Simpl. mod cats

Def. A bifunct. $- \otimes -$ betw. mod. cats is a left Quillen bifunct. if it preserves colims in both var. and if $\underbrace{- \hat{\otimes} -}_{\text{pushout bifunctor}}$ maps cofibs to cofib that is acyclic $\Leftrightarrow$ either of the domain are cofib.

Leibnitz construction

$- \otimes - : \mathcal{M} \times \mathcal{N} \to \mathcal{P}$, $\{-,-\} : \mathcal{M}^{op} \times \mathcal{P} \to \mathcal{N}$,

$\underline{hom}(-,-) : \mathcal{N}^{op} \times \mathcal{P} \to \mathcal{M}$

$$\mathcal{P}(m \otimes n, p) \cong \mathcal{N}(n, \{m, p\}) \cong \mathcal{M}(m, hom(n, p))$$

$- \hat{\otimes} - : \mathcal{M}^2 \times \mathcal{N}^2 \to \mathcal{P}^2$, $\{- \hat{,} -\} : (\mathcal{M}^2)^{op} \times \mathcal{P}^2 \to \mathcal{N}^2$

$\underline{\widehat{hom}}(\dots)$

$i : m \to m' \in \mathcal{M}^2$

$j : n \to n' \in \mathcal{N}^2$

$i \hat{\otimes} j$ is a pushout prod.

$$\begin{tikzcd}
m \otimes n \arrow[r, "i \otimes 1"] \arrow[d, "1 \otimes j"'] & m' \otimes n \arrow[d] \arrow[ddr, bend left] & \\
m \otimes n' \arrow[r] \arrow[drr, bend right] & \bullet \arrow[dr, "i \hat{\otimes} j"] & \\
 & & m' \otimes n'
\end{tikzcd}$$

$f : p \to p'$
$i : m \to m'$

$$\begin{tikzcd}
\{m', n'\} \arrow[r] \arrow[rr, bend left] \arrow[dr, bend right] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{m, p\} \arrow[d] \\
& \{m', p'\} \arrow[r] & \{m, p'\}
\end{tikzcd}$$

$\{i, \hat{f}\}$ is called a pullback-cotens.

Example. $\emptyset \to m$, $j : n \to n'$

$i \hat{\otimes} j : m \otimes n \to m \otimes n'$

Prop. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be classes of maps in M, N, P

$$\mathcal{A} \hat{\otimes} \mathcal{B} \boxslash \mathcal{C} \Leftrightarrow \mathcal{B} \boxslash \widehat{\{A, \mathcal{C}\}} \Leftrightarrow \mathcal{A} \boxslash \widehat{hom}(\mathcal{B}, \mathcal{C})$$

Prop. Left Quillen bifunctors are homotopical on the subcategory of cofibr. objects and preserve cofibr. obj.

Proof. $i : m \to m'$, $j : n \to n'$: $\mathcal{WE}$

$i \otimes j : \mathcal{WE}$

$$(i \otimes j : m \otimes n \to m' \otimes n') = m \otimes n \xrightarrow{i \otimes 1} m' \otimes n \xrightarrow{1 \otimes j} m' \otimes n'$$

$i \otimes 1$ is a pushout $i \hat{\otimes} (\emptyset \to n)$

i is a triv. cofib $\Rightarrow$ pushout of i is $\mathcal{WE}$ $\triangleleft$

Prop. If $(\otimes, \{\,,\,\}, \underline{hom})$ is a two-variable adj then $\otimes$ is a left Quillen adj $\Leftrightarrow$

$\iff \{\ ,\ \}$ is a right —"— $\iff \underline{\hom}$ is a righ -

Def. A simpl. model cat. is a model cat $\mathcal{M}$ that is tens., cotens., simplicially enr. and s.t. $(\otimes, \{\cdot,\cdot\}, \underline{\hom})$ is a Quillen two-variable adj

Final functors in unenriched cats

Lemma. Supp. $\mathcal{D}$ has a term. object t and $F : \mathcal{D} \to \mathcal{M}$

$$\operatorname{colim}_{\mathcal{D}} F \cong Ft$$

▷ Consider a cocone $F \to x$

$\exists\ \tau : Ft \to x$ — by def. of a cocone

$$\begin{tikzcd}
Fd_1 \arrow[r] \arrow[dr] \arrow[ddr, bend right] & Fd_2 \arrow[d] \arrow[dd, bend left] \\
& Ft \arrow[d, "\tau"] \\
& x
\end{tikzcd}
\qquad
\begin{tikzcd}
Ft \arrow[r] \arrow[dr] \arrow[ddr, bend right] & Ft \arrow[d, "id"'] \arrow[dd, bend left, "\tau"] \\
& Ft \arrow[d, "\rho"] \\
& x
\end{tikzcd}$$

