The following are the discrete-time expression for the processes $(S_t,\, X_t,\, Y_t,\, I_t)_{t\in[0,T]}$:

$$
\begin{aligned}
S_{t+\Delta t} &= S_t \exp\left(\left(r+\sigma_S\lambda_S-\frac{1}{2}\sigma_S^2\right)\Delta t+\sigma_S\Delta W_t\right)\\
&\quad Y_{t+\Delta t} = Y_t \exp\left(\left(\alpha-\frac{1}{2}\beta^2\right)\Delta t+\beta\Delta\tilde{W}_{t+\Delta t}\right)\\
&\quad X_{t+\Delta t} \approx X_t\left[\left(r+\pi_{St}\sigma_S\lambda_s\right)\Delta t+\pi_{St}\sigma_S\Delta W_t\right]+(Y_t-c_t)\Delta t\\
I_{t+\Delta t} &= I_t \exp\left(\int_t^{t+\Delta t}\left(\tilde{r}_s+\beta\lambda_{Is}-\frac{1}{2}\beta^2\right)\mathrm{d}s+\beta\Delta\tilde{W}_t\right)
\end{aligned}
$$

where $\Delta W_t \equiv (W_{t+\Delta t}-W_t)$, and $\tilde{W}=\rho W+(1-\rho^2)^{1/2}W_Y$

- optimization problem in discrete time at the terminal date $(T-1)$:

$$
\max_{c_{T-1},\pi_{T-1}} \frac{c_{T-1}^{1-\gamma}}{1-\gamma}+\mathbf{E}_{T-1}\left[\varepsilon e^{-\delta}\frac{X_T^{1-\gamma}}{1-\gamma}\right]
$$

$$
\text{s.t.}\ \left\{\begin{array}{l} X_T = X_{T-1}(1+r+\pi_{T-1}(\ln(S_T/S_{T-1})-r))+(Y_{T-1}-c_{T-1})\\ X_T\geq 0;\ \ c_{T-1}\geq 0;\ \ \pi_{T-1}\in\mathbb{R}\end{array}\right.
$$

- 2015/04/01: the numerical procedure to determine (c_{T-1}^*, π_{T-1}^*) is not so simple; one alternativeis to work with

$$
S_T=\left\{\begin{array}{ll} uS_{T-1} & \text{with prob } p\\ dS_{T-1} & \text{with prob } 1-p\end{array}\right.
$$

  with $u>r>d$. In this case, the FOC are given by:

$$
\begin{aligned}
c_{T-1}\ &:\ c_{T-1}^{-\gamma}-\mathbf{E}_{T-1}\left[\left(X_{T-1}(1+r+\pi_{T-1}(\ln(S_T/S_{T-1})-r))+(Y_{T-1}-c_{T-1})\right)^{-\gamma}\right]=0\\
&:\ c_{T-1}^{-\gamma}-p\left(X_{T-1}(1+r+\pi_{T-1}(\ln(u)-r))+(Y_{T-1}-c_{T-1})\right)^{-\gamma}+(1-p)\left(X_{T-1}(1+r+\pi_{T-1}(\ln(d)-r))+(Y\right.\\
\pi_{T-1}\ &:\ \mathbf{E}_{T-1}\left[\varepsilon e^{-\delta}X_T^{-\gamma}(\ln(S_T/S_{T-1})-r)\right]=0\\
&:\ p(X_T^u)^{-\gamma}(\ln(u)-r)+(1-p)(X_T^d)^{-\gamma}(\ln(d)-r)=0
\end{aligned}
$$

  in order to let us verify the discretization of the complete mkts without problem in Bick et al. (2013): 1 source of uncertainty wo port constraints.

  - in the complete market case, with liquidity unconstrained optimal wealth (X_t^* to be defined below); i.e., where policies (c,π) are admissible if satisfy

$$
X_t+\mathbf{E}_t\left[\int_t^T \xi_{t,s}Y_s\mathrm{d}s\right]\geq 0,
$$

    where ξ is the stochastic discount factor (SDF) that satisfies:

$$
\mathrm{d}\xi_t=-\xi_t(r\mathrm{d}t+\theta\mathrm{d}W_t),
$$

    with $\theta=\lambda_S$, it follows that:

$$
c_t^*=(y\xi_t e^{\delta t})^{-1/\gamma},\quad X_T^*=(y\xi_T\varepsilon^{-1}e^{\delta T})^{-1/\gamma}
$$

