International Hedge of Fixed-Income Contracts

Replication of domestic immunization may be too costly.

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The literature on the role of duration and convexity in managing the interest rate risk of fixed-income contracts is concerned primarily with single-currency domestic applications. The dominant view seems to be that fixed-income risk management in the international setting involves a mere replicating of domestic immunization according to Redington [1952] and Bierwag, Kaufman, and Toevs [1983]; see, for example, Thomas and Willner [1997].

We argue that interest risk reduction by separate duration matching of assets and liabilities in each country is generally not optimal, and it is unduly costly to implement. The pursuit of this strategy is also likely to raise the cost of hedging against currency risk.

We extend the domestic, single-currency immunization methodology to the international setting.[1] Based on reasonably weak assumptions, our dedicated international immunization model yields a hedging strategy that is less restrictive, more economical, and possibly more efficient than a mere replication of domestic immunization. The strategy requires that duration be matched only between the overall portfolios of assets and liabilities held in a domestic and foreign country, rather than between assets and liabilities held in each country.

THEORETICAL BACKGROUND

Consider two portfolios, assets and liabilities held by a U.S.-based investor, each portfolio including two currencies, U.S. dollars and a foreign currency. For further simplicity, assume in each country a flat yield curve subject only to parallel shifts. Findings reported in the literature strongly suggest a negligible adverse effect of these restrictions on the quality of immunization.

Bierwag, Kaufman, and Latta [1987], Reitano [1992], and Carcano and Foresi [1997] show that, even when the yield curve's slope is taken into account, immunization is often defeated by the stochastic nature of interest rates coupled with frequent changes in the shape of the yield curve itself. Furthermore, Balbás and Ibáñez [1998] show that simple duration-matching methods work well even when shifts of the term structure of interest rates are not parallel.

A further justification for our simplifying assumptions is provided by Lee and Cho [1992] and Reitano [1996] who show that a more general treatment intended to immunize against a wider class of shifts would have the downside of more restrictive necessary conditions. Immunization against the parallel shifts assumed here requires constraints on the modified duration of assets and liabilities and on convexities.

Denote the value of assets and liabilities held in the two currencies by

$$A_d = \sum_{i=1}^{n} A_{id}(1+r_d)^{-i} \tag{1}$$

$$A_f = \sum_{i=1}^{n} A_{if}(1+r_f)^{-i} \tag{2}$$

$$L_d = \sum_{i=1}^{n} L_{id}(1+r_d)^{-i} \tag{3}$$

$$L_f = \sum_{i=1}^{n} A_{if}(1+r_f)^{-i} \tag{4}$$

where A_{id}, L_{id}, A_{if}, and L_{if} are the receipts and payments in the i-th period (i = 1, 2, ..., n) denominated in the domestic (dollars) and foreign currency, respectively. Total assets are represented by $A = A_d + SA_f$ and total liabilities by $L = L_d + SL_f$, where S is the fixed exchange rate of dollars per unit of foreign currency. Without loss of generality, assume that the net worth, $N = A - L$, is initially zero.

Based on Macaulay's [1938] definition, the duration of assets and liabilities held in each country is:

$$D_{A_d} = \sum_{i=1}^{n} iA_{id}(1+r_d)^{-i} / A_d \tag{5}$$

$$D_{A_f} = \sum_{i=1}^{n} iA_{if}(1+r_f)^{-i} / A_f \tag{6}$$

$$D_{L_d} = \sum_{i=1}^{n} iL_{id}(1+r_d)^{-i} / L_d \tag{7}$$

$$D_{L_f} = \sum_{i=1}^{n} iL_{if}(1+r_f)^{-i} / L_f \tag{8}$$

and the respective convexity of these holdings is:

$$C_{A_d} = \tfrac{1}{2}\frac{1}{(1+r_d)^2}\sum_{i=1}^{n} i(i+1)A_{id}(1+r_d)^{-i} / A_d \tag{9}$$

