Final Report of the Inverse Problem

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1. Introduction

Inverse problems are known as the process of Finding out unknown information from external or indirect measurement through the model of the system. In this report, we mainly focus on the heat transfer inverse problem. In industrial production process, iron ore is chemically reacted with the fuels(coal for example) under high temperature and pressure with a molten at 1500℃. With the aim of maintain stable operation and better output, a good control of inside heat and pressure is very necessary. However, because the internal temperature is quite high may reach over 2000℃, using normal direct measurement is impossible.

One potential solution is by using inverse problems and data assimilation to solve this kind of issue[1]. The heat flux flowing into the inner surface of the refractory of molten iron can be calculated using 1D heat transfer model(shown at Fig1.1-a) and time series data of the refractory temperature(shown at Fig1.1-b) measured by the thermocouple. By using this model and data, it is possible to observe the changes in the heat flux which is difficult to measure only by checking the value measured by the thermometer. And this makes it possible to diagnose internal blast furnace phenomena.

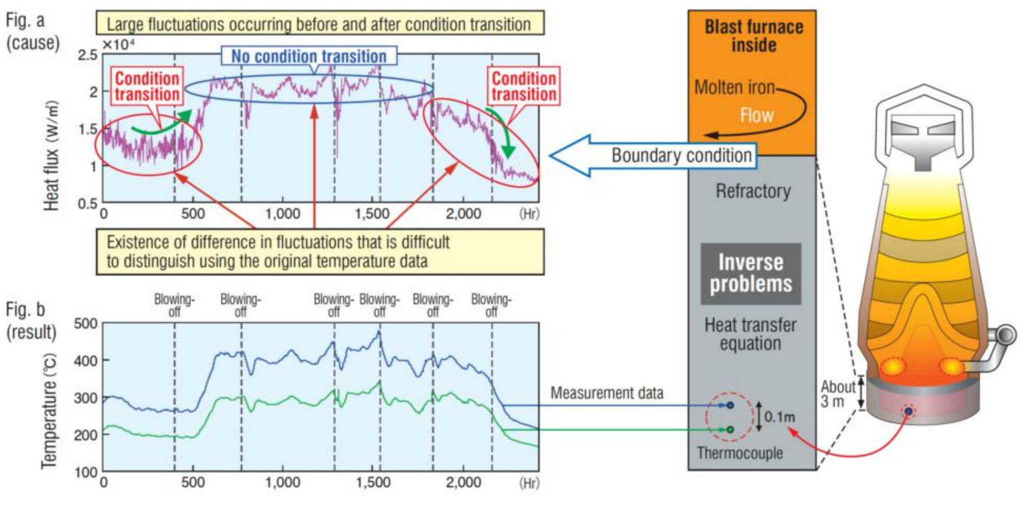


Figure 1.1 Inverse Problem Approach Being Applied to Heat Transfer in Blast-furnace

2. Inverse Problem Setting

- Unknown: heat flux q1(0, t) t=0~30 days every 3hours
- Measurement: temperature TC1(t),TC2(t) at 2 points t=0~30 days every 3hours
- Model: 2D heat transfer boundary value problem

$$\mathrm{p} \bullet \mathrm{C_p} \frac{\partial T}{\partial t} = k \frac{\partial T}{\partial x^2} \tag{1}$$

The unknown heat flux is a 240 × 1(30 x 24 /3) column vector X, and the measurements can be written as a 480×1 column vector Y, whose first 240 data are from $TC1$ and the left 240 data are from $TC2$. By using finite-difference methods or basis function methods, the transfer matrix A(shape is 480x240) between these two column vectors can be calculated. Then we have:

$$Y_{480} = A_{480\times 240} X_{240} \tag{2}$$

Our goal is to estimate X from measurements Y.

