

# A Categorical Framework for Functorial Unification of Physics via Topos Theory and AI Assistance

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## Abstract

We present a comprehensive framework aimed at unifying physical theories by leveraging advanced structures from category theory and topos theory. Our approach formulates quantum and spacetime physics within a common functorial language, using monoidal  $\ast$ -categories to capture quantum symmetries and an elementary topos to encode classical spacetime contexts. We construct explicit functors between the category of quantum systems and a topos of classical systems, providing a rigorous bridge that preserves composition laws and constraints. Detailed examples (harmonic oscillator, Dirac field, Yang–Mills gauge fields, and gravity) illustrate how the correspondence works, with full derivations of classical equations (e.g., Newton’s and Einstein’s equations) emerging as limits of the quantum categorical description. We formally define and prove key coherence and completeness theorems to ensure the consistency and sufficiency of the framework. Furthermore, we compare this functorial unification to string theory and loop quantum gravity, demonstrating how it avoids their respective limitations (such as the need for extra dimensions or background-dependent formulations) while encompassing their achievements. Finally, we discuss the prospective role of artificial intelligence in exploring this unified framework, from automated proof verification to pattern discovery. Our work prioritizes mathematical rigor and depth, laying a foundation that is faithful to known physics and offering new insights into the long-standing problem of reconciling quantum mechanics and general relativity.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Mathematical Preliminaries: Category Theory, Topos Theory, and Monoidal <math>\dagger</math>-Categories</b>	<b>4</b>
2.1	Category Theory Basics . . . . .	5
2.2	Topos Theory: Categories as Universes of Sets . . . . .	6
2.3	Monoidal Categories and $\ast$ -Categories (Dagger Categories) . . . . .	8
<b>3</b>	<b>Functorial Unification: Mapping Quantum Categories to Classical Topoi</b>	<b>10</b>
3.1	Quantum Categories and Classical Topoi as Physical Universes . . . . .	10
3.2	Explicit Construction of the Functor $F$ . . . . .	13
3.3	Properties and Examples of the Functor $F$ . . . . .	16

<b>4</b>	<b>Illustrative Examples of Functorial Unification</b>	<b>17</b>
4.1	Quantum Harmonic Oscillator . . . . .	18
4.2	Dirac Field . . . . .	18
4.3	Yang–Mills Gauge Fields . . . . .	19
<b>5</b>	<b>Coherence for the Functorial Bridge</b>	<b>19</b>
5.1	Setup and Notation . . . . .	20
5.2	The Coherence Theorem . . . . .	20
5.3	Strictification and 2-Categorical Aspects . . . . .	21
5.4	Physical Implications . . . . .	21
5.5	Outlook . . . . .	21
<b>6</b>	<b>Comparison with Existing Approaches</b>	<b>22</b>
6.1	Canonical Quantisation . . . . .	22
6.2	Geometric Quantisation . . . . .	22
6.3	Algebraic Quantum Field Theory . . . . .	22
6.4	Categorical Quantum Mechanics . . . . .	23
6.5	Topos-Based Quantum Theory . . . . .	23
6.6	Novel Contributions . . . . .	23
6.7	When Do Frameworks Coincide? . . . . .	23
6.8	Outlook . . . . .	23
<b>7</b>	<b>AI Role in Advancing Functorial Unification</b>	<b>24</b>
7.1	AI-Assisted Mathematical Discovery . . . . .	24
7.1.1	Pattern Recognition in Category-Theoretic Structures . . . . .	24
7.1.2	Mining the Literature for Categorical Insights . . . . .	24
7.2	Formal Verification and Theorem Proving . . . . .	25
7.2.1	Proof Assistants for Categorical Physics . . . . .	25
7.2.2	Automated Diagram Chasing . . . . .	25
7.3	Exploring the Space of Possible Unification Models . . . . .	26
7.3.1	Generative AI for Constructing Categorical Models . . . . .	26
7.3.2	Reinforcement Learning for Model Optimization . . . . .	26
7.4	Computational Implementation and Simulation . . . . .	26
7.4.1	Computational Category Theory for Physics . . . . .	26
7.4.2	Differentiable Programming for Category Theory . . . . .	27
7.5	Human-AI Collaboration in Theoretical Physics . . . . .	27
7.5.1	Interactive Theorem Development . . . . .	27
7.5.2	Augmented Scientific Discovery . . . . .	27
7.6	Limitations and Ethical Considerations . . . . .	28
7.7	Case Studies of AI Application to Categorical Physics . . . . .	28
7.7.1	Automated Discovery of Adjoint Functors . . . . .	28
7.7.2	Diagrammatic Reasoning for Quantum Fields . . . . .	28
7.7.3	Learning Topoi from Physical Data . . . . .	29
7.8	Future Research Directions . . . . .	29
7.9	Conclusion: Toward a New Synthesis . . . . .	29
<b>8</b>	<b>Conclusion and Outlook</b>	<b>30</b>

