

Def. Nerve Realization Context is a functor

$$F: \mathcal{L} \xrightarrow{\text{small}} \mathcal{D}$$

locally small

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{F} & \mathcal{D} \\ \downarrow \eta & & \uparrow \bar{F} \\ [\mathcal{L}^{\text{op}}, \text{Set}] & & [\mathcal{L}^{\text{op}}, \text{Set}] \end{array}$$

Theorem (Univ property of  $[\mathcal{L}^{\text{op}}, \text{Set}]$ )

1) There is  $\text{Lan}_{\mathcal{L}'} F : \text{Lan}_{\mathcal{L}'} \dashv -\circ \mathcal{L}'$ ,

the unit of this adjunction is invertible

2) The ess. image of  $\text{Lan}_{\mathcal{L}'}$  consists of those

$F : [\mathcal{L}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$  that preserves all colimits

3) If  $\mathcal{D} = [\mathcal{L}^{\text{op}}, \text{Set}]$ , this ess. image is equivalent to the subcat of left adjoints  $F : [\mathcal{L}^{\text{op}}, \text{Set}] \rightarrow [\mathcal{L}^{\text{op}}, \text{Set}]$

Proof. 1)  $\forall F : \mathcal{L} \rightarrow \mathcal{D}$  extends uniq. to  $\bar{F} : [\mathcal{L}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$   
(up to isomorphism)

- $\exists$  an adjunction  $\text{Lan}_{\mathcal{L}'} \rightleftarrows - \circ \mathcal{L}'$   
that has invertible unit  $\eta$

$\bar{F} \Rightarrow (F \mapsto \bar{F})$  is a functor

$\forall G : [\mathcal{L}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{G \circ \mathcal{L}'} & \mathcal{D} \\ \downarrow \eta & & \uparrow \bar{G} \\ [\mathcal{L}^{\text{op}}, \text{Set}] & & [\mathcal{L}^{\text{op}}, \text{Set}] \end{array}$$

$\exists \varepsilon: \text{Lan}_{\mathcal{L}'}(GK) \rightarrow G$  is the counit of our adj?

$$\eta: \text{Lan}_{\mathcal{L}'} G \circ \mathcal{L}' \xrightarrow{\sim} G$$

Lemma.  $\text{Lan}_K F \circ K \cong F$

$$\triangleright \text{Lan}_K F(d) = \int \mathcal{D}(Kc, d) \cdot Fc$$

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E} \\ \downarrow & \nearrow \text{copower or tensor prod} & \downarrow \\ K & \xrightarrow{\text{fully faithful}} & \mathcal{D}, \text{Lan}_K^F \end{array}$$

$$(\text{Lan}_K F)(K(c)) = \int_{x \in \mathcal{E}_0} \mathcal{D}(Kx, Kc) \cdot Fx \cong$$

$$\cong \int_{x \in \mathcal{E}_0} \mathcal{E}(x, c) \cdot Fx \xrightarrow{\mathcal{L}} Fc$$

$\Rightarrow \text{Lan}_K F: [\mathcal{E}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$  is continuous  $\Rightarrow \square$

2) Suppose that  $G: [\mathcal{E}^{\text{op}}, \text{Set}] \rightarrow \mathcal{D}$  is invertible

$$\Rightarrow (\varepsilon_G: \text{Lan}_{\mathcal{L}'}(G \circ \mathcal{L}') \rightarrow G)$$

$$\text{Lan}_{\mathcal{L}'}(G \circ \mathcal{L}') (P) \cong \int [\mathcal{E}^{\text{op}}, \text{Set}] (\mathcal{L}'c, P) \cdot G\mathcal{L}'c \cong$$

$$\cong \int_{c \in \mathcal{E}} P_c \cdot G\mathcal{L}'c \xrightarrow{\sim} G \left( \int_{c \in \mathcal{E}} P_c \cdot \mathcal{L}'c \right) \cong GP$$

