

The 2-category of quasi-categories

- The adjunction

$$h: \text{sSet} \rightleftarrows \text{Cat} : N$$

- The counit is an isomorphism

Lemma. The functor $h: \text{sSet} \rightarrow \text{Cat}$ preserves finite products

- ▷ • $(h-)$ & $h(- \times -)$ preserve colimits
- sSet & Cat are cartesian closed
- $h\Delta^n \times h\Delta^m \stackrel{?}{\cong} h(\Delta^n \times \Delta^m)$
- $\Delta^n = N(\tilde{n})$ where \tilde{n} is some category $hN = \mathcal{E}$ is iso
- $(h\Delta^n) \times (h\Delta^m) \cong (hN\tilde{n}) \times (hN\tilde{m}) \cong \tilde{n} \times \tilde{m} \cong hN(\tilde{n} \times \tilde{m}) \cong h(N\tilde{n} \times N\tilde{m})$ ◀

- Hence, h and N are strong monoidal
- $h_* : \text{SCat} \xrightleftharpoons[\perp]{\quad} \mathcal{Q}\text{Cat} : N_*$

Def. $q\text{Cat}_\infty \hookrightarrow \underline{\text{sSet}} \rightsquigarrow q\text{Cat}_2 := h_* q\text{Cat}_\infty$
 $\downarrow \downarrow \downarrow$
 $\mathcal{Q}\text{-category of quasi-categories}$

$\text{Ob}(q\text{Cat}_2) = \{\text{quasi-categories}\}$

1-cells of $q\text{Cat}_2 = \{\text{maps of quasi-categories}\}$

2-cells of $q\text{Cat}_2 = \{\text{homotopy classes of homotopies}\}$

2-cell $\alpha: f \Rightarrow g \rightsquigarrow 1\text{-simplex } \tilde{\alpha}: f \rightarrow g \text{ in } Y^X$
 $\alpha_1 \sim \alpha_2 \Leftrightarrow \tilde{\alpha}_1 \sim \tilde{\alpha}_2 \text{ as 1-simplices in } Y^X$

Prop. qCat_2 is cartesian closed

- $h : \text{qCat} \rightarrow \underline{\text{Cat}}$ is a 2-functor $h^* : \text{qCat}_2 \rightarrow \underline{\text{Cat}}$

$$h(Y^X) \times hX \cong h(Y^X \times X) \xrightarrow{h(\text{ev})} hY$$

\uparrow adj

$$h(Y^X) \rightarrow hY^{hX}$$

} On hom-cats

- The aim: the category theory of quasi-categories
- qCat_2 has finite products (see above)
- 2-limits theory
- Cotensors with the walking arrow category \mathcal{Z}

Weak limits in $\text{qCat}_\mathbb{Z}$

$$\mathcal{F}: \text{qCat}_\mathbb{Z}(A, X^\mathbb{Z}) \cong h((X^\mathbb{Z})^A) \stackrel{?}{\cong} h((X^A)^\mathbb{Z}) \stackrel{?}{\cong} (h(X^A))^\mathbb{Z} \cong$$

$$\cong (\text{qCat}_\mathbb{Z}(A, X))^\mathbb{Z}$$

↑
 It defines a notion of cotensoring by \mathbb{Z}

cotensors
 commute with
 internal hom

- We require

$$h(X^\mathbb{Z}) \cong (hX)^\mathbb{Z} \longrightarrow \cdot$$

- In qCat_∞ , $X^{\Delta^1} \in \text{qCat}$. Also, $N\mathbb{Z} = \Delta^1$ and $h\Delta^1 = \mathbb{Z}$
 $h(X^{\Delta^1}) \longrightarrow (hX)^{h\Delta^1} \cong (hX)^\mathbb{Z}$ is not iso

• weak cotensor with \mathbb{Z}

Lemma (the universal property) The canonical comparison functor

$$h(X^{\Delta^1}) \rightarrow (hX)^{\mathbb{Z}}$$

is subjective on objects
full & conservative
reflects iso

it's smothering

- ▷ • Surjectivity: every arrow in hX is represented by a 1-simplex in X

- Fullness: find a morphism from $h(X^{\Delta^1})$ to

$$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ f \downarrow & & \downarrow g \\ \bullet & \xrightarrow{b} & \bullet \end{array}$$

$$\begin{array}{ccc}
 & \xrightarrow{a} & \\
 \bullet & \searrow \text{---} \swarrow & \bullet \\
 f \downarrow & \text{---} \sim & \downarrow g \\
 \bullet & \xrightarrow{b} & \bullet
 \end{array}
 \quad
 \begin{aligned}
 R &\sim ga \quad \text{in } hX \\
 R &\sim bf
 \end{aligned}$$

