relevant even without perfect spanning.

Note that for our benchmark parameters with zero income-asset correlations and a zero stock-bond correlation, Proposition 1 will deliver exactly the same value of human wealth as in the case with a locally risk-free income studied above. Moreover, Proposition 2 will deliver exactly the same consumption and investment strategy as for locally risk-free income. Therefore, Figs. 2–5 also illustrate how investments and expected wealth vary over the life-cycle when that strategy is applied with the benchmark parameters.

4. Unspanned income uncertainty and liquidity constraints

We now consider the case where income is risky and not spanned by traded assets so that $\|\rho_{yP}\| < 1$ and, in addition, we impose a liquidity constraint on the individual so that the financial wealth has to stay non-negative at all points in time. We will also consider the effects of imposing stricter constraints so that the investor is restricted to non-negative positions in both the bond, the stock, and the bank account. Due to the unspanned income and the investment constraints, it is no longer possible to derive a closed-form solution to the utility maximization problem, so we have to resort to numerical solution techniques.

Our numerical method is based on a finite difference backwards iterative solution of the HJB equation with an optimization over feasible consumption rates and portfolios at each (time, state) node in the lattice. Before retirement, the original formulation of the problem has three state variables: financial wealth, interest rate, and income rate. In order to simplify the implementation of the numerical solution algorithm, we exploit a homogeneity property that reduces the number of state variable by one. It follows from the power utility assumption, the linearity of the wealth dynamics in (8), and the income dynamics in (7) that if the consumption and investment strategy (c_s, θ_s) is optimal for $s \geq t$ with time t wealth and income (W_t, y_t), then $(k(t)c_s, k(t)\theta_s)$ is optimal with time t wealth and income $(k(t)W_t, k(t)y_t)$. The value function is thus homogeneous of degree $1-\gamma$ in (W, y), i.e.,

$$J(k(t)W_t, r_t, k(t)y_t, t) = k(t)^{1-\gamma} J(W, r, y, t)$$
$$\Rightarrow \quad J(W, r, y, t) = k(t)^{\gamma-1} J(k(t)W_t, r_t, k(t)y_t, t).$$

Using this with $k(t) = e^{-\beta t}/y_t$, where β is some constant to be discussed, we obtain[9]

$$J(W_t, r_t, y_t, t) = y_t^{1-\gamma} e^{-\beta(\gamma-1)t} J(x_t, r_t, e^{-\beta t}, t) \equiv y_t^{1-\gamma} F(x_t, r_t, t), \tag{26}$$

where we have defined $x_t = e^{-\beta t} W_t / y_t$. By substitution of (26) into the HJB-equation (10), we get that $F = F(x,r,t)$ solves the non-linear partial differential equation (PDE)

$$\begin{aligned}\hat{\delta}(r,t)F = \sup_{\hat{c},\pi}\Bigg\{ & \frac{\hat{c}^{1-\gamma}}{1-\gamma} + F_t + F_r(\kappa[\overline{r} - r] + (1-\gamma)\rho_{yr}\sigma_y(t)\sigma_r) \\ & + \frac{1}{2}\sigma_r^2 F_{rr} + F_x((1-\hat{c})e^{-\beta t} + x[(1-\xi_1)r - \beta - \xi_0(t) \\ & + \gamma\sigma_y(t)^2 + \pi^\top \Sigma(\lambda - \gamma\sigma_y(t)\rho_{yP})]) + \frac{1}{2}x^2 F_{xx}(\pi^\top \Sigma\Sigma^\top \pi \\ & + \sigma_y(t)^2 - 2\sigma_y(t)\pi^\top \Sigma \rho_{yP}) - xF_{xr}\sigma_r(\pi^\top \Sigma \mathbf{e}_1 + \rho_{yr}\sigma_y(t))\Bigg\},\end{aligned} \tag{27}$$

where we have introduced

$$\hat{\delta}(r,t) = \delta + (\gamma-1)(\xi_0(t) + \xi_1 r) - \tfrac{1}{2}\gamma(\gamma-1)\sigma_y(t)^2$$

and $\hat{c}_t = c_t / y_t$ is the consumption-to-income ratio and $\pi_t = \theta_t / W_t$ is the vector of portfolio weights. Assuming that any portfolio constraints are non-binding in the retirement phase, the value function at the retirement date $\tilde{T}$ is given by $J(W, r, y, \tilde{T}) = g(r, \tilde{T})^\gamma W^{1-\gamma}/(1-\gamma)$, where g is given by (18). It follows that[10]

$$F(x, r, \tilde{T}) = \frac{1}{1-\gamma} e^{-\beta(\gamma-1)\tilde{T}} g(r, \tilde{T})^\gamma x^{1-\gamma}. \tag{28}$$

We set up a lattice in (x,r,t) and solve the PDE (27) numerically using a backward iterative procedure starting from the retirement date $\tilde{T}$. At each time t_n in the lattice we first guess on the optimal controls $\hat{c}(x_i, r_j, t_n), \pi(x_i, r_j, t_n)$ and solve (27) using an implicit finite difference technique for $F(x_i, r_j, t_n)$, which is then a guess on the value function at time t_n. Using that in the first-order conditions for the maximization in (27), we can derive a new guess on the optimal controls, which can again be used to find a new guess on the value function. We continue these iterations until the guess on the value function at t_n is stable, and we can then move on to the previous time step t_{n-1}.

The wealth/income ratio can have a very high drift rate initially since the individual wants to save for retirement and, in particular when initial wealth is low,

[8] The loss is computed by comparing the expected utility from following the sub-optimal strategy, approximated by Monte Carlo simulations, and an upper bound on the maximum expected utility with unspanned income, which is computed analytically using the idea of artificially completing the market introduced by Cvitanić and Karatzas (1992). The loss reported is therefore an upper bound on the true loss.

[9] Of course, this is also true in the complete market case studied previously. There we have

$$F(x_t, r_t, t) = \frac{1}{1-\gamma} g(r_t, t)^\gamma (x_t e^{\beta t} + M(r_t, t))^{1-\gamma},$$

where $M(r,t)$ is given by (14) and $g(r,t)$ by (18).

[10] If some constraint is binding for some state in the retirement phase, $J(W, r, \cdot, \tilde{T})$ has to be found numerically as well, by similar backwards iterations from the terminal date T and back to $\tilde{T}$. Here, the separation $J(W, r, \cdot, \tilde{T}) = G(r,t)W^{1-\gamma}/(1-\gamma)$ reduces the problem to solving

$$\begin{aligned}(\delta + r(\gamma-1))G = \sup_{\tilde{c},\pi}\{ & \tilde{c}^{1-\gamma} + G_t - (\gamma-1)G(\pi^\top \Sigma\lambda - \tilde{c}) + G_r\kappa(\overline{r} - r) \\ & + \frac{1}{2}\sigma_r^2 G_{rr} + \frac{1}{2}\gamma(\gamma-1)G\pi^\top \Sigma\Sigma^\top \pi + (\gamma-1)G_r \pi^\top \Sigma \mathbf{e}_1 \sigma_r\}\end{aligned}$$

with $G(r, T) = \varepsilon$. Here, $\tilde{c} = c/W$.

to get away from the unpleasant liquidity constraint. The wealth/income ratio can therefore reach values far exceeding the initial value, which makes it hard to approximate the dynamics of the wealth/income ratio by transitions in a lattice with a fairly low number of grid points. For appropriate values of the constant β, the scaled wealth/income ratio $x_t = e^{-\beta t} W_t / y_t$ is more stable and easier to handle numerically. The appropriate value of β will depend on the parameters affecting the dynamics of wealth and income and is found experimentally for each parameter constellation we consider.

Our solution technique is closely related to that applied by Brennan, Schwartz, and Lagnado (1997) and is very similar to the well-documented Markov Chain Approximation Approach, which has previously been used to study various consumption/investment problems, cf. Fitzpatrick and Fleming (1991), Hindy, Huang, and Zhu (1997), and Munk (1999, 2000). Additional information concerning the numerical procedure can be found in Appendix D.

We ensure that financial wealth W, and hence the scaled wealth-income ratio x, stay non-negative by restricting the individual's choice of consumption and portfolio whenever x=0 to a zero investment in the risky assets and to a consumption level which is smaller than the current income, i.e., $\pi(0,r,t) = 0$ and $\hat{c}(0,r,t) \leq 1$. If we do not further restrict the consumption and portfolio choice, the optimal choice for a strictly positive value of x is given by the first-order conditions from (27), i.e., that if we substitute the expression for F in the complete market case given in Footnote 9 into (30), we get (24).

$$\hat{c}_t = (F_x)^{-1/\gamma} e^{\beta t/\gamma}, \tag{29}$$

$$\pi_t = -\frac{F_x}{xF_{xx}} (\Sigma^\top)^{-1} (\lambda - \gamma \sigma_y(t) \rho_{yP}) + \frac{F_{xr}}{xF_{xx}} \sigma_r (\Sigma^\top)^{-1} \mathbf{e}_1 + \sigma_y(t) (\Sigma^\top)^{-1} \rho_{yP}. \tag{30}$$

Imposing the liquidity constraint makes lower levels of financial wealth worse. Without the liquidity constraint, a consumer-investor with a large human capital is not that concerned with a fall in financial wealth from a low level to zero, in fact the financial wealth can go negative. With the liquidity constraint, the consequences of losing financial wealth from a low level are more severe. If you end up at a zero financial wealth, you have to stay away from the risky assets and keep consumption below current labor income. You can only return to strictly positive wealth and risky positions by consuming strictly less than your income. We therefore expect significantly less risky positions at near-zero financial wealth levels relative to the case without the liquidity constraint.

Imposing a strict no-borrowing condition for all wealth levels, we must have $\pi_{Bt} + \pi_{St} \leq 1$. Of course, if the portfolio given by (30) satisfies this condition, it is still the optimal portfolio, but if the constraint is binding, we maximize in (27) over portfolios (π_{Bt}, π_{St}) with $\pi_{Bt} + \pi_{St} = 1$ and get

$$\begin{aligned} \pi_{Bt} = \frac{1}{\sigma_B^2 + \sigma_S^2 - 2\rho_{SB}\sigma_S\sigma_B} \Bigg\{ & -\frac{F_x}{xF_{xx}} (\sigma_B [\lambda_r - \gamma \sigma_y(t) \rho_{yB}] \\ & -\psi + \gamma \sigma_S \sigma_y(t) \rho_{yS}) + \frac{F_{xr}}{xF_{xx}} (\sigma_B - \sigma_S \rho_{SB}) \\ & + \sigma_S^2 - \rho_{SB} \sigma_S \sigma_B + \sigma_y(t) [\sigma_B \rho_{yB} - \sigma_S \rho_{yS}] \Bigg\} \end{aligned}$$

and, of course, $\pi_{St} = 1 - \pi_{Bt}$. In a similar manner we can impose non-negativity constraints on the portfolio weights.

Below we illustrate and discuss various aspects of the optimal strategies and the corresponding financial wealth over the 30-year period where the individual is assumed to be active in the labor market. Unless otherwise mentioned, we will use the benchmark parameters of Tables 1 and 2 and assume that the initial income rate and the initial financial wealth are identical and equal to one, i.e., W_0=y_0=1, and that the initial interest rate is equal to its long-term level of 2%. We assume a relative risk aversion of $\gamma = 4$ and a time preference rate of $\delta = 0.03$. We solve the appropriate HJB equations numerically as discussed above to find the optimal state- and age-dependent consumption rate and portfolio over the entire time span. Then we simulate the financial wealth, the income rate, and the interest rate forward from time zero to retirement using those consumption rates and portfolios in the wealth dynamics. We average over 1,000 paths to obtain an estimate of the typical life-cycle pattern in consumption, investments, and wealth. We will investigate how the optimal behavior over the life-cycle depends on constraints and variations in key parameter values. Throughout we study the impact of the interest rate sensitivity of expected income growth and show results both for $\xi_1 = 1$ (pro-cyclical income) and $\xi_1 = 0$ (non-cyclical income) with ξ_0 set so that the expected income growth at the mean interest rate, $\xi_0 + \xi_1 \bar{r}$, equals 3%. Again we note that the parameter estimates of Table 3 show that ξ_1-values of 0 and 1 are not at all extreme. We will first assume that the growth rate of labor income is not age-dependent to get a clearer picture of the effects of constraints and parameter values. Subsequently, we consider the case with an age-dependent labor income growth as discussed in Section 2.3.

