A Functorial Reformulation of Spacetime and Global Constraints
in Modern Physics

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Abstract

The reconciliation of local quantum field theories with large-scale geometric constraints has long been a central puzzle. Many physical phenomena—from topological phases of matter to gravitational boundary conditions in cosmology—involve global constraints that are not always evident when looking at local dynamics alone. In this paper, we propose a *functorial physics* approach that unifies local processes with global constraints by treating spacetime regions, boundaries, and even topological data as objects and morphisms in a (higher) category. By embedding quantum amplitudes into functors from a "spacetime cobordism" category to a category of state spaces, we show how global constraints naturally emerge from local rules. We provide the conceptual and mathematical underpinnings, give illustrative equations, and argue that this perspective clarifies the interplay of local field content with boundary conditions, anomalies, and topological constraints.

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1 Introduction

Understanding how local quantum descriptions reconcile with global geometric or topological constraints remains one of the major challenges in theoretical physics. In quantum field theory (QFT) and general relativity (GR), one encounters scenarios where the bulk, local dynamics cannot be fully understood without specifying boundary conditions or global topological data. For example:

- *Topological phases of matter* (quantum Hall states, topological insulators) depend crucially on boundary excitations and global invariants.
- *Spacetime boundaries in gravitational settings*, such as asymptotically AdS or FRW cosmologies, necessitate boundary terms and constraint equations that can drastically affect the global dynamics.

- *Gauge anomalies and homotopy classes* of gauge fields highlight how local gauge transformations fail to capture full consistency conditions unless extended globally.

A unifying framework that properly accounts for local physics while simultaneously encapsulating global constraints is therefore indispensable. *Functorial physics*—originally introduced in the context of topological quantum field theories (TQFTs)—provides a powerful lens through which spacetime and global constraints can be seen as natural consequences of certain functorial assignments.

This paper details how a *functorial* perspective reinterprets *spacetime regions*, *boundaries*, and *global constraints* as objects and morphisms in a suitable category. By generalizing the Atiyah–Segal axioms for TQFT, we incorporate local quantum theories and show how boundary conditions, anomalies, or other global features manifest functorially as coherence conditions. We then discuss how these ideas shed new light on issues such as:

(i) **Gauge invariance and anomalies,**

(ii) **Gravitational boundary terms,**

(iii) **Symmetry-protected topological phases** at finite temperature or curvature.

2 Spacetime as a Category and Its Global Constraints

2.1 Cobordisms and Regions

A key insight, pioneered by the work on TQFT, is that *spacetimes with boundaries (cobordisms)* can be treated as morphisms between boundary components. Concretely, one can define:

$$\mathcal{C} \;=\; \text{(Category of cobordisms)},$$

where

- **Objects** are $(d-1)$-dimensional manifolds Σ (possible spatial boundaries),
- **Morphisms** are d-dimensional cobordisms M such that $\partial M = \Sigma_{\text{in}} \cup \Sigma_{\text{out}}$,
- Composition of morphisms corresponds to gluing cobordisms along matching boundaries.

In topological quantum field theory, a TQFT is then a functor

$$Z : \mathcal{C} \;\longrightarrow\; \mathcal{D},$$

where $\mathcal{D}$ is typically a category of vector spaces (or more advanced algebraic structures). By assigning a vector space $Z(\Sigma)$ to each boundary Σ and a linear map $Z(M) : Z(\Sigma_{\text{in}}) \to Z(\Sigma_{\text{out}})$ to each cobordism M, Z encapsulates how global constraints (e.g. topology of M) encode quantum amplitudes.

2.2 Extending Beyond TQFT: Local Fields + Boundary Data

While TQFT focuses on purely topological data, one can enrich $\mathcal{C}$ to incorporate local field content (e.g. gauge fields, metric data). For instance:

$$\mathcal{C}_{\text{Grav}} : \quad \text{Objects} = \{\Sigma,\ \text{possible 3-metrics, boundary conditions}\}, \quad \text{Morphisms} = \{\text{4D cobordisms, local g}$$

The presence of nontrivial curvature or local degrees of freedom means the resulting functor Z (or a more general $\mathcal{F}$) encodes not just topological invariants but also boundary constraints that must be satisfied by the local fields. This systematically merges local PDE constraints (Einstein equations, gauge field equations) with boundary conditions and global anomalies (e.g. gravitational anomalies).