$\rho \circ id = \tau$ ◁

Def. A functor $K : \mathcal{C} \to \mathcal{D}$ is final if for any $F : \mathcal{D} \to \mathcal{M}$

$$\operatorname{colim}_{\mathcal{C}} FK \xrightarrow{\cong} \operatorname{colim}_{\mathcal{D}} F$$

K is initial if $\forall\ F : \mathcal{D} \to \mathcal{M}$

$$\lim_{\mathcal{D}} F \xrightarrow{\cong} \lim_{\mathcal{C}} FK$$

Example $t : \mathbb{1} \longrightarrow \mathcal{D}$, where image is a term. obj

t is final

Lemma. $K : \mathcal{C} \to \mathcal{D}$ is final $\Longleftrightarrow$ $\forall d \in \mathcal{D}_0$

d/K is non-empty and connected

$$\bullet \longleftrightarrow \bullet \longrightarrow \bullet \longleftarrow \cdots$$

$\forall$ any two obj $\overset{\text{def}}{\Longrightarrow}$ the cat is called connected

$\pi_0 : Cat \longrightarrow Set$

Proof $\Leftarrow$ • $F : \mathcal{D} \to M$

Cone under F $\rightsquigarrow$ cone over FK

• But converse is also true

$\lambda_c : FKc \longrightarrow m$

$\exists d \longrightarrow Kc$ as the cat d/K is non-empt. and connected

$Fd \xrightarrow{F-} FKc \xrightarrow{\lambda_c} m \rightsquigarrow$ we have a cocone $Fd \to m$

$$\begin{tikzcd}
& Fd \arrow[dl] \arrow[dr] & \\
FKc_1 \arrow[rr] \arrow[dr] & & FKc_2 \arrow[dl] \\
& m &
\end{tikzcd}$$

This diagram means that the set of all $\underbrace{\text{cones under } FK}_{\text{(cocone)}}$ is isom. to the one for F

$\Rightarrow$ we have iso betw. colim's

$\Rightarrow$ $X : \mathcal{C} \to Set$

$$\pi_0(\mathcal{C} \int X) \cong \operatorname{colim}_{\mathcal{C}} X$$

$$(\mathcal{C}\int X)_0 = \{(c, x) \mid c \in \mathcal{C}_0,\ x \in Xc\}$$

$$(\mathcal{C}\int X)_1 = \{u : c \to c' \mid X(u)(x) = x'\}$$

$$\pi_0(d/K) \cong \pi_0(\mathcal{C} \int \mathcal{D}(d, K-)) \cong$$

$$\cong \operatorname{colim}_{\mathcal{C}} \underbrace{\mathcal{D}(d, K-)}_{\mathcal{D}(d,-)\circ K} \cong \operatorname{colim}_{\mathcal{D}} \mathcal{D}(d,-) \cong *$$

$\triangleleft$

$\Rightarrow$ d/K is non-empty and connected

Corollary. $\forall$ cat $\mathcal{D}$ $\operatorname{colim}_{\mathcal{D}} 1_{\mathcal{D}} = t$

Conv., any termin. obj. defines a $\operatorname{colim}_{\mathcal{D}} 1_{\mathcal{D}}$

Proof. Let t be a term. for $\mathcal{D}$

$t : 1 \to \mathcal{D}$ is final $\Rightarrow$ $\operatorname{colim}_{\mathcal{D}} 1_{\mathcal{D}} \cong \operatorname{colim}_{\mathcal{D}} t \circ 1_{\mathcal{D}} \cong t$ $\triangleleft$

Another impl. is exercise

Homotopy final functor

Def. $K : \mathcal{C} \to \mathcal{D}$ is a homotopy final if $N(d/K)$ is contractible for all $d \in \mathcal{D}_0$ and it is homotopy initial if $N(K/d)$ is contractible

Lemma. If $\mathcal{D}$ has a init or term obj $\Rightarrow N\mathcal{D}$ is contr.