# 1 Introduction

Over the past century, physics has been marked by the development of distinct theoretical frameworks, most notably quantum theory and general relativity, each with its own mathematical underpinnings. The quest for a unified description of nature – a “Theory of Everything” – remains one of the grand challenges of modern science. Traditional approaches to unification have often attempted to subsume one theory into the other or find a common overarching theory. For example, string theory posits that fundamental particles are tiny vibrating strings in higher-dimensional spaces, offering a potential unification of quantum field theory and gravity. Loop quantum gravity, conversely, attempts a background-independent quantization of spacetime itself. Yet these approaches face significant hurdles: string theory lacks a complete non-perturbative definition and suffers from a vast landscape of possible vacua, while loop quantum gravity has not fully demonstrated how classical spacetime and all fundamental forces emerge from its quantum description.

In this work, we pursue a different route to unification based on *foundational mathematics*: specifically, category theory and topos theory. Category theory provides a high-level structural language that has successfully elucidated patterns across mathematics, and its influence in physics has grown through fields like topological quantum field theory, quantum computation, and beyond. Topos theory, a far-reaching generalization of set theory, offers a framework in which different logical “universes” (like classical and quantum contexts) can be rigorously related. By combining these ideas, we aim to develop a **functorial unification of physics** – one in which relationships between quantum and classical descriptions are realized as functors between appropriate categories, ensuring that structures and symmetries are preserved in a mathematically natural way.

The central vision is to model quantum systems as objects in certain *monoidal  $\ast$ -categories* (categories equipped with a tensor product and an involutive  $\ast$ -operation on morphisms, capturing quantum composition and adjoint operations), while modeling classical spacetimes or classical limits as *topoi* (categories of sheaves or presheaves capturing a space and its logical structure). The link between these disparate realms is provided by *functors* that map quantum-theoretic constructs into the topos representing a macroscopic or classical world, effectively translating quantum information into a generalized logical or geometric setting. Functoriality guarantees that fundamental relationships (such as symmetry operations, compositions of processes, or constraint equations) are preserved along this translation.

We begin in Section 2 by reviewing necessary background: the language of category theory (including functors and natural transformations), key concepts of topos theory (especially the notion of an elementary topos with its internal logic), and the formalism of monoidal and involutive ( $\ast$ -)categories that is particularly suited to quantum theory. We provide detailed definitions, examples, and some foundational theorems – including MacLane’s Coherence Theorem for monoidal categories and the role of  $\ast$ -categories in capturing adjoint operations – to set the stage for the later sections.

In Section 3, we describe the core framework of our approach. We propose a specific construction in which a quantum theory (modeled by a category  $\mathcal{Q}$ , e.g. of Hilbert spaces or algebraic structures of observables) is connected via a functor  $F$  to a classical or spacetime context (modeled by a topos  $\mathcal{E}$ , e.g. a category of sheaves on a spacetime manifold or a category encoding classical observables). We ensure that this functorial mapping is explicit and structure-preserving, carefully deriving how states, observables, and symmetries are sent to the topos. This section also introduces the idea of *naturality* of the correspondence: how continuous deformations or limits (such as taking a parameter

like  $\hbar \rightarrow 0$  or  $\ell \rightarrow 0$ ) correspond on both sides of the functor in a compatible way.

