Stochastic Volatility

Jacques Burrus [jacques.burrus@bfi.lat]

Table of Contents

Table of Contents

Table of Contents

Empirical observations about the volatility

- The Black-Scholes model makes the simplifying assumptions that volatility was a constant (or at least deterministic) quantity.
- Since Black Monday it is clear that this hypothesis was no longer able to satisfactorily explain the smile.
- Multiple empirical studies focus on the behavior of volatility and highlight the stochastic nature of volatility.
- In his 1988 article "Volatility Persistence abd Stock Valuations: Some Empirical Evidence Using Garch", Chou suggests that:
  - Volatility is mean-reverting: it tends to oscillate around a long-term equilibrium value.
  - Volatility is persistent: it is a predictable random variable (unlike stock returns in efficient markets).
  - Also, volatility tends to be related to the returns: generally periods of negative returns (financial crisis) are associated with a high volatility.

Stock Index

- Chou describes the US stock market with a proxy consisting of a NYSE stock index S_t.
  - This index reinvests dividends so as to remove the jumps associated with dividends.
  - The proportion of each stock is weighted according to the firm's capitalization.
  - The index is normalized to the arbitrary value of 100 at the beginning of the data in 1962.

Stock Index

- The index S increased more than tenfolds between 1962 and 1985.

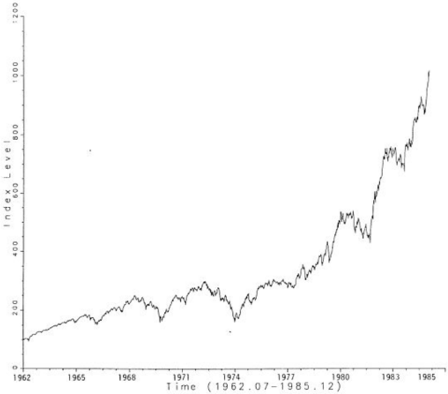


Figure 1. NYSE stock index

Index Returns

- Clearly, the stock index is prone to cointegration.
  - For example, one could explain the value of the stock index S_t as a function of the size X_t of a tree planted in 1962.
- Therefore Chou focuses on the index returns R_t defined as:

$$R_{t+1} = \frac{(S_{t+1} + D_t) - S_t}{S_t}$$

  - The term D_t is the amount of dividends paid between t and $t+1$ and needs to be added to the capital gains.
  - For the sake of studying the volatility, this definition of the returns is a good proxy for the log-returns.

$$R_{t+1} \approx \ln \frac{S_{t+1} + D_t}{S_t}$$

Index Returns

- As expected the returns R look quite hectic.

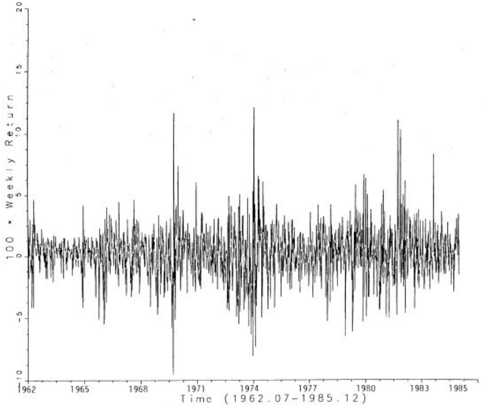


Figure 2. NYSE stock return

Index Variance

- Chou aims at explaining the conditional variance of the returns, i.e. the variance ν_t at a given date t.
- The variance is an observable quantity that describes the fluctuations of the returns.
  - Thus, Chou cannot get as many independent observations of the variances as there are published stock returns.
  - Therefore he divides the data set into subperiods of N points of length ΔT and assumes a constant variance ν for each subperiod. The variance ν is estimated as:

$$\nu_j = \left(\frac{1}{N} \sum_{t=t_j}^{t_j+N} R_t^2 \right) - \left(\frac{1}{N} \sum_{t=t_j}^{t_j+N} R_t \right)^2$$

    - A week corresponds to $N = 5$ business days and $\Delta T = 5/252$.
    - The variance ν increases with N unlike to volatility $\sigma = \sqrt(\nu/\Delta T)$.

Index Variance

- The variance ν of the index is not constant but seems smoother than the returns R.