We have

$$\Delta^1 \times \Delta^1 \longrightarrow X \rightsquigarrow \Delta^1 \longrightarrow X^{\Delta^1}$$

It represents the desired arrow in $h(X^{\Delta^1})$

- Conservativity: omit it!



$$\cdot q\text{Cat}_2(A, X^{\Delta^1}) \cong h((X^{\Delta^1})^A)$$

$$\cong h((X^A)^{\Delta^1}) \rightarrow \left(h(X^A)\right)^2 = q\text{Cat}_2(A, X)^2$$

natural in A

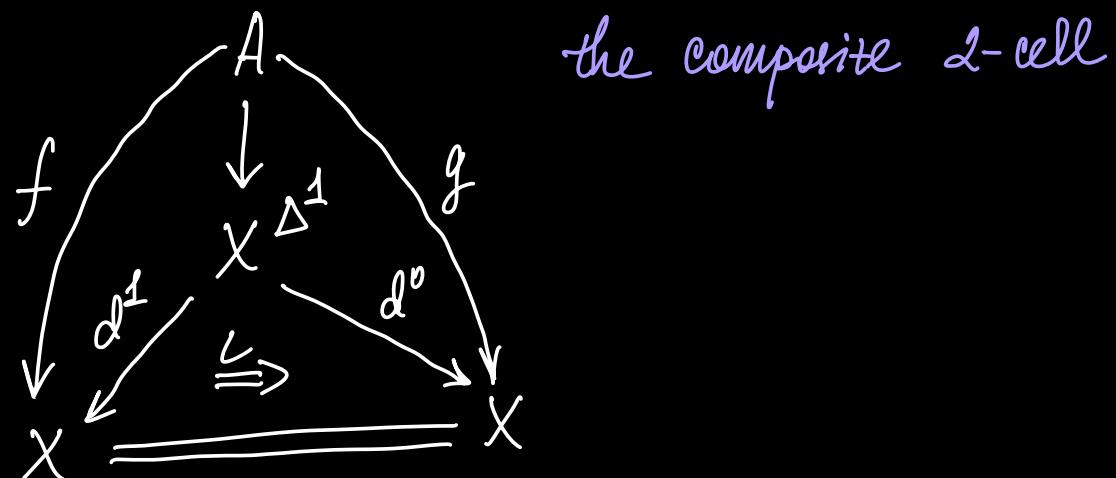
• Put $A = X^{\Delta^1}$ and image of id will be:

$$\begin{array}{ccc}
 & X^{\Delta^1} & \\
 d^1 \swarrow & \Downarrow & \searrow d^0 \\
 X & \xlongequal{\quad} & X
 \end{array}$$

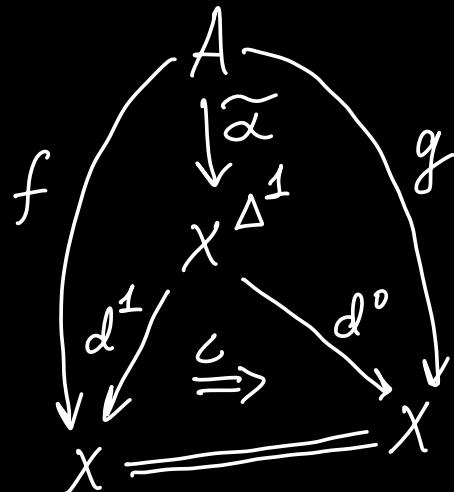
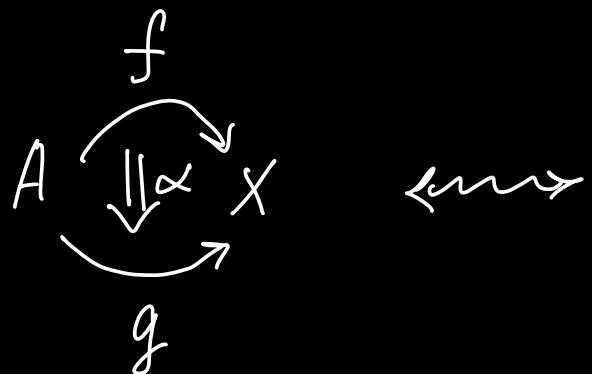
In general case:

$$A \rightarrow X^{\Delta^1} \in \text{Ob}(\text{qCat}(A, X^{\Delta^1}))$$

a homotopy from f to g



Surjectivity of $q\text{Cat}_2(A, X^{\Delta^1}) \rightarrow (q\text{Cat}_2(A, X))^2$ says:



$$l \cdot \tilde{\alpha} = \alpha$$

whiskering

- $\tilde{\alpha}$ is not unique: X^{Δ^1} is only a weak cotensor by \mathbb{Z}
- The universal property defines the arrow quasi-categories up to equivalence