4.1. Numerical illustrations

Fig. 9 depicts how the liquidity constraint and short-sales constraints affect the optimal investment in stocks and bonds, the optimal consumption rate, and the financial wealth over the active phase of the individual. The curves marked by diamonds represent the case with unconstrained strategies, the unmarked curves represent the case where the liquidity constraint is imposed, while the curves marked by boxes are for the case where the liquidity constraint is imposed *and* short sales are not allowed. In every panel there are two curves of each style, a thick curve for $\xi_1 = 1$ and a thin curve for $\xi_1 = 0$. Each panel therefore contains a total of six curves but in some cases two or more curves are indistinguishable. The unconstrained strategy is determined by substituting the benchmark correlation parameter values of Tables 1 and 2 into the explicit expressions of Proposition 2. While this procedure does not give the precise optimal unconstrained strategies for non-spanned income, the strategies can be expected to be near-optimal as discussed in Section 3.4. Panels (a) and (b) of the figure show that imposing a liquidity constraint will significantly lower the

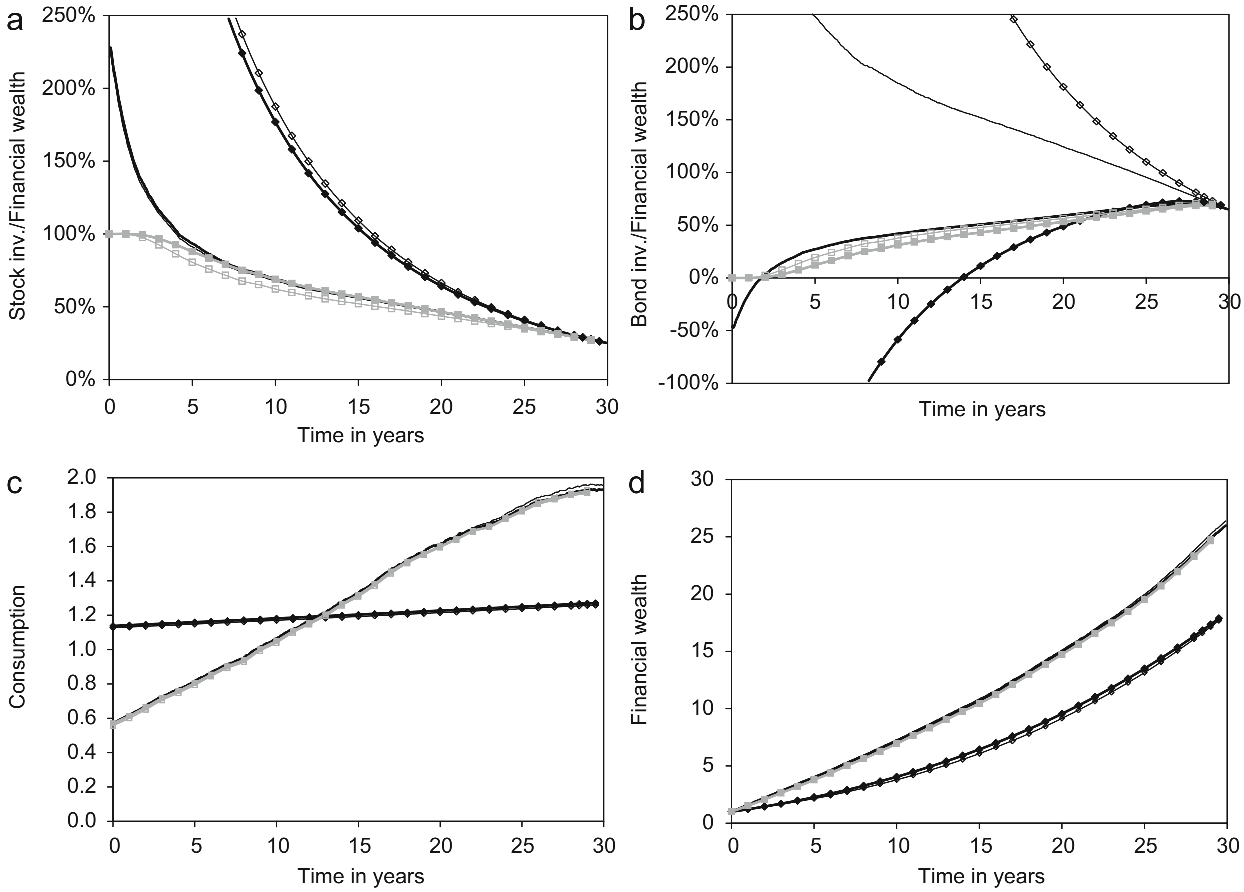

Fig. 9. Sensitivity to constraints on consumption and portfolios. The figure illustrates how imposing the liquidity constraint and short-sales constraints affect the optimal investment in stocks and bonds, the optimal consumption rate, and the financial wealth. The black curves marked by diamonds are for the case where strategies are unconstrained, the unmarked curves are for the case where strategies are liquidity constrained, and the gray curves marked by boxes are for the case with a liquidity constraint and prohibition of short sales. Thick curves are for $\xi_1 = 0$ and thin curves are for $\xi_1 = 1$. In Panels (c) and (d), the thick and the thin curves referring to the same type of constraints are almost indistinguishable. The graphs are drawn assuming that the initial annual income y_0 and the initial financial wealth W_0 are identical and equal to one, an initial risk-free rate of $r_0 = \bar{r}$, and a 30-year income stream with an age-independent average income growth rate of $\xi_0 + \xi_1\bar{r} = 3\%$. The individual has a relative risk aversion of $\gamma = 4$, a subjective time preference rate of $\delta = 0.03$, and a utility weight $\varepsilon = 3$ on terminal wealth. For other parameters, the benchmark values from Table 1 are used. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) The stock weight. (b) The bond weight. (c) The consumption rate. (d) Financial wealth.

investment in the stock and the bond throughout the working period. Intuitively the liquidity constraint will lower the value the individual associates with a given future income stream since it limits the use of future income for consumption smoothing. Hence, the speculative asset demands and the interest rate hedge demand for the bond will be dampened relative to the unconstrained case. As can be seen from Panels (c) and (d), consumption will be scaled down in early years in order to build up a considerable financial wealth that makes the liquidity constraint less pertinent in the future. In later years consumption can be increased relative to the unconstrained case. Note that after imposing the liquidity constraint, the business cycle dependence of income growth remains very important for the optimal bond investment (and the risk-free position, not shown in the figure). Of course, the liquidity constraint will have larger effects on the optimal strategies for individuals with lower financial wealth compared to labor income.

When short-sales constraints are superimposed, the optimal investments are dampened further. For the first 25 years or so, the optimal liquidity and short-sales constrained portfolio only contains the stock and the bond, while a small risk-free investment becomes optimal in the last five years before retirement. In the very early years, stocks are so attractive (as reflected by the high unconstrained positions) that the investor typically has 100% in stocks, but after a few years the long-term bond enters due to its hedging qualities. Note that the optimal stock-bond-cash allocation is roughly consistent with typical investment advice. Furthermore, note that the degree of cyclicality of income growth remains important for the optimal bond investment and, as long as the stock and the bond investments add up to 100% of financial wealth, it follows that the optimal stock investment will also depend on the cyclicality of expected income growth, in fact more than in the unconstrained case and the case with liquidity constraints only. The effect of income cyclicality on optimal constrained behavior will become clearer in the following figures. Compared to the case with liquidity constraints only, the addition of short-sales constraints reduces further the present value of future income and thus, the effective total wealth of the individual, and the individual will therefore slightly reduce consumption throughout life.

In the following we will focus on the case where investors face both a liquidity and a short-sales constraint. Fig. 10 plots the optimal strategies and wealth for three different coefficients of relative risk aversion, namely $\gamma = 2$ (black curves with diamonds), the benchmark value $\gamma = 4$ (unmarked curves), and $\gamma = 6$ (gray curves with boxes). As expected, higher risk aversion is associated with a lower investment in stocks, a higher investment in bonds, a lower consumption rate, and financial wealth which is higher in early years and lower near retirement. We can also see from Panels (a) and (b) that the degree of income growth cyclicality is more important to highly risk-averse investors, which is natural since they are more concerned with the intertemporal hedging qualities of the bond.

Fig. 11 compares the optimal behavior and wealth for three values of the volatility of the labor income rate, σ_y, namely 10% (black curves with diamonds), the benchmark value of 20% (unmarked curves), and 30% (gray curves with boxes). Higher income uncertainty induces a lower stock investment, a higher bond investment, higher financial wealth, and a consumption rate which is lower when young and higher near retirement. The importance of the slope ξ_1 of expected income growth for the optimal portfolio is clear whenever the bond is optimally included. Young investors with a low income volatility will optimally invest 100% in the stock independently of the