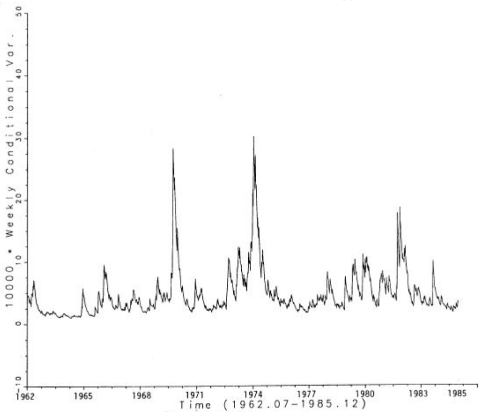


Figure 3. NYSE stock variance

Table of Contents

Generalized Auto-Regressive Conditional Heteroskedasticity

- Chou explores the variance with a Generalized Auto-Regressive Conditional Heteroskedasticity (Garch) model.
  - Conditional: The variance ν is not observable directly. It represents the fluctuation of the noise conditional on a equation for the returns R.
  - Heteroskedasticity: The variance follows a stochastic process and may as such vary over time. The statistician must explicitly the heteroskedasticity model for the variance ν.
  - Auto-Regressive: The variance is assumed mean-reverting. Said differently, the variance at date $t+1$ depends partly on its value at date t. This is a specification of the model of the variance ν.
  - Generalized: The statistician has latitude to define the models for the return R and the variance ν.

Generalized Auto-Regressive Conditional Heteroskedasticity

- The model for the return is:

$$R_{t+1} = r_f + \delta \cdot \nu_t + \epsilon_{t+1}$$

  - The variance is defined indirectly as:

$$\epsilon_{t+1} | I_t \sim N(0, \nu_t)$$

  - Note that this is a one-factor model that explains the extra return $R - r_f$ in terms of the variance. This is the risk-return trade-off.

- The model for the variance reads:

$$\nu_t = \alpha_0 + \alpha \cdot \epsilon_t^2 + \beta \cdot \nu_{t-1}$$

  - The coefficient β is responsible for the persistence.
  - The long-term equilibrium value is: $\nu_\infty = \frac{\alpha_0}{1-\alpha-\beta}$
  - Therefore, the coefficients α and β must satisfy: $0 \leq \alpha + \beta \leq 1$

Generalized Auto-Regressive Conditional Heteroskedasticity

- The Garch model parameters are r_f, α_0, α, β, and δ.
- In order to estimate the value of these parameters, the statistician can e.g. maximize the log-likelyhood of the observed data wrt the aforementioned parameters.

Weekly results

- Chou obtained the following results for subperiods of length one week ($N = 5$).

Table I. Full and sub-sample estimation of the GARCH(1,1)-M model using weekly returns of NYSE value weighted index

$$R_{t+1} = r_f + \delta V_t + e_{t+1}$$
$$V_t = \alpha_0 + \alpha e_t^2 + \beta V_{t-1}$$
with $e_{t+1} \mid I_t \sim N(0, V_t)$,

Sample period	$r_f \times 10^2$	$\alpha_0 \times 10^4$	α	β	δ	$\alpha + \beta$	$\sigma^2 \times 10^4$[a]
July 1962–December 1985 No. obs. = 1225	0·122 (1·49)[b]	0·096 (2·83)	0·151 (7·17)	0·835 (38·85)	4·50 (1·94)	0·986	4·074
July 1962–December 1985 No. obs. = 1225	—	0·099 (2·83)	0·151 (7·15)	0·834 (38·64)	4·56 (3·28)	0·985	4·090
July 1962–December 1973 No. obs. = 599	0·159 (1·85)	0·092 (2·18)	0·217 (6·12)	0·778 (24·13)	5·05 (1·78)	0·995	3·574
January 1974–December 1985 No. obs. = 626	0·008 (0·04)	0·159 (2·03)	0·084 (3·14)	0·882 (23·67)	6·15 (1·32)	0·966	4·567

[a] The estimate is computed using the residuals of the last step iteration. This estimate is more stable than that given by equation (8) if $\alpha + \beta$ is close to one.
[b] Numbers in parentheses are t-ratios.
[c] Weekly excess returns are used for estimation hence no constant terms are necessary.

Weekly results

- The parameters of the model describing the returns R are not statistically significant at the 95% confidence interval (t-stat lower than 3).
    - Predicting the level of returns is usually a tedious and thankless task.
    - However, the parameter r_f has the typical dimension of a risk-free rate. Also, the estimated coefficient δ is positive so that the greater the risk ν_t, the greater the return R_{t+1}.
- For the sake of option pricing, these parameters are irrelevant.
    - The model is continuous and the Girsanov theorem states that the empirical drift of the return R is irrelevant for the transition to the Equivalent Martingale Measure.
- Fortunately, the parameters of the heteroskedasticity model are statistically significant at the 95% confidence interval (t-stat higher than 3).
    - These parameters are truly important for option pricing.

Weekly results

- The persistence is quite strong as the parameter β is greater than 0.778.
- The surprise variance α is about 0.15.
- As required the sum $\alpha + \beta$ is positive but smaller than 1.
- The variance ν_t henceforth reverts to a defined long-term value ν_∞.

Weekly results

- Note that the residuals are not auto-correlated.