Lemma Let \mathcal{Z} be in qCat , s.t. \exists natural transformation

$$h(\mathcal{Z}^A) \rightarrow (h(X^A))^2$$

is smothering (i.e., surjective on objects, full and conservative)

Then

$$\mathcal{Z} \cong X^{\Delta^1}$$

$\triangleright \bullet \exists A = \mathcal{Z}$, the image of $1_{\mathcal{Z}} \in h(\mathcal{Z}^{\mathcal{Z}})$ is

$$\begin{array}{ccc} & \mathcal{Z} & \\ e^1 \swarrow & \Rightarrow & \searrow e^0 \\ X & \xlongequal{\quad} & X \end{array}$$

• Surjectivity implies:

$$A \xrightarrow{\begin{matrix} f \\ \Downarrow \alpha \\ g \end{matrix}} X \iff$$

$$\begin{array}{ccccc} & A & & & \\ & \downarrow \alpha & & & \\ & \mathcal{Z} & & & \\ e^1 \swarrow & \Rightarrow \alpha & \searrow e^0 & & \\ X & \xlongequal{\quad} & X & & \end{array}$$

- Apply the weak univ. prop. of \mathcal{L} to \mathcal{R} :

$$\tilde{\mathcal{R}}: \mathcal{Z} \rightarrow X^{\Delta^1}$$

$$\begin{array}{c} \mathcal{Z} \\ \downarrow \tilde{\mathcal{R}} \\ X \xrightarrow{\cong} X \end{array}$$

$\mathcal{L} \cdot \tilde{\mathcal{R}} = \mathcal{R}$

- Apply the univ. prop. of \mathcal{R} to \mathcal{L} :

$$\tilde{\mathcal{L}}: X^{\Delta^1} \rightarrow \mathcal{Z}$$

$$\begin{array}{c} X^{\Delta^1} \\ \downarrow \tilde{\mathcal{L}} \\ \mathcal{Z} \\ \downarrow \mathcal{R} \\ X \xrightarrow{\cong} X \end{array}$$

- $X^{\Delta^1} \xrightarrow{\tilde{\mathcal{L}}} \mathcal{Z} \xrightarrow{\tilde{\mathcal{R}}} X^{\Delta^1}$

gives a factorization \mathcal{L}

through itself



by conservativity & fullness

$$\mathcal{F}(\tilde{\mathcal{R}}\tilde{\mathcal{L}}) \cong \mathcal{F}(1_{X^{\Delta^1}}) \Rightarrow \tilde{\mathcal{R}}\tilde{\mathcal{L}} \cong 1_{X^{\Delta^1}} \text{ in } h((X^{\Delta^1})^{X^{\Delta^1}})$$

$$\mathcal{F}: h((X^{\Delta^1})^{X^{\Delta^1}}) \rightarrow h(X^{(X^{\Delta^1})^2})$$

- Similarly,

$$\widetilde{R} \cong \mathbb{L}_{\mathbb{Z}} \text{ in } h(\mathbb{Z}^{\mathbb{Z}})$$

- These iso's are represented by

$$\mathbb{I} \rightarrow (X^{\Delta^1})^{X^{\Delta^1}} \quad \& \quad \mathbb{I} \rightarrow \mathbb{Z}^{\mathbb{Z}}$$

where $\mathbb{I} = \mathcal{N}(\bullet \cong \bullet)$

and

Recall $f: \Delta^1 \rightarrow X$ is an iso in a quasi-category $\Leftrightarrow \exists$ an extension to \mathbb{I}

- So, we have $X^{\Delta^1} \times \Delta^1 \xrightarrow{\cong} X^{\Delta^1}$ by adjunction
 $\mathbb{Z} \times \Delta^1 \xrightarrow{\cong} \mathbb{Z}$

△

Remark It can be generalized to any category freely generated by a graph categories

