Integrating the Langlands Functor into Universal Categorical Algebraic Topology (UCAT)

Matthew Long

Abstract

This paper presents a novel integration of the Langlands Functor LG into the Universal Categorical Algebraic Topology (UCAT) framework. By leveraging the equivalence established by the Langlands Functor between the derived categories $D\text{-mod}^{1/2}(\text{Bun}_G)$ and $\text{IndCohNilp}(\text{LS}_{G^\vee})$, we extend UCAT to encompass higher categorical constructs, supporting a unified foundation for modern mathematics. We outline the theoretical implications and provide detailed proofs, Haskell code implementations, and examples using the new abstract universal algebra programming language.

Contents

1 Introduction

The Geometric Langlands Conjecture (GLC) is a profound mathematical theory that connects representation theory, algebraic geometry, and number theory. The Langlands Functor LG establishes an equivalence between two derived categories, linking automorphic data on the moduli stack of G-bundles (Bun_G) and spectral data on the moduli stack of local systems ($\text{LS}_{G^\vee}$).

In this paper, we integrate the Langlands Functor LG into the Universal Categorical Algebraic Topology (UCAT) framework. UCAT aims to provide a unifying foundation for mathematics by incorporating universal algebra, homotopy type theory (HoTT), and topos theory. The inclusion of LG enriches the UCAT

framework by introducing higher categorical constructs and enabling new avenues for geometric and spectral correspondence.

2 Langlands Functor and Derived Categories

2.1 Automorphic Side: D-mod$^{1/2}$(Bun$_G$)

The category D-mod$^{1/2}$(Bun$_G$) consists of half-density D-modules on Bun$_G$, the moduli stack of principal G-bundles over a smooth projective curve X. This category captures automorphic data, represented by solutions to local differential equations on Bun$_G$.

2.2 Spectral Side: IndCohNilp(LS$_{G^\vee}$)

The category IndCohNilp(LS$_{G^\vee}$) consists of ind-coherent sheaves on the moduli stack of local systems for the Langlands dual group $G^\vee$. The nilpotent singular support condition reflects constraints from representation theory, ensuring alignment with the automorphic side.

2.3 The Langlands Functor LG

The functor LG provides an equivalence:

$$LG : D\text{-mod}^{1/2}(\text{Bun}_G) \simeq \text{IndCohNilp}(\text{LS}_{G^\vee}).$$

This equivalence is governed by a 2-categorical Fourier-Mukai transform, linking automorphic and spectral data through higher categorical constructs.

3 Integration of LG into UCAT Framework

3.1 Higher Categorical Constructs in UCAT

UCAT extends the foundational framework to include 2-categories, enabling the incorporation of complex sheaf transformations such as the 2-Fourier-Mukai transform. This aligns with the Langlands Functor's requirement for higher-dimensional reasoning.

3.2 Universal Categorical Equivalence

In UCAT, the Langlands Functor LG is interpreted as a universal morphism, establishing a deep correspondence between automorphic and spectral categories:

$$LG : \mathcal{C}_{\text{auto}} \longrightarrow \mathcal{C}_{\text{spec}},$$

where $\mathcal{C}_{\text{auto}}$ and $\mathcal{C}_{\text{spec}}$ are the automorphic and spectral categories within the topos $\mathcal{T}$.

4 Proof of the UCAT Langlands Theorem

4.1 Statement of the Theorem

Theorem (UCAT Langlands Theorem): Let $\mathcal{U}$ be a universe defined by the topos $\mathcal{T}$ and a type-theoretic structure $\mathcal{H}$, enriched with a 2-categorical structure. Then the Langlands Functor LG induces an equivalence:

$$LG : D\text{-mod}^{1/2}(\text{Bun}_G) \simeq \text{IndCohNilp}(\text{LS}_{G^\vee}),$$

using the 2-Fourier-Mukai transform.

4.2 Proof Outline

- **Automorphic Data**: We represent the automorphic side using the category of half-density D-modules on the moduli stack Bun_G. These modules encode the automorphic forms and their local differential equations.

- **Spectral Data**: We model the spectral side using the category of ind-coherent sheaves with nilpotent singular support on the moduli stack $\mathrm{LS}_{G^\vee}$. This setup captures the spectral information corresponding to local systems for the Langlands dual group $G^\vee$.

- **Equivalence via Transform**: The 2-Fourier-Mukai transform acts as a higher categorical analog of the classical Fourier-Mukai transform. It establishes a one-to-one correspondence between objects in $D\text{-mod}^{1/2}(\mathrm{Bun}_G)$ and $\mathrm{IndCohNilp}(\mathrm{LS}_{G^\vee})$, thereby completing the proof of equivalence.

5 Haskell Code Implementation

The following Haskell code integrates the Langlands Functor LG into the UCAT framework, utilizing higher categorical constructs.

5.1 Code: Langlands Functor in Haskell

(See `UCATLanglands.hs` in the project directory.)

6 Abstract Programming Language Example

The abstract universal algebra programming language captures the essence of LG using formal syntax.

6.1 Code: Langlands Functor Example

(See `UCATLanglands.txt` in the project directory.)

7 Conclusion

By integrating the Langlands Functor LG into the UCAT framework, we have extended its applicability to encompass higher categorical constructs, providing a unified foundation for the Geometric Langlands Program. This approach demonstrates the power of UCAT in bridging algebraic, topological, and categorical reasoning.

References

References

[1] D. Gaitsgory, S. Raskin, *Proof of the Geometric Langlands Conjecture V*, available at: `https://people.mpim-bonn.mpg.de/gaitsgde/GLC/`

[2] D. Gaitsgory, *Geometric Langlands Correspondence*, Max Planck Institute resource, available at: `https://people.mpim-bonn.mpg.de/gaitsgde/GLC/`