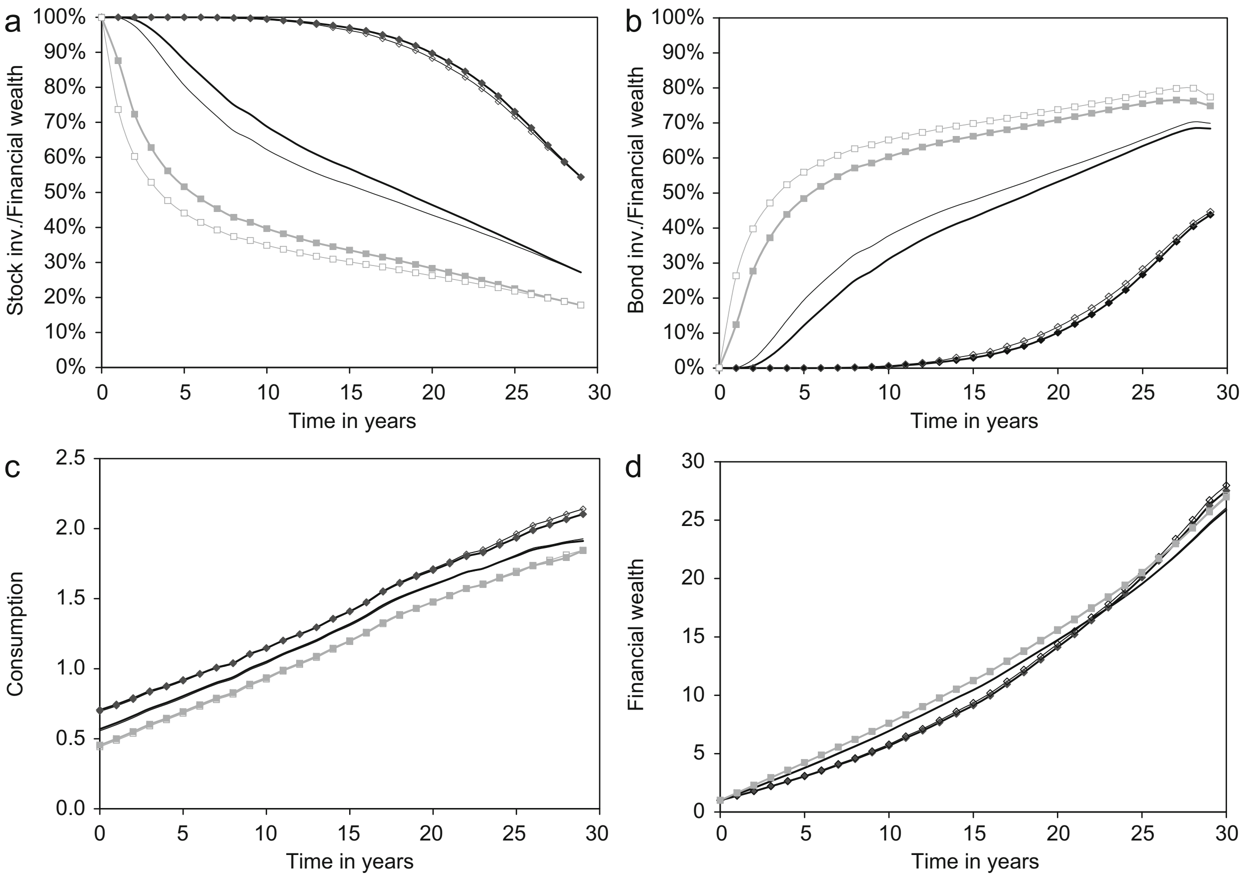

Fig. 10. Sensitivity to the relative risk aversion. The figure illustrates the sensitivity of the optimal strategies and financial wealth to the relative risk-aversion coefficient γ, when liquidity and short-sales constraints have been imposed. The black curves marked by diamonds are for $\gamma = 2$, the unmarked curves are for $\gamma = 4$, and the gray curves marked by boxes are for $\gamma = 6$. Thick curves are for $\xi_1 = 0$ and thin curves are for $\xi_1 = 1$. In Panels (c) and (d), the thick and the thin curves referring to the same γ are almost indistinguishable. The graphs are drawn assuming that the initial annual income y_0 and the initial financial wealth W_0 are identical and equal to one, an initial risk-free rate of $r_0 = \bar{r}$, and a 30-year income stream with an age-independent average income growth rate of $\xi_0 + \xi_1 \bar{r} = 3\%$. The individual has a subjective time preference rate of $\delta = 0.03$ and a utility weight $\varepsilon = 3$ on terminal wealth. For other parameters, the benchmark values from Table 1 are used. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) The stock weight. (b) The bond weight. (c) The consumption rate. (d) Financial wealth.

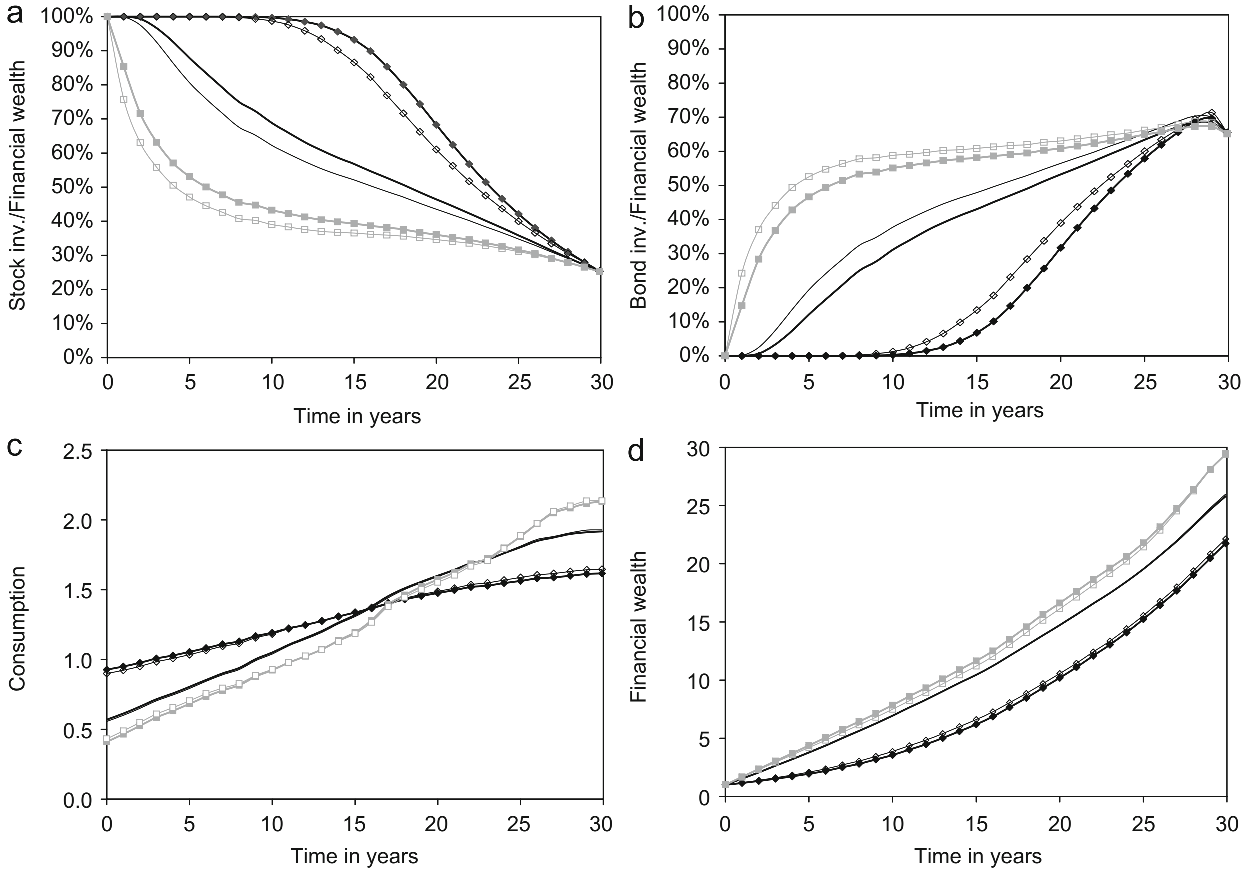

Fig. 11. Sensitivity to income volatility. The figure illustrates the sensitivity of the optimal strategies and financial wealth to the income volatility σ_y, when liquidity and short-sales constraints have been imposed. The black curves marked by diamonds are for $\sigma_y = 0.1$, the unmarked curves are for $\sigma_y = 0.2$, and the gray curves marked by boxes are for $\sigma_y = 0.3$. Thick curves are for $\xi_1 = 0$ and thin curves are for $\xi_1 = 1$. In Panels (c) and (d), the thick and the thin curves referring to the same σ_y are almost indistinguishable. The graphs are drawn assuming that the initial annual income y_0 and the initial financial wealth W_0 are identical and equal to one, an initial risk-free rate of $r_0 = \overline{r}$, and a 30-year income stream with an age-independent average income growth rate of $\xi_0 + \xi_1 \overline{r} = 3\%$. The individual has a relative risk aversion of $\gamma = 4$, a subjective time preference rate of $\delta = 0.03$, and a utility weight $\varepsilon = 3$ on terminal wealth. For other parameters, the benchmark values from Table 1 are used. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) The stock weight. (b) The bond weight. (c) The consumption rate. (d) Financial wealth.

value of ξ_1, but even in that case the optimal consumption is (slightly) depending on ξ_1 since that parameter affects the implicit present value of future income.

Fig. 12 shows how the optimal strategies are affected by the correlation ρ_{yr} between shocks to labor income and shocks to the short-term interest rate. We consider $\rho_{yr} = -0.25$ (black curves with diamonds), the benchmark value of $\rho_{yr} = 0$ (unmarked curves), and $\rho_{yr} = 0.25$ (gray curves with boxes). Consistent with the analysis in earlier sections, the optimal bond investment is increasing in this correlation. For a negative (positive) ρ_{yr}, the labor income stream will implicitly represent a long (short) investment in the bond. In the complete markets case, this correction of the bond demand due to the contemporaneous correlation with income is balanced by an offsetting cash position (same magnitude, opposite sign), but with short-sales constraints such behavior is often impossible so that a positive correction term for the bond has to be balanced by a lower demand for the stock. We can see from Panels (a) and (b) that this is indeed the case for ρ_{yr} equal to zero or 0.25, where the optimal portfolio consists only of the stock and the bond, except when the individual is close to retirement. Therefore, the optimal stock investment has to be decreasing in the correlation ρ_{yr}. A correlation of $\rho_{yr} = -0.25$ leads an unconstrained investor to lower bond demand offset by a positive cash position and this is still possible when short sales are prohibited. In that case the slope ξ_1 of the income drift has a bigger effect on the bond and is offset by a change in the cash position in the opposite direction, exactly as in the unconstrained case. Consumption (and wealth, not illustrated) is only slightly sensitive to ρ_{yr}.

Fig. 13 illustrates the role of the initial value of the short-term interest rate for the case where $\xi_1 = 0.5$ so that expected income growth is modestly related to the interest rate level. A lower initial short rate will thus lower expected income growth. Even though the interest rate will revert quite rapidly to the long-term average of 2% and thus, the expected income growth rate will return to the higher, "normal" level, the early low growth rates

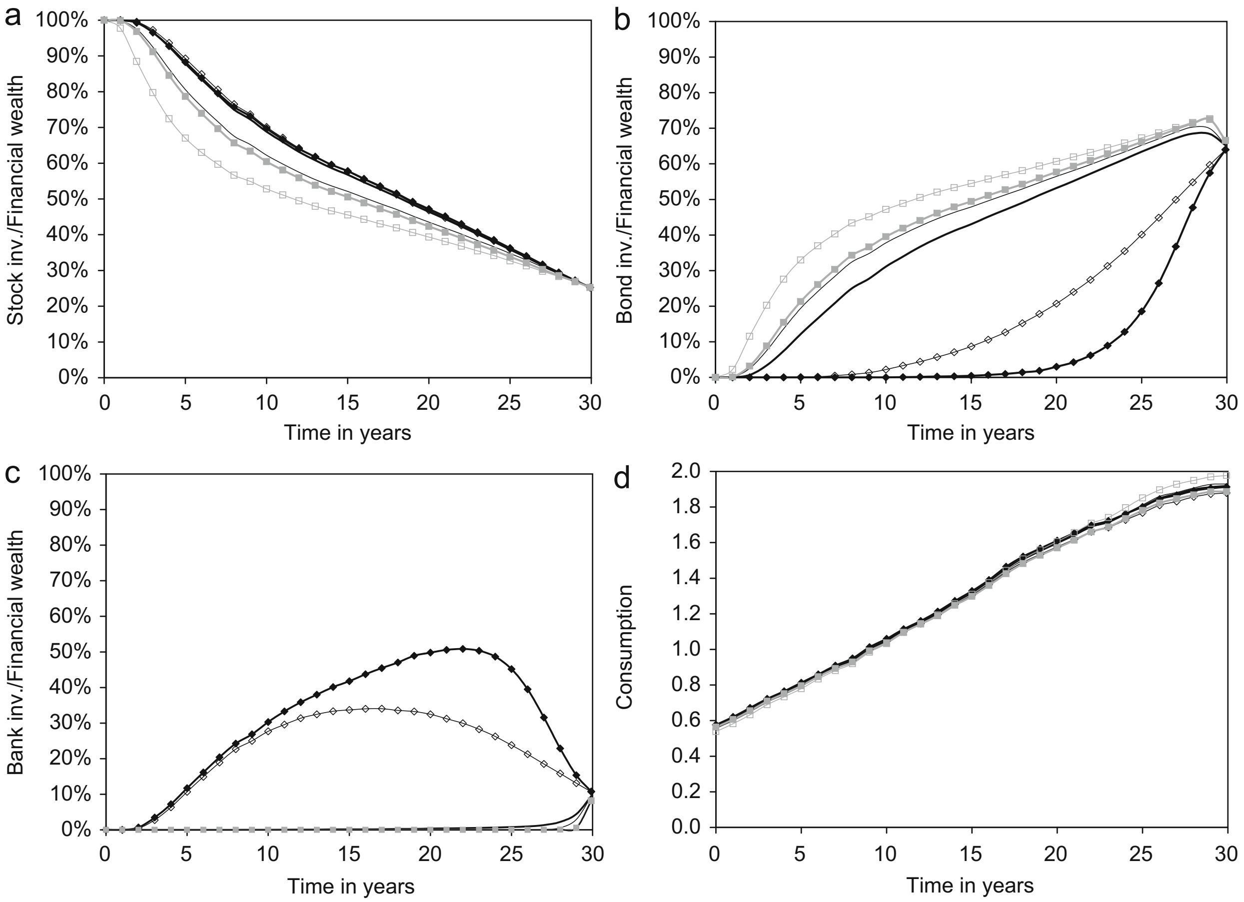

Fig. 12. Sensitivity to the correlation between income and interest rates. The figure illustrates the sensitivity of the optimal strategies to the instantaneous correlation between labor income and the short-term interest rate. The black curves marked by diamonds are for $\rho_{yr}=-0.25$, the unmarked curves are for $\rho_{yr}=0$, and the gray curves marked by boxes are for $\rho_{yr}=0.25$. Thick curves are for $\xi_1=0$ and thin curves are for $\xi_1=1$. Some of the thick and the thin curves referring to the same ρ_{yr} are almost indistinguishable. The graphs are drawn assuming that the initial annual income y_0 and the initial financial wealth W_0 are identical and equal to one, an initial risk-free rate of $r_0=\bar{r}$, and a 30-year income stream with an age-independent average income growth rate of $\xi_0+\xi_1\bar{r}=3\%$. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For other parameters, the benchmark values from Table 1 are used. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) The stock weight. (b) The bond weight. (c) The consumption rate. (d) Financial wealth.