Lemma \mathcal{V} diagram
 $E \xrightarrow{p} B \leftarrow \begin{matrix} q \\ F \end{matrix}$ a fibration in Joyal's model
structure
cofibrant objects

of quasi-categories with q an isofibration

$$h(E \times_B F) \rightarrow hE \times_{hB} hF$$

is Smothering functor

Comma quasi-categories

$E \xrightarrow{p} B \xleftarrow{q} F$ is a diagram of quasi-categories

$$\begin{array}{ccc}
 p \downarrow q & \longrightarrow & B^{\Delta^1} \\
 \downarrow & \lrcorner & \downarrow \\
 E \times F & \xrightarrow{pxq} & B \times B \cong B^{\partial\Delta^1}
 \end{array}$$

— an isofibration (i.e., fibration
in Joyal's
model structure)

Corollary The canonical functor

$$h(p \downarrow q) \rightarrow h(E \times F) \times_{h(B \times B)} h(B^{\Delta^1}) \rightarrow (hE \times hF) \times_{hB \times hB} (hB)^{\Delta^1} = h(p) \downarrow h(q)$$

is smothering

|||||||||||
this is the usual comma
category in Cat

- The functor

$$h((p \downarrow q)^A) \rightarrow h(E^A) \times h(F^A) \times h(B^A)^2 = h(p^A) \downarrow h(q^A)$$

$\downarrow h(B^A) \times h(B^A)$

is also smothering

- The Weak universal property of $p \downarrow q$

$$\begin{array}{ccc} & p \downarrow q & \\ d_1 \swarrow & \Downarrow \chi & \searrow d_0 \\ E & & F \\ \searrow p & B & \swarrow q \\ & & \end{array}$$

the image of the identity at $p \downarrow q$

Surjectivity gives

$$\begin{array}{ccc} e_1 \swarrow & A & \searrow e_0 \\ E & \Downarrow \chi & F \\ \searrow p & B & \swarrow q \\ & & \end{array} = \begin{array}{ccc} & A & \\ \downarrow & \downarrow & \downarrow \\ E & p \downarrow q & F \\ \searrow g & B & \swarrow f \\ & & \end{array}$$

- By fullness & conservativity, one can derive that

if $\chi \cdot \underset{\text{whiskering}}{\overset{\leftarrow}{\circ}} (f, g : A \rightrightarrows p \downarrow q) = \alpha$

Then \exists an iso f

$$A \begin{array}{c} \Downarrow \cong \\ \circlearrowright \end{array} p \downarrow q$$

g

represented by a map $A \times \mathcal{I} \rightarrow p \downarrow q$

- $\Rightarrow p \downarrow q$ is unique up to equivalence

The definition of ∞ -cosmos

Def. An ∞ -cosmos is a simplicially enriched category K :

- objects by def. are ∞ -categories
- hom's are quasi-categories
- there are a subcategory of isofibrations $A \rightarrow\!\!\!> B$

s.t. the following axioms hold:

(a) Completeness. - K possesses a terminal object 1 :

cotensors A^T , $T \in \text{Set}$;

pullbacks of isofibrations along any functor

(b) Isofibrations. - The class of isofibrations \supset isomorphisms

$$!: A \begin{array}{c} \cup \\ \rightarrow \end{array} 1$$

- This class is stable under pullback along all functors

- $\exists p: E \rightarrow B$, $i: U \hookrightarrow V \Rightarrow i \hat{\wedge} p: E^V \rightarrow E^U \times_{B^U} B^V$
 $\text{in } \mathbf{Set}$

the Leibnitz cotensor

- $\forall X \in \mathcal{O}\mathcal{C}(K)$, $\forall p: E \rightarrow B \Rightarrow \text{fun}(X, p): \text{fun}(X, E) \rightarrow \text{fun}(X, B)$

(c) Equivalences. $f: A \xrightarrow{\sim} B$ is an equivalence when

$$\text{fun}(X, f): \text{fun}(X, A) \rightarrow \text{fun}(X, B)$$

is an equivalence of quasi-categories $\forall X \in K$

(d) Cofibrancy. All objects (say, A) are cofibrant

$$\begin{array}{ccc} & \exists & \rightarrow E \\ A & \nearrow & \downarrow 2 \\ & \longrightarrow B & \end{array}$$