The geometry, material properties and conditions are shown in figure below(Figure2.1)

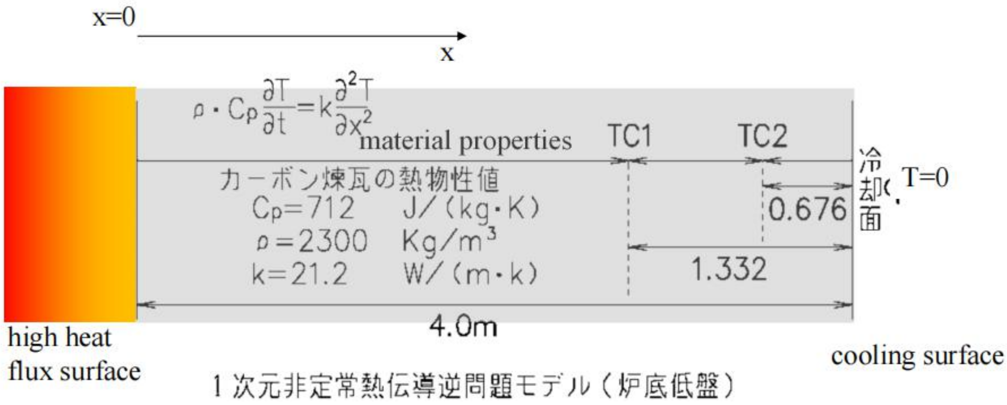


Figure 2.1 1D unsteady heat transfer inverse problem

We also have two boundary settings here:

- initial condition: T(x,0)=0 (℃)
- boundary condition T(4m,t)=0 (℃)

3. Discretization

3.1 Finite-Element Analysis(FEM)

The finite element method (FEM) is a widely used method for numerically solving differential equations arising in engineering and mathematical modeling. Typical problem areas of interest include the traditional fields of structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential.

What we are going to do next is to simulate the continuous time differential transformation into a finite discrete differential transformation, so the differential of T over time can be expressed in the following form:

$$\frac{\partial T}{\partial t} \approx \frac{\Delta T}{\Delta t} = \frac{T_i^{n+1} - T_i^n}{t^{n+1} - t^n} \tag{3}$$

In equation 3, n is the id number of the timestamp, while the i is the x-ray location. So following this kind of discretization process, The spatial derivative of equation (1) is replaced by a central finite difference approximation:

$$\frac{\partial T^2}{\partial^2 x}=\frac{\partial}{\partial x}(\frac{\partial T}{\partial x})\approx\frac{\frac{T_{i+1}{}^{n}-T_i^n}{\Delta x}-\frac{T_i^n-T_{i-1}{}^{n}}{\Delta x}}{\Delta x}=\frac{T_{i+1}{}^{n}-2T_i^n+T^n{}_{i-1}}{\Delta x^2} \quad (4)$$

In the equation1 , because the thermal conductivity k, and the density ρ and the heat capacity Cp are constant numbers, so we assume that :

$$M=\frac{k}{pCP} \quad (5)$$

Then the equation1 can be simplified as:

$$\frac{\partial T}{\partial t}=M\frac{\partial T}{\partial x^2}. \quad (6)$$

Next step let us merge all the equations from (6), (4) and (3), we have the equation just like the follows:

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=M(\frac{T_{i+1}{}^{n}-2T_i^n+T^n{}_{i-1}}{\Delta x^2}) \quad (7)$$

After organization, we can get the following discretized equation:

$$T_i^{n+1}=T_i^n+M\Delta t(\frac{T_{i+1}^n-2T_i^n+T_{i-1}^n}{\Delta x^2}) \quad (8)$$

Because the temperature at the current time step (n) is known for us, so we can use this equation to compute the new temperature without solving any additional equations.

Next step is to calculate the impulse response of the whole system.The boundary condition is set as follows:

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0}=q(t)=\delta(t) \quad (9)$$

Here the $\delta(t)$ is the Dirichlet function. We can use the boundary condition to compute the new temperature of the point at x=0, which is :

$$T_0^{n+1}=T_0^n+M\Delta t(\frac{2T_1^n-2T_0^n}{\Delta x^2})+\frac{\Delta x}{k}\delta(t) \quad (10)$$

So by given an impulse of heat flux and settings with $\Delta x=0.676$, the temperatures of the time serials of TC1 and TC2 can be plotted as follows:0,3h,6h.....,240x3h (Figure3.1). In the figure, the blue line is the temperature of TC1, and the red line is the temperature variance of TC2.

