Plan

1. Weighted colimits in unenriched case
2. Enriched WC

$\Big\}$ (Chapter 7)

3. Weighted colimits as left Quillen functors (Chapter 11)
4. Reedy model structures (Chapter 14)

See Emily Riehl "Categorical Homotopy Theory" and Gambino'10 work

Def. $F: \mathcal{C} \to \mathcal{M}$, $W: \mathcal{C}^{op} \to Set$ — weight

$\mathrm{colim}^W F$ is defined by isomorphism

$$\mathcal{M}(\mathrm{colim}^W F, m) \cong Set^{\mathcal{C}^{op}}(W, \mathcal{M}(F-, m))$$

If $\mathcal{M}$ is cocomplete

$$Set^{\mathcal{C}^{op}}(W, \mathcal{M}(F-, m)) \overset{\not{L}}{\cong} \int_{c\in\mathcal{C}} Set(Wc, \mathcal{M}(Fc, m)) \cong$$

$$\cong \int_{c\in\mathcal{C}} \mathcal{M}(Wc\cdot Fc, m) = \mathcal{M}\left(\int^{c\in\mathcal{C}} Wc\cdot Fc, m\right)$$

$$\Rightarrow \boxed{\int^{c\in\mathcal{C}} Wc\cdot Fc \cong \mathrm{colim}^W F}$$

Example. $\mathrm{colim}^* F = \int^{c\in\mathcal{C}} *(c)\cdot Fc \cong \mathrm{colim}_{\mathcal{C}} F$ ← muted coend

Example. $\mathrm{colim}^{\mathcal{C}(-,c)} F \overset{\not{L}}{\cong} \int^{x\in\mathcal{C}} \mathcal{C}(x,c)\cdot Fc \cong Fc$

$\mathrm{colim}^{\mathcal{C}(-,=)} F \cong F$

Example. $Lan_K F \cong \int^{c\in\mathcal{C}} \mathcal{D}(Kc, d)\cdot Fc \cong \mathrm{colim}^{\mathcal{D}(K-,d)} F$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{M} \\ \mathcal{D} \arrow[ur, dashed, "Lan_K F"'] & \end{tikzcd}$$

$$W \Rightarrow * \rightsquigarrow \underset{\substack{\| \\ \operatorname{colim} F}}{\operatorname{colim}^* F} \longleftarrow \operatorname{colim}^W F$$ — comparison map

$$\operatorname{colim}^W F = \operatorname{coeq}\left[\coprod_{c \to c' \in \mathcal{C}_1} Wc \cdot Fc' \rightrightarrows \coprod_{c \in \mathcal{C}_0} Wc \cdot Fc \right]$$

Weighted colimits in enriched case

Def. $F : \underline{\mathcal{D}} \to \underline{\mathcal{M}}$, $W : \underline{\mathcal{D}}^{op} \to \underline{\mathcal{V}}$ — $\mathcal{V}$-functors

$\mathcal{V}$ is monoidal cat

$\operatorname{colim}^W F \in \mathcal{M}$, s.t.

$$\underline{\mathcal{M}}(\operatorname{colim}^W F, m) \cong \underline{\mathcal{V}}^{\underline{\mathcal{D}}^{op}}(W, \underline{\mathcal{M}}(F-, m))$$

Example. $\mathbb{1} = \{ \bullet \circlearrowleft^{*} \}$

$\mathcal{V}$-functor $\mathbb{1} \to \underline{\mathcal{M}}$ is an object of $\mathcal{M}$

$$\underline{\mathcal{V}}^{\mathbb{1}} \cong \underline{\mathcal{V}}$$

$$n : \mathbb{1} \to \underline{\mathcal{M}}, \quad \underbrace{v : \mathbb{1} \to \underline{\mathcal{V}}}_{\text{weight}}$$

$$\underline{\mathcal{M}}(m, \lim^v n) \cong \underline{\mathcal{V}}(v, \underline{\mathcal{M}}(m, n))$$

$$\Rightarrow \lim^v n = \{v, n\} = n^v$$

$$\operatorname{colim}^v m = v \otimes m$$

End in unenriched case

$$\int_{\mathcal{D}} \mathcal{M}(Fd, Gd) \cong$$

$$\cong eq\left[\prod_d \mathcal{M}(Fd, Gd) \rightrightarrows \prod_{d,d'} \prod_{\mathcal{D}(d,d')} \mathcal{M}(Fd, Gd')\right]$$

