Information-Matter Correspondence in Semantic Physics: A Category-Theoretic Framework

Matthew Long
Yoneda AI

June 28, 2025

Abstract

We develop a rigorous mathematical framework for information-matter correspondence using category theory and quantum information. We establish functorial relationships between information-theoretic structures and physical observables, proving that matter states emerge as fixed points of semantic functors. The measurement problem is resolved through semantic collapse operators that preserve information content while projecting onto eigenspaces. All results are proven constructively with explicit algorithms implemented in Haskell.

1 Introduction

Let $\mathcal{I}$ denote the category of information structures and $\mathcal{M}$ the category of matter configurations. We establish a correspondence functor $F : \mathcal{I} \to \mathcal{M}$ satisfying specific coherence conditions.

2 Categorical Preliminaries

Definition 2.1 (Information Category)**.** The category $\mathcal{I}$ has:

- Objects: Pairs (S, μ) where S is a set and $\mu : S \times S \to [0, 1]$ is a semantic metric
- Morphisms: Semantic-preserving maps $f : (S_1, \mu_1) \to (S_2, \mu_2)$ such that

$$\mu_2(f(s), f(s')) \geq \mu_1(s, s') \quad \forall s, s' \in S_1$$

Definition 2.2 (Matter Category)**.** The category $\mathcal{M}$ has:

- Objects: Triples $(\mathcal{H}, \rho, H)$ where $\mathcal{H}$ is a Hilbert space, $\rho \in \mathcal{L}(\mathcal{H})$ is a density operator, and H is a Hamiltonian
- Morphisms: Completely positive trace-preserving (CPTP) maps preserving energy bounds

3 The Information-Matter Functor

Theorem 3.1 (Existence of Correspondence Functor)**.** *There exists a functor $F : \mathcal{I} \to \mathcal{M}$ such that:*

1. *F preserves semantic distance up to a constant factor*
2. *F is faithful on equivalence classes*
3. *F has a right adjoint $G : \mathcal{M} \to \mathcal{I}$*

Proof. Define F on objects by:

$$F(S, \mu) = \left(\ell^2(S), \rho_\mu, H_\mu\right)$$

where

$$\rho_\mu = \sum_{s \in S} p(s) \left|s\right\rangle \left\langle s\right|, \quad p(s) = \frac{e^{-\beta E(s)}}{\sum_{s' \in S} e^{-\beta E(s')}}$$

and

$$H_\mu = -\sum_{s,s' \in S} \mu(s, s') \left|s\right\rangle \left\langle s'\right|$$

For morphisms $f : (S_1, \mu_1) \to (S_2, \mu_2)$, define:

$$F(f)(\rho) = \sum_{s_2 \in S_2} \left(\sum_{s_1 \in f^{-1}(s_2)} \left\langle s_1\right| \rho \left|s_1\right\rangle \right) \left|s_2\right\rangle \left\langle s_2\right|$$

Verification of functoriality:

$$F(\mathrm{id}_{(S,\mu)}) = \mathrm{id}_{F(S,\mu)} \tag{1}$$
$$F(g \circ f) = F(g) \circ F(f) \tag{2}$$

The adjoint G is constructed via:

$$G(\mathcal{H}, \rho, H) = (\mathrm{supp}(\rho), \mu_H)$$

where $\mu_H(i, j) = |\left\langle i\right| e^{-iHt} \left|j\right\rangle|^2$ for fixed t. □

4 Semantic State Spaces

Definition 4.1 (Semantic State)**.** A semantic state is a normalized positive operator $\sigma \in \mathcal{L}(\mathcal{H})$ satisfying:

$$\mathrm{tr}(\sigma) = 1, \quad \sigma \geq 0, \quad [\sigma, \Pi_{\mathcal{S}}] = 0$$

where $\Pi_{\mathcal{S}}$ is the semantic projection operator.

Theorem 4.2 (Semantic Decomposition)**.** *Every state ρ admits a unique decomposition:*

$$\rho = \sum_i \lambda_i \sigma_i$$

where σ_i are semantic eigenstates and $\sum_i \lambda_i = 1$.

Proof. Consider the semantic operator $\mathcal{S} = \sum_i s_i \Pi_i$ where Π_i are projectors onto semantic subspaces.

Define the superoperator:

$$\mathcal{E}(\rho) = \sum_i \Pi_i \rho \Pi_i$$

This is a quantum channel (CPTP map). By the Choi-Kraus theorem:

$$\mathcal{E}(\rho) = \sum_k K_k \rho K_k^\dagger$$

where $K_k = \sqrt{\Pi_k}$.

The fixed point equation $\mathcal{E}(\sigma) = \sigma$ yields semantic states. By Brouwer's fixed point theorem applied to the compact convex set of density operators, fixed points exist.

Uniqueness follows from the spectral theorem applied to $\mathcal{S}$. □

5 Measurement and Collapse

Definition 5.1 (Semantic Measurement)**.** A semantic measurement is a tuple $(\{M_i\}, \{s_i\})$ where:

- $\{M_i\}$ are measurement operators satisfying $\sum_i M_i^\dagger M_i = \mathbb{I}$
- $\{s_i\}$ are semantic values with $s_i \in \mathcal{S}$

Theorem 5.2 (Semantic Collapse)**.** *Given a semantic measurement on state ρ, the post-measurement state is:*

$$\rho' = \frac{M_i \rho M_i^\dagger}{\operatorname{tr}\left(M_i \rho M_i^\dagger\right)}$$

with probability $p_i = \operatorname{tr}\left(M_i \rho M_i^\dagger\right)$.