$$C_{A_f} = \tfrac{1}{2}\frac{1}{(1+r_f)^2}\sum_{i=1}^{n} i(i+1)A_{if}(1+r_f)^{-i} / A_f \tag{10}$$

$$C_{L_d} = \tfrac{1}{2}\frac{1}{(1+r_d)^2}\sum_{i=1}^{n} i(i+1)L_{id}(1+r_d)^{-i} / L_d \tag{11}$$

$$C_{L_f} = \tfrac{1}{2}\frac{1}{(1+r_f)^2}\sum_{i=1}^{n} i(i+1)L_{if}(1+r_f)^{-i} / L_f \tag{12}$$

The overall durations of total assets and total liabilities held in both countries are:

$$D_A = \frac{A_d}{A}D_{A_d} + \frac{SA_f}{A}D_{A_f} \tag{13}$$

$$D_L = \frac{L_d}{L}D_{L_d} + \frac{SL_f}{L}D_{L_f} \tag{14}$$

Finally, from (13) and (14), the investor's net worth is:

$$N = A - L = (A_d + SA_f) - (L_d + SL_f) \qquad (15)$$

initiated at N = 0. Substitution of (1)-(4) in (15) yields:

$$N = \left[\sum_{i=1}^{n} A_{id}(1+r_d)^{-i} + S\sum_{i=1}^{n} A_{if}(1+r_f)^{-i}\right] - \left[\sum_{i=1}^{n} L_{id}(1+r_d)^{-i} + S\sum_{i=1}^{n} L_{if}(1+r_f)^{-i}\right] \qquad (16)$$

Equation (16) is used to derive necessary and sufficient conditions for immunizing the two-currency portfolio.

Theorem: A two-currency portfolio is simultaneously immunized against foreign and domestic interest rate risk if conditions as follows hold:

1. First-order: The average duration of total assets held in both currencies equals that of total liabilities ($D_A = D_L$).
2. First-order: The value of foreign assets held equals that of foreign liabilities ($A_f = L_f$).
3. Second-order: The convexity of assets is greater than that of liabilities in either currency ($C_{A_d} > C_{L_d}$ and $C_{A_f} > C_{L_f}$).

A comprehensive proof is provided in the appendix.

The first-order condition (1) of simultaneous immunization in both currencies stipulates that the duration of total assets held in the two currencies be set equal to the duration of total liabilities. This condition of dedicated international immunization is clearly less onerous than the corresponding condition of replicated domestic immunization, that duration of assets and liabilities be matched in *each currency*. Since the first-order condition (2) and second-order condition (3) are the same for both immunization strategies, we conclude that the conditions of our dedicated international immunization are less restrictive and therefore less costly to meet than those of replicated domestic immunization.

SIMULATION

We use simulation to examine the practical implications of the first- and second-order conditions of international immunization, and compare the performance of the two competing strategies. Recall that replication of domestic immunization in a two-currency setting requires that in each currency 1) the duration of assets be equal to that of liabilities, and 2) the convexity of assets be greater than that of liabilities. Our strategy requires by contrast that 1) across currencies, the duration of total assets be equal to that of liabilities, and 2) in each currency, the convexity of assets be greater than that of liabilities.

Data

Our interest rate data cover the period from January 1984 through April 1998, including daily observations of Euro rates for three and six months, and one, two, three, and five years in the United States, Britain, Germany, and Switzerland.[2] Also included are spot exchange rates of these countries against the U.S. dollar. Exhibit 1 provides descriptive statistics of these data.

Methodology

We start by testing the assumptions underlying our conditions of two-currency immunization by setting as null hypotheses the assumption $dS(1 + r_d) - dF(1 + r_f) \cong 0$ for the first-order condition, and the assumption $dSdr_f \cong 0$ for the second-order condition. The results displayed in Exhibit 2 empirically confirm both assumptions in failing to reject either hypothesis at the 1% level for the three country pairs in all maturities.