3 Resolving Global Constraints Functorially

3.1 Local to Global Consistency

In a purely local quantum field theory, one might write down an action or Hamiltonian density $\mathcal{L}(\phi, \partial_\mu \phi, \dots)$, deriving field equations or correlation functions. However, specifying boundary conditions or topological sectors typically happens *ad hoc*. In the functorial approach:

$$\mathcal{F} \ : \ \mathcal{C}_{\text{Spacetime}} \longrightarrow \mathcal{C}_{\text{States}}, \tag{1}$$

the data of allowed boundary conditions, anomalies, or zero-modes is forced to *commute* with gluing operations. Hence, the resulting global constraints are not separate from local equations but an inevitable property of how $\mathcal{F}$ assigns state spaces to boundaries. Composition of spacetimes yields the composition of maps in $\mathcal{C}_{\text{States}}$, which can be linear operators, groupoid morphisms, or even ∞-morphisms.

3.2 Anomalies as Obstructions to Natural Transformations

Gauge or gravitational anomalies can be seen as obstructions to extending $\mathcal{F}$ consistently across all morphisms in the cobordism category. If a global symmetry fails to be realized at the quantum level, it implies $\mathcal{F}$ cannot be upgraded to a symmetry-transformed version $\mathcal{F}'$ in a way that is fully natural. Diagrammatically, certain diagrams in $\mathcal{C}$ fail to commute in $\mathcal{C}_{\text{States}}$, manifesting the anomaly as the mismatch in amplitude assignments for “looped” or “twisted” boundary identifications.

4 Illustrative Equations and Setup

4.1 Functorial TQFT Generalization

For a standard d-dimensional TQFT, we have the assignment:

$$Z(\Sigma) \ = \ \text{a vector space (Hilbert space) } H_\Sigma, \quad Z(M) : H_{\Sigma_{\text{in}}} \ \to \ H_{\Sigma_{\text{out}}}.$$

If we incorporate local gauge fields A and curvature R, the amplitude might look like

$$Z(M;A,R) \;=\; \int_{\substack{\phi,\gamma\in \\ \text{boundary conditions}}} \exp\Big(iS[M,\phi,A,R]\Big), \tag{2}$$

where the path integral or functional measure is restricted by boundary conditions $\phi\big|_{\partial M} = \varphi$ and gauge constraints. The result is an element of $\mathrm{Hom}(H_{\Sigma_{\text{in}}},\, H_{\Sigma_{\text{out}}})$.

4.2 Local PDE, Global Glueing

Local PDE constraints (e.g. the Einstein equations in a region M) imply that only certain fields ϕ or metrics g are admissible. But requiring the boundary Σ to connect consistently with another manifold M' in a glued manifold $M \cup_\Sigma M'$ imposes further constraints (like matching gauge potentials or metric data at Σ). From a functorial standpoint:

$$Z(M \cup_\Sigma M') \;=\; Z(M') \circ Z(M),$$

so any mismatch in boundary data would break the composition law and thus break the functor property. This is precisely how global constraints arise from local PDE solutions glued across boundaries.

5 Interpretational Clarity

The advantage of a functorial viewpoint lies in:

- ◇ **Unified PDE + Boundary Approach:** One no longer sees boundary conditions as external picks but as morphological data in the category itself.
- ◇ **Geometric / Topological Phases as Objects:** Different phases or topological classes appear as objects in the "extended" category, revealing that transitions between them are morphisms requiring or disallowing certain boundary identifications.
- ◇ **Anomaly Detection as Diagrammatic Failure:** Instead of "secret constraints," anomalies emerge as the impossibility of making certain diagrams commute, i.e. a fundamental mismatch in amplitude assignment across global loops.

Thus, what might seem mysterious or scattered in a local-only viewpoint finds a coherent explanation as soon as we consider "spacetime + boundary" as morphisms in a structured category, with quantum states or amplitude maps realized as functors out of this category.

6 Conclusion and Outlook

By reinterpreting *spacetime regions, boundaries, gauge fields, and constraints* as part of a functorial tapestry, we:

(a) Enforce local PDE constraints by restricting object and morphism content,

(b) Derive global constraints naturally from composition laws in the category,

(c) Clarify anomalies as the non-existence of certain natural transformations,

(d) Provide a robust setting for bridging classical boundary value problems (like GR with specific asymptotics) and quantum phenomena (like TQFT or boundary excitations in topological phases).

Looking ahead, several key directions emerge:

- **Higher-Categorical Structures:** Many realistic situations require 2-categories or ∞-categories to capture extended operators, corners, or fracton-like excitations.
- **Applications to Quantum Gravity:** Attempting a fully quantum gravitational theory might demand a functor from a 4D cobordism category (with boundary data, possibly an ∞-category) to a category of states that includes gravitational degrees of freedom, topological transitions, and black hole boundaries.
- **Interfacing with AdS/CFT:** Global constraints in holography can be recast in boundary-bulk functors, bridging local bulk PDE solutions with boundary CFT operators in a rigorous compositional manner.

In short, *Functorial Physics* offers a unifying language where *Spacetime + Global Constraints* become not an afterthought but an inherent feature of how amplitudes, states, and boundaries compose.

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