Proof. $\emptyset \Rightarrow 1$

$\alpha : F \Rightarrow G$ $\rightsquigarrow$ $\mathcal{C} \times 2 \to \mathcal{D}$

$F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{D}$

$$N(\mathcal{C}) \times \Delta^1 \longrightarrow N(\mathcal{D})$$

◁

We have a contr. homotopy

Def. $\mathcal{C}$ is filtered if $\forall$ diagram $\exists$ cocone of this diagram

Lemma. If $K : \mathcal{C} \to \mathcal{D}$ is homotopy final $\Rightarrow$ K is final. Conversely, if $\mathcal{C}$ is filtered, $K : \mathcal{C} \to \mathcal{D}$ is final $\Rightarrow$ K is homotopy final

Proof. ($\Rightarrow$) $\cdot \ \forall \mathcal{C}$ $\pi_0 \mathcal{C} \cong \pi_0 N\mathcal{C}$

- If K is homotopy final $\Rightarrow N(d/K)$ is contr. $\Rightarrow$

$\Rightarrow \pi_0(d/K) \cong \pi_0(N(d/K)) \cong *$ (by prev. lemma)

d/K is non-empty and connected

($\Leftarrow$)] $K : \mathcal{C} \to \mathcal{D}$ is final

- d/K is filtered (exercise)
- Use the classical result that nerve of filtered cats is contr.

◁

Theorem (homotopy fin. th. or motivation for hom. fin. funct)

$\exists\, F : \mathcal{D} \to \mathcal{M}$ — diagram in simpl. mod. cat.

If $K : \mathcal{C} \to \mathcal{D}$ is homotopy final then

$$\operatorname*{hocolim}_{\mathcal{C}} FK \simeq \operatorname*{hocolim}_{\mathcal{D}} F$$

Proof

$$\begin{array}{ccc} N(d/K) & \xrightarrow{\ \sim\ } & N(d/\mathcal{D}) \\ \wr\wr & & \wr\wr \\ pt & & pt \end{array}$$

- $N(-/K) \cong \mathcal{B}(*, \mathcal{C}, \mathcal{D}(-, K-))$
- $N(-/\mathcal{D}) \cong \mathcal{B}(*, \mathcal{D}, \mathcal{D})$, $\mathcal{D} = \mathcal{D}(-,-)$
- $\mathcal{B}(-, \mathcal{D}, QF)$ preserves WE

$$\begin{tikzcd}
\mathcal{B}(\underbrace{\mathcal{B}(*,\mathcal{C},\mathcal{D}(-,K-))}_{\simeq N(-/K)}, \mathcal{D}, QF) \arrow[r, "\simeq WE"] \arrow[d, "\cong"'] & \mathcal{B}(\underbrace{\mathcal{B}(*,\mathcal{D},\mathcal{D})}_{\simeq pt}, \mathcal{D}, QF) \arrow[d, "\cong"'] \\
\mathcal{B}(*, \mathcal{C}, \mathcal{B}(\mathcal{D}(-,K-), \mathcal{D}, QF)) \arrow[r] \arrow[d, "\cong"'] & \mathcal{B}(*, \mathcal{D}, \mathcal{B}(\mathcal{D},\mathcal{D},QF)) \arrow[d, "\cong"'] \\
\mathcal{B}(*, \mathcal{C}, QFK) \arrow[r] \arrow[d, "\cong"'] & \mathcal{B}(*, \mathcal{D}, QF) \arrow[d, "\cong"'] \\
\operatorname*{hocolim}_{\mathcal{C}} FK \arrow[r] & \operatorname*{hocolim}_{\mathcal{D}} F
\end{tikzcd}$$

$$B(\mathcal{D}(-,K-),\mathcal{D},QF) \xrightarrow{\simeq} QFK$$

By 2-of-3 property we obtain that

$$\operatorname{hocolim}_{\mathcal{C}} FK \xrightarrow{\simeq} \operatorname{hocolim}_{\mathcal{D}} F \quad \triangle$$

<u>Corollary.</u> (Quillen's theorem A)

If $K:\mathcal{C}\to\mathcal{D}$ is homotopy final

$$N\mathcal{C} \xrightarrow{\simeq} N\mathcal{D}$$

<u>Proof.</u> $|N\mathcal{C}| = B(*,\mathcal{C},*) \simeq \operatorname{hocolim}_{\mathcal{C}} *$

$$\begin{array}{l} \operatorname{hocolim}_{\mathcal{C}} (\underbrace{\mathcal{C} \xrightarrow{K} \mathcal{D} \xrightarrow{*} Top}_{*K}) \xrightarrow{\sim} \operatorname{hocolim}_{\mathcal{D}} * = |N\mathcal{D}| \\ \quad\| \\ |N\mathcal{C}| \end{array} \quad \triangle$$