We simulate portfolios comprising assets and liabilities held in each of the four currencies and the three currency pairs including U.S. dollars. The initial portfolios are constructed to satisfy the first- and second-order conditions. Under either immunization strategy, the initial value of assets in each currency is set equal to the value of liabilities ($A_d = L_d$ and $A_f = L_f$). For replicated domestic immunization, the duration of assets in each currency is set equal to that of liabilities ($D_{A_d} = D_{L_d}$ and $D_{A_f} = D_{L_f}$). For dedicated international immunization, the duration of total assets across currencies is set equal to that of total liabilities ($D_A = D_L$).

Under either strategy, the convexity of assets is greater than that of liabilities in each currency. Six dates are arbitrarily chosen for rebalancing the portfolios to satisfy the first- and second-order conditions.[3]

Immunization results based on the two competing models are reported in Exhibits 3, 4, and 5. All but the last column list the present value, duration, and convexity of assets and liabilities immediately after portfolio rebalancing. The last column in each exhibit shows the value of assets, liabilities, and profit at maturity. The last rows show the immunization profit in each of the six peri-

EXHIBIT 1
DESCRIPTIVE STATISTICS

	Mean	S.D.	Median	Minimum	Maximum
Exchange Rates					
US$/BP	1.4988	0.1766	1.500	1.0523	2.0030
US$/DM	0.4657	0.1219	0.4139	0.2887	0.7399
US$/SF	0.5550	0.1399	0.4922	0.3411	0.8961
Interest Rates					
U.S.					
3-month	0.07932	0.02239	0.08125	0.03000	0.12250
6-month	0.08118	0.02324	0.08250	0.03063	0.12813
1-year	0.08437	0.02413	0.08563	0.03188	0.13625
2-year	0.09085	0.02486	0.09250	0.03813	0.14125
3-year	0.09413	0.02466	0.09688	0.04188	0.14250
5-year	0.09840	0.02442	0.10000	0.04750	0.14625
U.K.					
3-month	0.09974	0.02357	0.10000	0.04938	0.15250
6-month	0.09950	0.02247	0.10000	0.04937	0.15500
1-year	0.10068	0.02139	0.10250	0.04937	0.15813
2-year	0.10283	0.01899	0.10750	0.05000	0.15125
3-year	0.10391	0.01788	0.10875	0.05250	0.14750
5-year	0.10589	0.01710	0.11250	0.05750	0.14500
Germany					
3-month	0.05507	0.01503	0.05500	0.02960	0.09750
6-month	0.05615	0.01505	0.05563	0.03000	0.09750
1-year	0.05758	0.01493	0.05688	0.03040	0.09750
2-year	0.06236	0.01401	0.06250	0.03375	0.09563
3-year	0.06562	0.01305	0.06625	0.03813	0.09438
5-year	0.06994	0.01133	0.07000	0.04125	0.09250
Switzerland					
3-month	0.10628	0.04279	0.10375	0.03170	0.40000
6-month	0.10857	0.04056	0.10625	0.03150	0.28000
1-year	0.10976	0.03980	0.10875	0.03150	0.22000
2-year	0.11092	0.03794	0.11125	0.03375	0.18500
3-year	0.11137	0.03686	0.11250	0.03750	0.18000
5-year	0.11218	0.03480	0.11250	0.04469	0.18000

US$/BP, US$/DM, and US$/SF represent exchange rates of the U.S. dollar against the British pound, German mark, and Swiss franc, respectively.