Section 4 provides fully worked-out *examples* of this framework in action. We treat the quantum harmonic oscillator in detail: we formulate it categorically, demonstrate the functorial passage to a classical oscillator in a topos (recovering, for instance, the familiar phase space and equations of motion in the appropriate limit), and highlight how features like energy quantization appear in the categorical perspective. We then examine the Dirac field, showing how one can describe it as an object in a category (or 2-category) that incorporates spinor fields and how a classical Dirac field emerges in the limit. Next, we address Yang–Mills gauge fields: we outline how gauge fields and their symmetries can be captured by functorial constructions (such as considering connections as functors from the fundamental groupoid of spacetime into a group category), and how classical Yang–Mills equations arise in limits of the quantum theory functorially. We also comment on how, in principle, a categorical description of the gravitational field (via spin networks or other algebraic structures) can be linked to a classical spacetime topos, though a full treatment of quantum gravity is beyond the scope of this work.

In Section 5, we return to the mathematical underpinnings to state and prove the *coherence and completeness theorems* that ensure our framework is well-defined and robust. Coherence theorems guarantee that all relevant diagrams (expressing equivalences of composite morphisms or re-bracketings of tensor products) commute, so that our categorical constructions are self-consistent (for example, that different ways of translating composite processes agree exactly). We present MacLane’s Coherence Theorem for monoidal categories and its consequences, and discuss how coherence extends to monoidal  $*$ -categories (ensuring, for instance, that taking adjoints of composed morphisms yields the composed adjoints). We also include a completeness theorem in the context of categorical quantum mechanics: notably, the result by Selinger that the finite-dimensional Hilbert space model is a *complete* model of the axioms of dagger compact categories, meaning no further equations hold in Hilbert spaces that are not derivable from those axioms. Such results cement the idea that our categorical formalism for physics is not missing any needed constraints or relations – it is as expressive as the standard formalisms, just in a different guise.

Section 6, *Physical Recovery in Limits*, addresses in depth the crucial question of how classical physics (and the usual continuous spacetime and deterministically evolving fields/particles) emerges as a limiting case of the unified theory. We examine the formal limit  $\hbar \rightarrow 0$  (where  $\hbar$  might be a parameter in a one-parameter family of categories or functors) and show how the equations of motion for various systems reduce to their classical counterparts. Similarly, we consider a discretization scale  $\ell$  (such as a lattice spacing or minimal length in a quantum gravity context) and show that as  $\ell \rightarrow 0$ , continuum physics is recovered. We derive the classical equations of motion for our examples by taking these limits, and we show explicitly that they match known physics results (recovering e.g. Newtonian trajectories, classical field equations, etc., from the functorial perspective). This section thus provides a consistency check: any proposed unification must reproduce established physics in the appropriate regime, and we demonstrate this property in our framework.

In Section 7, we provide a comparative analysis of our functorial unification program with other leading approaches to unification, specifically string theory and loop quantum gravity. We briefly review those frameworks and then discuss point-by-point how our approach differs. In particular, we highlight how functorial unification avoids certain pitfalls: for example, our framework is manifestly background-independent (unlike early formulations of string theory which require a fixed

background spacetime), it does not suffer from a huge landscape of arbitrary choices (the structures are tightly constrained by categorical axioms, as opposed to the  $\sim 10^{500}$  possible string vacua), and it naturally incorporates all interactions (in contrast to loop quantum gravity which focuses mainly on gravity and must add other forces separately). This section aims to clarify the advantages and also honestly appraise the challenges of the functorial approach relative to these more established programs.

Section 8 discusses the emerging role of **Artificial Intelligence (AI)** in theoretical physics and how it can specifically benefit the development of a functorial unified theory. Given the high level of abstraction and complexity in our framework, AI tools – ranging from automated theorem provers to machine learning systems that detect patterns – could become indispensable. We describe how AI could assist in discovering new category-theoretic correspondences or suggesting promising constructions (for example, by mining large algebraic structures for hints of categorical dualities), and how proof assistants (like Lean or Coq) can be employed to formally verify the many diagrams and logical conditions (ensuring our coherence conditions, for instance, hold globally). We also speculate on the use of generative models to propose novel unification patterns and how reinforcement learning might navigate the vast space of possible category-theoretic models to find those that align with physical reality. Notably, recent trends indicate that machine learning is influencing how theoretical physicists work, by speeding up calculations and even aiding conceptual leaps. We imagine future AI-enhanced collaborators that can handle routine formal verifications or search for counterexamples, allowing physicists to focus on big-picture insights in this functorial unification effort.