<u>Example.</u> $\mathcal{D}$ has a term. obj t

$t: \mathbb{1} \to \mathcal{D}$ is final, $\mathbb{1}$ is filtered $\Rightarrow$

$\Rightarrow t$ is homotopy final $\Rightarrow \operatorname{hocolim}_{\mathcal{D}} F \simeq$

$\simeq \operatorname{hocolim}_{\mathcal{D}} F\circ t \simeq Ft$

$\mathbb{2} \xrightarrow{f} \mathcal{D}$, $f(0)=X$, $f(1)=Y$, $f(0\to 1) = X \xrightarrow{f} Y$

$\operatorname{hocolim}_{\mathbb{2}} f \simeq Y \simeq Cyl(f)$

Example. $\vec{\Delta} \overset{i}{\hookrightarrow} \Delta$ is homotopy initial functor

$m/[n]$ is contractible $\forall m$

$S : m/[n] \circlearrowleft$

$\alpha : [k] \to [n] \longmapsto S\alpha : [k+1] \to [n]$

$E(-) \equiv ([0] \xrightarrow{0} [n])$ $\quad \begin{cases} S\alpha(0) = 0, \\ S\alpha(i) = \alpha(i-1), \quad i > 0 \end{cases}$

$id \Rightarrow S \Leftarrow E$ Apply $N(-)$

$N(m/[n])$ is contr.

Thus, $\operatorname{hocolim}_{\Delta^{op}} X \cong \operatorname{hocolim}_{\vec{\Delta}} X$

Fat realization

$$\begin{tikzcd} \vec{\Delta} \arrow[r, "\Delta^{\bullet}"] \arrow[d, hook] & sSet \\ sSet \arrow[ur, dashed] & \end{tikzcd}$$

Example. $\Delta : \Delta \to \Delta \times \Delta$ is homotopy initial

$\Rightarrow \operatorname{hocolim}_{\Delta^{op} \times \Delta^{op}} X_{\bullet,\bullet} \cong \operatorname{hocolim}_{\Delta^{op}} \operatorname{diag}(X_{\bullet,\bullet})$

Example. $f : 2 \to Top$

$N(2)$ is 1-skeletal $\qquad f : X \to Y$

So, $B_\bullet(*, 2, f)$ is 1-sk. too

$B_0 = X \sqcup Y, \quad B_1 = X^0 \sqcup X^f \sqcup Y^1$

$$\begin{tikzcd}[column sep=small]
X^0 \arrow[d, leftrightarrow] & \sqcup & X^f \arrow[d, leftrightarrow] & \sqcup & Y^1 \arrow[d, leftrightarrow] \\
0 \arrow[loop below] & & 0 \to 1 & & 1 \arrow[loop below]
\end{tikzcd}$$

$$\mathrm{hocolim}(f) \cong |B_\bullet(*, \mathcal{I}, f)| \cong$$

$$\cong \Delta^\bullet \underset{\Delta^{op}}{\otimes} B_\bullet(*, \mathcal{I}, f) \cong \int^{n \in \Delta} \Delta^n \otimes B_n(*, \mathcal{I}, f) \cong$$

$$\cong \mathrm{coeq}\left[\coprod_{f:[m]\to[n] \in \Delta_1} \Delta^m \otimes B_n \overset{f^*}{\underset{f_*}{\rightrightarrows}} \coprod_{[n] \in \Delta_0} \Delta^n \otimes B_n \right] \cong$$

$$\cong \mathrm{colim} \left(\begin{tikzcd}
\Delta^1 \otimes B_0 \arrow[r, "s_0"] \arrow[dr, "s^0"'] & \Delta^1 \otimes B_1 \\
\Delta^0 \otimes B_1 \arrow[ur, "d^0", shift left] \arrow[ur, "d^1"', shift right] \arrow[r, shift left] \arrow[r, shift right, "d_0, d_1"'] & \Delta^0 \otimes B_0
\end{tikzcd} \right) \cong \frac{X \times I \sqcup Y}{(x,1) \sim f(x)}$$

$f : [1] \to [0] \rightsquigarrow \Delta^1 \to \Delta^0$ is s^0

$\rightsquigarrow B_0 \to B_1 : s_0$

$g_{0,1} : [0] \to [1] \rightsquigarrow \Delta^0 \to \Delta^1 \quad d^0, d^1$

$\rightsquigarrow B_1 \to B_0 \quad d_0, d_1$