§1. Forcing $\mathcal{U}$-continuity

$$E : \mathbf{BornCoarse} \longrightarrow \mathcal{C}$$

a complete stable ∞-cat

$$E_{\mathcal{U}} : \mathbf{BornCoarse} \longrightarrow \mathcal{C}.$$

Usually, $E_{\mathcal{U}} \longrightarrow E$ — an equiv.

If E is good $\rightsquigarrow$ $E_{\mathcal{U}}$ is so

$$E_{\mathcal{U}}(X) := \operatorname*{colim}_{U \in \mathcal{C}} E(X_U)$$

$\mathbf{BornCoarse}^{\mathcal{C}}$

$Ob = \{ (X, U) \}$, $X \in$ BornCoarse, $U \in \mathcal{C}_X$

$Mor = \{ f : (X, U) \to (X', U') \mid f : X \to X'$ — a morphism s.t. $(f \times f)(U) \subseteq U' \}$

$$\begin{array}{ccccc} \mathbf{BornCoarse} & \xleftarrow{q} & \mathbf{BornCoarse}^{\mathcal{C}} & \xrightarrow{p} & \mathbf{BornCoarse} \\ X_U & \longleftarrow\!\shortmid & (X, U) & \longmapsto & X \end{array}$$

$$
\begin{tikzcd}
\mathrm{BornCoarse}^{\mathcal{C}} \arrow[r, "E\circ q"] \arrow[d, "p"'] \arrow[dr, phantom, "\Downarrow"] & \mathcal{C} \\
\mathrm{BornCoarse} \arrow[ur, dashed, "E_{\mathcal{U}} := \mathrm{Lan}_p E\circ q"'] \arrow[ur, bend right=60, "E"'] \arrow[ur, phantom, shift right=6ex, "\Searrow"]
\end{tikzcd}
$$

$q \to p$

$$
\begin{array}{ccc}
q(X,\mathcal{U}) & \longrightarrow & p(X,\mathcal{U}) \\
\| & & \| \\
X_{\mathcal{U}} & \longmapsto & X
\end{array}
$$

$$E\circ q \longrightarrow E\circ p$$

We get

$$E_{\mathcal{U}} \longrightarrow E$$

$$E_{\mathcal{U}}(X) \cong \operatorname*{colim}_{\mathrm{BornCoarse}^{\mathcal{C}}/X} E\circ q$$

$(X, \mathcal{U}, \mathrm{id}_X)$ is cofinal in $\mathrm{BornCoarse}^{\mathcal{C}}/X$

$\operatorname{colim} \iota\circ F = \operatorname{colim} F$

this condition $\Longleftrightarrow$ $\mathrm{BornCoarse}^{\mathcal{C}}/X$ is connected (zig-zags)

So, we get

$$E_{\mathcal{U}}(X) \cong \operatorname*{colim}_{U \in \mathcal{B}_X} E(X_U)$$

Prop. The foll. prop. are inherited by $E_{\mathcal{U}}$:

1. Coarse invariance
2. Excision
3. Vanishing on flasque born. coarse spaces

Proof: 1 Let E be coarsely inv.

$$E_{\mathcal{U}}(\underbrace{\{0,1\}}\otimes X) \longrightarrow E_{\mathcal{U}}(X) \text{ — an equiv.}$$

the max born.
the max coarse str.

Note that

$$(\{0,1\}\otimes X)_{\overline{U}} \cong \{0,1\}\otimes X_U$$

$$\overline{U} := \{0,1\}^2 \times U$$

$$E\left((\{0,1\}\otimes X)_{\overline{U}}\right) \cong E\left(\{0,1\}\otimes X_U\right) \cong E(X_U)$$

see above

as E is coarsely inv

$\{\bar{U}\}$ is cofinal

$$\operatorname*{colim}_{U\in\mathcal{C}_X}\left(\{0,1\}\otimes X_U\right) \cong \operatorname*{colim}_{\bar{U}\in\mathcal{C}_X}\left((\{0,1\}\otimes X)_{\bar{U}}\right) \cong$$

$$\cong \operatorname*{colim}_{U\in\mathcal{C}_X} E(X_U)$$

② —//— $\triangleleft$

Corollary. Assume that

1. E is coarsely inv.
2. E vanishes on flasque spaces
3. E satisfies excision

Then

1. $E_{\mathcal{U}}$ is a coarse homology theory
2. If E is coarse homology theory $\Rightarrow E_{\mathcal{U}}\to E$ will be an equiv.
3. If $\mathcal{C}_X$ is gen. by a single entourage $\Rightarrow E_{\mathcal{U}}(X)\to E(X)$ is an equiv.