    where $y>0$ solves

$$
\mathbf{E}_0\left[\int_0^T \xi_t c_t^*\mathrm{d}t+\xi_T X_T^*\right]=X_0+\mathbf{E}_0\left[\int_0^T \xi_t Y_t\mathrm{d}t\right]
$$

from where we obtain

$$y^{-1/\gamma} = \frac{X_0 + \mathbf{E}_0\left[\int_t^T \xi_t Y_t \mathrm{d}t\right]}{\mathbf{E}_0\left[\int_0^T \xi_t^{\varrho} e^{-(\delta/\gamma)t}\mathrm{d}t + \xi_T^{\varrho}\varepsilon^{1/\gamma}e^{-(\delta/\gamma)T}\right]}$$

To combine the martingale with the dyn prog approach, let us define the process (same as y, but for $t \in [0,T]$):

$$y_t := \left(\frac{X_t + \mathbf{E}_t\left[\int_t^T \xi_{t,s} Y_s \mathrm{d}s\right]}{\mathbf{E}_t\left[\int_t^T \xi_{t,s}^{\varrho} e^{-(\delta/\gamma)(s-t)}\mathrm{d}s + \varepsilon^{1/\gamma}e^{-(\delta/\gamma)(T-t)}\xi_{t,T}^{\varrho}\right]}\right)^{-\gamma} \quad \text{(dual process)}$$

where $\xi_{t,s} := \xi_s/\xi_t$ is the SDF that values cash flows from $s \geq t$ into time $t \in [0,T]$, and $\varrho := 1 - 1/\gamma$ is a constant parameter; it hence follows that the value function of the problem is:

$$\begin{aligned}
J(X_t, Y_t) \quad : \quad &= \sup_{(c,\pi)\in\mathcal{A}} \mathbf{E}_t\left[\int_t^T e^{-\delta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}\mathrm{d}s + \varepsilon e^{-\delta(T-t)}\frac{X_T^{1-\gamma}}{1-\gamma}\right] \quad \text{(primal problem)} \\
&= \mathbf{E}_t\left[\int_t^T e^{-\delta(s-t)}\frac{(c_s^*)^{1-\gamma}}{1-\gamma}\mathrm{d}s + \varepsilon e^{-\delta(T-t)}\frac{(X_T^*)^{1-\gamma}}{1-\gamma}\right] \quad \text{(opt sol in dual state var } \xi, y) \\
&= \mathbf{E}_t\left[\int_t^T e^{-\delta(s-t)}\frac{\left((y_t\xi_{t,s}e^{\delta(s-t)})^{-1/\gamma}\right)^{1-\gamma}}{1-\gamma}\mathrm{d}s + \varepsilon e^{-\delta(T-t)}\frac{((y_t\xi_{t,T}\varepsilon^{-1}e^{\delta(T-t)})^{-1/\gamma})^{1-\gamma}}{1-\gamma}\right] \\
&= \frac{y_t^{\varrho}}{1-\gamma}\mathbf{E}_t\left[\int_t^T e^{-\delta(s-t)}(\xi_{t,s}^{\varrho}e^{\varrho\delta(s-t)})\mathrm{d}s + \varepsilon^{1-\varrho}e^{-\delta(T-t)}(\xi_{t,T}^{\varrho}e^{\varrho\delta T})\right] \\
&= \frac{1}{1-\gamma}\left(\frac{X_t + \mathbf{E}_t\left[\int_t^T \xi_{t,s}Y_s\mathrm{d}s\right]}{\mathbf{E}_t\left[\int_t^T \xi_{t,t}^{\varrho}e^{-(\delta/\gamma)(s-t)}\mathrm{d}s + \xi_{t,T}^{\varrho}\varepsilon^{1/\gamma}e^{-(\delta/\gamma)(T-t)}\right]}\right)^{1-\gamma} \quad \text{(opt sol in primal state var } (X,Y) \\
&\quad \times\mathbf{E}_t\left[\int_t^T e^{-(\delta/\gamma)(s-t)}\xi_{t,s}^{\varrho}\mathrm{d}s + \varepsilon^{1/\gamma}e^{-(\delta/\gamma)(T-t)}\xi_{t,T}^{\varrho}\right] \\
&= \frac{\left(X_t + \mathbf{E}_t\left[\int_t^T \xi_{t,s}Y_s\mathrm{d}s\right]\right)^{1-\gamma}}{1-\gamma} \\
&\quad \times\mathbf{E}_t\left[\int_t^T e^{-(\delta/\gamma)(s-t)}\xi_{t,s}^{\varrho}\mathrm{d}s + \varepsilon^{1/\gamma}e^{-\delta(T-t)}\xi_{t,T}^{\varrho}\right]^{\gamma}.
\end{aligned}$$