For figure one, we create a transient thermal model for solving an rectangle domain problem. The x-axe is 0-4 for the variance of x, and the y-axe is 0-1, which is represent for the inner and outside surface of the rod. The whole figure can be modeled as a rectangle which shown at figure3.1 first graph. As we can see from the second figure of the Figure3.1. It shows the final time transient temperature . Like the boundary value suggests, when the x is 4, the temperature is 0. The sub-graph of figure 3.1 shows more the temperature change of the two sensor, blue is the TC1 and yellow is TC2.

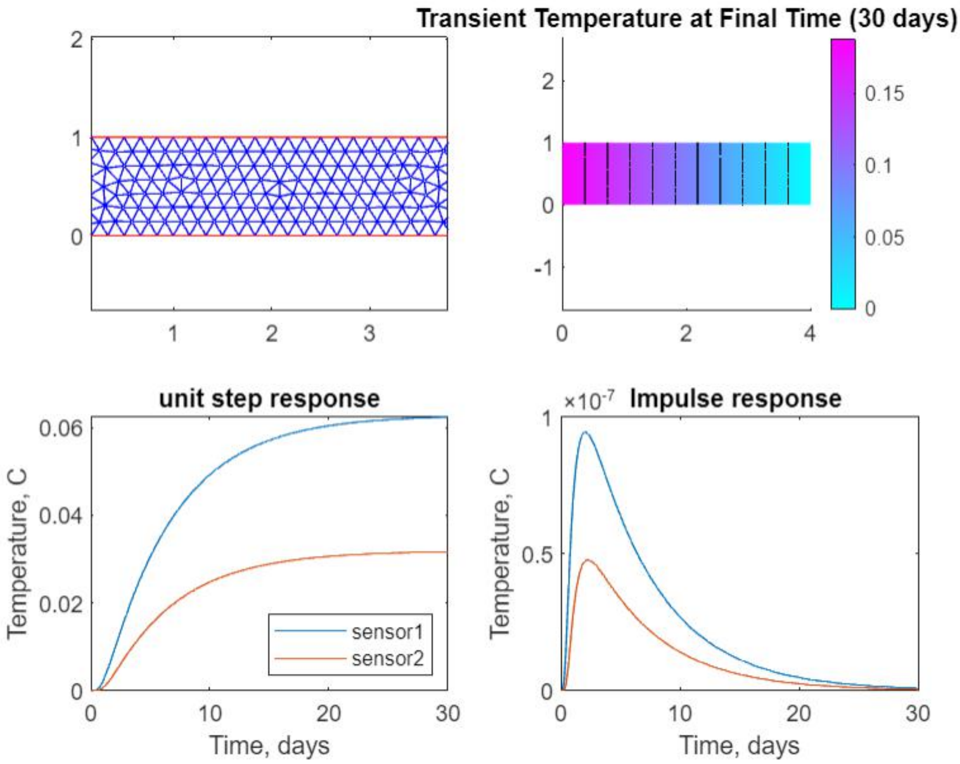


Figure 3.1 Unit Step response and impulse response graph

After modeling impulse response function, observation equation can be expressed with IFK.

$$TC1(t) = \int IR1(t-\tau)q1(\tau)d\tau$$

$$TC2(t) = \int IR2(t-\tau)q1(\tau)d\tau$$

discretization and matrix form,

TC1 = [A1]q1
TC2 = [A2]q1 ***y = Ax***

3.2 Construction of Transfer Matrix A

The transfer matrix A can be constructed from either FDM or basis function method. Here, basis function method is selected as an example. We denote the discretized kernel function $K(x, n\Delta t)$ at $TC1$ as $K1(n)$, kernel function $K(x, n\Delta t)$ at $TC2$ as $K2(n)$. The transfer matrix A can be constructed as follows:

$$A = \frac{\Delta t}{\rho C_p} \begin{bmatrix} K_1(0) & 0 & 0 & \cdots \\ K_1(1) & K_1(0) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ K_1(239) & K_1(238) & \cdots & K_1(0) \\ K_2(0) & 0 & 0 & \cdots \\ K_2(1) & K_2(0) & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ K_2(239) & K_2(238) & \cdots & K_2(0) \end{bmatrix}$$

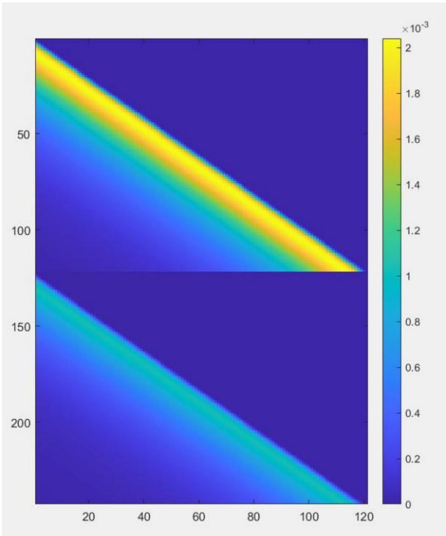


Figure 3.2 Visualization of Transfer Matrix.

