

Simpl. mod cats

Def. A bifunct. $- \otimes -$ betw. mod. cats is a left Quillen bifunct. if it preserves colims in both var. and if $\underbrace{- \hat{\otimes} -}_{\text{pushout bifunctor}}$ maps cofibs to cofib that is acyclic \Leftrightarrow either of the domain are cofib.

Leibnitz construction

$$- \otimes - : M \times N \rightarrow P, \quad \{-, -\} : M^{\text{op}} \times P \rightarrow N,$$

$$\underline{\text{hom}}(-, -) : N^{\text{op}} \times P \rightarrow M$$

$$P(m \otimes n, p) \cong N(n, \{m, p\}) \cong M(m, \text{hom}(n, p))$$

$$- \hat{\otimes} - : M^2 \times N^2 \rightarrow P^2, \quad \{-, -\} : (M^2)^{\text{op}} \times P^2 \rightarrow N^2$$

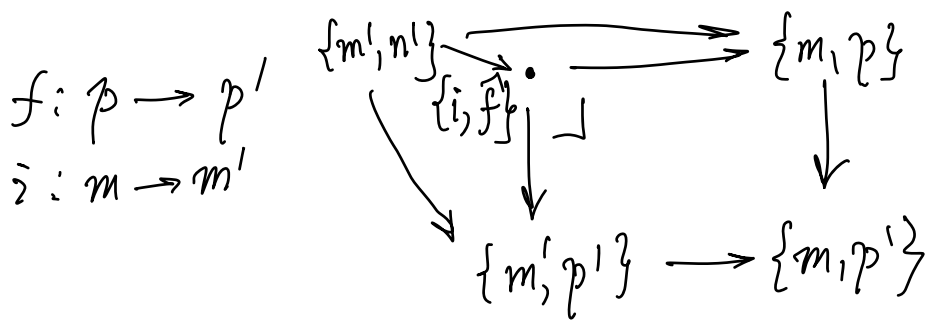
$$\underline{\text{hom}}(\dots)$$

$$i : m \rightarrow m' \in M^2$$

$$j : n \rightarrow n' \in N^2$$

$$\begin{array}{ccc} m \otimes n & \xrightarrow{i \otimes 1} & m' \otimes n \\ 1 \otimes j \downarrow & & \downarrow i \hat{\otimes} j \\ m \otimes n' & \xrightarrow{\quad \quad} & m' \otimes n' \end{array}$$

$i \hat{\otimes} j$ is a pushout prod.



$\{i, \hat{f}\}$ is called a pullback-cotens.

Example. $\emptyset \rightarrow m, j: n \rightarrow n'$

$$i \hat{\otimes} j: m \otimes n \rightarrow m \otimes n'$$

Prop. Let $\mathcal{H}, \mathcal{B}, \mathcal{L}$ be classes of maps in $\mathcal{M}, \mathcal{N}, \mathcal{P}$

$$\mathcal{A} \hat{\otimes} \mathcal{B} \triangleright \mathcal{L} \Leftrightarrow \mathcal{B} \triangleright \{ \mathcal{A}, \mathcal{L} \} \Leftrightarrow \mathcal{A} \triangleright \hat{\text{hom}}(\mathcal{B}, \mathcal{L})$$

Prop. Left Quillen bifunctors are homotopical on the subcategory of cofibr objects and preserve cofibr. obj.

Proof. $i: m \rightarrow m', j: n \rightarrow n' : \mathcal{W}\mathcal{E}$

$$i \otimes j: \mathcal{W}\mathcal{E}$$

$$(i \otimes j: m \otimes n \rightarrow m' \otimes n') = m \otimes n \xrightarrow{i \otimes 1} m' \otimes n \xrightarrow{1 \otimes j} m' \otimes n'$$

$i \otimes 1$ is a pushout $i \hat{\otimes} (\emptyset \rightarrow n)$

i is a riv. cofib \Rightarrow pushout of i is $\mathcal{W}\mathcal{E} \triangleleft$

Prop. If $(\otimes, \{, \}, \text{hom})$ is a two-variable adj then \otimes is a left Quillen adj \Leftrightarrow

$\Leftrightarrow \{ , \}$ is a right \dashv hom is a right

Def. A simpl. model cat. is a model cat M that is ten., coten., simplicially enr. and s.t. $(\otimes, \{ \cdot, \cdot \}, \text{hom})$ is a Quillen two-variable adj.

