

Towards Self-Correcting Topological Codes Through Functorial Physics: An Abstract

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Abstract

We propose a unification of Category Theory, Topology, and Thermodynamics as a framework for constructing and analyzing self-correcting topological quantum codes. Our formulation hinges on viewing physical models through functorial lenses, enabling us to map thermodynamic stability and error-correction properties to categorical invariants. Drawing on braided monoidal categories, extended topological quantum field theories, and Landau-like thermodynamic arguments, we demonstrate how protected quantum information might arise from topological phases in diverse dimensions and temperature regimes. We conclude with a discussion of open questions around finite-temperature stability, long-range entanglement, and the existence of truly self-correcting phases.

Contents

1	Introduction	2
1.1	Paper Outline	2
2	Preliminaries: Category Theory and Topological Quantum Field Theories	3
2.1	Braided Monoidal Categories	3
2.2	Extended TQFTs	3
3	Thermodynamic Landscapes and Self-Correction	3
3.1	Partition Functions and Free Energy	4
3.2	Toward a Functorial Thermodynamics	4
4	Functorial Architecture for Topological Codes	4
4.1	Categories of Lattice Systems	4
4.2	Braided Tensor Categories and Logical Operators	5
5	Examples and Model Calculations	5
5.1	The Toric Code	5
5.2	3D Codes and Self-Correction Attempts	5

1 Introduction

Quantum error correction is an essential prerequisite for scalable quantum computation. Among various strategies, *topological quantum error-correcting codes* (TQ ECCs) have drawn significant attention because they promise passive robustness against local errors by storing information in global topological degrees of freedom [1, 2]. However, such codes typically require cryogenic temperatures and impeccable isolation from the environment to maintain coherence. A long-standing goal is to devise *self-correcting* quantum memories, where the storage of quantum information would be robust over a range of nonzero temperatures and timescales without active error-correction procedures [3, 4].

In recent years, several theoretical developments have drawn on ideas from higher category theory and extended topological quantum field theories (TQFTs) to describe topological states of matter and their excitations [5, 6]. In parallel, progress in understanding the thermodynamics of topological phases (particularly the interplay of energy gaps, anyonic excitations, and entanglement renormalization flows) has offered new insights into stability conditions [7, 8].

This paper seeks to unify these approaches into a single “functorial physics” framework, leveraging the language of category theory to encode how local physical laws (Hamiltonians, interactions) give rise to global topological invariants and emergent error-correcting behavior. We propose that *functorial* mappings from suitable higher categories of quantum lattice systems into topological spaces—or TQFT-like data—can capture not only braiding and fusion structures but also the thermodynamic constraints for maintaining coherence.

1.1 Paper Outline

1. **Preliminaries in Category Theory and TQFT.** We review monoidal, braided, and modular tensor categories, extended TQFTs, and the concept of functoriality in physics.
2. **Thermodynamic Landscapes and Self-Correction.** We discuss the known thermodynamic constraints on topological codes and summarize existing no-go results for 2D self-correcting memories, along with prospects for 3D and higher dimensions.
3. **Functorial Architecture.** We propose a schematic approach for encoding Hamiltonians, partition functions, and error-correcting structures into a suitable categorical or TQFT-based language.
4. **Examples and Model Calculations.** We illustrate how to represent well-known topological codes (e.g., Kitaev’s toric code) as objects in a braided category and show how certain functorial invariants might encode error-correcting properties.

5. **Open Questions and Future Directions.** We address the problem of finite-temperature stability, possible pathways to higher-dimensional self-correcting codes, and how this functorial framework can inform hardware design.

2 Preliminaries: Category Theory and Topological Quantum Field Theories

We start by recalling the basic notions from category theory that are especially relevant for describing topological phases and quantum codes.

2.1 Braided Monoidal Categories

Let \mathcal{C} be a monoidal category with tensor product $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ and an associator $a_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$. A *braiding* is a natural isomorphism

$$c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X,$$

satisfying certain coherence conditions (the hexagon identities, see [6]). In a *modular tensor category* (MTC), one further demands non-degeneracy of the braiding and the existence of a twist (or ribbon) structure.

These categories can classify anyonic excitations in 2D topological states. For instance, the fusion and braiding statistics of quasiparticles in the *fractional quantum Hall* system are described by an MTC. The toric code can also be cast in this language, albeit in a simpler, Abelian category.

2.2 Extended TQFTs

A *topological quantum field theory* in dimension d is typically defined as a functor

$$Z : \text{Bord}_d \longrightarrow \mathcal{V},$$

where Bord_d is the category (or higher-category) of d -dimensional manifolds with boundaries (and possibly corners) and \mathcal{V} is a symmetric monoidal category of vector spaces, chain complexes, or more sophisticated algebraic objects [9, 10]. *Extended* TQFTs refine this further by assigning data not only to closed manifolds but also to their boundary manifolds and lower-dimensional strata.

In the context of quantum codes, Z can be construed as assigning Hilbert spaces (or state spaces) to boundary components and linear maps (or more structured morphisms) to cobordisms connecting these boundaries. Hence, topological invariants derived from Z can characterize robust ground states in a physical system.

3 Thermodynamic Landscapes and Self-Correction

A self-correcting quantum code demands that the effective error rate decreases with system size, naturally arising from energetically costly excitations that serve as errors. Intuitively,

a system with a large energy gap Δ and suppressed topological defect proliferation might provide such self-correction. However, no-go theorems indicate that in strictly 2D systems, thermal excitations proliferate enough to spoil self-correction [11, 12].