EXHIBIT 2
TESTING THE MODEL'S ASSUMPTIONS

	Interest Rates by Maturity					
Currency	**3 Months**	**6 Months**	**1 Year**	**2 Years**	**3 Years**	**5 Years**
	Assumption for First-Order Condition: $dS(1 + r_d) - dF(1 + r_f) \cong 0$					
US$/BP	0.27*	0.56*	0.38*	0.66*	0.55*	-0.15*
US$/DM	0.08*	0.09*	0.07*	0.23*	0.32*	0.58*
US$/SF	-6.25*	-6.15*	-5.24*	-4.07*	-3.61*	-2.85*
	Assumption for Second-Order Condition: $dSdr_f \cong 0$					
US$/BP	-1.91*	-14.24*	-11.60*	-10.22*	-2.86*	-3.68*
US$/DM	-1.61*	-1.55*	-1.45*	-1.88*	-2.12*	-2.10*
US$/SF	-15.75*	-14.24*	-11.60*	-10.22*	-9.73*	-8.82*

S and F represent the foreign exchange rate and the forward rate, respectively. Numbers are multiplied by 100,000.
*T-statistic does not reject the null hypothesis at the 1% significance level.

EXHIBIT 3
SIMULATION OF AN INTERNATIONAL PORTFOLIO: U.S. AND SWITZERLAND

	Portfolio Rebalancing Date						
	02/06/84	05/16/84	10/01/85	02/06/86	12/03/86	02/25/87	03/11/87
US$/SF	0.4531	0.4425	0.4569	0.4952	0.6055	0.6489	0.6408
r_d	0.1176	0.1377	0.0950	0.0839	0.0612	0.0631	0.0643
r_f	0.1551	0.1529	0.1301	0.1170	0.1000	0.0838	0.0838
Present Value							
US Assets (US$)	70,892	70,244	88,899	92,881	99,800	101,185	101,423
US Liabilities (US$)	70,892	69,508	87,739	91,592	98,419	99,765	100,000
Swiss Assets (SF)	128,025	133,906	167,680	177,277	194,949	199,384	200,000
Swiss Liabilities (SF)	128,025	133,906	167,680	177,277	194,949	199,384	200,000
Total Assets (US$)	128,900	129,497	165,512	180,668	217,842	230,565	229,583
Total Liabilities (US$)	128,900	128,761	164,352	179,379	216,461	229,146	228,160
Duration							
US Assets	2.5879	2.8968	1.1655	1.0269	0.2581	0.0384	
US Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Swiss Assets	3.7106	2.7272	1.7609	1.1576	0.2773	0.0384	
Swiss Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Total Assets	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Total Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Convexity							
US Assets	5.0729	4.8774	1.4719	0.9752	0.1563	0.0176	
US Liabilities	5.0679	4.1591	1.4669	0.9702	0.1513	0.0176	
Swiss Assets	6.9894	4.3568	2.2562	1.1209	0.1649	0.0170	
Swiss Liabilities	4.7442	4.0501	1.3773	0.9136	0.1408	0.0170	
Immunization Profit (US$)							
International	0.00	736.11	1,159.98	1,288.89	1,381.28	1,419.62	1423.00
Domestic	0.00	22.87	138.16	157.71	187.09	193.88	193.34

EXHIBIT 4
SIMULATION OF AN INTERNATIONAL PORTFOLIO: U.S. AND U.K.

	Portfolio Rebalancing Date						
	02/06/84	05/16/84	10/01/85	02/06/86	12/03/86	02/25/87	03/11/87
US$/BP	1.4265	1.3974	1.4055	1.3895	1.4310	1.5400	1.5955
r_d	0.1176	0.1377	0.0950	0.0839	0.0612	0.0631	0.0643
r_f	0.1089	0.1090	0.1092	0.1205	0.1113	0.1031	0.0980
Present Value							
US Assets (US$)	70,892	70,171	88,835	92,243	99,052	100,418	100,654
US Liabilities (US$)	70,892	69,508	87,739	91,592	98,419	99,765	100,000
British Assets (BP)	145,274	149,417	172,245	176,670	194,415	199,249	200,000
British Liabilities (BP)	145,274	149,417	172,245	176,670	194,415	199,249	200,000
Total Assets (US$)	278,125	278,966	330,925	337,727	377,260	407,261	419,754
Total Liabilities (US$)	278,125	278,303	329,829	337,075	376,627	406,608	419,100
Duration							
US Assets	2.5879	2.8698	1.1655	1.0304	0.2581	0.0384	
US Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
British Assets	3.2660	2.8059	1.5422	1.1130	0.2722	0.0384	
British Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Total Assets	3.0931	2.8219	1.4411	1.0904	0.2685	0.0384	
Total Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Convexity							
US Assets	10.1458	9.6266	2.9438	1.9618	0.3125	0.0352	
US Liabilities	10.1358	8.3181	2.9338	1.9403	0.3025	0.0352	
British Assets	13.2434	9.8120	4.0172	2.0902	0.3129	0.0327	
British Liabilities	10.2964	8.7550	2.8591	1.8156	0.2758	0.0327	
Immunization Profit (US$)							
International	0.25	662.87	1,096.70	651.59	633.36	652.63	654.00
Domestic	0.00	22.65	116.79	139.96	159.33	163.36	164.74