Finally, noticing that when $(r, \mu = r + \sigma\theta, \sigma = \sigma_S)$ are constants,

$$\mathbf{E}_t\left[\xi_{t,s}^{\varrho}\right] = \mathbf{E}_t\left[e^{\varrho(r-\theta^2/2)(s-t)+\varrho\theta(W_s-W_t)}\right] = e^{(\varrho r - \varrho\theta^2(1-\varrho^2)/2)(s-t)},$$

from where we have that:

$$
\begin{aligned}
& \mathbf{E}_t\left[\int_t^T e^{-(\delta/\gamma)(s-t)}\xi_{t,s}^{\varrho}\mathrm{d}s+\varepsilon^{1/\gamma}e^{-\delta(T-t)}\xi_{t,T}^{\varrho}\right] \\
= & \int_t^T e^{-(\delta/\gamma)(s-t)}\mathbf{E}_t\left[\xi_{t,s}^{\varrho}\right]\mathrm{d}s+\varepsilon^{1/\gamma}e^{-\delta(T-t)}\mathbf{E}_t\left[\xi_{t,T}^{\varrho}\right] \\
= & \int_t^T e^{-(\delta/\gamma)(s-t)}\left\{e^{(\varrho r-\varrho\theta^2(1-\varrho^2)/2)(s-t)}\right\}\mathrm{d}s+\varepsilon^{1/\gamma}e^{-\delta(T-t)}\left\{e^{(\varrho r-\varrho\theta^2(1-\varrho^2)/2)(T-t)}\right\} \\
= & \int_t^T e^{(\varrho r-\varrho\theta^2(1-\varrho^2)/2-\delta/\gamma)(s-t)}\mathrm{d}s+\varepsilon^{1/\gamma}e^{(\varrho r-\varrho\theta^2(1-\varrho^2)/2-\delta/\gamma)(T-t)}=:g^{\mathrm{com}}(t),
\end{aligned}
$$

which can be solved in closed form

– As for the term $H_t:=\mathbf{E}_t\left[\int_t^T\xi_{t,s}Y_s\mathrm{d}s\right]$, by direct calculation we have (if $|\rho|=1$):

$$
\begin{aligned}
& \mathbf{E}_t\left[\int_t^T\xi_{t,s}Y_s\mathrm{d}s\right]=Y_t\mathbf{E}\left[\int_t^T e^{(r-\theta^2/2)(s-t)+\theta(W_s-W_t)}e^{(\alpha-\beta^2/2)(s-t)+\beta(\tilde{W}_s-\tilde{W}_t)}\mathrm{d}s\right] \\
= & Y_te^{(r+\alpha-(\theta^2+\beta^2)/2+(\theta+\beta)^2/2)(T-t)}=:Y_tF^{\mathrm{com}}(t),
\end{aligned}
$$

and hence the value fn can be stated as

$$
J(X_t,Y_t)=\frac{1}{1-\gamma}\left(X_t+Y_tF^{\mathrm{com}}(t)\right)^{1-\gamma}\left(g^{\mathrm{com}}(t)\right)^{\gamma}
$$

Finally, the optimal policy in feed-back form, $(c^*(X_t,Y_t),\,\pi^*(X_t,Y_t))$, can be obtained from (notice that $c_s^*=(y_t\xi_{t,s}e^{\delta(s-t)})^{-1/\gamma}\ s\geq t\in[0,T]$):

$$
c^*(X_t,Y_t)=(y_t)^{-1/\gamma}=\left(\left(\frac{X_t+Y_tF^{\mathrm{com}}(t)}{g^{\mathrm{com}}(t)}\right)^{-\gamma}\right)^{-1/\gamma}=\frac{X_t+Y_tF^{\mathrm{com}}(t)}{g^{\mathrm{com}}(t)},
$$

while π^* can be recovered from the "$\mathrm{d}W_t$" term of $\mathrm{d}X_t^*$ (recall that, for any admissible (c,π), $\mathrm{d}X_t=X_t(1-\pi_t)r\mathrm{d}t+X_t\pi_t(\mu\mathrm{d}t+\sigma\mathrm{d}W_t)-(c_t-Y_t)\mathrm{d}t$, hence the "$\mathrm{d}W$" term of $\mathrm{d}X$ is $X_t\pi_t\sigma$), where X^* is the "optimal" wealth process (i.e., the one that finances c_t^* and X_T^*),