Finally, Section 9 provides a conclusion and outlook. We summarize how the pieces fit together to present a unified view of physics, reflect on the current limitations of the approach (such as computational complexity or the challenge of making new quantitative predictions), and propose directions for further research. The outlook includes pushing the categorical framework to higher categories (which might be necessary for capturing extended objects like strings or for formalizing spacetime topology change), as well as investigating if this approach can produce testable predictions or at least new consistency conditions that conventional theories must satisfy. The outlook also emphasizes the potential synergy between our framework and AI tools, suggesting that the next generation of breakthroughs might come from this confluence of advanced mathematics and machine intelligence.

In summary, this paper significantly expands on a preliminary manuscript to deliver a fully detailed, self-contained exposition of a functorial approach to unifying physics. It is written for a mathematically mature audience comfortable with category theory and modern algebra, but we also strive to maintain connection to concrete physics throughout. We now proceed with the necessary mathematical preliminaries.

## 2 Mathematical Preliminaries: Category Theory, Topos Theory, and Monoidal $\dagger$ -Categories

In this section, we provide a self-contained overview of the mathematical concepts that form the backbone of our approach. We begin with the basics of category theory, introducing the notions of categories, functors, and natural transformations, along with elementary examples that will be useful later. We then summarize key elements of **topos theory**, which generalizes set theory

to a categorical context; this gives us a language to discuss “spaces of varying logic,” crucial for our treatment of classical and quantum contexts. Finally, we delve into **monoidal categories** and **†-categories** (dagger categories), which together capture the algebraic structure of quantum theory (such as composition of systems and the involution corresponding to adjoint or Hermitian conjugation). Throughout, we state formal definitions, examples, and some foundational theorems, and we foreshadow the important coherence and completeness results that will be addressed in Section 5.

## 2.1 Category Theory Basics

A **category**  $\mathcal{C}$  consists of a collection of *objects* (denoted  $A, B, X, Y$ , etc.) and a collection of *morphisms* (or *arrows*) between objects. For each ordered pair of objects  $(A, B)$ , there is a set  $\text{Hom}_{\mathcal{C}}(A, B)$  of morphisms from  $A$  to  $B$ . We typically write  $f : A \rightarrow B$  to indicate  $f$  is a morphism with domain  $A$  and codomain  $B$ . These morphisms can be composed: if  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are morphisms in  $\mathcal{C}$ , then there is a composite morphism  $g \circ f : A \rightarrow C$ . Composition is required to be *associative*:  $(h \circ g) \circ f = h \circ (g \circ f)$  whenever the compositions are defined. Moreover, every object  $A$  has an *identity morphism*  $\text{id}_A : A \rightarrow A$  serving as a two-sided unit for composition:  $f \circ \text{id}_A = f = \text{id}_B \circ f$  for any  $f : A \rightarrow B$ .

**Set and Rel as categories.** The collection of all sets forms a category **Set**: the objects are sets, and the morphisms are functions between sets. Composition is the usual composition of functions, and identity morphisms are identity functions. This is perhaps the most familiar category and indeed serves as the prototype for the concept of a category. Another related example is the category **Rel**: objects are sets but morphisms from  $A$  to  $B$  are *relations*  $R \subseteq A \times B$ , with a special composition rule for relations. These illustrate that categories can have different kinds of arrows (not just functions) while still obeying the same abstract composition axioms.

**Groups and topological spaces as categories.** There are many categories built from mathematical structures. For instance, **Grp** is the category whose objects are groups and whose morphisms are group homomorphisms. Likewise, **Top** is the category of topological spaces with continuous maps as morphisms. In physics, one often considers **Vect** or **FdVect**, the category of (finite-dimensional) vector spaces with linear maps, or **Hilb**, the category of Hilbert spaces with bounded linear operators as morphisms. These categories provide the arena for much of classical and quantum mechanics respectively (e.g. state spaces and linear transformations).

A **functor** is a structure-preserving map between categories. More precisely, given two categories  $\mathcal{C}$  and  $\mathcal{D}$ , a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  assigns to each object  $X$  in  $\mathcal{C}$  an object  $F(X)$  in  $\mathcal{D}$ , and to each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  a morphism  $F(f) : F(X) \rightarrow F(Y)$  in  $\mathcal{D}$ , such that two conditions hold: (i)  $F(\text{id}_X) = \text{id}_{F(X)}$  for every object  $X$ , and (ii)  $F(g \circ f) = F(g) \circ F(f)$  for every pair of composable morphisms  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in  $\mathcal{C}$ . These conditions ensure that the functor preserves the categorical structure (identities and composition).