Vice Versa, if we have  $G \cong \text{Lan}_{\mathcal{L}'}(G\mathcal{L}')$  then

$$G(\underset{j}{\text{colim}} P_j) \cong \int_{c \in \mathcal{E}} \underset{j}{\text{colim}} ((P_j)_c) \cdot G(\text{hom}(-, c)) \cong$$

$$P: J \rightarrow [\mathcal{E}^{\text{op}}, \text{Set}] \xrightarrow{\text{Fubini theorem}} \underset{j}{\text{colim}} \underbrace{\int_{c \in \mathcal{E}} ((P_j)_c \cdot G(\text{hom}(-, c)))}_{(\text{Lan}_{\mathcal{L}'}(G\mathcal{L}'))(P_j)} \cong$$

$$\cong \underset{\mathcal{I}}{\operatorname{colim}} \mathcal{G}(P_j)$$

3)  $\mathcal{L}: [\mathcal{E}^{\text{op}}, \text{Set}] \rightarrow [\mathcal{E}^{\text{op}}, \text{Set}]$  — continuous functors

Then it has a right adj given by

$$\int_{C \in \mathcal{E}} [\mathcal{E}^{\text{op}}, \text{Set}](\mathcal{L}C, X) \cdot C$$

□

Example

$$\begin{array}{ccc} \Delta & \xrightarrow{\Delta^*} & \mathcal{E} \\ \mathcal{L}' \swarrow & \nearrow \perp & \\ \text{sSet} & & \end{array}$$

cocomplete

$$\mathcal{L}X := \int_{n \in \Delta} \text{sSet}(\Delta^n, X) \cdot \Delta^n \cong \int_{n \in \Delta} X_n \cdot \Delta^n \cong \text{coeq} \left( \bigsqcup_{[n] \hookrightarrow [m]} X_m \cdot \Delta^n \rightarrow \bigsqcup_{[n] \in \Delta} X_n \right)$$

We know that  $\perp \Delta^n \cong \Delta^n$

$\mathcal{L}$  is a colimit  $\Rightarrow \mathcal{L}$  commutes with colimits  $\Rightarrow \perp$  has

a right adj.  $R: \mathcal{E} \rightarrow \text{sSet}$

$$(Re)_n \cong \text{sSet}(\Delta^n, Re) \cong \mathcal{E}(\perp \Delta^n, e) \cong \mathcal{E}(\Delta^n, e)$$

$\Rightarrow n$ -simpl. of  $Re$  are maps in  $\mathcal{E}$  from  $\Delta^n$  to  $e$

Example 2.  $\Delta \rightarrow \text{Top}$ , s.t.  $\Delta^n$  is the stand. top.  $S^n$

$$|-|: \text{sSet}_{n \in \Delta} \rightarrow \text{Top}: S$$

$$|X| = \int_{n \in \Delta} X_n \times \Delta^n$$

Example 3.

$$\Delta \xrightarrow{\widetilde{sd}(\cdot)} \text{sSet}$$

$\Downarrow$  Lan $_{\mathcal{L}}$   $\widetilde{sd}(\cdot) = sd$

$sd \Delta^n$

$$\begin{matrix} & \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} & \\ \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} & \xrightarrow{\quad} & \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} \\ \left\{ \begin{matrix} 0 \\ 1, 2 \end{matrix} \right\} & \xrightarrow{\quad} & \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} \\ \left\{ \begin{matrix} 0 \\ 2 \end{matrix} \right\} & \xrightarrow{\quad} & \left\{ \begin{matrix} 2 \end{matrix} \right\} \end{matrix} \quad [2]$$

$$sd = \text{Lan}_{\mathcal{L}} \widetilde{sd}(\cdot) \quad sd: \text{sSet} \rightleftarrows \text{sSet}: ex$$

$sd \Delta^n \rightarrow \Delta^n$  is last vertex map

$sd \Rightarrow id$

colim of the  $\{\eta_{ex^n}\}$  defines a functor  $ex^\infty$

Functor tensor product

$$F: \mathcal{D} \rightarrow M, \quad G: \mathcal{D}^{\text{op}} \rightarrow \mathcal{V}$$

