

Reedy cats and Reedy structures: Kan's theorem and applications

References:

- Hirschhorn "Model cats and their localizations"
- Emily Riehl "Categorical homotopy theory"

Lemma (Hirschhorn's Lemma): A, B, X, Y are \mathcal{D} -diagrams in $\mathcal{M}_{\text{mod cat}}$

\mathcal{D} is a Reedy cat

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & h \dashrightarrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

\downarrow see Hirschhorn's book,
chapter 15

where h is defined on $F^{n-1}B$

!

$$\begin{array}{ccc} A_\alpha \sqcup_{L_\alpha} B_\alpha & \longrightarrow & X_\alpha \\ \downarrow & H \dashrightarrow & \downarrow \\ B_\alpha & \longrightarrow & Y_\alpha \times_{M_\alpha} X \end{array}$$

$\alpha \in \mathcal{D}$
 $\deg(\alpha) = n$

There is a map $H: B_\alpha \rightarrow X_\alpha \quad \forall \alpha \deg(\alpha) = n \iff h$ can be extended over the restriction of B to the n -filtration

Def. Let \mathcal{D} be a Reedy cat, M is a modal cat

(1) $f: X \rightarrow Y$ is Reedy WE if $\forall \alpha \in \mathcal{D}$

$f_\alpha: X_\alpha \rightarrow Y_\alpha$ is a WE in M

(2) $f: X \rightarrow Y$ is a Reedy cofibration if $\forall \alpha \in \mathcal{D}$

$$X_\alpha \sqcup_{L_\alpha} Y_\alpha \rightarrow Y_\alpha$$

is a cofibration in M

(3) $f: X \rightarrow Y$ is a Reedy fibration if $\forall \alpha \in \mathcal{D}$

$$X_\alpha \rightarrow Y_\alpha \times_{M_\alpha Y} M_\alpha X$$

is a fibration in M

Theorem (D.Kan) (1) The cat $M^{\mathcal{D}}$ with the Reedy WE , Reedy cofib, Reedy fib is a model cat

(2) If M is a left proper then the modal cat $M^{\mathcal{D}}$

is also left proper model cat

Example. $M^{\Delta^{\text{op}}}$

Lemma. (1) If $\forall \beta \in \mathcal{D}, \deg \beta < \deg \alpha$

$$X_\beta \sqcup L_\beta X \rightarrow Y_\beta$$

has the LLP w.r.t. to S then

$$L_\alpha X \rightarrow L_\alpha Y$$

has the LLP w.r.t. to S

(2) —/—

Lemma. (1) If $\forall \alpha \in \mathcal{D}$

$$X_\alpha \sqcup L_\alpha X \rightarrow Y_\alpha$$

has the LLP vs. $S \Rightarrow \forall \alpha$ the map $f_\alpha: X_\alpha \rightarrow Y_\alpha$
has the LLP w.r.t. to every element of S (pointwise)

Proof: $(f_\alpha : X_\alpha \rightarrow Y_\alpha) = (X_\alpha \xrightarrow{\text{L}_\alpha X} X_\alpha \sqcup_{\text{L}_\alpha Y} Y_\alpha \xrightarrow{\text{R}_\alpha} Y_\alpha)$

$$\begin{array}{ccc} \text{L}_\alpha X \rightarrow \text{L}_\alpha Y & & \text{L}_\alpha P \\ \downarrow & \downarrow & \text{L}_\alpha P \\ X_\alpha \xrightarrow{f_\alpha} Y_\alpha & \xrightarrow{\exists!} & \text{as the pushout} \\ & & \text{of } \text{L}_\alpha X \rightarrow \text{L}_\alpha Y \end{array}$$

Prop. (1) If $f: X \rightarrow Y$ is a Reedy cofib \Rightarrow
 $\Rightarrow f$ is a pointwise cofib and $\text{L}_\alpha X \rightarrow \text{L}_\alpha Y$ is cofib in M