§2. Additivity & coproducts

$E : \mathrm{BornCoarse} \to \mathcal{C}$ — a coarse homology theory

complete

Def. E — strongly additive if $\forall \{X_i\}_{i\in I}$

$$E\Big(\coprod_{i\in I}^{\text{free}} X_i\Big) \longrightarrow \prod_{i\in I} E(X_i) \text{ — an equiv.}$$

Def. E is additive if $\forall\ I$

$$E\Big(\coprod_{I}^{\text{free}} *\Big) \longrightarrow \prod_{I} E(*) \text{ — an equiv.}$$

Lemma. If E — strongly additive, then $\exists$ a fiber sequence

$$\bigoplus_{i\in I} E(X_i) \longrightarrow E\Big(\coprod_{i\in I}^{\text{mixed}} X_i\Big) \longrightarrow \operatorname{colim}_{\substack{J\subseteq I\\ |J|<\infty}} \prod_{i\in I\setminus J} E(X_{i, \text{disc}})$$

Proof.

$$\bigoplus E(X_i) \longrightarrow E\Big(\coprod_{i\in I} X_i\Big) \qquad X_i \hookrightarrow \coprod X_i$$

$$\bigoplus E(X_i) \longrightarrow E\Big(\coprod_{i\in I}^{\text{mixed}} X_i\Big) \qquad X_i \hookrightarrow \coprod^{\text{mixed}} X_i$$

The fibres of the maps belong to $Sp\mathcal{X}\langle A_{disc}\rangle$

A – a set of objects in BornCoarse

$Sp\mathcal{X}\langle A\rangle$ denotes the minim. cocomplete stable full subcat. of $Sp\mathcal{X}$ containing $E(A)$

A_{disc} – the set of discr. born. coarse spaces

$$X := \coprod_{i\in I}^{\text{mixed}} X_i \qquad E(X) \simeq \operatorname*{colim}_{U\in\mathcal{C}} E(X_U)$$

Suppose that $U \supset \mathrm{diag}_X \Rightarrow \exists J \subset I$ & $U_j \in \mathcal{C}_j$ $\forall j\in J$ s.t.

$$U = \bigcup_{j\in J} U_j \cup \bigcup_{i\in I\setminus J} \mathrm{diag}(X_i)$$

$$\Rightarrow X_U \cong \coprod_{j\in J} X_{j,U_j} \sqcup \coprod_{i\in I\setminus J} X_{i,disc}$$

$$E(X_U) \cong \bigoplus_{j\in J} E(X_{j,U_j}) \oplus E\Big(\coprod_{i\in I\setminus J} X_{i,disc}\Big)$$

$$E\Big(\coprod_{\substack{i\in I\setminus J \\ J'\supset J}}^{\text{mixed}} X_{i,disc}\Big) \simeq \underbrace{E\Big(\coprod_{i\in I\setminus J'} X_{i,disc}\Big)}_{} \oplus$$

$$\oplus \bigoplus_{i\in I'\setminus J} E(X_{i,disc})$$

$$\Sigma^{-1}R \longrightarrow \bigoplus E(X_i) \to E(X) \to \underbrace{\operatorname*{colim}_{\substack{J\subseteq I \\ |J|<\infty}} E\Big(\coprod^{\text{mixed}} X_{i,disc}\Big)}_{R}$$

$$R \pitchfork \mathrm{Sp}\mathcal{X}\langle A_{disc}\rangle$$

$\lhd$

Def. $\bigoplus_i E(X_i) \longrightarrow E\Big(\coprod_{i\in I} X_i\Big)$ — an equiv.

§3. Coarse Ordinary Homology

Ch — the small cat of very small chain complexes

$H\mathcal{X}$

$\ell : Ch \longrightarrow Ch[W^{-1}] := Ch_\infty$

the Dwyer-Kan loc. functor

$Ch_\infty = Ch[W^{-1}]$

$Ho(Ch_\infty) = \mathcal{D}(Ab)$

Ch_∞ – stable, very small, complete & cocomplete

Define

$$C\mathcal{X} : BornCoarse \longrightarrow Ch$$
$$X \longmapsto \text{the chain complex of locally finite controlled chains}$$

Some definitions:

- $(x_0, \dots, x_n) \in X^{n+1}$ meets B if $\exists$ an index $i \in \{0, \dots, n\}$: $x_i \in B$

- $U \in \mathcal{C}_X$ $(x_0, \dots, x_n)$ is U-controlled if $\forall i, j \in \{0, \dots, n\}$ $(x_i, x_j) \in U$