$M$  is simpl. env.  
term. and exten

$$G \otimes F := \int_{\mathcal{D}} G \otimes F = \text{coeq} \left[ \bigsqcup_{f: d \rightarrow d'} (G_d \otimes F_d) \xrightarrow{\substack{f^* \\ f_*}} \bigsqcup_d (G_d \otimes F_d) \right]$$

Example.  $A: R^{\text{op}} \rightarrow \underline{\text{Ab}}$  this is a  $\underline{\text{Ab}}$ -functor

$$B: R \rightarrow \underline{\text{Ab}}$$

$$A \otimes B: R^{\text{op}} \times R \rightarrow \underline{\text{Ab}}$$

$$A \otimes_R B = \int_{\mathbb{Z}} A \otimes_{\mathbb{Z}} B$$

Example.  $* \otimes_{\mathcal{D}} F = \text{colim } F$

$*: \mathcal{D}^{\text{op}} \rightarrow \text{sSet}$  - const funct. to the term obj.

Example  $\text{Lan}_K F(d) = \mathcal{D}(K-, d) \otimes_{\mathcal{D}} F$

If  $X: \text{sSet}$  then  $|X| = \int^{n \in \Delta} X_n \times \Delta^n \cong X \otimes_{\Delta} \Delta \xrightarrow{K} \mathcal{D}$

Example  $X$  is bisimpl. set, i.e.  $X: \Delta^{\text{op}} \rightarrow \text{sSet}$

then  $|X| \in \text{sSet}$ ,  $|X| = \text{Diag}(X)$

$|X|_K = \int^{n \in \Delta} X(n, K) \times \Delta(K, [n]) = X(K, K)$

## Homotopical categories

Def. A left deform. on a homotopical cat  $M$  consists of an endofunctor  $Q$  together with a nat.

weak equiv:

$$\eta: Q \xrightarrow{\sim} 1$$

Def. A left def. for a functor  $F: M \rightarrow N$  consists of left deform. for  $M$ , s.t.  $F$  is homotopical on an associated subcat of cofib. objects

Theorem. If  $F: M \rightarrow N$  has a left deformation

$\eta: Q \xrightarrow{\sim} 1$  then  $L(F) := FQ$  is a left derived functor of  $F$

Def. Two-sided Bar-complex is a simpl. obj  
 $\mathcal{B}_*(G, \mathcal{D}, F) \in \mathcal{M}$  for  $F: \mathcal{D} \rightarrow \mathcal{M}$ ,  $G: \mathcal{D}^{\text{op}} \rightarrow \text{Set}$ ,  
 $\mathcal{M}$  is simpl. enr. tens. and catens.

$$\mathcal{B}_n(G, \mathcal{D}, F) = \bigcup \left[ Gd_n \otimes Fd_0 \right]$$

$$\vec{d}: [n] \rightarrow \mathcal{D}$$

$\vec{d}: [n] \rightarrow \mathcal{D} \rightsquigarrow d_0 \rightarrow d_1 \rightarrow \dots \rightarrow d_n \text{ in } \mathcal{D}$

Example  $\mathcal{B}(*, \mathcal{D}, *) = \mathcal{ND}$   
 $\mathcal{M} = \text{Set}$ ,  $\mathcal{V} = \text{Set}$ ,  $F = *$ ,  $G = *$

Def.  $\mathcal{B}(G, \mathcal{D}, F) = \{ \mathcal{B}_*(G, \mathcal{D}, F) \} =$

$$= \Delta \otimes_{\Delta^{\text{op}}} \mathcal{B}_*(G, \mathcal{D}, F)$$

$$\begin{array}{ccc} \Delta^n \rightarrow * & \xrightarrow{\text{nat. transf.}} & \Delta \Rightarrow * \\ - \otimes_{\Delta^{\text{op}}} \mathcal{B}_*(G, \mathcal{D}, F) & \rightsquigarrow & \mathcal{B}(G, \mathcal{D}, F) \rightarrow \underset{\mathcal{D}}{G \otimes F} \\ & & \parallel \qquad \parallel \text{Enriched} \\ & & | \mathcal{B}_*(G, \mathcal{D}, F) | \qquad \text{colim } \mathcal{B}_*(G, \mathcal{D}, F) \end{array}$$