End in enriched case

We should replace

$$\prod_{\mathcal{D}(d,d')} \mathcal{M}(Fd, Gd') \text{ by } \underline{\mathcal{M}}(Fd, Gd')^{\mathcal{D}(d,d')}$$

$$\int_{\underline{\mathcal{D}}} \underline{\mathcal{M}}(Fd, Gd) \cong$$

$$\cong eq\left[\prod_d \mathcal{M}(Fd, Gd) \rightrightarrows \prod_{d,d'} \mathcal{V}(\mathcal{D}(d,d'), \underline{\mathcal{M}}(Fd, Gd'))\right]$$

Theorem ($\mathcal{V}$–Yoneda lemma)

$F : \underline{\mathcal{D}} \longrightarrow \underline{\mathcal{V}}$, $d \in \mathcal{D}$

$$Fd \xrightarrow{\cong} \underline{\mathcal{V}}^{\underline{\mathcal{D}}}(\underline{\mathcal{D}}(d, -), F)$$

Example. (Enriched $\mathcal{V}$-cat in $\underline{\mathcal{V}}$) $\quad \mathcal{V}$ – closed mod. cat

$W, F : \underline{\mathcal{D}} \rightrightarrows \underline{\mathcal{V}}$

$$\lim^W F \cong \underline{\mathcal{V}}(*, \lim^W F) \cong \underline{\mathcal{V}}^{\underline{\mathcal{D}}}(W, \underline{\mathcal{V}}(*, F)) \cong$$

unit in $\mathcal{V}$

$$\cong \underline{\mathcal{V}}^{\underline{\mathcal{D}}}(W, F)$$

So, $\lim^W F$ is the object of $\mathcal{V}$–nat. transf. $W \Rightarrow F$

From this example we conclude

$$\underline{M}(m, \lim^{W} F) \cong \underline{\mathcal{V}}^{\underline{\mathcal{D}}}\big(W, \underbrace{\underline{M}(m, F-)}_{\underline{\mathcal{D}} \to \underline{\mathcal{V}}}\big) \cong \lim^{W} M(m, F-)$$

$$\underline{\mathcal{D}} \to \underline{\mathcal{V}}$$

Theorem. When $\underline{M}$ is tens. and cotens., $F : \underline{\mathcal{D}} \to \underline{M}$, $W : \underline{\mathcal{D}}^{op} \to \underline{\mathcal{V}}$

$$\operatorname{colim}^{W} F \cong W \otimes_{\underline{\mathcal{D}}} F, \quad \lim^{W} F \cong \{W, F\}^{\underline{\mathcal{D}}}$$

Weighted colimits as left Quillen bifunctors

Theorem (Gambino '10). Let M be a simpl. mod cat. Suppose that the cat of weights $sSet^{\mathcal{D}^{op}}$ is equipped with the inj mod structure and the cat of diagrams $M^{\mathcal{D}}$ is equipped with the proj mod structure. Then the functor

$$- \otimes_{\mathcal{D}} - : sSet^{\mathcal{D}^{op}}_{\underset{\text{Proj}}{\text{Inj}}} \times M^{\mathcal{D}}_{\underset{\text{Inj}}{\text{Proj}}} \to M$$

is a left Quillen functor in 2 variables

[$\Phi : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is said to be a left Quillen bifunctor in 2 variables if

- Φ is cocont in each variable
- If f: CoFib, g: CoFib $\Rightarrow f \otimes g$: CoFib]

Proof. The following conditions are equiv.

(i) $\Phi : sSet^{\mathcal{D}^{op}}_{Inj} \times \mathcal{M}^{\mathcal{D}}_{proj} \longrightarrow \mathcal{M}$ is a left Quillen bifunctor

(ii) $\Theta : (\mathcal{M}^{\mathcal{D}})^{op}_{Proj} \times \mathcal{M} \longrightarrow sSet^{\mathcal{D}^{op}}_{Inj}$ is a right

(iii) $\Psi : (sSet^{\mathcal{D}^{op}})^{op}_{Inj} \times \mathcal{M} \to \mathcal{M}^{\mathcal{D}}_{Proj}$ is a right