(2) $f: X \rightarrow Y$ is a Reedy fib $\Rightarrow f$ is a pointwise fib in M

fib in M and $M_\alpha X \rightarrow M_\alpha Y$ is a fib in M

Proof: f is a Reedy cofib $\Leftrightarrow X_\alpha \sqcup_{\text{L}_\alpha Y} Y_\alpha$ is a cofib.

in $M \Leftrightarrow$ the latter map has the LFP vs. the set of
 acyclic fibrations in $M \Leftrightarrow f_\alpha: X_\alpha \rightarrow Y_\alpha$ has the LFP

vs. the set of acyclic fibrations in M

Corollary (from the prop.) (1) If X is Reedy cofib

$\Rightarrow \forall \alpha \in \mathcal{D}_0$ both X_α and $\text{L}_\alpha X$ are cofib in M .

In particular, Reedy cofibrant diagrams are pointwise cofib

(2) $\dashv \vdash$

Prop. (1) If $X_\alpha \sqcup L_\alpha Y \rightarrow Y_\alpha$ is a trivial cofib
 $\Rightarrow \forall \alpha \in \mathcal{D}_\beta$ both $L_\alpha X$, f_α and $L_\alpha X \rightarrow L_\alpha Y$ are triv.
 cofibes

(2) $\longrightarrow //$

1

▷ Obvious

Prop. (1) If $f: X \rightarrow Y$ cofib $\cap W\mathcal{E}$
 Then the maps $f_\alpha: X_\alpha \rightarrow Y_\alpha$, $L_\alpha f: L_\alpha X \rightarrow L_\alpha Y$,
 $X_\alpha \sqcup L_\alpha Y \rightarrow Y_\alpha$ are trivial cofibes

$L_\alpha X$

(2) $\longrightarrow //$

Proof: If $L_\alpha X \rightarrow L_\alpha Y$ is $W\mathcal{E}$ $\Rightarrow X_\alpha \sqcup L_\alpha X \rightarrow Y_\alpha$
 $L_\alpha X$ will be a $W\mathcal{E}$

$X_\alpha \rightarrow X_\alpha \sqcup L_\alpha Y \rightarrow Y_\alpha$
 $W\mathcal{E}$ $W\mathcal{E}$ by 2-of-3 property

- The base. If $\deg(\alpha) = 0 \Rightarrow L_\alpha X \rightarrow L_\alpha Y$ is an identity map
- The induction step. Suppose that $L_\beta X \rightarrow L_\beta Y$ is a cofib $\cap W\mathcal{E}$ $\forall \beta$, $\deg(\beta) < n$, let α be an object of

$\mathcal{D}, \deg(\alpha) = n$

$X_\beta \cup L_\beta Y \rightarrow Y_\beta$ is a trivial cofib $\forall \beta, \deg(\beta) < n$

$L_\beta X$
 $\Rightarrow L_\beta X \rightarrow L_\beta Y$ is a triv. cofib by Hirschhorn's Lemma \square

Theorem. (1) $f: X \rightarrow Y$ is in $\text{Cofib} \cap \text{WE}$

$\Leftrightarrow \forall \alpha \in \mathcal{D}_0, X_\alpha \cup L_\alpha Y \rightarrow Y_\alpha$ is a trivial cofib
in M

(2) \longrightarrow (The Description of the trivial fibrations)
 \downarrow see Emily Riehl's Book,
 Chapter 14

Example. $0 \leftarrow 1 \rightarrow 2$

$$X : (0 \leftarrow 1 \rightarrow 2) \longrightarrow M$$

$X \in M^{\mathcal{D}}$

$L_0 X = \emptyset, L_1 X = \emptyset$

$$\overrightarrow{\mathcal{D}}_{<0}/_0 \quad \overrightarrow{\mathcal{D}}_{<1}/_1$$

is cofibrant in the Reedy model if a, b, c is cofib in M and g is cofib in M

$$L_2 X = \text{colim} \left(\overrightarrow{\mathcal{D}}_{<2}/_2 \xrightarrow{X^\top} M \right) = \alpha = X(1)$$