$$
\begin{aligned}
X_t^* &= \mathbf{E}_t\left[\int_t^T\xi_{t,s}(c_s^*-Y_s)\mathrm{d}s+\xi_{t,T}X_T^*\right] \\
&= \mathbf{E}_t\left[\int_t^T\xi_{t,s}(y\xi_se^{\delta s})^{-1/\gamma}\mathrm{d}s+\xi_{t,T}(y\varepsilon^{-1}\xi_Te^{\delta T})^{-1/\gamma}\right]-Y_tF^{\mathrm{com}}(t) \\
&= \xi_t^{-1/\gamma}\int_t^T\mathbf{E}_t\left[\xi_{t,s}^{\varrho}\right](ye^{\delta s})^{-1/\gamma}\mathrm{d}s+\xi_t^{-1/\gamma}\mathbf{E}_t\left[\xi_{t,T}^{\varrho}\right](y\varepsilon^{-1}e^{\delta T})^{-1/\gamma}-Y_tF^{\mathrm{com}}(t) \\
&= (y\xi_te^{\delta t})^{-1/\gamma}\left(\int_t^T\mathbf{E}_t\left[\xi_{t,s}^{\varrho}\right](e^{\delta(s-t)})^{-1/\gamma}\mathrm{d}s+\varepsilon^{1/\gamma}\mathbf{E}_t\left[\xi_{t,T}^{\varrho}\right]e^{-(\delta/\gamma)(T-t)}\right)-Y_tF^{\mathrm{com}}(t) \\
&= (y\xi_te^{\delta t})^{-1/\gamma}g^{\mathrm{com}}(t)-Y_tF^{\mathrm{com}}(t)
\end{aligned}
$$

hence,

$$
\begin{aligned}
X_t^*\pi_t^*\sigma &= \frac{\theta}{\gamma}(y\xi_te^{\delta t})^{-1/\gamma}g^{\mathrm{com}}(t)-\beta\rho Y_tF^{\mathrm{com}}(t) \\
\pi^*(X_t,Y_t) &= \frac{\mu-r}{\gamma\sigma^2}\frac{(X_t+Y_tF^{\mathrm{com}}(t))}{X_t}-\frac{\beta\rho}{\sigma}\frac{Y_t}{X_t}F^{\mathrm{com}}(t).
\end{aligned}
$$

- 2015/04/08: the discrete-time version of the model can be accomodated in a CRR binomial-tree model

  - there is an almost one-to-one mapping between the continuous-time (CT) and the discrete-time (DT) versions of the model, with the exception of π^*, which in the DT version has to be determined from the delta hedge expression,
$$\pi^*_{t_n} = \frac{\Delta X^*_{t_{n+1}}/X_{t_n}}{\Delta S_{t_{n+1}}/S_{t_n}},$$
which is the solution of the standard binomial tree replication problem
$$\begin{cases} X_{t_n}(1-\pi^*_{t_n})e^{r\Delta t} + X_{t_n}\pi^*_{t_n}u = X^{*,u}_{t_{n+1}}, \\ X_{t_n}(1-\pi^*_{t_n})e^{r\Delta t} + X_{t_n}\pi^*_{t_n}d = X^{*,d}_{t_{n+1}}, \end{cases}$$
where
$$\begin{aligned} \Delta X^*_{t_{n+1}} &= X^{*,u}_{t_{n+1}} - X^{*,d}_{t_{n+1}}, \;\; \Delta S_{t_{n+1}}/S_{t_n} = (u-d), \\ X^{*,u}_{t_{n+1}} &= (y\xi^u_{t_{n+1}}e^{\delta t_{n+1}})^{-1/\gamma}g^{\mathrm{com}}(t_{n+1}) - Y^u_{t_{n+1}}F^{\mathrm{com}}(t_{n+1}), \\ X^{*,d}_{t_{n+1}} &= (y\xi^d_{t_{n+1}}e^{\delta t_{n+1}})^{-1/\gamma}g^{\mathrm{com}}(t_{n+1}) - Y^d_{t_{n+1}}F^{\mathrm{com}}(t_{n+1}), \end{aligned}$$
with $\Delta t = T/N$ and $t_n = nT/N$, and $n = \{0, \dots, N\}$.
  - we hence can compare the discrepancies and convergence rate of $J(X_{t_n}, Y_{t_n})$ and $c^*(X_{t_n}, Y_{t_n})$ with its CT analogs, as well as, the time-discretized version of $\pi^*(X_t, Y_t)$ with the DT delta-hedge optimal inv policy, $\pi^*_{t_n}$.
  - a key issue is how to calibrate the DT model to ensure the convergence from the DT to the CT version. The table below (taken from Broadie and Detemple, 2004) shows three possible alternatives to calibrate the binomial tree used in the literature.

