# The Sheaf Condition: Mathematical Explanation

Let $X$ be a topological space. A *presheaf* $\mathcal{F}$ of sets (or groups, rings, etc.) on $X$ assigns:

- to each open set $U \subseteq X$, a set $\mathcal{F}(U)$ (called the sections of $\mathcal{F}$ over $U$),
- to each inclusion of open sets $V \subseteq U$, a restriction map

$$\rho_{V,U} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V),$$

  such that $\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}$ whenever $W \subseteq V \subseteq U$ are open in $X$.

A *sheaf* is a presheaf $\mathcal{F}$ that satisfies the following additional **sheaf condition** (often stated in two parts):

1. **Local identity:** If $U \subseteq X$ is open and $\{U_i\}_{i \in I}$ is an open cover of $U$, then for any two sections $s, t \in \mathcal{F}(U)$, if

$$s\big|_{U_i} = t\big|_{U_i} \quad \text{for all } i \in I,$$

   then $s = t$. In other words, a section is determined by its restrictions to an open cover.

2. **Gluing (local data give a global section):** If $U \subseteq X$ is open and $\{U_i\}_{i \in I}$ is an open cover of $U$, and if for each $i$ we have a section $s_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$,

$$s_i\big|_{U_i \cap U_j} = s_j\big|_{U_i \cap U_j},$$

   then there is a *unique* section $s \in \mathcal{F}(U)$ such that

$$s\big|_{U_i} = s_i \quad \text{for each } i \in I.$$

In more intuitive terms:

> *The sheaf condition says that if you have local data (sections) that agree on every overlap of the covering sets, then you can "glue" them all together to get one global section on the whole set, and this global section is unique once you fix its local restrictions.*

## Diagrammatic Illustration (for two open sets)

When the open cover consists of two sets $U_1$ and $U_2$ covering $U = U_1 \cup U_2$, the *gluing* condition can be visualized with the following diagram:

$$\begin{array}{ccccc} & & & \mathcal{F}(U) & \\ & \swarrow & & \searrow & \\ \mathcal{F}(U_1) & & \longleftrightarrow & & \mathcal{F}(U_2) \\ & \searrow & & \swarrow & \\ & & \mathcal{F}(U_1 \cap U_2) & & \end{array}$$

Given $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ such that their restrictions agree on $U_1 \cap U_2$, the sheaf condition asserts the existence of a *unique* $s \in \mathcal{F}(U)$ that restricts to $s_1$ on $U_1$ and $s_2$ on $U_2$.