will lead to permanently lower income. In addition, the expected return on all investments will initially be lower than in the benchmark situation. Therefore, the individual lowers the consumption rate. Conversely, a higher initial short rate leads to permanently higher labor income, wealth, and consumption. Clearly, these effects depend on the sign and magnitude of the parameter ξ_1. In all cases we have considered, the initial short rate has a very limited effect on the portfolio weights. Fig. 13 shows that when the initial short rate is changed from 2% to either 0% or 4%, the average consumption, income, and financial wealth changes by less than 2.5% for our benchmark parameters.

Above we have considered values of the parameter ξ_1 between zero and one but, as indicated by the estimates in Table 3, both higher and lower values appear relevant for some individuals. In Fig. 14 we vary ξ_1 between -4 and +4. The figure confirms that ξ_1 is important for all portfolio weights, while the impact on consumption (and wealth; not illustrated) is small. Again note that while the portfolio weight of the stock is independent of ξ_1 in the unconstrained complete market solution of Proposition 2 (at least for a fixed human wealth H_t), this is not true when portfolio constraints are introduced, cf. the discussion above.

4.2. Incorporating life-cycle variations in labor income

We will now consider the case where the expected income growth rate, $\xi_0(t)$, depends on age. We adopt estimation results from Cocco, Gomes, and Maenhout (2005) and Campbell and Viceira (2002) and assume that the average labor income profile over the working life (age 20 to 65 years) of an individual can be described by a third-order polynomial in age, $P^i(t) = a^i + b^i\,t + c^i t^2 + d^i t^3$. The coefficients of the polynomial depend on the educational level, categorized as "No high school," "High

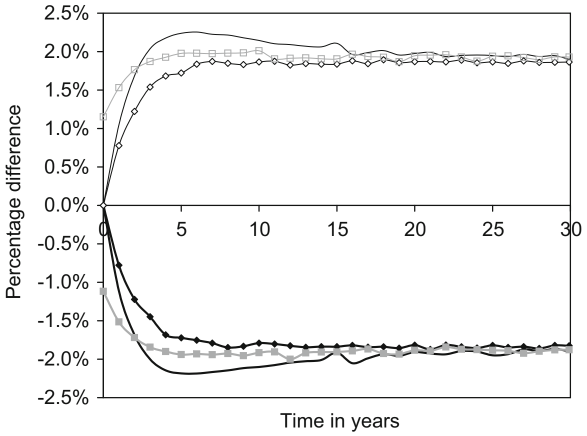

Fig. 13. Sensitivity to the initial short-term interest rates. The figure shows percentage changes in income (black curves with diamonds), wealth (unmarked curves), and consumption (gray curves with boxes) when the initial short-term interest is changed from $r_0 = 2\%$ to 0% (thick curves) or to $r_0 = 4\%$ (thin curves). The graphs are drawn assuming that the initial annual income y_0 and the initial financial wealth W_0 are identical and equal to one, and a 30-year income stream with an age-independent average income growth rate with $\xi_1 = 0.5$ and $\xi_0 = 0.02$. The individual has a relative risk aversion of $\gamma = 4$, a subjective time preference rate of $\delta = 0.03$, and a utility weight $\varepsilon = 3$ on terminal wealth. For other parameters, the benchmark values from Table 1 are used. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies.

school," and "College." In the discrete-time framework of Cocco, Gomes, and Maenhout (2005), it is assumed that the retirement income level for age 66 and higher is proportional to the income level at age 65 with a replacement rate, $\overline{P}^i$, where the superscript i indicates education category. In our continuous-time setting we simply linearly interpolate the income level in the one-year period between age 65 and age 66. As discussed in more detail in Appendix A, the expected income growth level can then be written as

$$\xi_0(t) = \begin{cases} \overline{\xi}_0 + b^i + 2c^i t + 3d^i t^2 & \text{if } 20 \le t \le 65, \\ -(1-\overline{P}^i) & \text{if } 65 < t < 66, \\ 0 & \text{if } t \ge 66, \end{cases} \tag{31}$$

where $\overline{\xi}_0$ captures a general, age-independent real wage improvement over time. The estimated polynomial life-cycle income profiles are illustrated in Fig. 15, and the polynomial coefficients used in the figure are reproduced from Cocco, Gomes, and Maenhout (2005) in Table 4. We will use $\overline{\xi}_0 = 0.02$ and the coefficients listed in Table 4 representing the three different levels of education. We take the income rate volatility to be $\sigma_y(t) = 0.20$ for $t \le 65$ and $\sigma_y(t) = 0$ for $t \ge 65$. Note that these assumptions imply that the retired individual receives a fixed income equal to the fraction $\overline{P}^i$ of her income immediately before retirement. For simplicity we continue to assume that the agent has a fixed terminal date, which we set to age 80. We still assume a relative risk aversion of four, a time preference rate of 0.03, and W_0=y_0=1. We take the median estimates of ξ_1 in each educational group, which can be found in Table 3. We impose liquidity constraints and short-selling constraints. As above, we first find the optimal strategies by numerically solving the constrained HJB equation and then we simulate forward 1,000 paths using those strategies. The graphs below depict averages over these 1,000 paths. In order to keep the presentation clear, we focus only on college graduates and individuals without high school education. The results for individuals with high school, but no college, education are mostly in between the results for the two other educational groups.

Fig. 16 plots the average income (unmarked curves), consumption (black curves with diamonds), and financial wealth (gray curves with boxes) over the life-cycle for a college graduate (thin curves) and for an individual with no high school education (thick curves). Clearly, the income increases more rapidly and reaches a considerably higher level for the college graduate which generally leads to higher consumption and higher financial wealth throughout life.[11] Note, however, that the financial wealth of the college graduate in the early years drops below the financial wealth of the poorly educated individual, which is due to the higher initial consumption of the college graduate who foresees the higher future income.

Fig. 17 compares the average portfolio weights over the life-cycle for an individual with no high school education (upper panel) and an individual with a college degree (lower panel). They follow similar investment strategies in the retirement phase. The stock weight (lower gray area) is decreasing from roughly 60% at retirement to 26% at the terminal date, while the bond weight (white area) is approximately 30% at retirement, increases to around 38%, and then decreases rapidly in the final years as there is less wealth to insure against interest rate risk (recall that we assume a zero bond risk premium). Cash (upper shaded area) becomes increasingly important in the final years and ends up at around 70% of total investments. Before retirement the optimal portfolios differ substantially. The college graduate invests more in the stock market than the poorly educated investor, in particular for an age between 30 and 50. For example, a 35-year (40-year) old average college graduate invests 96% (84%) in stocks compared to 80% (68%) for the individual with no high school education. The biggest differences, however, are in the allocations to bonds and cash. The college graduate does not hold cash until close to retirement, while cash enters the optimal portfolio of the less educated investor early and has weight of approximately 20% from ages 40 to 60. Until age 40 or so, the two investor types have very similar bond positions (e.g., 10% for both at age 38), but then bond position of the college graduate increases much more rapidly with age than that of the investor with no high school education. At age 50, the bond weights are 36% and 21%, and at age 60 they are 41% and 22%, respectively. These differences can be explained by the differences in the estimate of the ξ_1 parameter. For the individual with no high school education, the median estimate is -0.2429, while it is 0.4912 for the college

[11] The income curves in Fig. 16 differ from those in Fig. 15 due to the inclusion of the age-independent term $\overline{\xi}_0$ in the income drift (31).

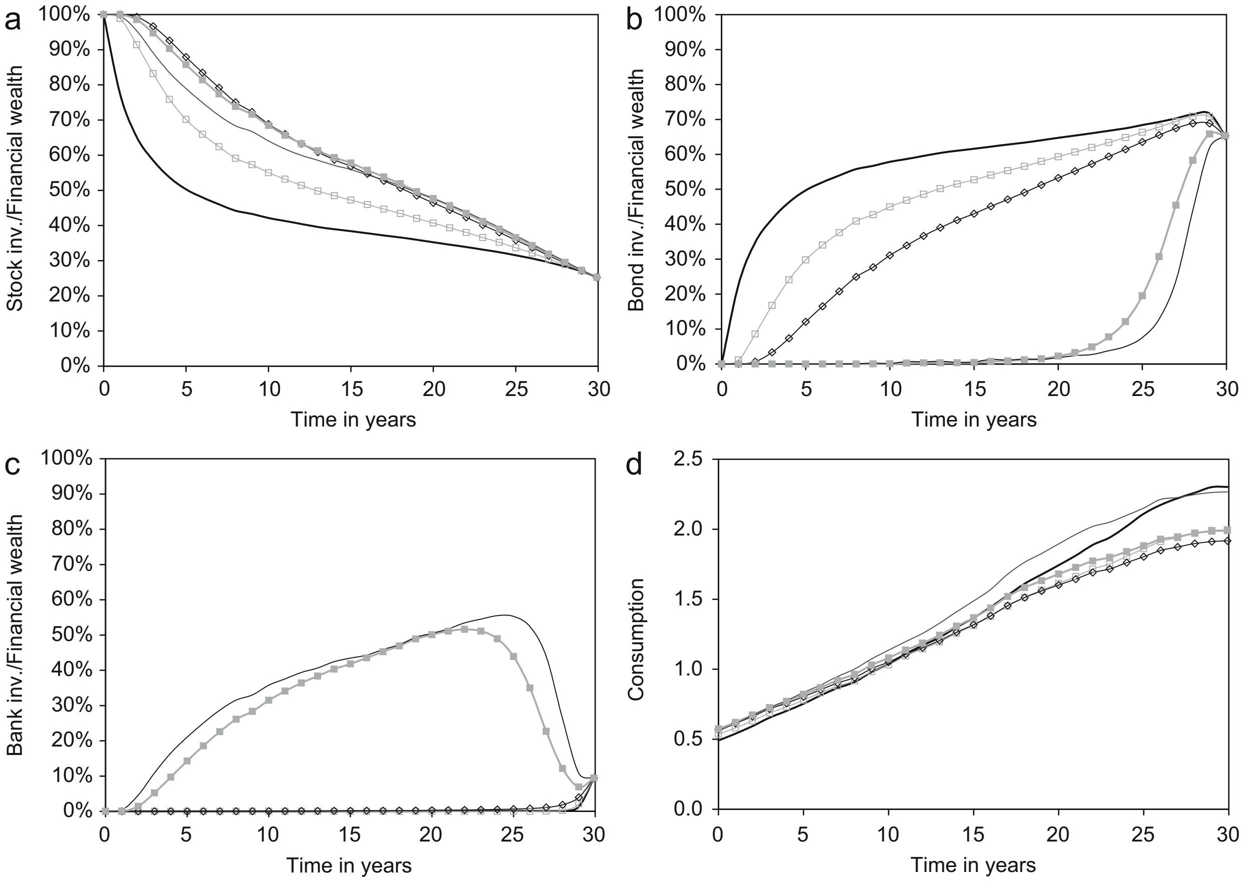

Fig. 14. Sensitivity to the parameter ξ_1. The figure illustrates the sensitivity of the optimal strategies to the parameter ξ_1 that measures the business cycle sensitivity of expected income growth. The thick unmarked curve is for $\xi_1=4$, the gray curve with open boxes for $\xi_1=2$, the curve marked by diamonds for $\xi_1=0$, the gray curve with filled boxes for $\xi_1=-2$, and the thin unmarked curve for $\xi_1=-4$. The graphs are drawn assuming that the initial annual income y_0 and the initial financial wealth W_0 are identical and equal to one, an initial risk-free rate of $r_0=\overline{r}$, and a 30-year income stream with an age-independent average income growth rate of $\xi_0+\xi_1\overline{r}=3\%$. The individual has a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=3$ on terminal wealth. For other parameters, the benchmark values from Table 1 are used. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) The stock weight. (b) The bond weight. (c) bank account weight. (d) The consumption rate.