Table 1 Standard Lattice Methods

Lattice	Outcome	Probability
CRR (1979) (1)	$x_1 = \sigma\sqrt{h}$	$p_1 = \dfrac{e^{(r-\delta)h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$
	$x_2 = -\sigma\sqrt{h}$	$p_2 = 1 - p_1$
Jarrow and Rudd (1983) (2)	$x_1 = (r-\delta-\sigma^2/2)h + \sigma\sqrt{h}$	$p_1 = \dfrac{e^{\sigma^2 h/2} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$
	$x_2 = (r-\delta-\sigma^2/2)h - \sigma\sqrt{h}$	$p_2 = 1 - p_1$
Amin (1991) (3)	$x_1 = (r-\delta-\ln(\cosh(\sigma\sqrt{h})))h + \sigma\sqrt{h}$	$p_1 = 1/2$
	$x_2 = (r-\delta-\ln(\cosh(\sigma\sqrt{h})))h - \sigma\sqrt{h}$	$p_2 = 1/2$
Boyle (1986) (4)	$x_1 = \lambda\sigma\sqrt{h}$	$p_1 = \dfrac{1}{2\lambda^2} + \dfrac{(r-\delta-\sigma^2/2)\sqrt{h}}{2\lambda\sigma}$
	$x_2 = 0$	$p_2 = 1 - 1/\lambda^2$
	$x_3 = -\lambda\sigma\sqrt{h}$	$p_3 = \dfrac{1}{2\lambda^2} - \dfrac{(r-\delta-\sigma^2/2)\sqrt{h}}{2\lambda\sigma}$

  a key different of our exercise is that, unlike relative a standard derivative pricing execise, we are trying to match the replicating portfolio of the contingent claim (c^*, X^*_T), and not only its (rational) price; hence we should be careful about matching the "right" elements that ultimately drive π^*.

- 2015/04/15: lets solve for $n = 3$

- 2015/06/05: from Detemple and Sundaresan (1999) we can guess that the consumption-portfolio problem with convex portfolio constraints can be reduced to an uncontrained problem with endogenous dividend process for the risky asset.
  - the discrete-time binomial model version of the problem entails solving:

$$\max_{(c_{t_n},\pi_{t_n})_{n=0}^{N-1}} \mathbf{E}_{t_0}\left[\sum_{n=0}^{N-1} e^{-\alpha(t_n-t_0)}u(c_{t_n}) + \varepsilon e^{-\alpha(t_N-t_0)}u(X_{t_N})\right]$$

$$\text{s.t.} \left\{ \begin{array}{l} X_{t_{n+1}} = (X_{t_n}+Y_{t_n}-c_{t_n})\left[r+\pi_{t_n}\left\{\begin{array}{c} u-r \\ d-r \end{array}\right.\right], \\ (X_{t_0},Y_{t_0}) \text{ given}, \;\; a \le \pi_{t_n} \le b, \end{array}\right.$$

    where $n=\{0,1,\dots,N-1\}$, and $t_n = 0+nT/(N-1)$
  - by the dynamic programing principle, it follows that the optimization problem can be written as:

$$V_{t_n}(X_{t_n},Y_{t_n}) = \max_{(c_{t_n},\pi_{t_n})} u(c_{t_n}) + e^{-\delta\Delta t_n}\mathbf{E}_{t_n}\left[V_{t_{n+1}}(X_{t_{n+1}},Y_{t_{n+1}})\right]$$

$$\text{s.t.} \left\{ \begin{array}{l} X_{t_{n+1}} = (X_{t_n}+Y_{t_n}-c_{t_n})\left[r+\pi_{t_n}\left\{\begin{array}{c} u-r \\ d-r \end{array}\right.\right] \\ (X_{t_n},Y_{t_n}) \text{ given}, \;\; a \le \pi_{t_n} \le b, \\ V(X_{t_N},Y_{t_N},t_N) = \varepsilon \cdot u(X_{t_N}), \end{array}\right.$$