Ψ is defined pointwise by $[-,-] : sSet^{op} \times \mathcal{M} \to \mathcal{M}$

$f : m_1 \to m_2$

$g : n_1 \to n_2$

$$\begin{tikzcd} & {[f,g]} \\ \mathcal{M}(m_2,n_1) \arrow[ur] \arrow[r] \arrow[d] & \vdots \\ {} & \end{tikzcd}$$

◁

Corollary. Let M be a simpl. mod cat, $\mathcal{D}$ be a small cat. If $\mathcal{M}$ is cofibr. generated then the proj cofib. replacement defines a left deformation for $\operatorname{colim} : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}$
$\Rightarrow$ hocolim can be computed as the colimit of any proj. cofibr. repl.

▷ $* \underset{\mathcal{D}}{\otimes} - : sSet^{\mathcal{D}^{op}}_{Inj} \times \mathcal{M}^{\mathcal{D}}_{Proj} \longrightarrow \mathcal{M}$ ◁

Lemma. Let $A \to B$ be any cofib in a model cat $\mathcal{M}$. Then the induced map
$$\mathcal{D}(d,-)\cdot A \longrightarrow \mathcal{D}(d,-)\cdot B$$
is proj. cofibration in $\mathcal{M}^{\mathcal{D}}$

$$\mathcal{D}(d,-)\cdot A \cong \coprod_{\mathcal{D}(d,-)} A$$

$\triangleright$
$$\begin{tikzcd} \mathcal{D}(d,-)\cdot A \arrow[r] \arrow[d] & F \arrow[d] \\ \mathcal{D}(d,-)\cdot B \arrow[r] \arrow[ur, dashed] & G \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d] & Fd \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Gd \end{tikzcd}$$
$\triangleleft$

$\mathcal{D}(d,-) \dashv ev_d$ by Yoneda Lemma

Corollary. Any retract of a transitive composite of pushouts of coproducts of the maps of the lemma above is a proj. cofibr.

Example. $\exists\, \mathcal{D} = (b \xleftarrow{f} a \xrightarrow{g} c)$

$\exists\, F : \mathcal{D} \to \mathcal{M}$ is cofibr. gener. mod. cat

Use the (Inj., Proj.) – model structure

$\exists\, Fa$ is cofibr., Ff, Fg are cofibr. $\Rightarrow$ $\underline{\underline{F \text{ is proj. cofibr.}}}$

$$\operatorname{hocolim} F \cong \operatorname{colim}(Fb \xleftarrow{Ff} Fa \xrightarrow{Fg} Fc)$$

$$\emptyset \longrightarrow F \qquad (\mathcal{D}(b,-)\cdot Fa, \mathcal{D}(a,-)\cdot Fa) \cong (\mathcal{D}(b,-), (Fa, \mathcal{D}(a,-)\cdot Fa)) \cong$$

$$\cong (Fa, \mathcal{D}(a,b)\cdot Fa) \ni id_{Fa}$$

$$\begin{tikzcd}
 & \emptyset \arrow[d] & \\
\mathcal{D}(b,a)\cdot Fa \arrow[r, "u"] \arrow[d, tail, "\text{Cofib}"'] & \mathcal{D}(a,-)\cdot Fa \arrow[d, tail] & \\
\mathcal{D}(b,-)\cdot F(b) \arrow[r] & F' \arrow[d] \arrow[ul, phantom, "\ulcorner"] & \mathcal{D}(c,-)\cdot Fa \arrow[l, "v"'] \arrow[ul, dashed] \arrow[d] \\
 & F & \mathcal{D}(c,-)\cdot Fc \arrow[l]
\end{tikzcd}$$

$$\mathrm{coeq}\Big(\coprod_{d_1 \to d_2} \mathcal{D}(d_1,-)\cdot Fd_2 \rightrightarrows \coprod_{d\in\mathcal{D}} \mathcal{D}(d,-)\cdot Fd \Big) \cong F$$

$$\underbrace{\emptyset \longrightarrow F}_{\text{cofib}} \cong \cdot \rightarrowtail \cdot \rightarrowtail \cdot$$

Example (mapping telescope)

$$\omega = 0 \to 1 \to 2 \to \dots \qquad F = X_0 \to X_1 \to X_2 \to \dots$$

$$F : \omega \to \mathcal{M}$$

$$\widetilde{X}_0 \xrightarrow{\sim} \widetilde{X}_1 \xrightarrow{\sim} \widetilde{X}_2 \xrightarrow{\sim} \dots$$