3.1 Partition Functions and Free Energy

Consider a lattice Hamiltonian H encoding a topological phase. At finite temperature T , the partition function is

$$Z(\beta) = \text{Tr } e^{-\beta H}, \quad \text{where } \beta = \frac{1}{k_B T}.$$

If H is gapped and topologically ordered, we expect a degeneracy of the ground state that is stable under small local perturbations. Nonetheless, the free-energy landscape $F(\beta) = -\frac{1}{\beta} \ln Z(\beta)$ can develop local minima corresponding to excited states that effectively behave as logical errors when they proliferate.

3.2 Toward a Functorial Thermodynamics

In a *functorial* viewpoint, each physical configuration (or boundary condition) can be mapped to partition-function-like data (vector spaces, chain complexes, etc.), and evolution under thermodynamic transformations (e.g., raising or lowering T) could be interpreted as natural transformations of these functors. The challenge is to formulate a higher categorical framework that captures entropic and energetic considerations—where topological properties (braiding, twisting) couple to thermal excitations in a controlled manner.

4 Functorial Architecture for Topological Codes

We propose that a “functorial physics” architecture can unify:

1. The *local data* of physical interactions (encoded by a Hamiltonian on a discretized manifold).
2. The *global topological data* that emerges in the low-energy sector (described by an extended TQFT functor).
3. The *thermal excitations and error processes* in the finite-temperature regime.

4.1 Categories of Lattice Systems

Let us define a category \mathcal{L} whose objects are $(d-1)$ -dimensional lattices (possible “space slices” of our system), and morphisms are local rearrangements or “time evolutions” of these lattices (potentially including local error processes). A second category \mathcal{T} consists of topological data: for instance, objects are topological boundary conditions, and morphisms are cobordisms.

A *functorial physics* approach posits a functor

$$\Phi : \mathcal{L} \longrightarrow \mathcal{T},$$

that assigns to each lattice (and local Hamiltonian) a topological boundary data object in \mathcal{T} , and to each local evolution an appropriate cobordism or topological morphism capturing how excitations transform. The essence of self-correction is then to show that any local error chain with energy cost E is “topologically trivializable” within \mathcal{T} , or that the cost grows sufficiently with system size so that such errors become exponentially suppressed.

4.2 Braided Tensor Categories and Logical Operators

In many 2D TQEC models (like the toric code), logical operators correspond to global string or membrane operators that wrap nontrivial topological cycles. In braided tensor category terms, these correspond to braiding or fusion paths among anyonic excitations. To show that errors are correctable, one needs to argue that any local error process that attempts to realize a nontrivial braiding or fusion pathway requires creating and transporting excitations over macroscopic distances, incurring large energy costs.

In 3D and higher dimensions, there is room for more exotic excitations (string operators, membrane operators, fractons, etc.) [13, 14]. Our formalism suggests analyzing them as morphisms in a higher or “stratified” categorical framework, thereby generalizing the notion of braiding and fusion.

5 Examples and Model Calculations

5.1 The Toric Code

The simplest example is Kitaev’s toric code, defined on a 2D lattice with qubits on edges. The Hamiltonian is

$$H = - \sum_v A_v - \sum_f B_f,$$

where A_v and B_f are star and plaquette operators, respectively [1]. Each violation of these constraints corresponds to an anyonic excitation. In the category-theoretic viewpoint, the ground state subspace corresponds to a *Drinfeld center* (or a suitable representation) of the finite group \mathbb{Z}_2 in a braided category. Logical operators are Wilson loops that wrap nontrivial cycles.

5.2 3D Codes and Self-Correction Attempts

In 3D, one might consider the *3D toric code* [7] or the *Haah code* [13]. The latter is of particular interest for self-correction because of the fractal structure of logical operators. In principle, large system sizes could suppress logical errors at finite temperature. However, a complete “functorial TQFT” interpretation of fracton codes remains an active area of research. Preliminary attempts place fractons in a higher category framework, but the nature of the excitations is more rigid than simple anyon braiding.

6 Open Questions and Future Directions

- **Finite-temperature Stability.** Despite rigorous frameworks, no *proven* 2D self-correcting code is known, and 3D models remain challenging. Can a purely topological code maintain coherence over arbitrarily long times at finite T ?
- **Holographic and Higher-Dimensional Insights.** Methods from holographic dualities and higher form symmetries might elucidate the interplay of boundary/bulk degrees of freedom in higher-dimensional codes.
- **Functorial Thermodynamics.** A fully fleshed-out category that encodes partition functions, excitations, and topological invariants in the same structure remains elusive.
- **Experimental Realization.** Even if theoretically possible, implementing a large-scale, truly self-correcting code poses significant hardware challenges, such as engineering large energy gaps and controlling boundary conditions in 3D or 4D structures.

7 Conclusion

We have outlined a unifying “functorial physics” framework aimed at describing topological quantum codes through categorical and topological data, while taking into account the thermodynamic constraints relevant to *self-correction*. By leveraging extended TQFT structures and higher monoidal categories, one can systematically track how local errors translate into global topological actions. Although significant theoretical and experimental challenges remain, this viewpoint offers a coherent language in which to pursue the long-sought goal of *passively* stable quantum memories.

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