    ?

    with $\Delta t_n =?$
  - from the dynamic budget equation it follows that there is a one-to-one mapping between $X_{t_{n+1}}$ and π_{t_n}. In particular, we have that:

$$\pi_{t_n} = \frac{X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r}{u-r} = \frac{X^d_{t_{n+1}} - -(X_{t_n}+Y_{t_n}-c_{t_n})r}{d-r}.$$

    Hence, in terms of $X_{t_{n+1}}$, the constraints on π_{t_n} are equivalent to:

$$\begin{array}{rcl} X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r & \ge & a(u-r), \;\; X^d_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r \ge a(d-r), \\ X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r & \le & b(u-r), \;\; X^d_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r \le b(d-r). \end{array}$$

    Likewise, the condition

$$\frac{X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r}{u-r} = \frac{X^d_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r}{d-r}$$

    translates into:

$$\begin{aligned} \frac{u-r}{u-d}\left(X^d_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r\right) + \frac{r-d}{u-d}\left(X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r\right) = 0 \\ \Leftrightarrow (1-q)\left(X^d_{t_{n+1}} - X_{t_n}r - (Y_{t_n}-c_{t_n})\right) + q\left(X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r\right) = 0 \\ \Leftrightarrow \frac{(1-q)}{(1-p)}\frac{1}{r}(1-p)\left(X^d_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r\right) + \frac{q}{p}\frac{1}{r}p\left(X^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n})r\right) = 0 \\ \Leftrightarrow \frac{(1-q)}{(1-p)}\frac{1}{r}(1-p)X^d_{t_{n+1}} + \frac{q}{p}\frac{1}{r}pX^u_{t_{n+1}} - (X_{t_n}+Y_{t_n}-c_{t_n}) = 0 \\ \Leftrightarrow \mathbf{E}_{t_n}\left[\xi_{t_n,t_{n+1}}X_{t_{n+1}}\right] = X_{t_n}+Y_{t_n}-c_{t_n}. \end{aligned}$$

Hence, it follows that the optimization problem can be written as:

$$V_{t_n}(X_{t_n}, Y_{t_n}) = \max_{(c_{t_n}, X_{t_{n+1}})} u(c_{t_n}) + e^{-\delta \Delta t_n} \mathbf{E}_{t_n}\left[V_{t_n}(X_{t_{n+1}}, Y_{t_{n+1}})\right]$$

$$\text{s.t.} \begin{cases} \mathbf{E}_{t_n}\left[\xi_{t_n,t_{n+1}} X_{t_{n+1}}\right] = X_{t_n} + Y_{t_n} - c_{t_n}, \;\; (X_{t_n}, Y_{t_n}) \text{ given}, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - a(u-r) \geq 0, \;\; X^d_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - a(d-r) \geq 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - b(u-r) \leq 0, \;\; X^d_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - b(d-r) \leq 0, \\ V(X_{t_N}, Y_{t_N}, t_N) = \varepsilon \cdot u(X_{t_N}) \end{cases}$$

– The optimality conditions are then given by:

$$\begin{cases} V'_{t_n}(X^u_{t_{n+1}}, Y_{t_{n+1}}) = y_{t_n} \xi^u_{t_n,t_{n+1}} + (\lambda^u_{b,t_n} - \lambda^u_{a,t_n})/p, \\ V'_{t_n}(X^d_{t_{n+1}}, Y_{t_{n+1}}) = y_{t_n} \xi^d_{t_n,t_{n+1}} + (\lambda^d_{b,t_n} - \lambda^d_{a,t_n})/(1-p), \\ u'(c_{t_n}) = y_{t_n}, \\ \mathbf{E}_{t_n}\left[\xi_{t_n,t_{n+1}} X_{t_{n+1}}\right] = X_{t_n} + Y_{t_n} - c_{t_n}, \;\; y_{t_n} > 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - a(u-r) \geq 0, \;\; \lambda^u_{a,t_n} \geq 0, \\ X^d_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - a(d-r) \geq 0, \;\; \lambda^d_{a,t_n} \geq 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - b(u-r) \leq 0, \;\; \lambda^u_{b,t_n} \geq 0, \\ X^d_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - b(d-r) \leq 0, \;\; \lambda^d_{b,t_n} \geq 0, \\ \left[X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - a(u-r)\right] \cdot \lambda^u_{a,t_n} = 0, \\ \left[X^d_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - a(d-r)\right] \cdot \lambda^d_{a,t_n} = 0, \\ \left[X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - b(u-r)\right] \cdot \lambda^u_{b,t_n} = 0, \\ \left[X^d_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n})\, r - b(d-r)\right] \cdot \lambda^d_{b,t_n} = 0. \end{cases}$$