Table II. Auto-correlation for residuals and normalized squared residuals of the GARCH(1,1)-M model (1962.07–1985.12)

$$R_{t+1} = r_f + \delta V_t + e_{t+1}$$
$$V_t = \alpha_0 + \alpha e_t^2 + \beta V_{t-1}$$
with $e_{t+1} \mid I_t \sim N(0, V_t)$,

lag	1	2	3	4	5	6	7	8	9	10
acf[a] of e_t	0·050[b]	−0·024	0·046	0·026	−0·043	0·048	0·025	−0·056	−0·007	−0·003
acf of u_t^2[c]	0·012	0·040	−0·047	0·004	−0·011	0·006	−0·021	0·005	−0·022	−0·032
lag	11	12	13	14	15	16	17	18	19	20
acf of e_t	−0·005	0·019	0·028	0·012	−0·023	0·014	0·023	0·006	0·004	0·056
acf of u_t^2	0·008	−0·023	−0·005	0·033	0·001	−0·040	0·003	−0·036	−0·032	−0·015
lag	21	22	23	23	25	26				
acf of e_t	0·018	0·042	0·044	0·012	−0·018	0·008				
acf of u_t^2	−0·001	0·022	0·018	0·000	−0·035	0·028				

[a] acf: auto-correlation function.

[b] $2(\text{std. error of acf}) = \frac{2}{\sqrt{\text{no. observations}}} = \frac{2}{\sqrt{1225}} = 0\cdot057.$

[c] $u_t^2 = \frac{e_t^2}{V_t}$: normalized squared residuals.

[d] The Box–Pierce Q statistics for e_t and u_t^2 are respectively 29·62 and 18·12, which are both less than the 5 per cent critical value of 38·90.

[e] Lagrange multiplier test statistics for GARCH(1,2) and GARCH(2,1) are respectively 0·074 and 0·355, while the 5 per cent critical value is 3·84.

Effect of the length of the subperiod

- The previous conclusions drawn for $N = 5$ are still valid.
  - Note that the heteroskedasticity model is a simpler I-Garch model where $\alpha + \beta = 1$.

$$\nu_t = \alpha_0 + \alpha \cdot \epsilon_t^2 (1 - \alpha) \cdot \nu_{t-1}$$

Table III. Parameter estimates and tests of the IGARCH(1,1)-M model for temporal aggregated data

$$R_{t+1}^N = r_f + \delta V_t + e_{t+1}$$
$$V_t = \alpha_0 + \alpha e_t^2 + (1 - \alpha) V_{t-1}$$

with $e_{t+1} \mid I_t \sim N(0, V_t)$,

where R_t^N is the N-trading-day return.

N	$r_f \times 10^2$	$\alpha_0 \times 10^4$	α	δ	llf1	llf2[a]	LR[b] (p-value)
1	0·04	0·003	0·089	6·09	−6089·23	−6088·49	1·48
	(3·30)[c]	(6·41)	(18·14)	(2·52)			(0·224)
2	0·07	0·012	0·104	4·72	−4402·61	−4401·17	2·88
	(2·40)	(5·51)	(12·03)	(1·96)			(0·089)
3	0·10	0·017	0·097	4·03	−3416·68	−3415·68	2·00
	(2·24)	(3·56)	(8·37)	(1·80)			(0·158)
5	0·13	0·053	0·139	3·87	−2407·68	−2407·24	0·88
	(1·75)	(3·28)	(6·82)	(1·74)			(0·348)
10	0·17	0·182	0·148	4·13	−1446·91	−1445·31	3·20
	(0·97)	(2·51)	(4·73)	(1·83)			(0·074)
20	0·25	0·793	0·210	3·62	−847·70	−846·09	3·22
	(0·64)	(1·88)	(3·22)	(1·59)			(0·073)
50[d]	—	5·870	0·406	3·64	−388·09	−386·80	2·58
		(1·30)	(2·26)	(3·77)			(0·108)
250	—	—	—	—	—	−93·22	−0·55[e]
							(0·250)
Weekly	0·130	0·072	0·166	4·09	−2493·98	−2493·49	0·98
	(1·69)	(2·83)	(7·94)	(1·87)			(0·322)
Monthly	—	0·427	0·138	3·36	−801·92	−799·96	3·92
		(1·57)	(2·78)	(2·75)			(0·048)
Monthly (1926–1985)	0·92	0·507	0·157	0·96	−2136·41	−2125·31	2·20
	(4·29)	(3·04)	(8·22)	(1·20)			(0·276)