4. Calculation of simulated data

The heat flux data q1 and generated phantom data T1 T2 by make_data.m, and the temprature variances of $T\,C1$ and $T\,C2$ are plotted respectively as follows（FIgure4-1）

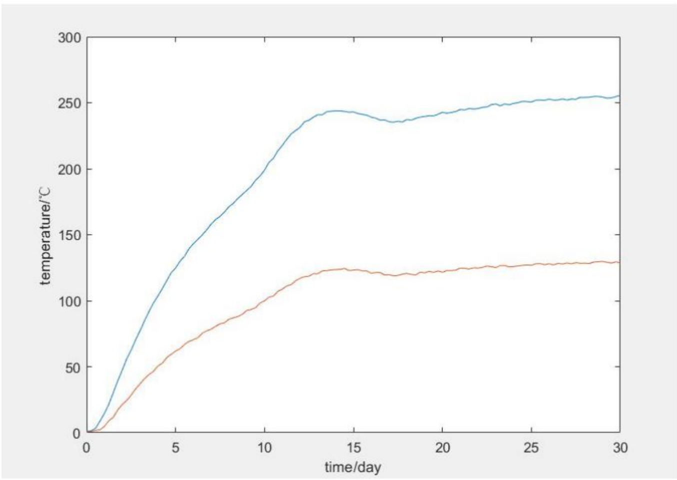


Figure4.1 Generated Phantom data of T1 and T2

The phantom data generation function is y = Ax+e, here I use the error level =5 to get the phantom data.

4. Reconstruction with LSM

Since the target function is shown like below:

$$Y_{480} = A_{480\times240}X_{240} \qquad (2)$$

The LSM is trying to find the least squares solution , which satisfies the conditions like follows:

$$x^* = \arg\min_x \left(\|AX - Y\|^2\right) \qquad (11)$$

So in theory, the LS solution $\boldsymbol{X}^*$ can be calculated as follows:

$$X^* = (A^T A)^{-1} A^T y$$

Here using the pseudo inverse of A to get the best solution.

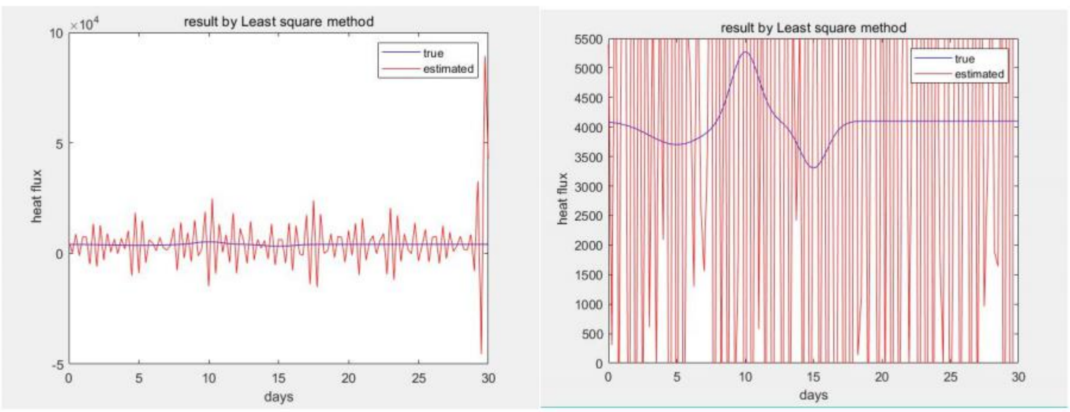


Figure5 Estimated Result by LSM

As we can see from the result, the estimation is not so good directly using the LM methods. That is because there is huge noise in the input data.