- $c : X^{n+1} \longrightarrow \mathbb{Z}$ is an n-chain on X if it is a function

$$\mathrm{supp}(c) := \{x \in X^{n+1} \mid c(x) \neq 0\}$$

- $X \in \mathrm{BornCoarse}$, $U \in \mathcal{C}_X$, $c : X^{n+1} \to \mathbb{Z}$
  - c is locally finite if $\forall B \in \mathcal{B}_X$ the set $\mathrm{supp}(c)$ which meet B is finite
  - c — U-controlled if $\forall$ point in $\mathrm{supp}(c)$ is U-controlled

Def. $C\mathcal{X}_n(X)$ to be the abelian group of locally finite & controlled n-chaines

$$\sum_{x \in X^{n+1}} c(x)x \text{ — an } n\text{-chain}$$

$$\partial_i : X^{n+1} \to X^n$$

$$\partial_i(x_0, \dots, x_n) := (x_0, \dots, \hat{x}_i, \dots, x_n)$$

$$\rightsquigarrow \partial_i : C\mathcal{X}_n(X) \to C\mathcal{X}_{n-1}(X)$$

Define $\partial := \sum_{i=0}^{n} (-1)^i \partial_i$, $\partial \circ \partial = 0$

So, $$C\mathcal{X}(X) := ((C\mathcal{X}_n(X))_{n \in \mathbb{N}}, \partial)$$

$f : X \to X' \rightsquigarrow C\mathcal{X}(f) : C\mathcal{X}_n(X) \to C\mathcal{X}_n(X')$

Def. $H\mathcal{X} : \mathrm{BornCoarse} \longrightarrow \mathrm{Ch}_\infty$

$$H\mathcal{X} := \ell \circ C\mathcal{X}$$

Theorem. $H\mathcal{X}$ is a Sp-valued coarse homology theory

Prop 1. $H\mathcal{X}$ is coarsely inv.

▷ $f, g : X \to X'$ – close

$H\mathcal{X}(f)$ & $H\mathcal{X}(g)$ are equiv?

Construct a chain homotopy from $C\mathcal{X}(g)$ to $C\mathcal{X}(f)$

$i \in \{0, \dots, n\}$

$h_i : X^{n+1} \longrightarrow X^{n+2}$

$(x_0, \dots, x_n) \mapsto (f(x_0), \dots, f(x_i), g(x_i), \dots, g(x_n))$

$h_i : C\mathcal{X}_n(X) \longrightarrow C\mathcal{X}_{n+1}(X')$ $\quad (f(x_i), g(x_j)) \in \underset{\uparrow\, \mathcal{E}_X}{U}$

Define $h := \sum_{i=0}^{n} (-1)^i h_i$

$$\partial \circ h + h \circ \partial = C\mathcal{X}(g) - C\mathcal{X}(f)$$

$\triangleleft$

Prop. 2. $H\mathcal{X}$ sat. excision

$\triangleright$ $X \in \mathrm{BornCoarse}$

$(Z, \mathcal{Y})$ — a complement. pair

$$H\mathcal{X}(\mathcal{Y}) := \operatorname{colim}_i H\mathcal{X}(Y_i) := \operatorname{colim}_i \ell C\mathcal{X}(Y_i) \simeq$$

$$\simeq \ell \operatorname{colim}_i C\mathcal{X}(Y_i)$$

$$C\mathcal{X}(X)$$

$$C\mathcal{X}(Z) \hookrightarrow C\mathcal{X}(X)$$

$$\begin{tikzcd}
0 \arrow[r] & C\mathcal{X}(Z \cap \mathcal{Y}) \arrow[r, hook] \arrow[d, hook] & C\mathcal{X}(Z) \arrow[r] \arrow[d, hook] & \frac{C\mathcal{X}(Z)}{C\mathcal{X}(Z \cap \mathcal{Y})} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C\mathcal{X}(\mathcal{Y}) \arrow[r] \arrow[ur, phantom, "?\ \ulcorner"] & C\mathcal{X}(X) \arrow[r] & \frac{C\mathcal{X}(X)}{C\mathcal{X}(\mathcal{Y})} \arrow[r] & 0
\end{tikzcd}$$

Verify that the right handside map is an iso

Injectivity: $\exists\, c \in \mathcal{C}\chi(Z)$

$[c] \longmapsto 0$

Then $c \in \mathcal{C}\chi(Y) \cap \mathcal{C}\chi(Z) = \mathcal{C}\chi(Y \cap Z) \Rightarrow$