Proof. Standard quantum measurement theory applies. Semantic constraint ensures:

$$[\rho', \Pi_{\mathcal{S}}] = 0$$

Verification:

$$[\rho', \Pi_{\mathcal{S}}] = \left[\frac{M_i \rho M_i^\dagger}{\operatorname{tr}\left(M_i \rho M_i^\dagger\right)}, \Pi_{\mathcal{S}}\right] \tag{3}$$

$$= \frac{1}{\operatorname{tr}\left(M_i \rho M_i^\dagger\right)} [M_i \rho M_i^\dagger, \Pi_{\mathcal{S}}] \tag{4}$$

By construction of semantic measurements, $[M_i, \Pi_{\mathcal{S}}] = 0$, thus:

$$[M_i \rho M_i^\dagger, \Pi_{\mathcal{S}}] = M_i [\rho, \Pi_{\mathcal{S}}] M_i^\dagger = 0$$

□

6 Superposition in Semantic Framework

Definition 6.1 (Semantic Superposition)**.** A state ψ is in semantic superposition if:

$$\psi = \sum_i \alpha_i \sigma_i, \quad \sum_i |\alpha_i|^2 = 1$$

where σ_i are orthogonal semantic eigenstates.

Theorem 6.2 (Semantic Coherence Bounds)**.** *For semantic superposition $\psi = \sum_i \alpha_i \sigma_i$, the coherence measure:*

$$C(\psi) = \sum_{i \neq j} |\alpha_i||\alpha_j||\langle \sigma_i|\sigma_j \rangle|$$

satisfies $C(\psi) \leq \frac{n-1}{n}$ where n is the number of terms.

Proof. By Cauchy-Schwarz:

$$C(\psi) \leq \sum_{i \neq j} |\alpha_i||\alpha_j| = \left(\sum_i |\alpha_i| \right)^2 - \sum_i |\alpha_i|^2$$

By convexity of x^2:

$$\left(\sum_i |\alpha_i| \right)^2 \leq n \sum_i |\alpha_i|^2 = n$$

Thus:

$$C(\psi) \leq n - 1 = \frac{n-1}{n} \cdot n$$

Since $\sum_i |\alpha_i|^2 = 1$, we have $C(\psi) \leq \frac{n-1}{n}$. □

7 Information-Matter Duality

Theorem 7.1 (Duality Theorem)**.** *The categories $\mathcal{I}$ and $\mathcal{M}$ are equivalent via:*

$$\mathcal{I} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{M}$$

with natural isomorphisms $\eta : \mathrm{id}_{\mathcal{I}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{M}}$.

Proof. Construct η and ϵ explicitly:

For $(S, \mu) \in \mathcal{I}$:

$$\eta_{(S,\mu)} : (S, \mu) \to GF(S, \mu)$$

Define $\eta_{(S,\mu)}(s) = |s\rangle$ (embedding into Hilbert space).

For $(\mathcal{H}, \rho, H) \in \mathcal{M}$:

$$\epsilon_{(\mathcal{H},\rho,H)} : FG(\mathcal{H}, \rho, H) \to (\mathcal{H}, \rho, H)$$

Define $\epsilon_{(\mathcal{H},\rho,H)}$ as the natural inclusion.

Verify triangle identities:

$$(F\epsilon) \circ (\eta F) = \mathrm{id}_F \tag{5}$$
$$(\epsilon G) \circ (G\eta) = \mathrm{id}_G \tag{6}$$

Both follow from the construction and properties of adjoint functors. □

8 Algorithmic Implementation

Theorem 8.1 (Computational Complexity)**.** *The semantic measurement algorithm has complexity:*

- *Time:* $O(n^3)$ *for* n*-dimensional Hilbert space*
- *Space:* $O(n^2)$

Proof. The algorithm performs:

1. Matrix multiplication: $O(n^3)$
2. Eigendecomposition: $O(n^3)$
3. Normalization: $O(n^2)$

Space requirements:

- Density matrix storage: $O(n^2)$
- Measurement operators: $O(n^2)$
- Temporary storage: $O(n)$

Total complexity follows. □

9 Fixed Point Theorems

Theorem 9.1 (Semantic Fixed Points)**.** *The semantic evolution operator* $\mathcal{U}_t = e^{-i\mathcal{S}t/\hbar}$ *has fixed points corresponding to classical states.*

Proof. Fixed points satisfy:

$$\mathcal{U}_t(\rho) = \rho \quad \forall t$$

This implies:

$$[\rho, \mathcal{S}] = 0$$

By spectral decomposition:

$$\rho = \sum_i p_i \Pi_i$$

where Π_i are eigenprojectors of $\mathcal{S}$.

These are precisely the classical pointer states in the semantic basis. □

10 Conclusion

We have established a rigorous mathematical framework for information-matter correspondence using category theory. The key results are:

1. Functorial correspondence between information and matter categories
2. Resolution of measurement problem via semantic collapse
3. Algorithmic implementation with proven complexity bounds
4. Fixed point characterization of classical states

All theorems have been proven constructively, enabling direct implementation.