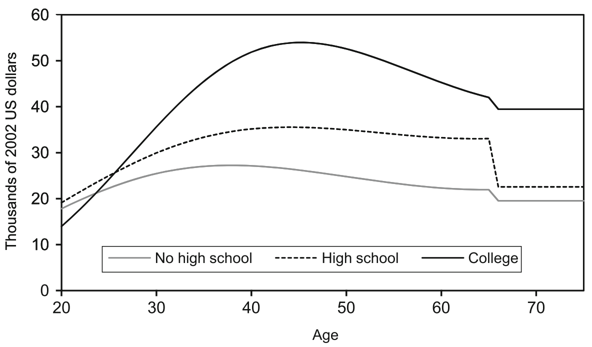

Fig. 15. Calibrated life-cycle income profiles. The average labor income profile over the working life (age 20 to 65 years) of an individual is modeled by a third-order polynomial in age, $P(t)=a+bt+ct^2+dt^3$. In retirement (age 66 and higher), income is proportional to the income level at age 65 with a replacement rate, $\overline{P}$. The figures are drawn using the parameters estimated for three different educational groups by Cocco, Gomes, and Maenhout (2005) based on data from the Panel Study of Income Dynamics (PSID) survey over the period 1970–1992. These parameter estimates are reproduced in Table 4.

graduate. The human wealth, therefore, represents an implicit levered position in the long-term bond for the poorly educated, but a roughly even mix of cash and the long-term bond for the college graduate. The explicit investments are adjusted accordingly. As retirement approaches, the human wealth and the differences in the explicit investments fade away.

Within each educational group there are also significant differences in the parameter ξ_1 as can be seen from Table 3. Fig. 18 gives a final view of the importance of ξ_1 by comparing the life-cycle variations in optimal portfolios for two college graduates who only differ with respect to their ξ_1–parameter. With $\xi_1=3$ (upper panel), human wealth resembles a short position in the bond and a long position in cash so that bond plays a dominant role in the optimal portfolio before retirement, while cash is uninteresting. With $\xi_1=-3$ (lower panel), human wealth is similar to a highly levered position in the bond, so that the bond plays no role in the explicit investments before retirement, while cash does enter with a significant weight. Again, note the stock positions are different in the two cases due to the portfolio constraints imposed.

5. Concluding remarks

This paper has demonstrated that the relative allocation to stocks, bonds, and cash is significantly affected by the presence of uncertain labor income and the variations in labor income over the business cycle. Our model allows the expected labor income growth rate to be an affine function of the real short-term interest rate in order to encompass such business cycle variations, and our calibration using PSID income data supports such a relation with a substantial variation across individuals in the slope of that affine function. Our analysis demonstrates the importance of this slope for the interest rate risk inherent in the human capital and thus, for the allocation of financial wealth to different asset classes.

We provide a closed-form solution for the optimal consumption and investment strategies of an unconstrained power utility investor for the case where the labor income contains no unspanned risk. Here, the asset allocation implications of the labor income variations over the business cycle become clear. If the expected labor income growth is non-cyclical (zero slope), the human capital substitutes a long-term coupon bond so that the optimal financial investment is tilted away from long-term bonds and more towards cash. If the income is pro-cyclical with a slope exactly equal to one, the human capital will substitute for cash only and financial investments should involve more bonds and less cash. When the slope is between zero and one, the human capital is like a mix of cash and the long-term bond. When the slope is above one, human capital is like a levered position in the bond, and when the slope is negative, the human capital is like short-selling the bond and depositing the proceeds at the short-term risk-free rate with the obvious consequences for the financial investments in bonds and cash. Given the natural path of human capital over the life-cycle, the quantitative impact on the optimal financial investments is typically large for young individuals and

Table 4
Coefficients in the life-cycle income profiles.

The average labor income profile over the working life (age 20 to 65 years) of an individual is modeled by a third-order polynomial in age, $P(t)=a+bt+ct^2+dt^3$. In retirement (age 66 and higher), income is proportional to the income level at age 65 with a replacement rate, $\overline{P}$. The table reproduces the estimates from Cocco, Gomes, and Maenhout (2005) of the coefficients a, b, c, d, and $\overline{P}$ together with the average income level at age 20 for different educational levels, categorized as "No high school," "High school," and "College." The estimates are based on data from the Panel Study of Income Dynamics (PSID) survey over the period 1970–1992.

	No high school	High school	College
a^i	−2.1361	−2.1700	−4.3148
b^i	0.1684	0.1682	0.3194
c^i	−0.00353	−0.00323	−0.00577
d^i	0.000023	0.000020	0.000033
$\overline{P}^i$	0.88983	0.68212	0.93887
Initial age 20 level (in 2002 US dollars)	17,763	19,107	13,912

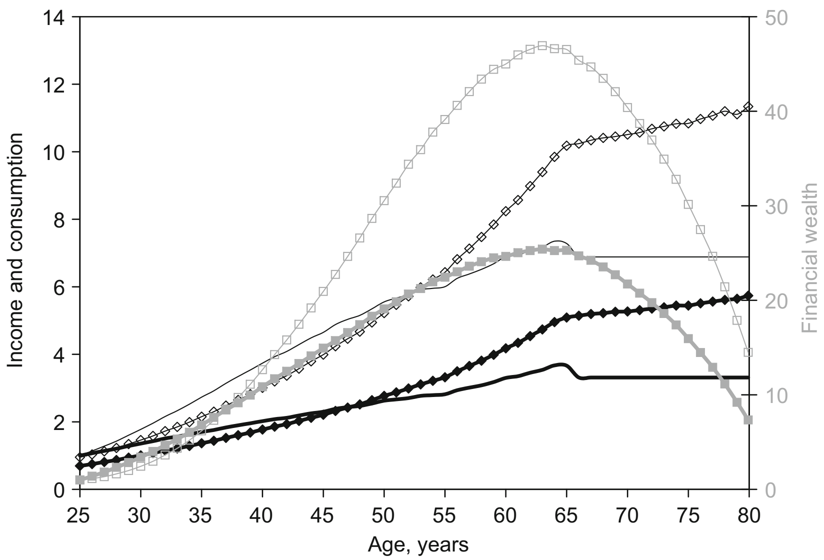

Fig. 16. Income, consumption, and wealth over the life-cycle. The figure illustrates the average income (unmarked curves), wealth (gray curves with boxes), and consumption (black curves with diamonds) over the life-cycle. Thin curves are for an individual with a college degree, thick curves are for an individual with no high school degree. The initial annual income y_0 and the initial financial wealth W_0 are assumed identical and equal to one for both investors, the initial short-term interest rate is 2%, the fixed component of expected income growth is $\overline{\xi}_0=2\%$, and the individuals are assumed to earn the risky income until age 65 after which they live on to age 80 receiving a fixed income equal to the fraction $\overline{P}^i$ of their income immediately before retirement. The estimated life-cycle income polynomials for each educational group as illustrated in Fig. 15 are used. The individuals have a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=1$ on terminal wealth. For all other parameters we apply the benchmark values from Table 1. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies.

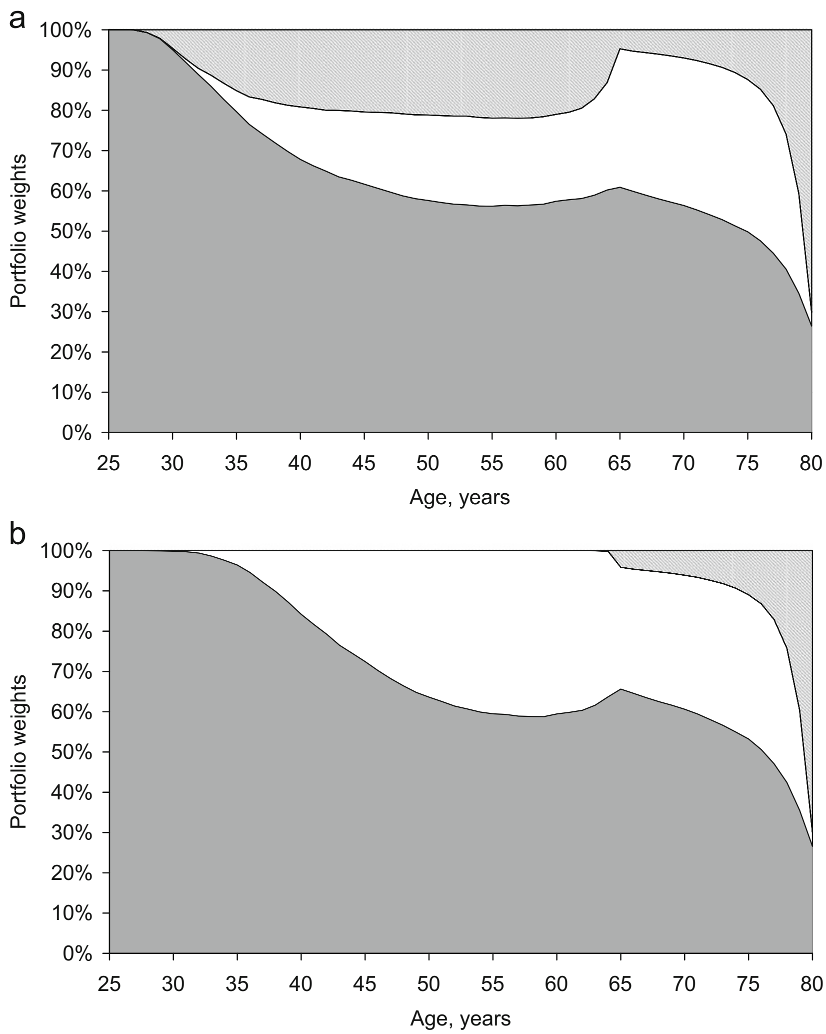

Fig. 17. Optimal portfolios over the life-cycle for different education levels. The upper (lower) panel shows how the optimal asset allocation of an individual with no high school education (a college degree) varies over the life-cycle. The different areas show the optimal fraction of financial wealth invested in the stock (lower gray area), in the bond (white area), and in the bank account (upper shaded area). The initial annual income y_0 and the initial financial wealth W_0 are assumed identical and equal to one for both investors, the initial short-term interest rate is 2%, the fixed component of expected income growth is $\bar{\xi}_0 = 2\%$, and the individuals are assumed to earn the risky income until age 65 after which they live on to age 80 receiving a fixed fraction of their income immediately before retirement. The estimated life-cycle income polynomials for each educational group as illustrated in Fig. 15 are used. The individuals have a relative risk aversion of $\gamma = 4$, a subjective time preference rate of $\delta = 0.03$, and a utility weight $\varepsilon = 1$ on terminal wealth. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) No high school education. (b) College degree.

then decreases until retirement. In the unconstrained case, the optimal stock position is only affected due to the fact that the valuation of the human capital depends on this slope parameter.