ods measured by the difference between the incremental value of assets and liabilities. Our immunization strategy, as stipulated by the three conditions of the underlying theorem, is designed to hedge against losses and possibly generate profit in the presence of changing interest rates.

The results show that both immunization strategies generate profit in the three binational cases in all six study periods. In all cases our dedicated international strategy consistently outperforms the domestic strategy, generating profits that are at least four times greater. These results understate the overall advantage of our strategy by ignoring its greater flexibility and lower direct and indirect costs.

Based on real data, this preliminary evidence suggests the robustness of our model despite the simplifying assumptions inherent in Macaulay's duration. Successful immunization based on the assumption of parallel shifts in the term structure is reported by Balbás and Ibáñez [1998] and several others.

CONCLUDING REMARKS

The traditional approach of replicating domestic immunization to control interest rate risk in the international setting is expensive. We derive a more efficient immunization strategy in this setting. Under this strategy, international firms match the duration of the overall portfolios of assets and liabilities, and set the convexity of assets held in each currency to be greater than that of liabilities. This strategy avoids the stringent and often costly requirement of separately matching the duration of assets and liabilities in each currency, and has the added

EXHIBIT 5
SIMULATION OF AN INTERNATIONAL PORTFOLIO: U.S. AND GERMANY

	Portfolio Rebalancing Date						
	02/06/84	05/16/84	10/01/85	02/06/86	12/03/86	02/25/87	03/11/87
US$/DM	0.3645	0.3654	0.3739	0.4191	0.5048	0.5469	0.5384
r_d	0.1176	0.1377	0.0950	0.0839	0.0612	0.0631	0.0643
r_f	0.0776	0.0777	0.0608	0.0519	0.0527	0.0400	0.0607
Present Value							
US Assets (US$)	70,892	70,171	88,743	92,627	99,412	100,788	101,025
US Liabilities (US$)	70,892	69,508	87,739	91,592	98,419	99,765	100,000
German Assets (DM)	158,704	161,956	183,688	189,266	197,262	199,699	200,000
German Liabilities (DM)	158,704	161,956	183,688	189,266	197,262	199,699	200,000
Total Assets (US$)	128,740	129,350	157,424	171,949	198,990	210,004	208,705
Total Liabilities (US$)	128,740	128,686	156,420	170,913	197,997	208,981	207,680
Duration							
US Assets	2.5879	2.8519	1.1823	1.0337	0.2581	0.0384	
US Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
German Assets	3.7123	2.7790	1.7755	1.1566	0.2789	0.0384	
German Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Total Assets	3.0931	2.8186	1.4411	1.0904	0.2685	0.0384	
Total Liabilities	3.0932	2.8192	1.4411	1.0904	0.2685	0.0384	
Convexity							
US Assets	10.1458	9.5415	2.9962	1.9725	0.3125	0.0352	
US Liabilities	10.1358	8.3181	2.9338	1.9403	0.3025	0.0352	
German Assets	16.0696	10.2469	5.1699	2.5244	0.3635	0.0368	
German Liabilities	10.9022	9.2701	3.1261	2.0601	0.3073	0.0368	
Immunization Profit							
International	0.00	663.28	1,004.61	1,035.70	993.26	1,022.69	1025.00
Domestic	0.00	22.65	128.61	143.54	162.97	167.13	166.62

benefit of greater flexibility in managing side-by-side interest and currency risks.