Example.  $BG = |\mathcal{B}_*(*, G, *)|$

Theorem Let  $\mathcal{M}$  be simpl. mod. cat. Then  
 $\mathcal{B}(\mathcal{D}, \mathcal{D}, Q-): \mathcal{M}^{\mathcal{D}} \rightarrow \mathcal{M}^{\mathcal{D}}$ ,  $\mathcal{B}(\mathcal{D}, \mathcal{D}, Q-) \xrightarrow{\epsilon_Q} Q \xrightarrow{q} 1$

is a left deformation for colimit:  $M^D \rightarrow M$

$$\underline{\text{What is } \varepsilon_Q?} \quad B(D, D, F) \xrightarrow{\text{colim}_{d \in D}} B_*(D, D, F) \cong \\ \cong \underset{D}{\coprod} D(-, d) \otimes F = \int D(d', d) \otimes F^{d'} \cong Fd$$

$$\varepsilon_Q: B(D, D, F) \longrightarrow F$$

Corollary If  $M$  is simpl. mod cat and  $D$  is any small cat then

$$\text{hocolim}_D := \coprod_D \text{colim}_D \cong B_*(*, D, Q-)$$

$$\text{holim}_D := \lim_D \cong C_*(*, D, R-)$$

Proof of cor.

$$\coprod_D \text{colim}_D(-) \cong \underset{d \in D}{\text{colim}} B(D, D, Q-) \cong * \underset{D}{\otimes} B(D, D, Q-)$$

$$* \underset{D}{\otimes} F \cong \int * (d) \otimes F_d \xrightarrow{\text{mutual coend}} \text{colim}_D F$$

$$* \underset{D}{\otimes} B(D, D, Q-) \cong B_*(*, D, Q-)$$

$$\text{So, } \coprod_D \text{colim}_D (-) \cong B_*(*, D, Q-)$$

△

Proof the main theorem.

$\varepsilon: B(\mathcal{D}, \mathcal{D}, F) \Rightarrow F$  is pointwise weak equiv, as  
 $\mathcal{D} \rightarrow *$  was a weak equiv  
Hence, the comp.  $B(\mathcal{D}, \mathcal{D}, F) \xrightarrow{\varepsilon_Q} Q \xrightarrow{q} 1$  is a left deform. on  $M$

What do we want?

[ $\varepsilon \circ q$  defines a left deform. of  $\operatorname{colim}$ ] ?

It remains to verify that  $\operatorname{colim}$  preserves weak equiv. between diagrams in full subcat. of  $M^{\mathcal{D}}$  which contains the image of  $B(\mathcal{D}, \mathcal{D}, Q-)$

Lemma.]  $(Q, q)$  is a left deform. on  $M$ ,

$F: M \rightarrow N$ . If 

- $FQ$  is homotopical
- $Fq: FQ \Rightarrow FQ$  is nat. weak equiv.

then  $(Q, q)$  is a left deform. for  $F$

▷ Let  $\varphi$  be a map  $\varphi: M_1 \rightarrow M_2$

$Q\varphi \in \mathcal{W} \xrightarrow{?} FQ\varphi \in \mathcal{W}$

We know  $FQ(Q\varphi) \in \mathcal{W}$  as  $FQ$  is homotopical

$\subset \mathcal{W}$

$$\begin{array}{ccc}
 FQ^2M_1 & \xrightarrow{FQ^2\varphi} & FQ^2M_2 \\
 \downarrow \tau_{M_1} \quad \quad \quad & & \downarrow \tau_{M_2} \in \mathcal{W} \\
 FQM_1 & \xrightarrow{FQ\psi} & FQM_2
 \end{array}$$

$\Rightarrow$  from 3-out-of-2  $FQ\psi \in \mathcal{W}$

△

It remains to prove that the functor  
 $B(\ast, \mathcal{D}, Q-)$  is homotopical and that the

composite

$$\begin{array}{ccc}
 B(\mathcal{D}, \mathcal{D}, QB(\mathcal{D}, \mathcal{D}, QF)) & \xrightarrow{\text{colim}^E} & \text{colim}_{\mathcal{D}} QB(\mathcal{D}, \mathcal{D}, F) \rightarrow \\
 \xrightarrow{\text{colim}^q} \text{colim}_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, QF)
 \end{array}$$