$X \rightarrow Y$ is a cofib in M^D $\Leftrightarrow L_\alpha Y \sqcup X_\alpha \rightarrow Y_\alpha$ is a cofib in M

Take X to be \emptyset

$$Y \text{ is cofib.} \Leftrightarrow L_\alpha Y \sqcup \emptyset \rightarrow Y_\alpha$$

\emptyset
 \parallel
 L_α

$L_0 X = \emptyset \rightarrow b = X(0) \Rightarrow b$ should be cofibr.

$L_1 X = \emptyset \rightarrow a = X(1) \Rightarrow a \dashrightarrow$

$L_2 X = a \rightarrow c = X(2) \Rightarrow a \rightarrow c$ is a cofib.

Theorem. $- \otimes_D - : s\text{Set}^D \times M^D \xrightarrow{\text{op}} M$

is a left Quillen bifunctor w.r.t. to the Reedy

model cat

$- \amalg - : \{-, -\}^D : (s\text{Set}^D)^{\text{op}} \times M^D \rightarrow M$

cotensor product

Corollary.

$$\begin{array}{ccccc} & f & a & \xrightarrow{g} & c \\ & \swarrow & \downarrow & \searrow & \\ b & \xrightarrow{\quad} & r & \downarrow & \\ & \downarrow & \downarrow & \downarrow & \\ & a' & \xrightarrow{i} & c' & \\ & \searrow & \downarrow & \swarrow & \\ & e & \xrightarrow{\quad} & ! & \end{array}$$

$\square * \otimes_D M^D \rightarrow M$

$* : D^{\text{op}} \rightarrow s\text{Set}$

$$\operatorname{colim}_{\Delta} : {}_0 M^{\Delta} \rightarrow M$$



Remark. In the left proper model cat setting
the cofibrant objects condition can be dropped (Exercise)

What is a characterization of those compl. sets
that are Reedy cofibrant?

- Fix $X \in \text{Set}^{\Delta}$
 $x \in X^n$ is non-degen. if it is not in the image of
any monomorphism of Δ
 $x = \partial \tilde{x}$ where ∂ is a monomorphism in Δ
 $\Rightarrow \tilde{x}$ is non-degen. simplex

- The n^{th} latching map is a monomorphism when
each such expr. is unique (See details in Riehl)

- if $x = \partial \tilde{x} = \partial' \tilde{x} \Rightarrow \partial' = \partial$
- $d^0, d^1 : [0] \rightrightarrows [1]$ — the exceptions from some rule

Def. If \forall degen. simpl. in X is uniquely expr.

as the image of a non-degen. simplex under a
monomorphism we say that X has the Eilenberg-Zilber
property

Prop. A cosimpl. object has the Eilenberg-Zilker prop.

\Leftrightarrow it is unaugmentable, that is

$$\text{eq}^o(X^o \rightrightarrows X^1) = \emptyset$$

Lemma. Any bisimpl. set is Reedy cofibrant

Lemma. If a *compl.* object X is *unaugmentable*,

then $L_m X \rightarrow X$ is a monomorphism

If X and Ψ are both renaugmentable, then $\Psi \circ X$ is also.

Sketch of Proof: A colimpl. object in a set-valued

Sketch of Proof: The functor cat has the Eilenberg-Silber property \Leftrightarrow ??

does pointwise

does pointwise
 Example $f: A \hookrightarrow \text{Set}$, $L \in \text{sSet}^A$

Example. $\mathcal{L}: \Delta \hookrightarrow \text{sSet}$, \hookrightarrow
 $\text{eq}(\Delta^0 \rightarrow \Delta^1) = \emptyset \Rightarrow \mathcal{L}$ is cofibrant in the
 Reedy model structure in sSet^Δ