For the more realistic case with unspanned labor income risk and liquidity and short-sales constraints, our extensive numerical results show that the slope of the relation between income growth and the interest rate level remains an important parameter for the optimal investment decisions. Naturally, the quantitative impact of the features of labor income is smaller when constraints on the use of future income are imposed, but there are still substantial differences in the optimal investment strategies of individuals who earn income with different business cycle sensitivity but are otherwise identical. Due to short-sales constraints, this sensitivity will now have a bigger effect on the optimal stock investments than in the unconstrained case, but the sensitivity is still mainly important for the allocation between cash and long-term bonds.

The results of the present paper highlight the need for precise information on the labor income risk characteristics when formulating asset allocation advice to households. Identification of these characteristics for different occupational categories would be an interesting empirical project. On the modeling side, our analysis can be extended in several interesting directions albeit with added computational complexity. One extension would be to allow the investor to invest in residential real estate that can serve as collateral for a mortgage loan, thus

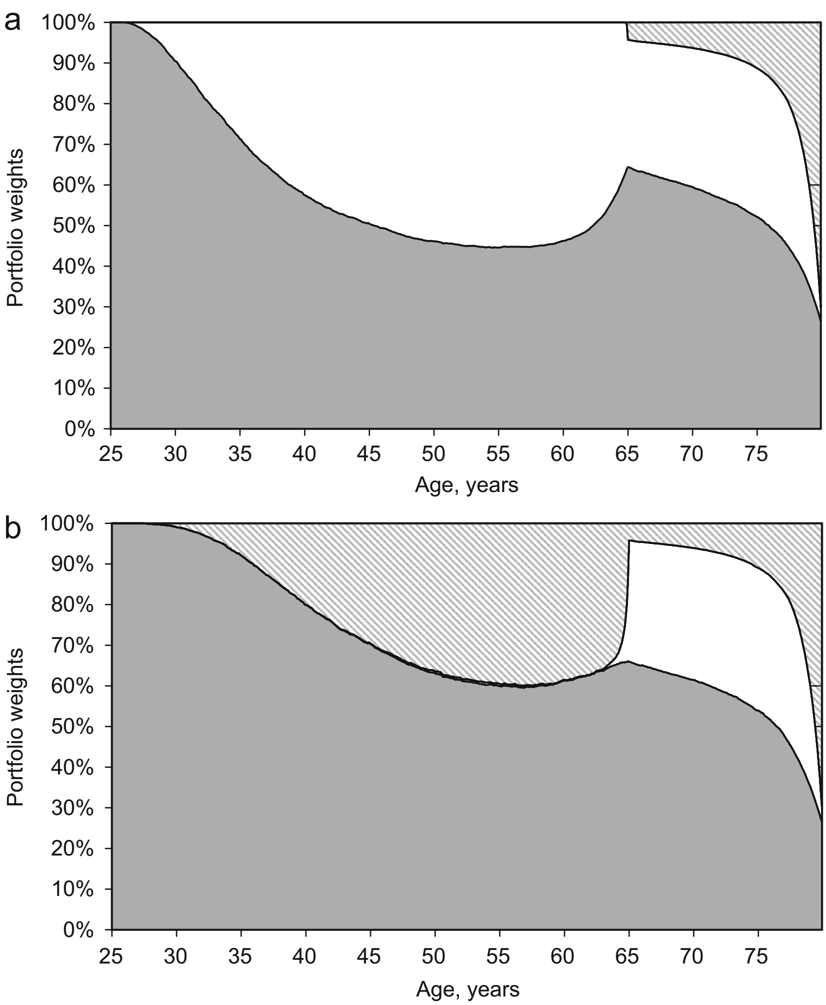

Fig. 18. Optimal portfolios over the life-cycle for college graduates with different ξ_1- values. The different areas show the optimal fraction of financial wealth invested in the stock (lower gray area), in the bond (white area), and in the bank account (upper shaded area). The value of the interest rate sensitivity ξ_1 of the expected income growth is +3 (resp. -3) in the upper (lower) panel with the insensitive part ξ_0 adjusted so that the two individuals have the same income growth rate when the short-term interest rate is at its long-term level. The initial annual income y_0 and the initial financial wealth W_0 are assumed identical and equal to one for both investors, the initial short-term interest rate is 2%, and the individuals earn a risky income until age 65 after which they live on to age 80 receiving a fixed fraction of their income immediately before retirement. The estimated life-cycle income polynomials for each educational group as illustrated in Fig. 15 are used. The individuals have a relative risk aversion of $\gamma=4$, a subjective time preference rate of $\delta=0.03$, and a utility weight $\varepsilon=1$ on terminal wealth. For all other parameters we apply the benchmark values from Table 1. The graphs show averages over 1,000 simulated paths using the numerically computed optimal strategies. (a) College degree $\xi_1=3$. (b) College degree $\xi_1=-3$.

avoiding a strict no borrowing constraint. This would basically call for a combination of our model with a set-up like that considered by Yao and Zhang (2005) and Van Hemert (2009). Another extension is to allow for several stocks. Presumably a larger fraction of the income rate variations can be hedged using multiple risky assets, but little is known about the typical correlations between a labor income stream and individual stocks and, hence, it is unclear how large a fraction of the risk of a typical labor income stream that can be hedged in the financial markets.[12]

Appendix A. Details of the calibration of the model

In line with Cocco, Gomes, and Maenhout (2005) (henceforth referred to as CGM) and Campbell and Viceira (2002), the log-income is decomposed into a personal idiosyncratic component and a common labor income

[12] Allowing for multiple stocks may "help" as indicated by a small example. Assume n stocks that are similar in the sense that they have identical expected rates of return ψ_i, identical volatilities σ_i, identical

(*footnote continued*)
correlations ρ_{iB} with the bond, identical correlations ρ_{yi} with the labor income rate, and all pairwise stock–stock correlations are equal to ρ. Then the income process is spanned whenever

$$(\rho_{yi}-\rho_{iB}\rho_{yB})^2=(1-\rho_{yB}^2)\left(\frac{1-\rho}{n}+\rho-\rho_{iB}^2\right).$$

The value of ρ_{yi} for this equation to hold is decreasing in n, the number of stocks.

component,

$$\log y_t^i = P^i(t) + u_t + v_t^i, \tag{32}$$

where the deterministic component $P^i(t)$ reflects the expected life-cycle income of investors with the same characteristics as individual i, u_t is a common stochastic income factor, and v_t^i is an individual-specific stochastic component which is assumed uncorrelated across individuals.

In order to have the model in a form consistent with the income process in (10), we assume that

$$du_t = (\xi_{u0} + \xi_{u1} r_t - \tfrac{1}{2}\sigma_u^2)\,dt + \sigma_u\,dz_{ut} \tag{33}$$

and

$$dv_t^i = (\xi_{v0}^i + \xi_{v1}^i r_t - \tfrac{1}{2}{\sigma_v^i}^2)\,dt + \sigma_v^i\,dv_{vt}^i. \tag{34}$$

Furthermore, following CGM, in our calibration approach we assume that the deterministic personal income component $P^i(t)$ is described by a third-order polynomial in time (age),

$$P^i(t) = a^i + b^i t + c^i t^2 + d^i t^3,$$

where a^i, b^i, c^i, and d^i are constant parameters.[13] The polynomial form of $P^i(t)$ only applies for age 20 until 65 (years). In the discrete-time framework of CGM, it is thus assumed that the retirement income level for age 66 and higher is proportional to the income level at age 65 with a replacement rate, $\overline{P}^i$. We will adopt this form, and in our continuous-time setting we simply linearly interpolate the income level in the one-year period between age 65 and age 66. Since most of our calibrations are based on the PSID data used by CGM, we will adopt their estimated third-order polynomial structures and replacement rates for groups characterized by three different educational backgrounds: "No high school," "High school," and "College." The polynomial life-cycle income profiles are illustrated in Fig. 15, and the polynomial coefficients used in the figure are reproduced from CGM in Table 4.

With the above decomposition and assumptions, it can be inferred (by an application of Ito's lemma) that the individual income process before retirement is of the form (7) with $\xi_0^i(t) = \xi_{u0} + \xi_{v0}^i + dP^i(t)/dt$, and when we include the smooth transition to the fixed retirement income we have

$$\xi_0^i(t) = \begin{cases} \xi_{u0} + \xi_{v0}^i + b^i + 2c^i t + 3d^i t^2 & \text{if } 20 \le t \le 65, \\ -(1-\overline{P}^i)P^i\ (65) & \text{if } 65 < t < 66, \\ 0 & \text{if } t \ge 66. \end{cases}$$

The income process has a constant volatility in the active phase with ${\sigma_y^i}^2 = \sigma_u^2 + {\sigma_v^i}^2$, and the income volatility then drops to zero at retirement. Furthermore, let l_t^i denote the stochastic part of the income process, $l_t^i = \log y_t^i - P^i(t) = u_t + v_t^i$. Then,

$$dl_t^i = (\xi_0^i + \xi_1^i r_t - \tfrac{1}{2}{\sigma_y^i}^2) + \sigma_y^i\,dz_{yt}^i, \tag{35}$$

where $\xi_0^i = \xi_{u0} + \xi_{v0}^i$.

As noted earlier, we will carry out two kinds of calibrations. Some of the calibrations are based on individual time-series of income data. In these we estimate the parameters of the residual income process in (35) separately for all individual households in the PSID database. These calibrations will give some insight into the dispersion of the parameter values of expected income growth, interest sensitivity, and income volatility (reflected in the parameters ξ_0, ξ_1, and σ_y) across individuals with different characteristics.

However, following ideas of, e.g., CGM, we also carry out calibrations based on aggregating data across individuals that basically assume that the parameters ξ_0, ξ_1, and σ_y are the same across individuals in different educational groups. Formally, assume that the idiosyncratic component in (34) is a random-walk component in the sense that $\xi_{v0}^i = \xi_{v1}^i = 0$. This implies by assumption that all individual income processes have the same drift parameters as the common stochastic component u_t. Now, by averaging across individuals in any given year, one can construct the aggregate stochastic income component $\bar{l}_t = u_t$ (using that the idiosyncratic components v_t^i average out). We have,

$$d\bar{l}_t = (\xi_0 + \xi_1 r_t - \tfrac{1}{2}\sigma_u^2) + \sigma_u\,dz_{ut}, \tag{36}$$

where $\xi_0 = \xi_{u0}$ and $\xi_1 = \xi_{u1}$.