Simulations based on actual transaction data reveal that the proposed immunization strategy outperforms the traditional replicated domestic immunization. Although our model relies on Macaulay's duration and does not recognize sloping yield curves and non-parallel shifts of those curves, it performs as well as the costlier and more rigid model based on domestic immunization.

Further research is needed to explore potential benefits from more complicated models that would account for more complex yield curves and their shifts, stochastic interest rates, internationally diversified portfolios, and default risk.

APPENDIX
PROOF OF THEOREM

First-Order Conditions

First-order conditions are attained by expanding (16) as a first-order Taylor series, assuming N is a function of S, r_d, and r_f. According to definitions in the text, the expansion of the first-order condition yields:

$$dN = -\sum_{i=1}^{n} iA_{id}(1+r_d)^{-i-1}dr_d + dS\sum_{i=1}^{n} A_{if}(1+r_f)^{-i} - \sum_{i=1}^{n} iA_{if}(1+r_f)^{-i-1}dr_f + \sum_{i=1}^{n} iL_{id}(1+r_d)^{-i-1}dr_d - dS\sum_{i=1}^{n} L_{if}(1+r_f)^{-i} - \sum_{i=1}^{n} iL_{if}(1+r_f)^{-i-1}dr_f \quad \text{(A-1)}$$

We assume that $dr_d \to 0$, $dr_f \to 0$, and $dS \to 0$, so that $\rho = \sqrt{(dr_d)^2 + (dr_f)^2 + (dS)^2} \to 0$, and substitute (5)-(8) into (A-1), so that the change in the firm's net worth is:

$$dN = (1+r_d)^{-1}dr_d[-A_dD_{Ad} + L_dD_{Ld}] + S(1+r_f)^{-1}dr_f[-A_fD_{Af} + L_dD_{Lf}] + dS(A_f - L_f) \quad \text{(A-2)}$$

Since A = L, reorganization of (A-2) yields:

$$dN = (1+r_d)^{-1}dr_dA[-(A_d / A)D_{Ad} + (L_d / A)D_{Ld}] + S(1+r_f)^{-1}dr_fA[-(A_f / A)D_{Af} + (L_d / A)D_{Lf}] + dS(A_f - L_f) \quad \text{(A-3)}$$

We further assume that covered interest rate parity holds, i.e., $S(1 + r_d) = F(1 + r_f)$, where F represents the forward rate, and differentiate (A-3) on the assumptions $dr_d \to 0$, $dr_f \to 0$, $dS \to 0$, and $dF \to 0$, to obtain:

$$dS(1+r_d) + Sdr_d = dF(1+r_f) + Fdr_f \quad \text{(A-4)}$$

and

$$dr_fS(1+r_f)^{-1} = dr_fF(1+r_d)^{-1} = [dS(1+r_d) + Sdr_d - dF(1+r_f) + Fdr_f](1+r_d)^{-1} \quad \text{(A-5)}$$

Substitution of (A-5) in (A-3) yields

$$dN = (1+r_d)^{-1}dr_dA\{[-(A_d / A)D_{Ad} + (L_d / A)D_{Ld}] + S[-(A_f / A)D_{Af} + (L_f / A)D_{Lf}]\} + [dS(1+r_d) - dF(1+r_f)](1+r_d)^{-1} \times A[-(A_f / A)D_{Af} + (L_f / A)D_{Ld}] + S(A_f - L_f) \quad \text{(A-6)}$$