From the axioms above one can derive the stability of Fib^{tr}

(d) Trivial fibrations. - They are defined to be in $WE \cap \text{Fib}$

- They define a subcat \supset Isomorphisms
- They are stable under pullback along all functors
- $i \pitchfork p: E^V \rightarrow E^U \times_{B^U} B^V \in \text{Fib}^{\text{tr}}$ $p: E \rightarrow B$
 $i: U \hookrightarrow V$

When $p \in \text{Fib}^{\text{tr}}$ or $i \in \text{Cofib}^{\text{tr}}$ in Joyal model structure on sSet

Brown factorization lemma

(e) Factorization. \forall functor $f: A \rightarrow B$ $f = pj$

$$\begin{array}{ccc} & q & \\ A & \xrightarrow{j} & N & \xrightarrow{p} & B \\ & f & \end{array}$$

(f) Cartesian closure. $- \times - : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ extends to

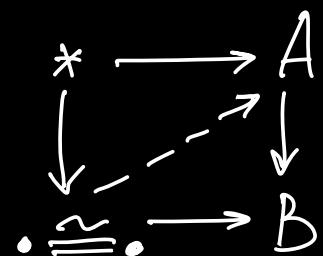
$$\text{fun}(A \times B, C) \cong \text{fun}(A, C^B) \cong \text{fun}(B, C^A)$$

Examples

M_{cf} enriched over the Joyal model structure on $sSet$

defines an ∞ -cosmos

- Cat — the ∞ -cosmos of small categories
- Isofib — the usual isofibrations of Cat
- Equiv — the usual equivalences of Cat
- $q\text{Cat}$ — the ∞ -cosmos of quasi- Cat
- CSS — the ∞ -cosmos complete Segal spaces
- Segal — the ∞ -cosmos of Segal Cat s
- SSet_+ — the ∞ -cosmos of naturally marked simplicial sets

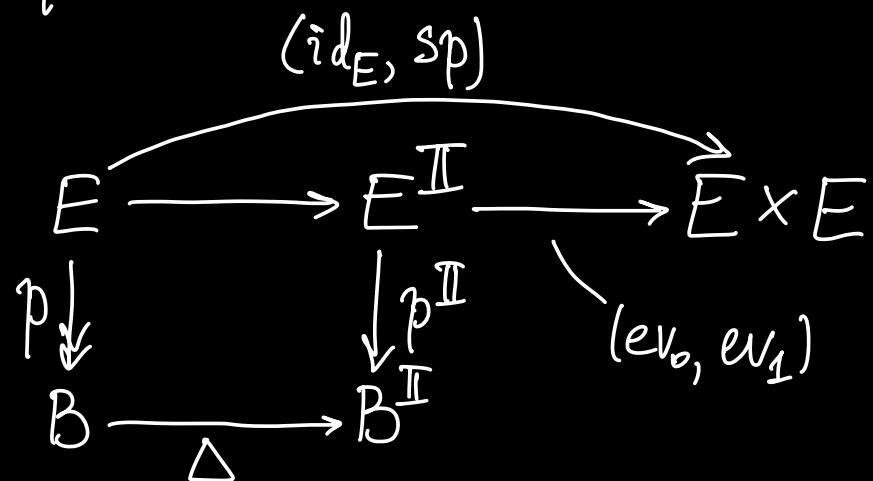


- $\mathbb{H}_n\text{-Sp}$ — the ∞ -cosmos of \mathbb{H}_n -spaces, a simplicial presheaf model of (∞, n) -categories
- $\text{Rezk}_{\mathcal{M}}$ — the ∞ -cosmos of Rezk objects in a nice model cat \mathcal{M}
 They are used to define iterated Segal spaces — another simplicial presheaf model of (∞, n) -categories

Lemma. The equivalences in an ∞ -cosmos are closed under retracts and satisfy 2-of-3 property

Lemma (trivial fibrations split)

The section defines a split fiber homotopy sequence



$$\begin{array}{ccc}
 E & \xrightarrow{s} & E \\
 \downarrow p & \lrcorner & \downarrow p \\
 B = B & &
 \end{array}$$

and conversely
A isofibration that
defines a split fiber
homotopy equivalence
is a trivial fibration

► If $p: E \xrightarrow{\sim} B$ is trivial $\Rightarrow p_*: \text{fun}(X, E) \xrightarrow{\sim} \text{fun}(X, B)$

