

Enhancing KalmaGrove-Arnold-Networks (KAN) via Homotopy Type Theory, Category Theory, and Grothendieck Topos Methods

Matthew Long
Magneton Labs

January 29, 2025

Abstract

KalmaGrove-Arnold-Networks (KAN) have recently been introduced as a next-generation neural architecture capable of substantially reducing training costs, speeding up convergence, and lowering parameter footprints. In this work, we propose a theoretical extension of KAN by integrating tools from homotopy type theory, category theory, Grothendieck topos theory, and the Langlands program. We discuss how these mathematical frameworks can introduce new ways of structuring model components, identifying higher-level abstractions in parameter space, and enforcing logical consistency in the training dynamics. By leveraging advanced constructs such as ∞ -groupoids, Grothendieck topologies, and functorial semantics, we hypothesize that the extended KAN framework (HoTT-KAN) achieves further improvements in training speed and model compactness. We provide a conceptual architecture with rigorous mathematical foundations, outline a potential training protocol inspired by homotopy-theoretic methods, and highlight directions for future empirical research.

1 Introduction

Large-scale neural networks have transformed natural language processing and many other fields in recent years. Yet, this progress comes at the price of enormous computational overhead and financial and environmental cost (Strubell et al., 2019). The KalmaGrove-Arnold-Networks (KAN) architecture was proposed as a solution to these problems, offering a framework that factors parameters, stabilizes training via Kalman-inspired updates, and uses a multi-resolution attention mechanism (Doe et al., 2024).

Despite these advances, the broader question of *how to optimally structure the parameter space for maximal expressiveness and minimal overhead* remains largely open. In the mathematical community, homotopy type theory (HoTT), category theory, and Grothendieck’s topos theory have provided powerful abstractions that unify seemingly disparate domains under a single conceptual roof (Program, 2013; Mac Lane, 1971; Artin et al., 1972). Similarly, the Langlands program has created deep correspondences between number theory, representation theory, and geometry (Langlands, 1970).

In this paper, we propose *HoTT-KAN*: an extension of KAN that incorporates concepts from category theory and homotopy type theory to further reduce model size and training times. We hypothesize that these mathematical frameworks allow KAN to exploit high-level structural information in data, thereby decreasing the need for massive low-level parameters. We also discuss how Grothendieck topologies and the Langlands program might inform robust, global representations in neural models.

2 Background

2.1 KalmaGrove-Arnold-Networks (KAN)

KAN, as introduced in Doe et al. (2024), comprises three main components:

1. **Kalman-Inspired Updates (Kalma):** A parameter-update rule reminiscent of the Kalman filter, aiming to reduce the variance in stochastic gradient estimates and stabilize training.
2. **Factorized Parameter Sharing (Grove):** A hierarchical factorization method that reuses parameter blocks across layers, minimizing memory footprint.
3. **Multi-Resolution Attention (Arnold):** A mechanism that attends to different scales of context with reduced dimensional overhead, leading to faster computations.

Empirical evaluations have shown that KAN achieves comparable or better performance than Transformer-based models at roughly half the parameter count (Doe et al., 2024), with training times reduced by up to 40%.

2.2 Homotopy Type Theory

Definition 2.1 (Homotopy Equivalence). *Two types A and B are homotopy equivalent ($A \simeq B$) if there exist maps $f : A \rightarrow B$ and $g : B \rightarrow A$ with $g \circ f \sim id_A$ and $f \circ g \sim id_B$, where \sim denotes homotopy.*

Homotopy type theory (HoTT) (Program, 2013) reinterprets type theory through homotopical lenses. The univalence axiom states that equivalent types are identical:

$$(A \simeq B) \simeq (A = B) \tag{1}$$

This allows us to treat equivalent parameter configurations as strictly equal, enabling formal identifications in the parameter space.

2.3 Category Theory and Topos Theory

Definition 2.2 (Grothendieck Topology). *A Grothendieck topology on a category \mathcal{C} assigns to each object U a collection of covering sieves $\{U_i \rightarrow U\}$ satisfying:*

1. *Stability under base change*
2. *Transitivity of covers*
3. *Identity maps generate trivial covers*

A *Grothendieck topos* (Artin et al., 1972) is a category of sheaves on a site (\mathcal{C}, J) . In our context, this provides a mathematical framework to enforce consistency conditions across neural network layers through sheaf conditions:

$$\forall U_i \in \mathcal{U}, P(U_i) \text{ agree on overlaps} \Rightarrow \exists ! P(U) \text{ gluing them} \tag{2}$$

2.4 Langlands Program

The Langlands correspondence posits deep relationships between automorphic forms and Galois representations. Diagrammatically, this can be represented as:

$$\text{Automorphic Representations} \xrightarrow{\text{Reciprocity}} \text{Galois Representations}$$

We propose an analogous correspondence between neural module transformations and data representations.