Notice that $\lambda^u_{a,t_n} > 0$ iff $\lambda^d_{a,t_n} > 0$, and likewise, $\lambda^u_{b,t_n} > 0$ iff $\lambda^d_{b,t_n} > 0$; in addition, $\lambda^u_{a,t_n} > 0$ implies $\lambda^u_{b,t_n} = 0$, and viceversa. Hence, we can set $\lambda^u_{a,t_n} = \lambda^d_{a,t_n}$ and $\lambda^u_{b,t_n} = \lambda^d_{b,t_n}$, and hence, use the change of variables[1]

$$\lambda^d_{b,t_n} - \lambda^d_{a,t_n} = \lambda^u_{b,t_n} - \lambda^u_{a,t_n} = y_{t_n} \frac{1}{r} \frac{1}{u-d} \Delta\lambda_{t_n}.$$

– By the martingale property of financial wealth we have:

$$\begin{aligned} X_{t_n} + Y_{t_n} - c_{t_n} &= \mathbf{E}_{t_n}\left[\xi_{t_n,t_{n+1}} X_{t_{n+1}}\right] \\ &= \frac{1}{r}\left[(1-q)\left(X^d_{t_{n+1}} - r\left\{X_{t_n} + Y_{t_n} - c_{t_n}\right\}\right) + q\left(X^u_{t_{n+1}} - r\left\{X_{t_n} + Y_{t_n} - c_{t_n}\right\}\right)\right] \\ &\quad + X_{t_n} + Y_{t_n} - c_{t_n} \\ &= \frac{1}{r}\left[\left\{(1-q) - y_{t_n}\frac{1}{r}\frac{1}{u-d}\Delta\lambda_{t_n}\right\}\left(X^d_{t_{n+1}} - r\left\{X_{t_n} + Y_{t_n} - c_{t_n}\right\}\right)\right. \\ &\quad \left. + \left\{q + y_{t_n}\frac{1}{r}\frac{1}{u-d}\Delta\lambda_{t_n}\right\}\left(X^u_{t_{n+1}} - r\left\{X_{t_n} + Y_{t_n} - c_{t_n}\right\}\right)\right] + X_{t_n} + Y_{t_n} - c_{t_n} \\ &= \frac{1}{r}\left[(1-q_{t_n})\left(X^d_{t_{n+1}} - r\left\{X_{t_n} + Y_{t_n} - c_{t_n}\right\}\right) + q_{t_n}\left(X^u_{t_{n+1}} - r\left\{X_{t_n} + Y_{t_n} - c_{t_n}\right\}\right)\right] \\ &\quad + X_{t_n} + Y_{t_n} - c_{t_n} \\ &= \mathbf{E}_{t_n}\left[\xi^\lambda_{t_n,t_{n+1}} X_{t_{n+1}}\right] \end{aligned}$$

[1] Notice that with the definition of $\Delta\lambda_{t_n}$, it follows that q_{t_n} can be seen as the risk-neutral probability in a financial market with $u_{t_n} = u + \Delta\lambda_{t_n}$ and $d_{t_n} = d - \Delta\lambda_{t_n}$ as the "up" and "down" movements of the risky asset.

– The optimal condicions are given by

$$\begin{cases} V'_{t_n}(X^u_{t_{n+1}}, Y_{t_{n+1}}) = y_{t_n} \frac{1}{r} q_{t_n}/p, \\ V'_{t_n}(X^d_{t_{n+1}}, Y_{t_{n+1}}) = y_{t_n} \frac{1}{r}(1-q_{t_n})/(1-p), \\ u'(c_{t_n}) = y_{t_n}, \\ X_{t_n} = \mathbf{E}_{t_n}\left[\xi^\lambda_{t_n,t_{n+1}} X_{t_{n+1}}\right], \quad y_{t_n} > 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r \geq a(u-r), \;\; q - q_{t_n} \geq 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r \leq b(u-r), \\ \left[X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r - a(u-r)\right] \cdot (q - q_{t_n}) \\ + \left[X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r - b(u-r)\right] \cdot (q - q_{t_n}) = 0, \end{cases}$$