Step 0: $q_0 : Q_0 \xrightarrow{\sim} X_0$

$$G^0 = \omega(0,-)\cdot Q_0$$

$$\begin{tikzcd}
G_0 \arrow[d, Rightarrow] & Q_0 \arrow[r, equal] \arrow[d, "q_0 \wr"'] & Q_0 \arrow[r, equal] \arrow[d, dashed] & Q_0 \arrow[r, equal] \arrow[d, dashed] & \dots \\
F & X_0 \arrow[r, "f_{01}"] & X_1 \arrow[r] & X_2 \arrow[r] & \dots
\end{tikzcd}$$

Step 1: $f_{01} q_0$

$$\begin{tikzcd}
Q_0 \arrow[r, tail, "g_{01}"] \arrow[d, "q_0"', "2"] & Q_1 \arrow[d, "q_1"', "2"] \\
X_0 \arrow[r, "f_{01}"] & X_1
\end{tikzcd}$$

$$\begin{tikzcd}
\omega(1,-)\cdot Q_0 \arrow[r] \arrow[d, tail, "g_{01}"'] & \omega(0,-)\cdot Q_0 = G^0 \arrow[d] \\
\omega(1,-)\cdot Q_1 \arrow[r] & G^1
\end{tikzcd}$$

$$\big(\omega(1,-)\cdot Q_0,\ \omega(0,-)\cdot Q_0\big) \cong \big(Q_0,\ \underbrace{\omega(0,1)}_{\text{is the set with 1 obj}}\cdot Q_0\big) \ni 1_{Q_0}$$

$$\begin{tikzcd}
Q_0 \arrow[r, tail, "g_{01}"] \arrow[d, "q_0"', "2"] & Q_1 \arrow[d, "q_1"', "2"] \arrow[r, equal] & Q_1 \arrow[d, dashed] \arrow[r, equal] & Q_1 \arrow[r, equal] & \cdots \\
X_0 \arrow[r, "f_{01}"] & X_1 \arrow[r, "f_{12}"] & X_2 \arrow[r] & \cdots
\end{tikzcd}$$

Step n: $f_{n-1,n} \cdot q_{n-1}$

$$\begin{tikzcd}
Q_0 \arrow[r, tail, "g_{01}"] \arrow[d, "q_0"', "2"] & Q_1 \arrow[r, tail, "g_{12}"] \arrow[d, "q_1"', "2"] & Q_2 \arrow[r, tail, "g_{23}"] \arrow[d, "q_2"', "2"] & Q_3 \arrow[r, tail] \arrow[d, "q_3"', "2"] & \cdots & G \arrow[d, Rightarrow, "q"', "\sim"] \\
X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots & F
\end{tikzcd}$$

$G = \operatorname{colim} G^n$

$$\emptyset \rightarrowtail G^0 \rightarrowtail G^1 \rightarrowtail G^2 \rightarrowtail \dots \rightarrowtail \operatorname{colim}_n G^n = G$$

Corollary. (from Gambino's th)

$N(-/\mathcal{D})$ is proj. cofibr. replacement for the const. weight in hocolim.

$N(\mathcal{D}/-)$ is inj. fibr. repl. for the const weigh in holim

Proof: Consider (Proj, Inj)– model structure

$* \otimes_{\mathcal{D}} F$, $N(-/\mathcal{D})$ is cofib. repl.

F should be pointwise cofibr. $\triangleleft$

$$\operatorname{hocolim} FK = \operatorname{colim}^{N(-/K)} F \longrightarrow \operatorname{colim}^{N(-/\mathcal{D})} F = \operatorname{hocolim} F$$

$N(-/K)$ is homotopy trivial

$N(-/\mathcal{D})$ is proj. cofib., hom. trivial

So, $N(-/K) \xrightarrow{\sim} N(-/\mathcal{D})$ is WE

We know that $\operatorname{colim}^{-} F$ sends trivial cofibs to trivial cofib. between cofibr. obj.

Hence, by Ken Brown's lemma

$$\operatorname{hocolim} FK \xrightarrow{\sim} \operatorname{hocolim} F \text{ is WE}$$

This gives the other proof of Quillen's Homotopy

Finality Theorem from the prev. talk