$\Rightarrow [c] = 0$

Surjectivity: — // —

Prop. If X — flasque $\Rightarrow H\chi(X) \cong 0$

▷ $f : X \to X$

$S : X^{n+1} \to \mathcal{C}\chi_n(X)$

$x \longmapsto \sum_{k \in \mathbb{N} \cup \{0\}} \mathcal{C}\chi(f^k)(x)$

$S : \mathcal{C}\chi(X) \longrightarrow \mathcal{C}\chi(X)$

$id_{\mathcal{C}\chi(X)} + \mathcal{C}\chi(f) \circ S = S$

Apply ℓ:

$$\mathrm{id}_{H\chi(X)} + \underbrace{H\chi(f)}_{\mathrm{id}_X} \circ \cancel{\ell(S)} \cong \cancel{\ell(S)}$$

f is close to $\mathrm{id}_X \Rightarrow H\chi(f) = \mathrm{id}$

$$H\chi(X) \cong 0$$

$\triangleleft$

Prop. 4. $H\chi$ is u-continuous

$\triangleright$ $H\chi(X) \cong \ell(\operatorname{colim} C\chi/X_{\mathcal{T}})) \cong$

$$\cong \operatorname*{colim}_{\mathcal{T}\in\mathcal{E}} \ell C\chi(X_{\mathcal{T}}) \cong \operatorname*{colim}_{\mathcal{T}\in\mathcal{E}} H\chi(X_{\mathcal{T}})$$

$\triangleleft$

Prop. 5. $H\chi$ is also strongly additive.

Prop. 6. $H\chi$ preserves coproducts

Example. X — with the max. structures

$$H\chi(X) \cong \ell(\mathbb{Z})$$

§4. Equivariant ordinary theory

$$\mathrm{Fun}(BG, \mathrm{BornCoarse})$$
$$\cup$$
$$G\mathrm{BornCoarse}$$

$\mathcal{E}^G$ is cofinal in $\mathcal{E}$

Example. $G \curvearrowright (X, d)$ isometrically

$\rightsquigarrow X_d \in G\mathrm{BornCoarse}$

Example. $\mathbb{Z} \curvearrowright (\mathbb{R}, d)$

$(n, x) \longmapsto 2^n x$

Example. (M, g)

M_{d_g, s_g} $\qquad \mathcal{B}_{s_g} := \{ B \subseteq M \mid \exists K \subseteq M$ s.t. $\inf_{B \setminus K} s_g > 0 \}$

Example. Assume that M admits an equiv. spin-structure

$\not{D}$ — the assoc. Dirac operator

index$\chi(\not{D})$ in $K\chi^G_{-\dim(M)}(M_{d_g, s_g})$

$K\chi^G := K_0 \, \mathcal{V}^G$

$\mathcal{V}^G : G\mathbf{BornCoarse} \longrightarrow C^*\mathbf{Cat}$

X — a G-Born

X-controlled Hilbert space

(H, ρ, p) $\qquad p = \{p_x\}_{x \in X}$

$\rho = \{\rho_g\} : G \to U(H)$

$A : (H, \rho, p) \longrightarrow (H', \rho', p')$

$A : H \to H'$ s.t. $\rho_g' A = A \rho_g$

$C^*((H,\rho,p),(H',\rho',p'))$ — the closure in $B(H,H')$

$\mathcal{V}^G(X)$

$Ob = \{ X\text{-controlled Hilbert spaces} \}$

$Mor = \{ C^*((H,\rho,p),(H',\rho',p')) \}$

Def. X & $\dim p_x = 1$

$$C^*(H,\rho,p) := \underbrace{\mathrm{End}_{\mathcal{V}^G(X)}(H,\rho,p)}_{\text{a Roe algebra associated to } X}$$

$$\mathcal{V}^G(f) : \mathcal{V}^G(X) \to \mathcal{V}^G(X')$$

$$(H,\rho,p) \mapsto (H,\rho,f_*p)$$

$$f_*p = \sum_{x \in f^{-1}(x')} p_x$$

$$K : C^*Cat \longrightarrow Sp$$

$$K\mathcal{X}^G := K \circ \mathcal{V}^G : GBornCoarse \to Sp$$

If $G = \{e\}$ & S_g admits a uniform pos. low bound

Then $\not{D}$ is Fredholm and its Fredholm index

$\operatorname{index}(\not{D}) \in K_{-\dim(M)}(\mathbb{C})$

$$\operatorname{index}(\not{D}) = p_* \operatorname{index}_{\chi}(\not{D})$$

$$K\chi_{-\dim(M)}(*) \cong \pi_{-\dim(M)}(KU)$$

$$p : M_{d_g, S_g} \to *$$