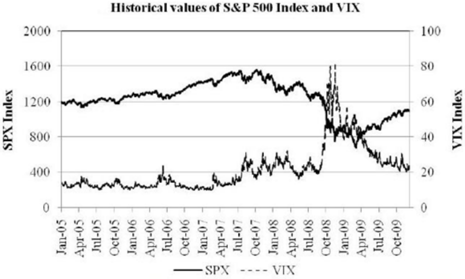
[a] llf1 and llf2 are respectively the log likelihood function estimates for the IGARCH-M and GARCH-M model.
[b] The likelihood ratio test statistic, $LR = 2 \times (\text{llf2} - \text{llf1})$.
[c] Numbers in parenthèses are t-ratios except for the last column where p-values are given.
[d] For monthly and $N = 50$, 250 cases, the excess returns are used hence the model is estimated without the constant term r_f. Negative estimates for the risk-free rate are obtained if total returns are used.
[e] The estimation for IGARCH does not converge for this data set hence a Wald-t statistic is used.

Burrus
Financial
Intelligence

Table of Contents

VIX index and SPX returns

- The previous Garch model did not contemplate a correlation between the stock return R_{t+1} and the concurrent variance ν_{t+1}.
- In fact, by taking the CBOE VIX index as a proxy for the volatility, one observes that crashes tend to be associated with higher values for VIX (see e.g. Manda 2010).



Daily closing levels of the S&P 500 Index (SPX) and the S&P 500 Volatility Index (VIX). The sample period is January 3, 2005 – December 11, 2009. Source: CBOE and Yahoo Finance

Table of Contents

Table of Contents

Stochastic Volatility Models

- Empirical studies suggest that volatility is persistent, mean-reverting, and related with the underlying level S.
- Therefore a natural extension of the Black-Scholes model is to treat the volatility σ or equivalently the instantaneous variance $\nu = \sigma^2$ as an additional random variable.

Physical measure

- In the empirical measure P, the dynamics of an underlying S under a general Stochastic Volatility (SV) model read:
- In the physical dynamics of the underlying S read:
  - Return model
$$dS_t/S_t = \mu^P dt + \sqrt{\nu_t} dW_{1t}^P$$
  - Volatility model:
$$d\nu_t = \kappa^P dt + \eta dW_{2t}^P$$
  - Dependence model:
$$dW_{1t}^P dW_{2t}^P = \rho dt$$

Physical measure

- The volatility model must be carefully chosen so that the variance ν remains always positive.
- Note that the SV model involves two distinct Wiener processes W_1^P and W_2^P that are linearly dependent through the instantaneous correlation ρ.
  - This will lead to an important caveat regarding SV models.

Table of Contents

Risk-neutral measure

- In order to switch to the risk-neutral measure Q, we assume that the market trades three assets:
  - Money Market Account (MMA) B compounded continuous at rate r:

$$B_t = B_0 e^{rt}$$

  - Stock S,
  - Another derivative C whose value at date t depends on the values of the state variables S_t and ν_t at date t.
- The dynamics of the MMA B and stock S are straightforward.
- The P&L of the derivative C are given by Itô's lemma:

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS^2 + \frac{\partial C}{\partial \nu} d\nu + \frac{1}{2} \frac{\partial^2 C}{\partial \nu^2} d\nu^2 + \frac{\partial^2 C}{\partial S \partial \nu} dS d\nu + o(dt)$$

- i.e.

$$\begin{aligned} dC &= \left[\frac{\partial C}{\partial t} + \mu^P S \frac{\partial C}{\partial S} + \frac{\nu S^2}{2} \frac{\partial^2 C}{\partial S^2} + \kappa^P \frac{\partial C}{\partial \nu} + \frac{\eta^2}{2} \frac{\partial^2 C}{\partial \nu^2} + \rho \eta \sqrt{\nu} S \frac{\partial^2 C}{\partial S \partial \nu} \right] dt \\ &+ \left[\sqrt{\nu} S \frac{\partial C}{\partial S} \right] dW_1^P + \left[\eta \frac{\partial C}{\partial \nu} \right] dW_2^P + o(dt) \end{aligned}$$

Risk-neutral measure

- For the sake of clarity we introduce the linear operation $\mathcal{A}$ defined as:

$$\mathcal{A} = \frac{\partial \cdot}{\partial t} + \frac{\nu S^2}{2}\frac{\partial^2 \cdot}{\partial S^2} + \frac{\eta^2}{2}\frac{\partial^2 \cdot}{\partial \nu^2} + \rho\eta\sqrt{\nu}S\frac{\partial^2 \cdot}{\partial S \partial \nu}$$

- The dynamics of the derivative C can then be written more concisely as:

$$dC = \left[\mathcal{A}C + \mu^P S\frac{\partial C}{\partial S} + \kappa^P\frac{\partial C}{\partial \nu}\right]dt + \left[\sqrt{\nu}S\frac{\partial C}{\partial S}\right]dW_1^P + \left[\eta\frac{\partial C}{\partial \nu}\right]dW_2^P + o(dt)$$