Finally, substitution of Macaulay's definition of duration using (13) and (14) into (A-6) yields:

$$dN = (1+r_d)^{-1}dr_dA(-D_A + D_L) + [dS(1+r_d) - dF(1+r_f)](1+r_d)^{-1} \times A[-(A_d / A)D_{Ad} + (L_d / A)D_{Ld}] + S(A_f - L_f) \quad \text{(A-7)}$$

Equation (A-7) enables us to prove the theorem. A sufficient first-order condition for solving (A-7) is $dS(1 + r_d) - dF(1 + r_f) \cong 0$. Namely, interest rate risk is significantly greater than currency risk.[4] The necessary conditions required to solve (A-7) are $D_A = D_L$ and $A_f = L_f$.

Second-Order Conditions

Second-order conditions are attained by expanding (16) as a second-order Taylor series. According to definitions in the text, the expansion of the second-order condition yields:

$$dN = dSN_s + Nr_ddr_d + Nr_fdr_f + \tfrac{1}{2}d^2SN_{ss} + \tfrac{1}{2}d^2r_dN_{r_dr_d} + \tfrac{1}{2}d^2r_fN_{r_fr_f} + dSdr_dN_{Sr_d} + dSdr_fN_{Sr_f} + dr_ddr_fN_{r_dr_f} \quad \text{(A-8)}$$

If $D_A = D_L$ and $A_f = L_f$ (namely, the first-order differential of N equals zero), (A-8) becomes:

$$dN = \tfrac{1}{2}d^2SN_{ss} + \tfrac{1}{2}d^2r_dN_{r_dr_d} + \tfrac{1}{2}d^2r_fN_{r_fr_f} + dSdr_dN_{Sr_d} + dSdr_fN_{Sr_f} + dr_ddr_fN_{r_dr_f} \quad \text{(A-9)}$$

where

$$N_{SS} = 0 \tag{A-10}$$

$$N_{r_d r_d} = \frac{\partial^2 (A_{r_d} - L_{r_d})}{\partial r_d r_d} = \sum_{i=1}^{n} i(i+1)A_{id}(1+r_d)^{-i-2} - \sum_{i=1}^{n} i(i+1)L_{id}(1+r_d)^{-i-2} \tag{A-11}$$

$$N_{r_f r_f} = \frac{\partial^2 (A_{r_f} - L_{r_f})}{\partial r_f r_f} = \sum_{i=1}^{n} i(i+1)A_{if}(1+r_f)^{-i-2} - \sum_{i=1}^{n} i(i+1)L_{if}(1+r_f)^{-i-2} \tag{A-12}$$

$$N_{Sr_d} = 0 \tag{A-13}$$

$$N_{Sr_f} = -\sum_{i=1}^{n} iA_{if}(1+r_f)^{-i-1} + \sum_{i=1}^{n} iL_{if}(1+r_f)^{-i-1} \tag{A-14}$$

$$N_{r_d r_f} = 0 \tag{A-15}$$

Substitution of (A-10)-(A-15) in (A-9) yields:

$$dN = \tfrac{1}{2} d^2 r_d \left[\sum_{i=1}^{n} i(i+1)A_{id}(1+r_d)^{-i-2} - \sum_{i=1}^{n} i(i+1)L_{id}(1+r_d)^{-i-2} \right] + \tfrac{1}{2} d^2 r_d \left[\sum_{i=1}^{n} i(i+1)A_{if}(1+r_f)^{-i-2} - \sum_{i=1}^{n} i(i+1)L_{if}(1+r_f)^{-i-2} \right] + dSdr_f \left[-\sum_{i=1}^{n} i(i+1)A_{if}(1+r_f)^{-i-1} + \sum_{i=1}^{n} i(i+1)L_{if}(1+r_f)^{-i-1} \right] \tag{A-16}$$