$\forall \infty\text{-cat } X$

by the stability
properties of Fib^{tr}

$$\begin{array}{ccc} \emptyset = \partial \Delta[0] & \longrightarrow & \text{fun}(B, E) \\ \downarrow & \dashrightarrow s & \downarrow p_* - \text{the trivial fibration} \\ 1 = \Delta[0] & \xrightarrow{\text{id}_B} & \text{fun}(B, B) \end{array}$$

After that we can
solve the problem

$$\begin{array}{ccc} 1 + 1 & \xrightarrow{(\text{id}_E, sp)} & \text{fun}(E, E) \\ \downarrow & \dashrightarrow \alpha & \downarrow \\ \mathbb{I} & \xrightarrow{!} 1 & \xrightarrow{p} \text{fun}(E, B) \end{array}$$

Recall: Formal category theory in a 2-category

- Objects are called ∞ -categories
- 1-cells $f: A \rightarrow B$ is said to be (∞ -)functors
- 2-cells $A \begin{array}{c} \xrightarrow{f} \\[-1ex] \Downarrow \alpha \\[-1ex] \xrightarrow{g} \end{array} B$ – (∞ -)natural transformations

Def. An adjunction between ∞ -categories:

- $f: B \rightarrow A, u: A \rightarrow B$
- $\eta: id_B \Rightarrow uf \quad \& \quad \varepsilon: fu \Rightarrow id_A$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & \xrightarrow{u} & B \\
 & f \searrow & \swarrow u \\
 & & A
 \end{array} & = & u \left(\begin{array}{c} \xrightarrow{\qquad B \qquad} \\[-1ex] \Downarrow idu \\[-1ex] \xrightarrow{\qquad A \qquad} \end{array} \right) u \\
 & & f \downarrow \eta \qquad u \downarrow \varepsilon \qquad f \searrow \qquad \swarrow idf \\
 B & \xrightarrow{id_B} & B \\
 & f \searrow & \swarrow idf \\
 & & A
 \end{array}$$

Prop. Adjunctions compose:

$$\begin{array}{ccc} C & \xleftarrow{\perp} & B \\ f' \downarrow & & \downarrow u \\ C & \xleftarrow{\perp} & B \\ \downarrow u' & & \downarrow f \\ A & \xleftarrow{\perp} & A \end{array} \rightsquigarrow \begin{array}{ccc} C & \xleftarrow{\perp} & A \\ ff' \downarrow & & \downarrow u'u \\ C & \xleftarrow{\perp} & A \\ \downarrow u'u & & \downarrow f \\ A & \xleftarrow{\perp} & A \end{array}$$

Def. An equivalence between ∞ -categories consists of:

- a pair of ∞ -categories A and B
- $f: A \rightarrow B$ & $g: B \rightarrow A$ — functors
 fg

- $A \xrightleftharpoons[\text{gf}]{\cong \Downarrow \alpha} A$ & $B \xrightleftharpoons[\text{B}]{{\beta} \Downarrow \cong} B$

The homotopy 2-category of an ∞ -cosmos

Def. This is a strict 2-cat K_2 or hK so that

- $Ob(K_2) = Ob(K)$ — ∞ -categories
 - 1-cells $f: A \rightarrow B$ of K_2 are the vertices $f \in \underline{\text{fun}}(A, B)$
— ∞ -functors f
 \uparrow
 $q\text{Cat}$
 - 2 cells $A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} B$ in K_2 are homotopy classes of
1-simplices
 $\alpha: f \rightarrow g \in \underline{\text{fun}}(A, B)$
- $K_2 := h_* K$
- $\begin{array}{ccc} \mathcal{2}\text{-Cat} & \begin{array}{c} \xleftarrow{h_*} \\ \perp \end{array} & \text{sSet-Cat} \end{array}$
- The cat $q\text{Cat}_2$ was first introduced by Joyal

Or, just $h\mathcal{K}$:

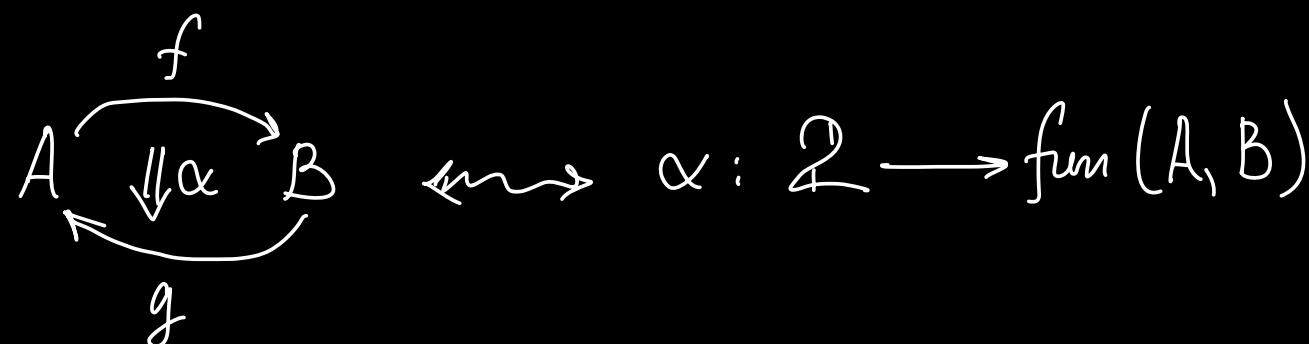
- $\text{ob}(h\mathcal{K}) := \text{ob}(\mathcal{K})$
- $\text{hom}(A, B) := h(\text{fun}(A, B))$

homotopy cat of quasi-cat

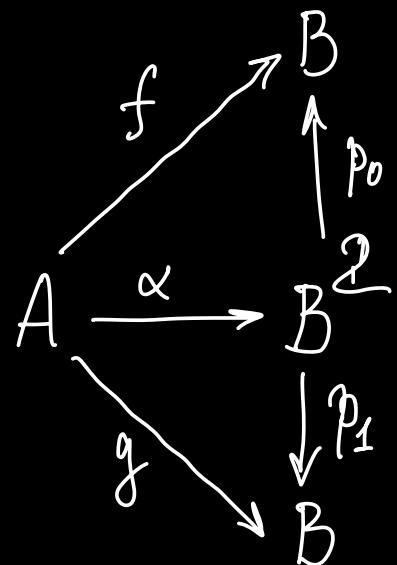
Def. The underlying cat of 2-cat — simply forgetting its 2-cells

$$u(\mathcal{K}) = u(h\mathcal{K})$$

Denote by $\mathcal{Q} := \mathcal{N}(\bullet \rightarrow \bullet) = \Delta^1$

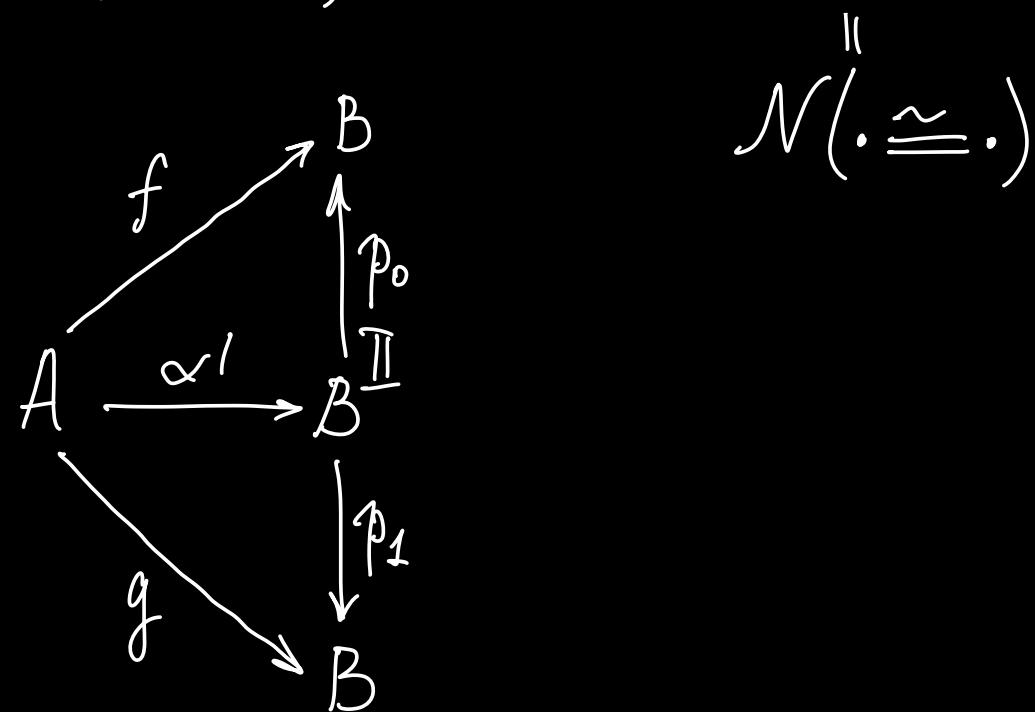


Transpose:



$A \xrightarrow{\downarrow \alpha} B$ is an iso in $\mathcal{K}_2 \Leftrightarrow \text{ho}(\alpha) : \text{ho} \mathbb{2} \rightarrow \text{ho} \text{fun}(A, B)$ is iso
 f
 g

$\Leftrightarrow \alpha : \mathbb{2} \rightarrow \text{fun}(A, B)$ extends to $\alpha' : \mathbb{I} \rightarrow \text{fun}(A, B)$



Lemma. (i) \forall 2-cell $A \xrightarrow{\text{f}} B$ in \mathcal{K}_2
 is repr. by a map in qCat in the homotopy 2-cat
 of an ∞ -cosmos

$$\begin{array}{ccc}
 \mathbb{I} + \mathbb{I} & \xrightarrow{(f,g)} & \text{fun}(A, B) \\
 \curvearrowleft & & \curvearrowright \\
 \mathbb{I} & \xrightarrow{\alpha} & \text{fun}(A, B) \\
 & \alpha \uparrow & \\
 & \text{qCat}_1 &
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{\lceil \alpha \rceil} & B^2 \\
 & \searrow (g,f) & \swarrow (p_1, p_0) \\
 & B \times B &
 \end{array}$$

(ii) \forall invertible 2-cell $A \xrightarrow{\cong, \text{f}, \text{g}} B$ in \mathcal{K}_2

$$\begin{array}{ccc}
 \mathbb{I} + \mathbb{I} & \xrightarrow{(f,g)} & \text{fun}(A, B) \\
 \curvearrowleft & & \curvearrowright \\
 \mathbb{I} & \xrightarrow{\alpha} & \text{fun}(A, B)
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \xrightarrow{\lceil \alpha \rceil} & B^{\mathbb{I}} \\
 & \searrow (g,f) & \swarrow (p_1, p_0) \\
 & B \times B &
 \end{array}$$

Functors between $h\mathcal{K}$

Lemma. \forall simplicial functor $F: \mathcal{K} \xrightarrow{\sim} \mathcal{L}$ induces
a \mathcal{L} -functor $F: \mathcal{K}_\bullet \rightarrow \mathcal{L}_\bullet$

▷ On objects and 1-cells — it's okay

$$\begin{array}{ccc} \text{fun}(A, B) & \xrightarrow{F} & \text{fun}(FA, FB) \\ f \downarrow \alpha & & Ff \downarrow F\alpha \\ A \xrightarrow{\quad g \quad} B & \mapsto & FA \xrightarrow{\quad Fg \quad} FB \end{array}$$

F is a morphism of simplicial sets

So, homotopic 1-cells map to homotopic ones



Cartesian closure and products

Prop. (i) \mathcal{K}_2 has 2-categorical products
(ii) \mathcal{K}_2 is cartesian closed as 2-cat

Theorem (Equivalences of ∞ -categories are 2-cat-equiv.)

In any ∞ -cosmos \mathcal{K} the F.A.E.:

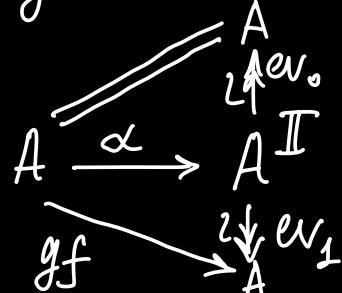
(i) $\forall X \in \mathcal{K}$

$$f_* : \text{fun}(X, A) \xrightarrow{\sim} \text{fun}(X, B)$$

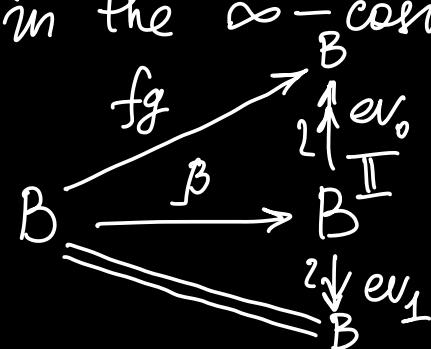
- equiv. of quasi-cat

(ii) $\exists g: B \rightarrow A$ & $\alpha: \text{id}_A \cong gf$, $\beta: fg \cong \text{id}_B$ in
the homotopy 2-cat

(iii) $\exists g: B \rightarrow A$ and maps in the ∞ -cosmos \mathcal{K}



&



Thank you!