Hedging portfolio

- We next construct a portfolio Π which is:
  - Long one unit of a derivative V,
  - Short α units of underlying asset S,
  - Short a MMA deposit of notional β ,
  - Short γ units of derivative C.
- The market value of this portfolio is therefore:

$$\Pi = V - \alpha S - \beta - \gamma C$$

Portfolio dynamics

- The composition of the portfolio Π are locally deterministic and its P&L reads:

$$d\Pi = dV - \alpha dS - d\beta - \gamma dC$$

- After replacing the P&L of the individual assets with their expressions, the P&L becomes:

$$\begin{aligned} d\Pi \quad &= \left[\left(\mathcal{A}V + \mu^P S\frac{\partial V}{\partial S} + \kappa^P\frac{\partial V}{\partial \nu}\right) - \alpha(\mu^P S) - r\beta - \gamma\left(\mathcal{A}C + \mu^P S\frac{\partial C}{\partial S} + \kappa^P\frac{\partial C}{\partial \nu}\right)\right] dt \\ &+ \sqrt{\nu}S\left[\frac{\partial V}{\partial S} - \alpha - \gamma\frac{\partial C}{\partial S}\right] dW_1^P + \eta\left[\frac{\partial V}{\partial \nu} - \gamma\frac{\partial C}{\partial \nu}\right] dW_2^P + o(dt) \end{aligned}$$

Portfolio composition

- The portfolio composition -i.e. the parameters α, β, and γ- are chosen so as to cancel out:
  - The volatility risk dW_2^P:
$$\gamma = \frac{\partial V}{\partial \nu} / \frac{\partial C}{\partial \nu}$$
  - The directional risk dW_1^P:
$$\alpha = \frac{\partial V}{\partial S} - \gamma \frac{\partial C}{\partial S}$$
  - And the theta leak dt:

$$\begin{aligned}
\beta &= \tfrac{1}{r}\left[\left(\mathcal{A}V + \mu^P S \tfrac{\partial V}{\partial S} + \kappa^P \tfrac{\partial V}{\partial \nu}\right) - \alpha\left(\mu^P S\right) - \gamma\left(\mathcal{A}C + \mu^P S \tfrac{\partial C}{\partial S} + \kappa^P \tfrac{\partial C}{\partial \nu}\right)\right] \\
&= \tfrac{1}{r}\left[\left(\mathcal{A}V + \kappa^P \tfrac{\partial V}{\partial \nu}\right) - \gamma\left(\mathcal{A}C + \kappa^P \tfrac{\partial C}{\partial \nu}\right)\right] \\
&= \tfrac{1}{r}\left[\mathcal{A}V - \gamma \cdot \mathcal{A}C\right]
\end{aligned}$$

Portfolio value

- Under this hedging strategy, the P&L is zero.

$$d\Pi = 0$$

- And the market value of the portfolio is preserved in time:

$$\Pi_t = \Pi_0 = 0$$

- Finally, this observation yields:

$$\beta = V - \alpha S - \gamma C$$

Risk-neutral PDE

- Replacing the MMA deposit β by its expression gives:

$$V - \alpha S - \gamma C = \frac{1}{r}\left[\mathcal{A}V - \gamma \cdot \mathcal{A}C\right]$$

- Then replacing α:

$$V - \left(\frac{\partial V}{\partial S} - \gamma \frac{\partial C}{\partial S}\right) S - \gamma C = \frac{1}{r}\left[\mathcal{A}V - \gamma \cdot \mathcal{A}C\right]$$

- Some basic manipulations:

$$\left(rV - \mathcal{A}V - rS\frac{\partial V}{\partial S}\right) = \gamma\left(rC - \mathcal{A}C - rS\frac{\partial C}{\partial S}\right)$$

Risk-neutral PDE

- Replacing γ shows that:

$$\left(rV - \mathcal{A}V - rS\frac{\partial V}{\partial S}\right) / \frac{\partial V}{\partial \nu} = \left(rC - \mathcal{A}C - rS\frac{\partial C}{\partial S}\right) / \frac{\partial C}{\partial \nu}$$

- At this stage it is critical to realize that the choice of the derivatives C and V is arbitrary.
- Therefore one can write that there exists a function κ^Q that depends only on the date t, and the state variables S and ν that verifies:

$$\left(rV - \mathcal{A}V - rS\frac{\partial V}{\partial S}\right) / \frac{\partial V}{\partial \nu} = \kappa^Q$$

  - This function κ^Q does not depend on the derivative but only on the state variables.
  - There is no additional information about κ^Q.
  - In particular, the function κ^Q is not unique.