Interesting question: What is the case when

$$\text{eq}(X^0 \Rightarrow X^1) \neq \emptyset$$

Corollary. If M is a simplicial model cat
 $|-| : M^{\Delta^{op}} \rightarrow M$

is left Quillen with respect to the Reedy model structure

Proof. $|-| = - \otimes_{\Delta} \mathcal{L} \Rightarrow |-| \text{ is a left Quillen as } \mathcal{L}' \text{ is cofibrant in the Reedy model structure and by the analog of Gambino's theorem}$

Homotopy limits in Top setting

theorem
1

- $|-| : \text{Top}^{\Delta^{\text{op}}} \rightarrow \text{Top}$ preserves weak between so-called split simplicial spaces even are not not Reedy cofibrant
- $\mathcal{B}_*(*, \mathcal{D}, -) = |\mathcal{B}_*(*, \mathcal{D}, -)|$

Def. A simplicial space X_* is split if $\bigcup_n X_n \hookrightarrow X_n$

$\forall n, \text{ s.t.}$

$$\bigsqcup_{[n] \rightarrow [x]} \bigcup_n X_n \hookrightarrow X_n$$

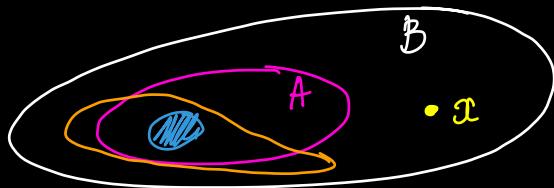
Example. $F : \mathcal{D} \rightarrow \text{Top}$
 $\mathcal{B}_*(*, \mathcal{D}, F)$ is the split simplicial space

$$d : \bigsqcup_{[\Sigma^n]} F d_n \rightarrow \mathcal{D}$$

When X_\cdot is split

$$\begin{array}{ccc} |\partial\Delta^n| \times N_n X & \longrightarrow & SK_{n-1}(X) \\ \downarrow & & \downarrow \\ |\Delta^n| \times N_n X & \xrightarrow{\Gamma} & SK_n(X) \end{array}$$

Def. $A \subset B$ is a relative T_1 inclusion
if $\forall T \subseteq A, \forall x \in B \setminus A \quad \exists V \subseteq B$



Lemma. If K is a compact

$$Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2 \hookrightarrow \dots$$

(Like small-object-argument - lemma)

$f: K \rightarrow \underset{n}{\text{colim}} Y_n$ factors through some Y_m

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Proof: Exercise

Prop. If $X_\cdot \rightarrow Y_\cdot$ is a pointwise weak of split

simpl. spaces, then $|X_\cdot| \rightarrow |Y_\cdot|$ is a weak

Prop: • X_\cdot is split $\Rightarrow |X_\cdot|$ is a colimit of a sequence

of relative T_1 -inclusions

• By the lemma above

$$\operatorname{colim}_n \pi_K(sK_n X_*) \xrightarrow{\cong} \pi_K(X_*)$$

$\forall k > 0$

• So, it suffice to show that $(sK_n X_*) \rightarrow (sK_n Y_*)$ are WE

$$\begin{array}{ccccc} |\Delta^n| \times N_n X & \leftarrow & |\partial\Delta^n| \times N_n X & \longrightarrow & sK_{n-1}(X) \\ WE \downarrow & & WE \downarrow & & WE \downarrow \\ |\Delta^n| \times N_n Y & \leftarrow & |\partial\Delta^n| \times N_n Y & \longrightarrow & sK_{n-1}(Y) \end{array}$$

Lemma (WE have local property in Top)

A map $f: X \rightarrow Y$ in Top is WE if

$\exists \mathcal{U}, \mathcal{V}$ - open cover of Y , s.t.

$$f^{-1}(\mathcal{U}) \rightarrow \mathcal{U}, \quad f^{-1}(\mathcal{V}) \rightarrow \mathcal{V}, \quad f^{-1}(\mathcal{U} \cap \mathcal{V}) \rightarrow \mathcal{U} \cap \mathcal{V}$$

are WE