6. Regularization

Here, TSVD method and TR regulation method are chosen for regularization to release the influence of the input noise.

For the TSVD method, we apply singular-value decomposition on matrix A and discard the small singular values which mainly represent noise which just like the equation follows:

$$A = U\sum V^T$$

$$X = (V^T)^{-1}\sum{}^{-1} U \cdot Y$$

where K is the truncated order. Here we use K = 50 for TSVD, the TSVD result can be seen as follows(Figure6.1):

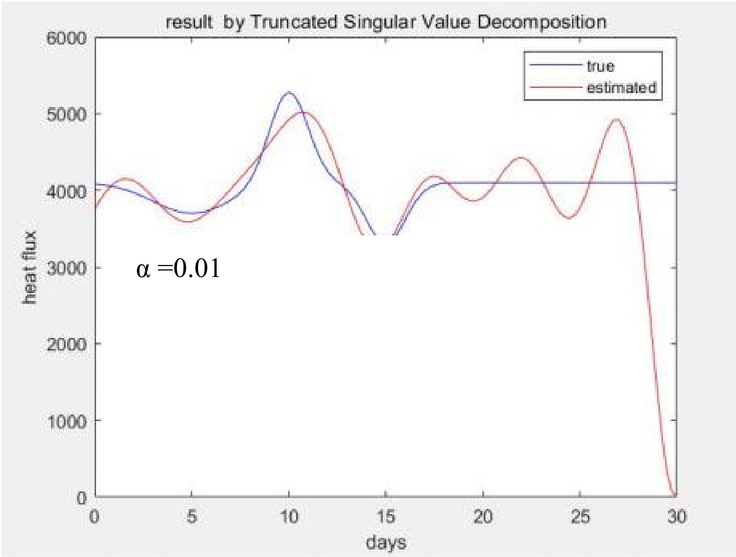


Figure6-1 Estimated Regularized Result by TSVD

In spite of the TSVD method, I also apply TR method for regularization. TR method , which full name is Tikhonov regulation method , it considers minimization with additional regularization term like follows:

$$X^* = \underset{X}{arg\ min}(\|Y - AX\|_2^2 + \alpha \|X\|_2^2)$$

$$X^* = (A^T A + \alpha I)^{-1} A^T y$$

In the equation, α is the regularization parameter. The refined result can be shown like follows:

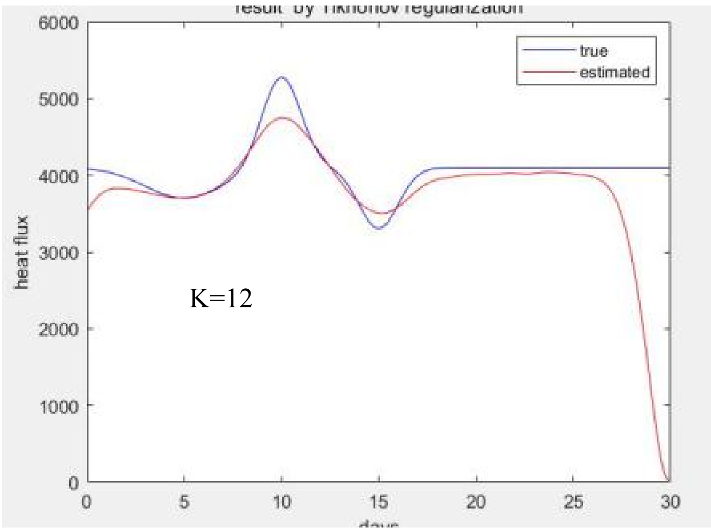
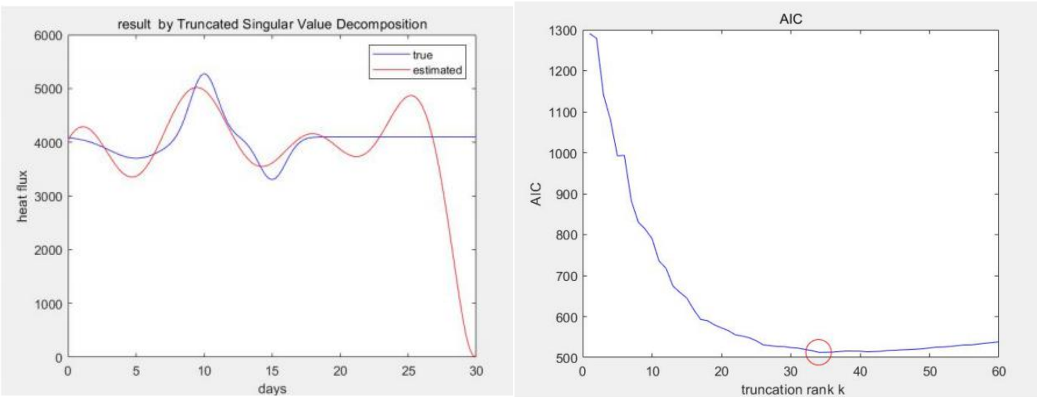