Risk-neutral PDE

- Eventually, the risk-neutral PDE becomes:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\nu S^2}{2}\frac{\partial^2 V}{\partial S^2} + \kappa^Q\frac{\partial V}{\partial \nu} + \frac{\eta^2}{2}\frac{\partial^2 V}{\partial \nu^2} + \rho\eta\sqrt{\nu}S\frac{\partial^2 V}{\partial S \partial \nu} - rV = 0$$

- Simple manipulations show that a dividend yield q affects only the convection term by replacing r by $r - q$.

Risk-neutral SDE

- Applying the Feynman-Kac theorem transforms the previous PDE into the risk-neutral SDE:
  - Return model:
  $$dS_t/S_t = (r-q)dt + \sqrt{\nu_t}dW_{1t}^Q$$
  - Volatility model:
  $$d\nu_t = \kappa^Q dt + \eta dW_{2t}^Q$$
  - Dependence model:
  $$dW_{1t}^Q dW_{2t}^Q = \rho dt$$
- As expected, the drift of the non-dividend paying stock S is the risk-free rate.

Table of Contents

Market incompleteness

- Market practitioners often relate the empirical and risk-neutral drifts κ^P and κ^Q as:

$$\kappa^Q = \kappa^P + \phi$$

- The function ϕ is interpreted as the market price for the variance risk.
- It is not uniquely defined when $\mid \rho \mid \neq 1$.
- Thus, the risk-neutral measure (i.e. the Arrow-Debreu density) is not uniquely defined.
- This means that the market is incomplete.
- Indeed SV models specify two Wiener processes (when the correlation ρ is ne) but describe only one risky asset S.

BFI Burrus Financial Intelligence

Market incompleteness

- In full rigor, quants should treat the market price of variance risk ϕ like a local volatility and extract its expression from market prices.
- However, practitioners usually make the simplifying assumptions that the function ϕ is a constant:

$$\phi(t, S, \nu) = \phi_0$$

- Quants specify the volatility model $\left(\kappa^Q, \eta\right)$ so that a calibration (e.g. minimizing the mismatch) has the least mismatch with market prices given the aforementioned simplifying assumption.

Remaining market risk

- Traders must realize that under SV models, the market risk cannot be entirely eliminated!
- A good SV model must therefore not only produce good Greeks but also explains the dynamics of the smile.

Hedging considerations

- Under SV models hedging requires:
  - A deposit
  - A directional risky asset such as stock, forward, future.
  - An a convex risky asset with preferably a strong and stable exposure to volatility (i.e. a stable gamma or equivalently a low speed) such as vanillas, straddles, or variance swaps.
- In practice, hedging volatility (that is the gamma risk) requires at least one convex instrument.

Table of Contents

Table of Contents

Heston model

- In 1993, Steven Heston published a SV model under which volatility σ is mean-reverting, persistent, correlated with the asset S, and remains positive.
- Specifically, in the empirical (or risk-neutral) measure, the variance ν follows a Cox-Ingersoll-Ross (CIR) process:

$$d\nu_t = \theta\left(\omega^{P/Q} - \nu_t\right) dt + \xi\sqrt{\nu_t} dW_{2t}^{P/Q}$$

  - θ controls the reversion-speed. The larger, the slower the mean-reversion.
  - ω is the equilibrium variance.
  - ξ is the volatility of the variance.
- The change of measure affects only the equilibrium variance ω which is assumed constant.

Variance properties

- The conditional mean of the variance ν at a date t in the future reads:

$$E_0[\nu_t] = \nu_0 e^{-\theta t} + \omega(1 + e^{-\theta t})$$

  - The mean of the variance reverts to its equilibrium ω within a typical period of $1/\theta$.

- The conditional variance is:

$$Var_0[\nu_t] = \frac{\xi^2}{2\theta}(1 - e^{-\theta t}) \left[2\nu_0 e^{-\theta t} + \omega(1 - e^{\theta t})\right]$$

  - The long term variance of the variance is $\frac{\xi^2 \omega}{2\theta}$.

Variance properties

- More exactly, it can be shown (using characteristic functions) that the scaled variance $2\alpha(t)\nu_t$ conditionally has a non-central χ^2-distribution where:
  - The scaling factor is: $\alpha(t) = \frac{2\theta}{\xi^2(1-e^{-\theta t})}$.
  - The degree of freedom is: $k = 4\theta\omega/\xi^2$
  - The non-centrality parameter is: $\lambda(t) = 2\alpha(t)e^{-\theta t}\nu_0$
- The variance ν is therefore guaranteed to remain positive.
- It is strictly positive only under the Feller condition: $k \geq 2$.

Table of Contents

European vanilla options

- Heston's model is popular because there exists a closed formula for the prices of European vanilla options that considerably simplifies the calibration.
- The formula for a European vanilla call option with strike K and time to maturity T combines two probabilities P_1 and P_2:

$$C_0 = S_0 e^{-qT} P_1 - K e^{-rT} P_2$$

- The put-call parity yields the price of the corresponding put.