Def. Simpl. obj. is called Reedy cofibrant  
if & inclusion from degenerate  $n$ -simplices  
into  $n$ -simplices is pointwise cofibration

(latching object)

Lemma.  $B_{\cdot}(\ast, \mathcal{D}, F)$  is Reedy cofibrant,  
when  $\mathcal{D}$  is a small cat,  $M$  is simpl. mod. cat.,

$F: \mathcal{D} \rightarrow M$  is a pointwise cofibration.

▷  $B_n(\ast, \mathcal{D}, F)$  is  $\underset{(ND)_n}{\text{colim}} Fd_n$ .

$n$ th latching obj. sits inside  $B_n(\ast, \mathcal{D}, F)$

But cofibr. are closed under colims and pushouts  $\triangleleft$

Theorem.  $M$  is simp. mod. cat.

$$|-|: M^{\Delta^{\text{op}}} \rightarrow M$$

is left Quillen with resp. to the Reedy model structure

Corollary. The functor  $B(Q, D, -)$  preserves weak equivalences between pointwise cofibr objects  
 $\Rightarrow B(*, D, Q-)$  is homotopical

Now we want to prove that the comp.

$$\cdot \xrightarrow{\text{colim } \varepsilon} \cdot \xrightarrow{\text{colim } \eta} \cdot \text{ is weak equiv (?)}$$

$$B(D, D, QB(D, D, QF)) \xrightarrow{\varepsilon_{QB} \in W} QB(D, D, QF)$$

$$B(D, D, B(D, D, QF)) \xrightarrow{\varepsilon_B \in W} B(D, D, QF)$$

$$* \otimes B(D, D, B(D, D, F)) \xrightarrow{* \otimes \varepsilon \in W} * \otimes B(D, D, F)$$

$$L$$

$$\cong$$

$$* \otimes \varepsilon \otimes B(D, D, F) \downarrow$$

$$* \otimes B(D, D, D) \otimes B(D, D, F) \xrightarrow{* \otimes 2 \otimes B(D, D, F)} * \otimes 2 \otimes B(D, D, F)$$

$$D L' \downarrow$$

$$\cong$$

$$B(*, D, D) \otimes B(D, D, F) \xrightarrow{\varepsilon \otimes B(D, D, F)} * \otimes B(D, D, F)$$

$$L' \downarrow$$

$$\cong$$

$$\mathcal{B}(\mathcal{B}(*, \mathcal{D}, \mathcal{D}), \mathcal{D}, F) \xrightarrow{\mathcal{B}(\mathcal{E}, \mathcal{D}, F)} \mathcal{B}(*, \mathcal{D}, F)$$

$\uparrow$  from  $\square$

Theorem. Let  $F: \mathcal{D} \rightarrow \mathcal{M}$

$$\mathcal{B}(*, \mathcal{D}, F) \cong \mathcal{N}(-/\mathcal{D}) \underset{\mathcal{D}}{\otimes} F = \int_{\mathcal{D}} \mathcal{N}(d/\mathcal{D}) \otimes F_d$$

$$\mathcal{C}(*, \mathcal{D}, F) \cong \int_{\mathcal{D}} \mathcal{D}(\mathcal{N}(\mathcal{D}/d), F_d)$$

▷ Exercise: Check that  $\mathcal{B}(*, \mathcal{D}, \mathcal{D}) \cong \mathcal{N}(-/\mathcal{D})$

Then

$$\begin{aligned} \mathcal{N}(-/\mathcal{D}) \underset{\mathcal{D}}{\otimes} F &\cong \mathcal{B}(*, \mathcal{D}, \mathcal{D}) \underset{\mathcal{D}}{\otimes} F \cong \\ &\cong \mathcal{B}(*, \mathcal{D}, \mathcal{D} \underset{\mathcal{D}}{\otimes} F) \stackrel{\text{nijja-Yoneda}}{\cong} \mathcal{B}(*, \mathcal{D}, F) \quad \square \end{aligned}$$