Figure6-2 Regularized Estimation Result by Using TR

7.Tuning of regularization parameter

In order to figure out the best regularization term α for TR method and the best regularization term K for TSVD method. Here method Generalized Cross-Validation is used for get the optimal α in TR method for better estimation.



(a) (b)

Figure 7-1 Generalized Cross-Validation

From the graph *b* ,we can see that when α is around 0.005, TR seems to have the best regularization outcome.

And as for fine-tuning of K in the TSVD method, here I mainly used the To apply discrepancy principle(DP) in TSVD method.

The figure(Figure7-2) below shows the result of DP applied TSVD method. From the DP analysis, the truncation rank equals 8 seems to be the most optimal for TSVD..

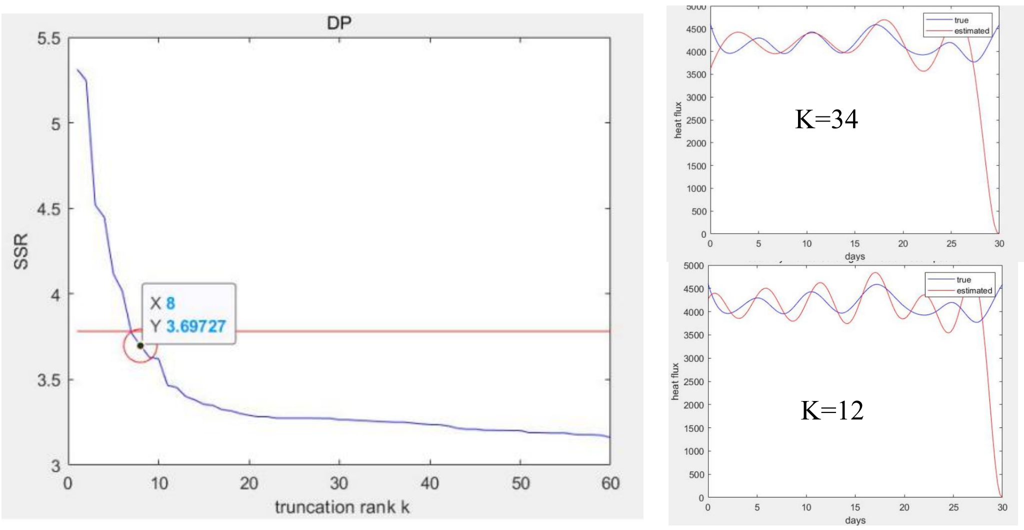


Figure7.2 Discrepancy Principle of TSVD

As we can see from right sub-figure of 7.2, after changing the truncation rank of TSVD from 34 to 12 based on the DP, the estimated result is more like the ground truth compared with previous LMS one.

8. Conclusion

In this report, we dealt with the inverse problem of estimating heat flux from the temperature measurements at two points using $1D$ heat transfer model.

We firstly use the FEM to build a differential model with the knowledge of the observed temperature and the heat flux. Then we discretized the data and obtained the system matrix A, successfully turning the heat estimation problem into an inverse problem.

We firstly directly use LSM to solving the problems, while a poor performance was obtained for the input noise influence. Then we use TSVD and TR method for regularization to get better performance. And in order to get the best regularization term, we apply DP in TSVD for better truncation K and Generalized Cross-Validation is used for get the optimal α in TR method for better estimation.

9. Reference

[1] www.nssmc.com/en/company/publications/monthly-nsc/pdf/2011090510442101723.pdf