European Vanilla options

- The probabilities P_j are the integrals:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_{\phi=0}^{+\infty} Re\left\{\frac{e^{-i\phi \ln K} f_j(\phi|x_0,\nu_0,T)}{i\phi}\right\} d\phi$$

- Where:

$$f_j(\phi|x_0,\nu_0,T) = exp\left[C_j(\phi|T) + D_j(\phi|T)\nu_0 + i\phi x_0\right]$$

$$C_j(\phi|T) = i\phi(r-q)T + \frac{a}{\xi^2}\left[(b_j - i\phi\rho\xi + d_j)T - 2\ln\frac{1-ge^{d_jT}}{1-g}\right]$$

$$D_j(\phi|T) = \left[\frac{b_j - i\phi\rho\xi + d_j}{\xi^2}\right]\left[\frac{1-e^{d_jT}}{1-ge^{d_jT}}\right]$$

$$g_j(\phi) = \frac{b_j - i\phi\rho\xi + d_j}{b_j - i\phi\rho\xi - d_j} \quad d_j(\phi) = \sqrt{(i\phi\rho\xi - b_j)^2 - \xi^2(2i\phi u_j - \phi^2)}$$

$$u_1 = \frac{1}{2} \quad u_2 = -\frac{1}{2} \quad a = \theta\omega \quad b_1 = \theta + \psi - \rho\xi \quad b_2 = \theta + \psi$$

$$\psi = \theta(\omega^P - \omega^Q) \quad x_0 = \ln S_0$$

Table of Contents

Example

- In order to illustrate Heston's model, let's consider the USD-CLP smile as of June 13th 2011.
- The spot price of the USD in CLP was:

$$S_0 = 481.84$$

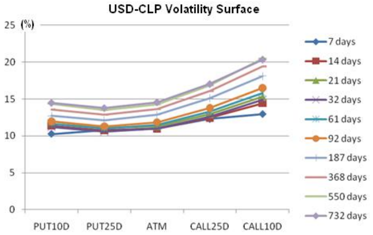
- A calibration yields the parameters:

$$\nu_0 = 0.01435 \approx (0.12)^2$$
$$\theta = 1.584$$
$$\omega = 0.03190$$
$$\xi = 0.6771$$
$$\rho = 0.3795$$

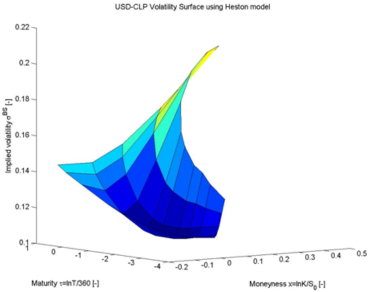
NB: The degree of freedom is:

$$k \sim 0.44 < 2$$

So that the Feller condition is not satisfied.



Example


USD-CLP Volatility Surface using Heston model
Implied volatility σ^BS [-]
Maturity τ=lnT/360 [-]
Moneyness x=lnK/S_0 [-]

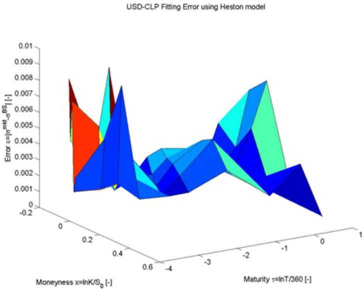

USD-CLP Fitting Error using Heston model
Error ε=|σ^mkt-σ^BS| [-]
Moneyness x=lnK/S_0 [-]
Maturity τ=lnT/360 [-]

Table of Contents

BFI Burrus Financial Intelligence

Table of Contents

Sabr model

- The Sabr model observes that the forward price (of let's say the EUR-CLP pair) $F_t(T^\star)$ at date t with maturity $T^\star$ is the ratio of the prices of two assets:

$$F_t(T*) = \frac{S_t D_t^{EUR}(T^\star)}{D_t^{CLP}(T^\star)}$$

  - $D_t^{CLP}(T^\star)$ is the current price in CLP of a Chilean pure discount bond with maturity $T^\star$.
  - $S_t D_t^{EUR}(T^\star)$ is the current price in CLP of a European pure discount bond with maturity $T^\star$.
- Therefore, the forward price $F_t(T^\star)$ is a martingale in the Equivalent Martingale Measure (EMM) using the Chilean pure discount bond with maturity $T^\star$ as a numeraire.
  - This measure is called the $T^\star$-forward measure.
  - The $T^\star$-forward measure is an EMM different from the risk-neutral measure.
- Sabr assumes the dynamics directly in the $T^\star$-forward measure.
- In fact, practioners also apply the Sabr model to fixed-income.