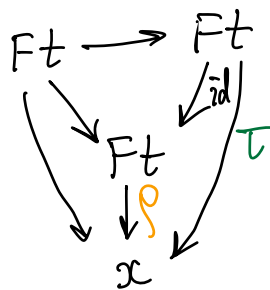
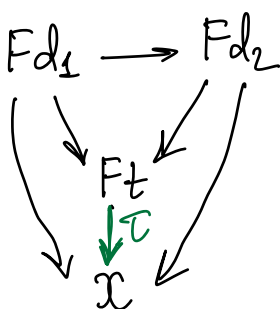
Final functors in unenriched cats

Lemma. Supp. \mathcal{D} has a term. object t and $F: \mathcal{D} \rightarrow M$

$$\text{colim}_{\mathcal{D}} F \cong Ft$$

▷ Consider a cocone $F \rightarrow x$

$\exists \tau: Ft \rightarrow x$ — by def. of a cocone



$$\rho \cdot \text{id} = \tau$$



Def. A functor $K: \mathcal{C} \rightarrow \mathcal{D}$ is final if for any

$$F: \mathcal{D} \rightarrow M \quad \text{colim}_{\mathcal{C}} FK \xrightarrow{\cong} \text{colim}_{\mathcal{D}} F$$

K is initial if $\forall F: \mathcal{D} \rightarrow M$

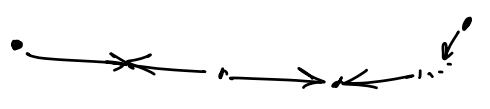
$$\lim_{\mathcal{D}} F \xrightarrow{\cong} \lim_{\mathcal{C}} FK$$

Example $t: \mathbb{1} \rightarrow \mathcal{D}$, where image is a term. obj
 t is final

Lemma. $K: \mathcal{C} \rightarrow \mathcal{D}$ is final $\Leftrightarrow \forall d \in \mathcal{D}_0$

\mathcal{d}/K is non-empty and connected

\forall any two obj $\xrightarrow{\text{def}}$ the cat is called connected



$\pi_0: \text{Cat} \rightarrow \text{Set}$

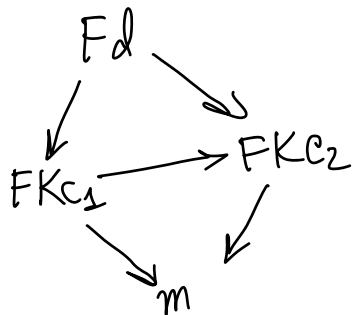
Proof \Leftarrow • $F: \mathcal{D} \rightarrow \mathcal{M}$
 Cone under $F \rightsquigarrow$ cone over FK

• But converse is also true

$$\lambda_c: FK_c \rightarrow m$$

$\exists d \rightarrow Kc$ at the cat \mathcal{d}/K is non-empt. and connected

$Fd \xrightarrow{F-} FKc \xrightarrow{\lambda_c} m \rightsquigarrow$ we have a cocone $Fd \rightarrow m$



This diagram means that the set of all cones under FK (cocone) is isom. to the one for F

\Rightarrow we have iso betw. colim's

$$\Rightarrow X: \mathcal{C} \rightarrow \mathbf{Set}$$

$$\pi_0(\mathcal{C} \int X) \cong \operatorname{colim}_{\mathcal{C}} X$$

$$(\mathcal{C} \int X)_0 = \{(c, x) \mid c \in \mathcal{C}_0, x \in Xc\}$$

$$(\mathcal{C} \int X)_1 = \{u: c \rightarrow c' \mid X(u)(x) = x'\}$$

$$\pi_0(d/K) \cong \pi_0(\mathcal{C} \int \mathcal{D}(d, K-)) \cong$$

$$\cong \operatorname{colim}_{\mathcal{C}} \mathcal{D}(d, K-) \cong \operatorname{colim}_{\mathcal{D}} \mathcal{D}(d, -) \cong *$$

$$\mathcal{D}(d, -) \circ K$$

$\Rightarrow d/K$ is non-empty and connected △

Corollary. $\forall \text{ cat } \mathcal{D} \quad \operatorname{colim}_{\mathcal{D}} 1_{\mathcal{D}} = t$
 Conv., any termin. obj. defines a $\operatorname{colim}_{\mathcal{D}} 1_{\mathcal{D}}$

Proof. Let t be a term. for \mathcal{D}
 $t: 1 \rightarrow \mathcal{D}$ is final $\Rightarrow \operatorname{colim}_{\mathcal{D}} 1_{\mathcal{D}} \cong \operatorname{colim}_{\mathcal{D}} t \circ 1_{\mathcal{D}} \cong t$ △

Another impl. is exercise

Homotopy final functor

Def. $K: \mathcal{C} \rightarrow \mathcal{D}$ is a homotopy final if
 $\mathcal{N}(d/K)$ is contractible for all $d \in \mathcal{D}_0$ and

it is homotopy initial if $N(K/d)$ is contractible

Lemma. If \mathcal{D} has a init or term obj $\Rightarrow N\mathcal{D}$ is contr.