– The system above is equivalent to:

$$\begin{cases} X^*_{t_{n+1}} = J(y_{t_n}\xi^\lambda_{t_n,t_{n+1}}, t_{n+1}), \quad c^*_{t_n} = I(y_{t_n}), \\ \mathbf{E}_{t_n}\left[\xi^\lambda_{t_n,t_{n+1}} J(y_{t_n}\xi^\lambda_{t_n,t_{n+1}}, t_{n+1})\right] = X_{t_n}, \quad y_{t_n} > 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r \geq a(u-r), \;\; q - q_{t_n} \geq 0, \\ X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r \leq b(u-r), \\ \left[X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r - a(u-r)\right] \cdot (q - q_{t_n}) \\ + \left[X^u_{t_{n+1}} - (X_{t_n} + Y_{t_n} - c_{t_n}) r - b(u-r)\right] \cdot (q - q_{t_n}) = 0, \end{cases}$$

where $I(\cdot)$ is the inverse of $u'(\cdot)$. The first two equations correspond to an unconstrained problem with time-varying coefficients, and the last three equations correspond to $\pi_{t_n}(\Delta\lambda_{t_n}) \geq a$, $\pi_{t_n}(\Delta\lambda_{t_n}) \leq b$ and $\pi_{t_n}(\Delta\lambda_{t_n}) \cdot \Delta\lambda_{t_n} = 0$.

– The optimal portfolio is obtained from the system:

$$\begin{cases} X^{*,u}_{t_{n+1}} = \left(X_{t_n} + Y_{t_n} - c^*_{t_n}\right)\left[r + \pi_{t_n}(u-r)\right] \\ X^{*,d}_{t_{n+1}} = \left(X_{t_n} + Y_{t_n} - c^*_{t_n}\right)\left[r + \pi_{t_n}(d-r)\right] \end{cases}$$

from where we obtain

$$\pi^*_{t_n} = \frac{X^{*,u}_{t_{n+1}} - X^{*,d}_{t_{n+1}}}{(X_{t_n} + Y_{t_n} - I(y_{t_n}))\{u-d\}}$$

where $y_{t_n} > 0$ solves $\mathbf{E}_{t_n}\left[\xi^\lambda_{t_n,t_{n+1}} J(y_{t_n}\xi^\lambda_{t_n,t_{n+1}}, t_{n+1})\right] = X_{t_n}$.

- Being the fictitious financial market, a complete one, it follows that the problem can be represented as a static-variational one. In particular, the problem

$$\max_{(c_{t_n},\pi_{t_n})_{n=0}^{N-1}} \mathbf{E}_{t_0}\left[\sum_{n=0}^{N-1} e^{-\alpha(t_n-t_0)} u(c_{t_n}) + \varepsilon e^{-\alpha(t_N-t_0)} u(X_{t_N})\right]$$

$$\text{s.t.} \begin{cases} X_{t_{n+1}} = (X_{t_n} + Y_{t_n} - c_{t_n})\left[r + \pi_{t_n}\left\{\begin{array}{l} u_{t_n} - r \\ d_{t_n} - r \end{array}\right.\right], \\ (X_{t_0}, Y_{t_0}) \text{ given}, \quad \pi_{t_n} \in \mathbb{R}, \end{cases}$$

can be written as

$$\max_{\left\{(c_{t_n})_{n=0}^{N-1}, X_{t_N}\right\}} \mathbf{E}_{t_0}\left[\sum_{n=0}^{N-1} e^{-\alpha(t_n-t_0)} u(c_{t_n}) + \varepsilon e^{-\alpha(t_N-t_0)} u(X_{t_N})\right]$$

$$\text{s.t. } \mathbf{E}_{t_0}\left[\sum_{n=0}^{N-1} \xi^\lambda_{t_0,t_n} c_{t_n} + \xi_{t_0,t_N} X_{t_N}\right] \leq X_{t_0} + \mathbf{E}_{t_0}\left[\sum_{n=0}^{N-1} \xi^\lambda_{t_0,t_n} Y_{t_n}\right],$$

where

$$\xi^{\lambda}_{t_0,t_n}(j,2^n) = \frac{1}{r^n}\left(\frac{1}{p}\right)^{j}\left(\frac{1}{(1-p)}\right)^{2^n-j}\left(\prod_{j=0}^{k} q_{t_j}\right)\left(\prod_{j=0}^{n-k}\right).$$

- The optimal pair $(c^*_{t_n}, X^*_{t_N})$ is given by