The parameters of the aggregate stochastic component of labor income, as well as correlations and parameters of real interest rates and stock prices, are estimated by maximum likelihood. The relevant dynamics are described by the real interest rate dynamics in (1), the stock dynamics of (5), and an income process of the form in (36) (or, equivalently, (35)). Let $Y_t = (\bar{l}_t, \log S_t, r_t)'$, then the relevant dynamics can be summarized by the linear stochastic differential equation,

$$dY_t = (A + BY_t)\,dt + V\,d\hat{z}_t \tag{37}$$

where

$$A = \begin{pmatrix} \xi_0 - \dfrac{1}{2}\sigma_u^2 \\ \psi - \dfrac{1}{2}\sigma_S^2 \\ \kappa\bar{r} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & 1 \\ 0 & 0 & -\kappa \end{pmatrix},$$

$$VV' = \begin{pmatrix} \sigma_u^2 & \rho_{uS}\sigma_u\sigma_S & \rho_{ur}\sigma_u\sigma_r \\ \rho_{uS}\sigma_u\sigma_S & \sigma_S^2 & \rho_{Sr}\sigma_S\sigma_r \\ \rho_{ur}\sigma_u\sigma_r & \rho_{Sr}\sigma_S\sigma_r & \sigma_r^2 \end{pmatrix},$$

and $\hat{z}_t$ is a three-dimensional standard Brownian motion. The discrete-time solution to the linear stochastic differential equation in (37) is a first-order vector autoregressive [VAR(1)] model of the form

$$Y_{t+\Delta} = A(\Psi,\Delta) + B(\Psi,\Delta)Y_t + \tilde{\varepsilon}_{t+\Delta}, \quad \tilde{\varepsilon}_{t+\Delta} \sim N(0, \Omega(\Psi,\Delta)), \tag{38}$$

[13] Our following estimation approach is also set up with an eye on controlling for family-specific fixed effects such as marital status and family size when relevant, as do CGM in their Table 1. For example, in our analysis of the individual household PSID income time-series, this is achieved by conditioning on the first income observation, which thus implies that we do not take subsequent changes in family-specific fixed effects into account when estimating the parameters of the individual household income dynamics.

where $\Psi=(\xi_0,\xi_1,\psi,\kappa,\overline{r},\sigma_y,\sigma_S,\sigma_r,\rho_{yS},\rho_{yr},\rho_{Sr})$ denotes the set of parameters to be estimated. The functions $A(\Psi,\Delta)$, $B(\Psi,\Delta)$, and $\Omega(\Psi,\Delta)$, which determine the relevant moments of the VAR(1)-model, can be obtained in closed-form using the general solution formula for linear stochastic differential equations; see, e.g., Karatzas and Shreve (1988). Based on the VAR(1)-model in (38), and the expressions for the moments, the relevant likelihood function for maximization with respect to the parameters in Ψ is formulated in a straightforward manner.[14]

In the calibration approach with aggregated income across individuals, the drift parameters for individuals are by assumption identical to those in the aggregated income process. The drift parameters can therefore be estimated from this constructed process. (Also, note that this aggregated PSID income process is conceptually very similar to a traditional macroeconomic income index, as for example tabulated in the NIPA tables of the National Economic Accounts.) In general, however, the volatility on aggregate income processes will understate the individual income uncertainty since the idiosyncratic noise components ν_t^i are canceled out. Following conceptual ideas of, e.g., CGM, one can define

$$\Delta l_t^i=(l_{t+\Delta}^i-l_t)-\mathrm{E}_t[(l_{t+\Delta}^i-l_t)],$$

where the expectations are calculated based on the drift parameter estimates of ξ_0 and ξ_1 obtained by maximum likelihood, as described above. We can now calculate the empirical version of $\mathrm{Var}(\Delta l_t^i)$ across all individuals and time periods and deduce the value of σ_y that this corresponds to given our assumed income process. This is our estimate of the individual income volatility. Furthermore, based on this estimate of σ_y and the estimates from the analysis of aggregate income, and assuming that the idiosyncratic income component is uncorrelated with the stock index and interest rates, one can deduce the implied correlation coefficients for individual household income: $\rho_{yS}=\rho_{uS}\sigma_u/\sigma_y$ and $\rho_{yr}=\rho_{ur}\sigma_u/\sigma_y$.

Appendix B. Proofs for Section 3

Proof of Proposition 1. The process $\hat{z}=(\hat{z}_r,\hat{z}_S)^\top$ defined by

$$\hat{z}_t=z_t+\lambda t$$

is a standard $(n+1)$-dimensional Brownian motion under the risk-neutral probability measure. Therefore, the risk-neutral dynamics of the short-term interest rate are

$$dr_t=(\hat{\phi}-\kappa r_t)\,dt-\sigma_r\,d\hat{z}_{rt},$$

where $\hat{\phi}=\kappa\overline{r}+\sigma_r\lambda_1$. This implies that

$$r_u=e^{-\kappa[u-t]}r_t+\frac{\hat{\phi}}{\kappa}(1-e^{-\kappa[u-t]})-\int_t^u\sigma_r e^{-\kappa[u-v]}\,d\hat{z}_{rv}.$$

Applying the Fubini rule for interchanging the order of integration, we get

$$\int_t^s r_u\,du=\left(\frac{r_t}{\kappa}-\frac{\hat{\phi}}{\kappa^2}\right)(1-e^{-\kappa[s-t]})+\frac{\hat{\phi}}{\kappa}(s-t)-\frac{\sigma_r}{\kappa}\int_t^s(1-e^{-\kappa[s-u]})\,d\hat{z}_{ru}. \tag{39}$$

The income dynamics under the risk-neutral probability measure are

$$dy_t=y_t[(\hat{\xi}_0(t)+\xi_1 r_t)\,dt+\sigma_y(t)\{\rho_{yB}\,d\hat{z}_{rt}+\hat{\rho}_{yS}^\top\,d\hat{z}_{St}\}],$$

where $\hat{\xi}_0(t)=\xi_0(t)-\sigma_y(t)\rho_{yP}^\top\lambda$ and $\hat{z}_t=z_t+\lambda t$, and hence,

$$y_s=y_t\exp\left\{\int_t^s\left(\hat{\xi}_0(u)+\xi_1 r_u-\frac{1}{2}\sigma_y(u)^2\right)du+\int_t^s\sigma_y(u)\rho_{yB}\,d\hat{z}_{ru}+\int_t^s\sigma_y(u)\hat{\rho}_{yS}^\top\,d\hat{z}_{Su}\right\}. \tag{40}$$

Combining (39) and (40) we get

$$\begin{aligned}y_s e^{-\int_t^s r_u\,du}=y_t\exp\Bigg\{&\int_t^s\left(\hat{\xi}_0(u)-\frac{1}{2}\sigma_y(u)^2\right)du\\&+(\xi_1-1)\left(r_t-\frac{\hat{\phi}}{\kappa}\right)b(s-t)\\&+\frac{\hat{\phi}}{\kappa}(\xi_1-1)(s-t)\\&+\int_t^s\sigma_y(u)\hat{\rho}_{yS}^\top\,d\hat{z}_{Su}\\&+\int_t^s(\sigma_y(u)\rho_{yB}-(\xi_1-1)\sigma_r b(s-u))\,d\hat{z}_{ru}\Bigg\}.\end{aligned}$$

The exponent on the right-hand side is normally distributed and applying the standard rule for expectations of exponentials of normal random variables, we get

$$\mathrm{E}_t^{\mathbb{Q}}\left[y_s e^{-\int_t^s r_u\,du}\right]=y_t e^{F(t,s)+(\xi_1-1)b(s-t)r_t}$$

for some easily computed function $F(t,s)$. Applying (2), we get

$$\mathrm{E}_t^{\mathbb{Q}}\left[y_s e^{-\int_t^s r_u\,du}\right]=y_t e^{F(t,s)-(\xi_1-1)a(s-t)}(B^s(r_t,t))^{1-\xi_1}.$$

Defining $\ln h(t,s)=F(t,s)-(\xi_1-1)a(s-t)$ and integrating over s, we arrive at (14).

Computation of expected human wealth: The stochastic differential equation (7) describing the income dynamics has the solution

$$y_t=y_0\exp\left\{\int_0^t\xi_0(u)\,du+\xi_1\int_0^t r_u\,du-\frac{1}{2}\int_0^t\sigma_y(u)^2\,du+\int_0^t\sigma_y(u)\rho_{yP}^\top\,dz_u\right\}.$$

The interest rate dynamics (1) implies that

$$\int_0^t r_u\,du=r_0t+(\overline{r}-r_0)(t-b(t))-\int_0^t\sigma_r b(t-u)\,dz_{ru}. \tag{41}$$

Substituting that into the preceding equation and taking expectations, we get

$$\mathrm{E}_0[y_t]=y_0\exp\left\{\int_0^t\xi_0(u)\,du+\xi_1 r_0 t\right.$$

[14] The numerical maximum likelihood estimation is carried out using the software program GAUSS. In the evaluation of the first- and second-order moments we used the analytical evaluation tools in the software program Mathematica and pasted the relevant analytical results into the GAUSS program.

$$+\xi_1\left(\overline{r}-r_0+\frac{\xi_1\sigma_r^2}{2\kappa^2}\right)(t-b(t))-\frac{\xi_1^2\sigma_r^2}{4\kappa}b(t)^2$$
$$-\xi_1\sigma_r\rho_{yB}\int_0^t\sigma_y(u)b(t-u)\,du\Big\}.$$

The time zero expectation of the human capital at time t is $E_0[H(y_t,r_t,t)]=E_0\,[y_tM(r_t,t)]$, where M is defined in Proposition 1. Numerical experiments reveal that M is relatively insensitive to the interest rate so that $M(r_t,t)\approx M(\overline{r},t)$ and, consequently, $E_0[H(y_t,r_t,t)]\approx M(\overline{r},t)E_0[y_t]$.

Computation of expected total wealth: From (22) and (23), the dynamics of total wealth W_t+H_t are

$$\frac{d(W_t+H_t)}{W_t+H_t}=\left(r_t+\frac{1}{\gamma}\|\lambda\|^2+\frac{\gamma-1}{\gamma}\sigma_r\lambda_rG(r_t,t)-\frac{1}{g(r_t,t)}\right)dt+\frac{1}{\gamma}\lambda^\top dz_t+\sigma_r\frac{\gamma-1}{\gamma}G(r_t,t)\,dz_{rt},$$

which implies that

$$\frac{W_t+H_t}{W_0+H_0}=\exp\Big\{\int_0^t r_u\,du+\frac{1}{\gamma}\|\lambda\|^2t+\frac{\gamma-1}{\gamma}\sigma_r\lambda_r\times\int_0^tG(r_u,u)\,du-\int_0^t\frac{1}{g(r_u,u)}\,du-\frac{1}{2\gamma^2}\|\lambda\|^2t$$
$$-\frac{1}{2}\sigma_r^2\left(\frac{\gamma-1}{\gamma}\right)^2\int_0^tG(r_u,u)^2\,du-\frac{\lambda_r\sigma_r(\gamma-1)}{\gamma^2}\times\int_0^tG(r_u,u)\,du+\frac{1}{\gamma}\lambda^\top\int_0^t dz_t+\sigma_r\frac{\gamma-1}{\gamma}\times\int_0^tG(r_u,u)\,dz_{ru}\Big\}.$$

It is not possible to compute the exact expectation in this expression, but numerical experiments have shown that the functions $g(r,u)$ and $G(r,u)$ are not very sensitive to the value of r. We therefore approximate $G(r_u,u)$ by $G(\overline{r},u)$ and similarly for g. Furthermore, we substitute in (41). Computing the expectation will then yield

$$\frac{E_0[W_t+H_t]}{(W_0+H_0)}\approx\exp\Big\{\left(r_0+\frac{1}{\gamma}\|\lambda\|^2\right)t+\left(\overline{r}-r_0+\frac{\sigma_r^2}{2\kappa^2}-\frac{\lambda_r\sigma_r}{\gamma\kappa}\right)(t-b(t))$$
$$-\int_0^t\frac{1}{g(\overline{r},u)}\,du+\frac{\gamma-1}{\gamma}\sigma_r\lambda_r\int_0^tG(\overline{r},u)\,du-\sigma_r^2\frac{\gamma-1}{\gamma}\int_0^tb(t-u)G(\overline{r},u)\,du\Big\},$$

where the integrals have to be evaluated numerically.