Based on (9)-(12) and the first-order conditions, (A-16) is rewritten as:

$$dN = \frac{A_d}{(1+r_d)^2} d^2 r_d [C_{A_d} - C_{L_d}] + \frac{A_f}{(1+r_f)^2} d^2 r_f [C_{A_f} - C_{L_f}] + dSdr_f \left[-\sum_{i=1}^{n} i(i+1)A_{if}(1+r_f)^{-i-1} + \sum_{i=1}^{n} i(i+1)L_{if}(1+r_f)^{-i-1} \right] \tag{A-17}$$

Sufficient second-order conditions required to ensure $dN \geq 0$ are obtained if the three terms in (A-17) are positive, or if the first and second terms are positive and greater than the absolute value of the third term. Following Hauser, Levy, and Yaari [1997], it is further assumed that $dSdr_f \cong 0$, so that the two-currency portfolio is immunized if in each currency the convexity of assets held is greater than that of liabilities ($C_{A_d} > C_{L_d}$ and $C_{A_f} > C_{L_f}$).

ENDNOTES

[1]A less comprehensive and less rigorous treatment is offered by Hauser, Levy, and Yaari [1997].

[2]The data are obtained from Israel's central bank. Missing interest rates for some maturities are calculated by interpolating neighboring rates. For example, the five-month spot rate is calculated by interpolating the contemporaneous three-month and six-month rates.

[3]The immunization strategy is designed to ensure that the total value of a portfolio of assets at the end of a specified period be equal to the expected value of the portfolio of liabilities. This suggests that the immunized portfolio must be frequently rebalanced if interest rates frequently change. Lee and Cho [1992] show that, in the presence of transaction costs, the immunization strategy works well even if the portfolio is rebalanced less frequently.

[4]The solution to (A-7) does not change by assuming instead that $Sdr_d \cong Fdr_f$ (that currency risk is significantly greater than interest rate risk). It can be shown that the same solution is attained by assuming $dr_f = Bdr_d$, where B denotes some measure of interest rate risk differential.

REFERENCES

Balbás, A. and A. Ibáñez. "When Can You Immunize a Bond Portfolio?" *Journal of Banking and Finance*, 22 (1998), pp. 1571-1596.

Bierwag, G.O., G.G. Kaufman, and C. Latta. "Bond Portfolio Immunization: Tests of Maturity, One and Two-Factor Duration Matching Strategies." *The Financial Review*, 22 (1987), pp. 203-219.

Bierwag, G.O., G.G. Kaufman, and A. Toevs. "Immunization Strategies for Funding Multiple Liabilities." *Journal of Financial and Quantitative Analysis*, 18 (1983), pp. 113-124.

Carcano, N., and S. Foresi. "Hedging Against Interest Rate Risk: Reconsidering Volatility-Adjusted Immunization." *Journal of Banking and Finance*, 21 (1997), pp.127-141.

Hauser, S., A. Levy, and U. Yaari. "Immunization Strategy for Multinational Fixed-Income Investments." In G. Klopfenstein, ed., *FX: Managing Global Currency Risk*. Chicago: Glenlake Publishing, 1997.

Lee, S.B., and H.Y. Cho. "A Rebalancing Discipline for an Immunization Strategy." *The Journal of Portfolio Management*, 18 (4) (Summer 1992), pp. 56-62.

Macaulay, F.R. *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields, and Stock Prices in the U.S. Since 1856*. New York: National Bureau of Economic Research, 1938.

Redington, F.M. "Review of the Principle of Life Office Valuations." *Journal of the Institute of Actuaries*, 18 (1952), pp. 286-340.

Reitano, R.R. "Non-Parallel Yield Curve Shifts and Immunization." *The Journal of Portfolio Management*, 18 (3) (Spring 1992), pp. 36-43.

——. "Non-Parallel Yield Curve Shifts and Immunization." *The Journal of Portfolio Management*, 22 (2) (Winter 1996), pp. 71-78.

Thomas, L. and R. Willner. "Measuring the Duration of an Internationally Diversified Bond Portfolio." *The Journal of Portfolio Management*, 24 (1) (Fall 1997), pp. 93-99.