Proof. $\emptyset \Rightarrow 1$

$$\begin{array}{ccc} \alpha: F \Rightarrow G & \rightsquigarrow & \mathcal{E} \times 2 \rightarrow \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{D} & & \mathcal{D} \end{array} \quad N(\mathcal{E}) \times \Delta^1 \rightarrow N(\mathcal{D})$$

We have a commutative homotopy

Def. \mathcal{E} is filtered if \forall diagram \exists cocone of this diagram

Lemma. If $K: \mathcal{E} \rightarrow \mathcal{D}$ is homotopy final \Rightarrow
 $\Rightarrow K$ is final. Conversely, if \mathcal{E} is filtered, $K: \mathcal{E} \rightarrow \mathcal{D}$
 is final $\Rightarrow K$ is homotopy final

Proof. $(\Rightarrow) \cdot \forall \mathcal{E} \quad \pi_0 \mathcal{E} \cong \pi_0 N\mathcal{E}$

\cdot If K is homotopy final $\Rightarrow N(d/K)$ is contr. \Rightarrow
 $\Rightarrow \pi_0(d/K) \cong \pi_0(N(d/K)) \cong *$
 d/K is non-empty and connected (by prev. lemma)

$(\Leftarrow) \cdot K: \mathcal{E} \rightarrow \mathcal{D}$ is final

$\cdot d/K$ is filtered (exercise)

\cdot Use the classical result that nerve of filtered cats is contr.

Theorem (homotopy fn. th. or motivation for hom. fin. funct)

$\square F: \mathcal{D} \rightarrow \mathcal{M}$ - diagram in simpl. mod. cat.

If $K: \mathcal{C} \rightarrow \mathcal{D}$ is homotopy final then

$$\operatorname{hocolim}_{\mathcal{C}} FK \cong \operatorname{hocolim}_{\mathcal{D}} F$$

Proof $\underset{\substack{\text{pt} \\ \downarrow}}{\mathcal{N}(d/k)} \xrightarrow{\sim} \underset{\substack{\text{pt} \\ \downarrow}}{\mathcal{N}(d/\mathcal{D})}$

• $\mathcal{N}(-/k) \cong \mathcal{B}(*, \mathcal{C}, \mathcal{D}(-, K-))$

• $\mathcal{N}(-/\mathcal{D}) \cong \mathcal{B}(*, \mathcal{D}, \mathcal{D})$

• $\mathcal{B}(-, \mathcal{D}, QF)$ preserves $W_{\mathcal{C}}^{\mathcal{E}}$

$$\begin{array}{ccc} \mathcal{B}(\mathcal{B}(*, \mathcal{C}, \mathcal{D}(-, K-)), \mathcal{D}, QF) & \xrightarrow{\cong W_{\mathcal{C}}^{\mathcal{E}}} & \mathcal{B}(\mathcal{B}(*, \mathcal{D}, \mathcal{D}), \mathcal{D}, QF) \\ \downarrow \cong & & \downarrow \cong \text{pt} \\ \mathcal{B}(*, \mathcal{C}, \mathcal{B}(\mathcal{D}(-, K-), \mathcal{D}, QF)) & \longrightarrow & \mathcal{B}(*, \mathcal{D}, \mathcal{B}(\mathcal{D}, \mathcal{D}, QF)) \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{B}(*, \mathcal{C}, QFK) & \longrightarrow & \mathcal{B}(*, \mathcal{D}, QF) \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{hocolim}_{\mathcal{C}} FK & \longrightarrow & \operatorname{hocolim}_{\mathcal{D}} F \end{array}$$

$$B(\mathcal{D}(-, K-), \mathcal{D}, QF) \xrightarrow{\cong} QFK$$

By 2-of-3 property we obtain that

$$\operatorname{hocolim}_{\mathcal{E}} FK \xrightarrow{\cong} \operatorname{hocolim}_{\mathcal{D}} F \quad \triangle$$

Corollary. (Quillen's theorem A)

If $K: \mathcal{E} \rightarrow \mathcal{D}$ is homotopy final

$$N\mathcal{E} \xrightarrow{\cong} N\mathcal{D}$$

Proof. $|N\mathcal{E}| = B(*, \mathcal{E}, *) \cong \operatorname{hocolim}_{\mathcal{E}} *$

$$\operatorname{hocolim}_{\mathcal{E}} (\underbrace{\mathcal{E} \xrightarrow{K} \mathcal{D} \xrightarrow{*} \operatorname{Top}}_{*K}) \xrightarrow{\sim} \operatorname{hocolim}_{\mathcal{D}} * = |N\mathcal{D}| \quad \triangle$$