Computation of expected financial wealth: The expectation of financial wealth equals the difference between expected total wealth and expected human wealth: $E_0[W_t]=E_0[W_t+H_t]-E_0[H_t]$, where the two expectations on the right-hand side are given above. □

Appendix C. Properties of the function $G(r,t)$

Lemma 1. *The function* $G(r,t)$ *defined by* (22) *has the following properties*:

(a) *$G(r,t)$ is increasing in T,*
(b) *$G(r,t)$ is decreasing in r if $\gamma>1$ and increasing in r if $\gamma<1$.*

In the proof we assume for notational simplicity that $\varepsilon=0$ so that the individual has no utility from terminal wealth.

Proof. (a) Computing the derivative $\partial G/\partial T$, we can see that it will be positive whenever

$$b(T-t)\int_t^T f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds>\int_t^T b(s-t)f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds,$$

which is indeed the case since b is increasing.

(b) Computing the derivative $\partial G/\partial r$, we see that it will be positive if and only if

$$(\gamma-1)\left[\left(\int_t^T f(s-t)b(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds\right)^2-\left(\int_t^T b(s-t)^2f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds\right)\times\left(\int_t^T f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds\right)\right]>0.$$

The term in the square brackets is negative due to the Cauchy–Schwarz Inequality:

$$\left(\int_t^T f(s-t)b(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds\right)^2=\left(\int_t^T[f(s-t)^{1/2}(B^s(r,t))^{\gamma-1/2\gamma}]\times[b(s-t)f(s-t)^{1/2}(B^s(r,t))^{\gamma-1/2\gamma}]\,ds\right)^2$$
$$\le\left(\int_t^T f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds\right)\times\left(\int_t^T b(s-t)^2f(s-t)(B^s(r,t))^{\gamma-1/\gamma}\,ds\right).$$

Consequently, $\partial G/\partial r$ is positive if $\gamma<1$ and negative if $\gamma>1$. □

Appendix D. Details on the numerical solution method

We solve the highly non-linear PDE (27) in the following way. We set up an equally spaced lattice in (x,r,t)-space defined by the grid points

$$\{(x_i,r_j,t_n)|i=0,1,\ldots,I,\ j=0,1,\ldots,J,\ n=0,1,\ldots,N\},$$

where $x_i=i\Delta x$, $r_j=r_0+j\Delta r$, and $t_n=n\Delta t$ for some fixed positive spacing parameters Δx, Δr, and Δt. Since wealth and income are restricted to non-negative values, $x_0=0$ is a natural lower bound for $x=W/y$, while the highest value x_I is an artificial upper bound. We place the long-term interest rate level $\overline{r}$ in the middle of the grid, $r_{J/2}=\overline{r}$, and since the model allows for interest rates of all real values, we introduce an artificial lower bound, r_0, and an artificial upper bound, r_J. Since the numerical approximation is likely to be relatively imprecise near the artificial bounds, we pick the values of these bounds so that it is highly unlikely that the state moves from the starting point (x,r), located near the center of the grid, to one of the imposed bounds. We denote the approximated value function in the grid point (x_i,r_j,t_n) by $F_{i,j,n}$ and use similar notation for the controls (portfolio weights and scaled consumption) and other state-dependent functions.

We apply a backward iterative procedure starting at the retirement date $\tilde{T}=N\Delta t$, where we first set

$$F_{i,j,N}=\frac{1}{1-\gamma}e^{-\beta(\gamma-1)\tilde{T}}g(r_j,\tilde{T})^{\gamma}x_i^{1-\gamma}.$$

At any earlier time t_n, we know the approximated value function at time t_{n+1}, i.e., $F_{i,j,n+1}$ for all $i=0,1,\ldots,I$ and all $j=0,1,\ldots,J$, and the optimal controls at time t_{n+1}, i.e., $\hat{c}_{i,j,n+1}$ and $\pi_{i,j,n+1}$ for all $i=0,1,\ldots,I$ and all $j=0,1,\ldots,J$. To find the approximated value function and the optimal controls at time t_n, we first make an initial guess of the optimal controls $\hat{c}_{i,j,n}$ and $\pi_{i,j,n}$ for all (i,j). Since optimal controls do not tend to vary dramatically over time, a good initial guess is $\hat{c}_{i,j,n}=\hat{c}_{i,j,n+1}$ and $\pi_{i,j,n}=\pi_{i,j,n+1}$. In the PDE (27), we can remove the sup-term if we substitute the optimal controls into the curly brackets, and then solve for the value function. Applying this idea, we set up a finite difference approximation of the PDE without the sup-operator. In the finite difference approximation we use so-called "up-wind" approximations of the derivatives, which tends to stabilize the approach. For each (i,j) in the interior of the grid, the relevant finite difference approximations of the derivatives are

$$D_t^+F_{i,j,n}=\frac{F_{i,j,n+1}-F_{i,j,n}}{\Delta t},$$

$$D_x^2F_{i,j,n}=\frac{F_{i+1,j,n}-2F_{i,j,n}+F_{i-1,j,n}}{(\Delta x)^2},$$

$$D_r^2F_{i,j,n}=\frac{F_{i,j+1,n}-2F_{i,j,n}+F_{i,j-1,n}}{(\Delta r)^2},$$

$$D_x^+F_{i,j,n}=\frac{F_{i+1,j,n}-F_{i,j,n}}{\Delta x},\quad D_x^-F_{i,j,n}=\frac{F_{i,j,n}-F_{i-1,j,n}}{\Delta x},$$

$$D_r^+F_{i,j,n}=\frac{F_{i,j+1,n}-F_{i,j,n}}{\Delta r},\quad D_r^-F_{i,j,n}=\frac{F_{i,j,n}-F_{i,j-1,n}}{\Delta r},$$

$$\begin{aligned}D_{xr}^+F_{i,j,n}=\frac{1}{2\Delta x\,\Delta r}(&2F_{i,j,n}+F_{i+1,j+1,n}+F_{i-1,j-1,n}\\&-F_{i+1,j,n}-F_{i-1,j,n}-F_{i,j+1,n}-F_{i,j-1,n}),\end{aligned}$$

$$\begin{aligned}D_{xr}^-F_{i,j,n}=-\frac{1}{2\Delta x\,\Delta r}(&2F_{i,j,n}+F_{i+1,j-1,n}+F_{i-1,j+1,n}\\&-F_{i+1,j,n}-F_{i-1,j,n}-F_{i,j+1,n}-F_{i,j-1,n}),\end{aligned}$$

and we obtain the equation

$$\begin{aligned}\hat{\delta}_{j,n}F_{i,j,n}=\;&\frac{\hat{c}_{i,j,n}^{1-\gamma}}{1-\gamma}+D_t^+F_{i,j,n}+D_r^+F_{i,j,n}(\kappa[\overline{r}-r_j]^+\\&-(1-\gamma)\rho_{yr}^-\sigma_{yn}\sigma_r)-D_r^-F_{i,j,n}(\kappa[\overline{r}-r_j]^-\\&-(1-\gamma)\rho_{yr}^+\sigma_{yn}\sigma_r)+\frac{1}{2}\sigma_r^2D_r^2F_{i,j,n}\\&+D_x^+F_{i,j,n}\Bigg\{e^{-\beta t}(1-\hat{c}_{i,j,n})^++x_i[((1-\xi_1)r_j)^+\\&+\xi_{0n}^-+\gamma\sigma_{yn}^2]+x_i\sigma_B(\pi_{i,j,n}^B(\lambda_r-\gamma\sigma_{yn}\rho_{yB}))^+\\&+x_i\sigma_S\left(\pi_{i,j,n}^S\left(\frac{\psi}{\sigma_S}-\gamma\rho_{yS}\sigma_{yn}\right)\right)^+\Bigg\}\\&-D_x^-F_{i,j,n}\Bigg\{e^{-\beta t}(1-\hat{c}_{i,j,n})^-+x_i[((1-\xi_1)r_j)^-\\&+\beta+\xi_{0n}^+]+x_i\sigma_B(\pi_{i,j,n}^B(\lambda_r-\gamma\sigma_{yn}\rho_{yB}))^-\\&+x_i\sigma_S\left(\pi_{i,j,n}^S\left(\frac{\psi}{\sigma_S}-\gamma\rho_{yS}\sigma_{yn}\right)\right)^-\Bigg\}\\&+\frac{1}{2}x_i^2D_x^2F_{i,j,n}(\pi_{i,j,n}^\top\Sigma\Sigma^\top\pi_{i,j,n}+\sigma_{yn}^2\\&-2\sigma_y(t)[\pi_{i,j,n}^B\sigma_B\rho_{yB}+\pi_{i,j,n}^S\sigma_S\rho_{yS}])\\&+x_i\sigma_rD_{xr}^+F_{i,j,n}((\pi_{i,j,n}^B)^-\sigma_B+(\pi_{i,j,n}^S\rho_{SB})^-\sigma_S\\&+\rho_{yr}^-\sigma_{yn})-x_i\sigma_rD_{xr}^-F_{i,j,n}((\pi_{i,j,n}^B)^+\sigma_B\\&+(\pi_{i,j,n}^S\rho_{SB})^+\sigma_S+\rho_{yr}^+\sigma_{yn}).\end{aligned}$$

Adding similar equations for all points of the boundary of the grid, we get a system of $(I+1)(J+1)$ equations (one for each grid point) that we can solve for the $(I+1)(J+1)$ values $F_{i,j,n}$; more information on the solution of the equations is given below. Given that solution, we compute a new guess on the optimal controls $\hat{c}_{i,j,n},\pi_{i,j,n}$ from the first-order conditions from the maximization in the PDE (27), i.e., the Eqs. (29) and (30) in the unrestricted case, again using finite difference approximations of the derivatives based on the newly computed guess on the value function $F_{i,j,n}$. Given the new guess on the optimal controls, we solve again the system of equations and obtain a new guess on the value function at time t_n. We continue these iterations until the largest relative change in the value function over all (i,j) relative to the previous iteration is below some small threshold (we use 0.1%). Then we continue to time t_{n-1}. Typically, the solution requires two to four iterations at each time step.

We can write the equation system that we have to solve in the form

$$\mathbf{M}_n\mathbf{F}_n=\mathbf{d}_n,$$

where $\mathbf{F}_n$ is the $(I+1)(J+1)$-dimensional vector of values $F_{i,j,n}$, $\mathbf{M}_n$ is a matrix of dimension $(I+1)(J+1)\times(I+1)(J+1)$, and $\mathbf{d}_n$ is an $(I+1)(J+1)$-dimensional vector of known values. The matrix will be a band matrix so that the equation system can be solved relatively fast. The width of the band depends on the order in which the points (i,j) are taken in the vector $\mathbf{F}_n$. The two natural candidates are

$(0,0),(1,0),(2,0),\ldots,(I,0),(0,1),(1,1),(2,1),\ldots,$
$(I,1),\ldots,(0,J),(1,J),(2,J),\ldots,(I,J)$

and

$(0,0),(0,1),(0,2),\ldots,(0,J),(1,0),(1,1),$
$(1,2),\ldots,(1,J),\ldots,(I,0),(I,1),(I,2),\ldots,(I,J).$

The first results in a matrix with a band width of $2I+5$, whereas the band width is $2J+5$ using the second enumeration of points. In the complete market case, we have observed that the value function and the optimal strategies are more sensitive to wealth and income than to the interest rate, and since we expect the same in the incomplete market case, we will use $I>J$. Therefore, the second enumeration is more efficient, both in relation to the number of non-zero values to be stored in the computer and in relation to the speed with which the equation system can be solved.

The numerical results stated in Section 4 are based on an implementation of the above procedure with $I=2000$, $J=50$, and 24 time steps per year. The imposed bounds are $x_I=20$, $r_0=-0.03$, and $r_J=0.07$.

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