Exercise 3.1: Dynamic Equations (5 Points)

In the convolution based DCM for ERP formalism, the post synaptic potential $v(t)$ arises from a convolution of the presynaptic firing $\sigma(t)$ with a convolution kernel $h(t)$, i.e.

$$v(t) = h(t) \otimes \sigma(t) = \int_{-\infty}^{t} h(t-\tau)\sigma(\tau)d\tau, \tag{1}$$

with

$$h(t) = \begin{cases} H\kappa t e^{-\kappa t} & t \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In this exercise, derive the generic second-order differential equations underlying the convolution based DCM for EEG models:

$$\ddot{v}(t) = H\kappa\sigma(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t) \tag{3}$$

Hint: Use Leibniz' rule for differentiation.

Exercise 3.2: Coupled harmonic oscillator (10 Points)

We will now try to better understand DCM for ERPs. This type of DCM is commonly used to infer on hidden parameters from EEG data and takes into account the causal interaction between neuronal cell populations (e.g. pyramidal cells, inhibitory interneurons, spiny stellate cells), with formalized dynamics. To understand these dynamics, we will look at a simpler variant, the harmonic oscillator (HO).

Let us first define a harmonic oscillator that is driven by an external force $u(t)$ with the following equation:

$$\ddot{x} = -f\dot{x} - \kappa^2 x + u(t) \tag{4}$$

(a) Convert the second-order differential equation of the harmonic oscillator into a first-order linear system to obtain the form (*2 Points*)

$$\dot{\vec{x}} = A\vec{x} + \vec{u}(t) \tag{5}$$

(b) Now we consider the problem of a coupled dynamic system. Here, the input $u(t)$ in Eq. 4 comes from the dynamics of a second HO ($z(t)$), i.e.

$$u(t) = a \cdot z(t) \tag{6}$$

with

$$\ddot{z} = -f_z \dot{z} - \kappa_z^2 z + u_z(t) \tag{7}$$

Again, convert the problem into a system of equations, such that

$$\dot{\vec{x}} = A\vec{x} + \vec{u}(t) \tag{8}$$

What are the components of A and $\vec{u}$ (*2 Points*)?

(c) Reconsider Eq. (3). Assume that

$$\sigma(t) = a \cdot s(v_z(t)) + u(t). \tag{9}$$

where $v_z(t)$ are the dynamics of a different population (also described by Eq.(3)) and

$$s(v) = \frac{1}{1 + exp(-rv)} - \frac{1}{2} \tag{10}$$

In the same line of thought as before, transform the system described by Eq. (3) into a system of first-order linear differential equations by linearizing $s(v)$ around $v = 0$. If you compare the resulting equation with the result in Exercise 2b: What is the analogy between the neural state equation of the DCM for ERP and the harmonic oscillator (How do the parameters/functions a, κ and f map onto the (linearized) DCM for ERP equations) (*5 Points*)?

(d) Draw the connectivity diagram (sources and connections in DCM-style) of this configuration (*1 Point*).

Exercise 3.3: Inference on network structure / parameter estimation *(20 Points)*

In this exercise, we will perform a reduced model inversion similar to how one could infer on the most likely modulation structure in an empirical question. For the solutions to exercises (a)-(c), please provide your Julia notebook.

Consider the following setup:

$$\dot{x} = Ax + Cu \tag{11}$$
$$x(t) = 0, \quad t < 0 \tag{12}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\kappa_1^2 & -f_1 & af & 0 \\ 0 & 0 & 0 & 1 \\ a_b & 0 & -\kappa_2^2 & -f_2 \end{pmatrix} \tag{13}$$

$$C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix} \tag{14}$$

$$u(t) = normpdf(t, \mu, \sigma) \tag{15}$$

normpdf refers to the gaussian probability density function and generates a brief gaussian pulse with mean μ and standard deviation σ.

(a) Integrate the system described by Eq. 11 over the interval $0 \le t \le 0.2s$. *Hint: You can use any integration scheme you like with adequate step-size. A simple Euler based integration scheme with $dt = 0.001s$ will work just fine.* Use the following settings and verify, that the integrated states x_1 and x_3 correspond to the data *x_condition_1*. (*10 Points*)

κ_1	80
κ_2	50
f_1	50
f_2	50
a_f	3000
a_b	1000
c	1
μ	0.05
σ	0.01

Where κ_1 and κ_2 are defined in a population-specific manner, a_f represents the weight of the forward connection, and a_b the weight of the backward connection.

(b) When looking at *x_condition_2*, it becomes apparent that something in the system has changed. In fact, we have changed one of the following parameter values: $\kappa_1, \kappa_2, a_f, a_b$. Try to find out which (*Hint: You can use a simple grid-search for each of the parameters*)! Compare the four different hypotheses in terms of the residual sum of squares or explained variance ($vE = 1 - var(y - yp)/var(y)$) *Hint: The true model should at least reach 98% of explained variance*). Which model best explains the data? What is the ensuing parameter estimate? (*8 Points*)

(c) We have told you to look at the output of states x_1 and x_3. What would be the analogy in terms of a leadfield matrix L, such that $\vec{y}$

$$\vec{y}(t) = L\vec{x}(t) \tag{16}$$

corresponds to the activity of these two states? (*2 Points*)