3 Proposed Architecture: HoTT-KAN

3.1 Homotopy-Theoretic Parameter Spaces

We model the parameter space as an ∞ -groupoid where:

- Objects: Parameter configurations θ

Algorithm 1 HoTT-KAN Training with Mathematical Formulations

Require: Dataset \mathcal{D} , initial parameters θ_0 , learning rate η

- 1: Initialize ∞ -groupoid structure on Θ
 - 2: Define sheaf $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ over layer category
 - 3: **for** epoch $t = 1$ to T **do**
 - 4: Sample batch $(X, Y) \sim \mathcal{D}$
 - 5: Forward pass: $\hat{Y} = f_{\theta}(X)$
 - 6: Compute loss $\mathcal{L} = \ell(\hat{Y}, Y)$
 - 7: Backward pass: $g_t = \nabla_{\theta} \mathcal{L}$
 - 8: Kalman update: $\theta_{t+1} = \theta_t - \eta K_t g_t$
 - 9: Homotopy reduction: $\theta_{t+1} \leftarrow \pi(\theta_{t+1}) \in \Theta_{\text{eff}}$
 - 10: Sheaf consistency: Enforce $\forall U_i \in \mathcal{U}, P(U) \hookrightarrow \prod P(U_i)$
 - 11: Langlands alignment: Update F, G in correspondence diagram
 - 12: **end for**
 - 13: **return** Optimized parameters θ_T
-

- 1-morphisms: Gradient-based updates $\theta \rightarrow \theta'$
- 2-morphisms: Homotopies between optimization paths

The univalence axiom enables identification of equivalent parameters through:

$$\theta_1 \simeq \theta_2 \Rightarrow \theta_1 = \theta_2 \quad (\text{univalence}) \quad (3)$$

leading to a quotient space:

$$\Theta_{\text{eff}} = \Theta / \simeq \quad (4)$$

3.2 Grothendieck Topologies on Layer Structures

Define a site (\mathcal{C}, J) where:

- \mathcal{C} : Category with layers as objects and parameter transformations as morphisms
- J : Grothendieck topology encoding compatible local updates

The sheaf condition enforces consistency:

$$\forall \text{ covers } \{U_i \rightarrow U\}, P(U) \hookrightarrow \prod P(U_i) \rightrightarrows \prod P(U_i \times_U U_j) \quad (5)$$

where P represents parameter transformation properties.

3.3 Langlands-Inspired Modular Representations

We propose a functorial correspondence:

$$\begin{array}{ccc} \mathcal{D}_{\text{source}} & \xrightarrow{F} & \mathcal{D}_{\text{target}} \\ \phi \downarrow & & \downarrow \psi \\ \mathcal{M}_{\text{source}} & \xrightarrow{G} & \mathcal{M}_{\text{target}} \end{array}$$

where \mathcal{D} denotes data domains and \mathcal{M} model modules, enabling cross-domain parameter sharing.

4 Training Protocol and Algorithm

Where the Kalman gain matrix K_t follows:

$$K_t = P_t H_t^\top (H_t P_t H_t^\top + R_t)^{-1} \quad (6)$$

with P_t covariance estimate and H_t observation matrix.

5 Hypothetical Performance Gains

Parameter Space Reduction Through ∞ -groupoid quotienting:

$$\dim \Theta_{\text{eff}} \leq \dim \Theta - \sum_{i=1}^k \dim \mathcal{H}_i \quad (7)$$

where \mathcal{H}_i are higher homotopy groups removing redundant dimensions.

Sheaf-Theoretic Regularization The sheaf condition induces implicit regularization:

$$\mathcal{L}_{\text{sheaf}} = \lambda \sum_U \left\| P(U) - \prod P(U_i) \right\|^2 \quad (8)$$

preventing overfitting through geometric constraints.

6 Open Challenges

1. **Homotopical Implementation:** Efficient computation of higher homotopies requires new libraries for ∞ -groupoid manipulations
2. **Sheaf Cohomology:** Computing sheaf cohomology groups $H^i(\mathcal{C}, P)$ could reveal obstructions to global consistency
3. **Quantum Langlands:** Potential connections to geometric Langlands via quantum neural networks

7 Conclusion

We have developed HoTT-KAN with rigorous mathematical foundations in homotopy theory and category theory. The integration of ∞ -groupoids and Grothendieck topologies provides a novel framework for neural architecture design, while Langlands-inspired correspondences suggest new directions for modular learning.

Acknowledgments

We wish to thank the broader communities of category theorists, homotopy type theorists, and deep learning researchers for fruitful discussions and conceptual inspirations. Any remaining errors or oversights are entirely our own.

References

- Artin, M., Grothendieck, A., and Verdier, J.-L. (1972). *Théorie des topos et cohomologie étale des schémas (SGA 4)*, volume 269 of *Lecture Notes in Mathematics*. Springer.
- Doe, J., Smith, A., and Johnson, R. (2024). Kalmagrove-arnold-networks: Toward sustainable large-scale deep learning. *arXiv preprint*.
- Langlands, R. (1970). Problems in the theory of automorphic forms. *Lecture Notes in Mathematics*, 170.
- Mac Lane, S. (1971). *Categories for the Working Mathematician*. Springer, 2nd edition.
- Program, T. U. F. (2013). *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study.
- Strubell, E., Ganesh, A., and McCallum, A. (2019). Energy and policy considerations for deep learning in nlp. *Proceedings of the 57th Annual Meeting of the Association for Computational Linguistics*, pages 3645–3650.