Forward price dynamics

- The dynamics of the forward price under Sabr are:

$$dF_t(T^\star) = \sigma_t F_t(T^\star)^\beta dW^\star_{1t}$$

- The volatility state variable σ is governed by:

$$d\sigma_t = \alpha\sigma_t dW^\star_{2t}$$

- Both Wiener processes are linearly related:

$$dW^\star_{1t} dW^\star_{2t} = \rho dt$$

- NB: The model draws its name from the parameters σ, α, β, ρ.

Spot price dynamics

- The student may feel more comfortable rewritting the model of the return in terms of the spot price S_t:

$$dS_t/S_t = (r-q)dt + \sigma_t \left[D_t^{CLP}(T^\star)/D_t^{EUR}(T^\star) \right]^{1-\beta} S_t^{\beta-1} dW_{1t}^\star$$

- Note that the instantaneous volatility of the spot S looks like a stochastic Constant Elasticity Volatility (CEV).
- The discount factor remain because the dynamics are written in the $T^\star$-forward measure.
- Crucially one cannot value a derivative with maturity greater than $T^\star$.
  - This is a reason why the risk-neutral measure is usually preferred.

Parameters

- The state variable σ controls the volatility of the spot price S.
- The parameter α is the volatility of the volatility state variable σ and is often called the volvol.
- The parameter β defines the elasticity or the skewness of the volatility of the spot price S.
- The parameter ρ is the instantaneous correlation between the spot return dS/S and the volatility variable σ.

Table of Contents

European vanilla options

- There exists an approximation for the Black-implied volatility for European options with strike K and maturity $T = T^\star$.
    - This implied volatility simplifies the calibration of the model.
    - But it restricts the maturities to $T^\star$.
- The first-order proxy (in $\epsilon = \alpha^2 T^\star$) for the Black-implied volatility reads:

$$\sigma^{Black}_{Sabr}(K, T^\star | F_0(T^\star), \sigma_0) = \alpha \frac{\ln F_0/K}{D(\zeta)} \left\{ 1 + \left[\frac{2\gamma_2 - \gamma_1^2 + 1/\chi}{24} \left(\frac{\sigma_0 C(\chi)}{\alpha} \right)^2 + \frac{\rho\gamma_1}{4} \frac{\sigma_0 C(\chi)}{\alpha} + \frac{2 - 3\rho^2}{24} \right] \epsilon + o(\epsilon) \right\}$$

Where

$$\chi = \sqrt{KF_0} \quad \zeta = \frac{\alpha}{\sigma_0(1-\beta)}(F_0^{1-\beta} - K^{1-\beta})$$

$$\gamma_1 = \frac{\beta}{\chi} \quad \gamma_2 = -\frac{\beta(1-\beta)}{\chi^2}$$

$$C(x) = x^\beta \quad D(x) = \ln \frac{\sqrt{1-2\rho x+x^2}+x-\rho}{1-\rho}$$

Table of Contents

BFI Burrus Financial Intelligence

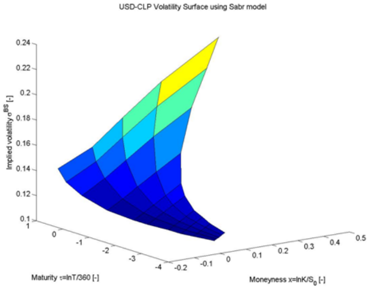
Example

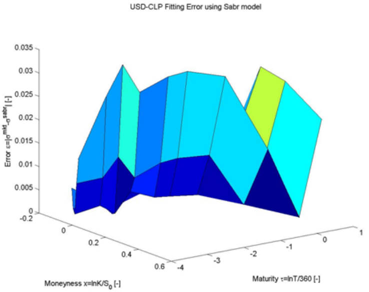
- A calibration of the Sabr model onto the same smile gives the parameters:

$$\sigma_0 = 83.03$$
$$\alpha = 0.4414$$
$$\beta = -0.0703$$
$$\rho = 0.8182$$

- Note that σ_0 is simply a state variable related to the volatility.
- The variable $\tilde{\sigma}_0 = \sigma_0 S_0^{\beta-1}$ has the order of magnitude of a volatility.
  - This is a good check for the validity of the calibration.
  - In this example: $\tilde{\sigma}_0 = 0.11$.
- NB: This calibration made the simplifying (and most probably incorrect) assumption that the parameters do not depend on the forward measure $T^\star$.

Example


USD-CLP Volatility Surface using Sabr model
Implied volatility σ^{BS} [-]
Maturity τ=lnT/360 [-]
Moneyness x=lnK/S_0 [-]


USD-CLP Fitting Error using Sabr model
Error $\varepsilon=|\sigma^{mkt}-\sigma^{sabr}|$ [-]
Moneyness x=lnK/S_0 [-]
Maturity τ=lnT/360 [-]