$|N\mathcal{E}|$

Example. \mathcal{D} has a term. obj t
 $t: 1 \rightarrow \mathcal{D}$ is final, 1 is filtered \Rightarrow
 $\Rightarrow t$ is homotopy final $\Rightarrow \operatorname{hocolim}_{\mathcal{D}} F \cong$

$$\cong \operatorname{hocolim}_{\mathcal{D}} F \circ t \cong F \circ t$$

$$2 \xrightarrow{\mathcal{D} f} \mathcal{D}, \quad f(0) = X, \quad f(1) = Y, \quad f(0 \rightarrow 1) = X \xrightarrow{f} Y$$

$$\operatorname{hocolim}_{\mathcal{D}} f \cong Y \cong \operatorname{Cyl}(f)$$

Example. $\vec{\Delta} \xrightarrow{i} \Delta$ is homotopy initial functor

$m/[n]$ is contractible $\forall m$

$$S: m/[n] \hookrightarrow$$

$$\alpha: [k] \rightarrow [n] \mapsto S\alpha: [k+1] \rightarrow [n]$$

$$E(-) \equiv ([0] \xrightarrow{0} [n]) \quad \begin{cases} S\alpha(0) = 0, \\ S\alpha(i) = \alpha(i-1), \quad i > 0 \end{cases}$$

$$\text{id} \Rightarrow S \Leftarrow E \quad \text{Apply } N(-)$$

$N(m/[n])$ is contr.

$$\text{Thus, } \text{hocolim}_{\Delta^{\text{op}}} X \cong \text{hocolim}_{\vec{\Delta}} X$$

Fast realization

$$\begin{array}{ccc} \vec{\Delta} & \xrightarrow{\Delta} & s\text{Set} \\ & \searrow & \nearrow \\ & s\text{Set} & \end{array}$$

Example. $\Delta: \Delta \rightarrow \Delta \times \Delta$ is homotopy initial

$$\Rightarrow \text{hocolim}_{\Delta^{\text{op}} \times \Delta^{\text{op}}} X_{\bullet, \bullet} \cong \text{hocolim}_{\Delta^{\text{op}}} \text{diag}(X_{\bullet, \bullet})$$

Example. $f: \mathcal{I} \rightarrow \text{Top}$

$N(\mathcal{I})$ is 1-skeletal

So, $B_*(\mathcal{I}, f)$ is 1-sk. too

$$f: X \rightarrow Y$$

$$B_0 = X \sqcup Y, \quad B_1 = X^0 \sqcup X^1 \sqcup Y^1$$

$$\begin{array}{ccc} \updownarrow & \updownarrow & \updownarrow \\ \bullet & \bullet \longrightarrow \bullet & \bigcirc^1 \end{array}$$

$$\text{hocolim}(f) \cong |\mathcal{B}.(*, \mathbb{Z}, f)| \cong_{n \in \Delta} \Delta^{\bullet} \otimes \mathcal{B}.(*, \mathbb{Z}, f) \cong \int \Delta^n \otimes \mathcal{B}_n(*, \mathbb{Z}, f) \cong$$

$$\cong \text{coeq} \left[\bigsqcup_{f: [m] \rightarrow [n] \in \Delta_1} \Delta^m \otimes \mathcal{B}_n \xrightleftharpoons[f_*]{f^*} \bigsqcup_{[n] \in \Delta_0} \Delta^n \otimes \mathcal{B}_n \right] \cong$$

$$\cong \text{colim} \left(\begin{array}{ccc} \Delta^1 \otimes B_0 & \xrightarrow{s_0} & \Delta^1 \otimes B_1 \\ & \searrow d^0 & \nearrow d^1 \\ \Delta^0 \otimes B_1 & \xrightarrow{s^0} & \Delta^0 \otimes B_0 \\ & \nearrow d_0, d_1 & \end{array} \right) \cong \frac{X \times I \sqcup Y}{(x, 1) \sim f(x)}$$

$$f: [1] \rightarrow [0] \rightsquigarrow \Delta^1 \rightarrow \Delta^0 \text{ is } s^0$$

$$\rightsquigarrow B_0 \rightarrow B_1 : s_0$$

$$g_{0,1}: [0] \rightarrow [1] \rightsquigarrow \Delta^0 \rightarrow \Delta^1 \quad d^0, d^1$$

$$\rightsquigarrow B_1 \rightarrow B_0 \quad d_0, d_1$$