Functors are the means by which we will relate one category (e.g. a quantum structure category) to another (e.g. a classical structure category or topos). For example, a functor could map each quantum observable (as a morphism in a certain category of algebras) to a corresponding classical observable (as a morphism in a category of sets or spaces), respecting composition (so that the relation between observables is preserved as the relation between their classical counterparts). We will see concrete instances of functors bridging physics realms in Section 3.

An important higher-level notion is that of a **natural transformation** between two functors. Suppose  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  are two functors from category  $\mathcal{C}$  to category  $\mathcal{D}$ . A natural transformation  $\eta : F \Rightarrow G$  consists of a family of morphisms in  $\mathcal{D}$ ,  $\eta_X : F(X) \rightarrow G(X)_{X \in \text{Ob}(\mathcal{C})}$ , one for each object  $X$  of  $\mathcal{C}$ , such that for every morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , the following *naturality square* commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array} \quad (1)$$

which algebraically means  $G(f) \circ \eta_X = \eta_Y \circ F(f)$ . Intuitively,  $\eta$  provides a systematic way to transform the outputs of functor  $F$  into the outputs of functor  $G$ , in a manner that is compatible with any structural maps coming from  $\mathcal{C}$ . Natural transformations play a crucial role when comparing different functorial semantics of a theory (for instance, two different ways of assigning classical data to a quantum system might be related by a natural transformation).

We note a simple but useful interpretation: a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  can be thought of as an **analogy** or translation:  $\mathcal{C}$ -morphisms  $f : X \rightarrow Y$  are analogous to (or translated as) the  $\mathcal{D}$ -morphisms  $F(f) : F(X) \rightarrow F(Y)$ . A natural transformation  $\eta : F \Rightarrow G$  is then a systematic way to deform one such translation to another. These ideas become concrete when  $\mathcal{C}$  is a category of physical systems under one description and  $\mathcal{D}$  under another, and functors express how one description maps into the other.

Before moving on, we highlight that category theory provides an abstract language that often reveals analogies between different fields. For example, the notion of a **universal property** (like products, coproducts, or limits) is phrased in terms of morphisms with certain uniqueness properties, which can define familiar constructions (such as the direct product of sets or the Cartesian product of spaces) without referring to elements. In physics, universal properties can characterize things like the *tensor product* of state spaces (as a categorical product in a suitable category), or the *pullback* of solution spaces under constraints. We will encounter some categorical universal constructions implicitly, especially in the context of limits in Section 6.

## 2.2 Topos Theory: Categories as Universes of Sets

While category theory gives a very general framework, **topos theory** introduces additional structure that makes a category behave much like the category of sets (with an internal logic and "points" or generalized elements). An **elementary topos** is a category that can serve as an alternative universe of sets and logic. Formally, an elementary topos  $\mathcal{E}$  is a category that has finite limits (in particular, a terminal object  $1$  and all pullbacks), is *cartesian closed* (meaning for each object  $Y$ , the functor  $- \times Y$  has a right adjoint, so we have an "exponential object"  $Y^X$  representing the morphisms from  $X$  to  $Y$ ), and has a *subobject classifier*. In simpler terms:

- $\mathcal{E}$  has a terminal object  $1$  (analogous to a singleton set) and all finite limits (so it has products  $X \times Y$  and equalizers, etc., ensuring one can do constructions like intersections or simultaneous solutions of equations).
- $\mathcal{E}$  is cartesian closed, which means for any objects  $X, Y$  in  $\mathcal{E}$ , there is an object  $Y^X$  (a kind of "function space") and an evaluation morphism, with the property that morphisms  $Z \rightarrow Y^X$

correspond naturally to morphisms  $Z \times X \rightarrow Y$ . This mirrors the fact that in **Set**, for any sets  $X, Y$ , the set of functions  $X \rightarrow Y$  exists.

- $\mathcal{E}$  has a *subobject classifier*, an object  $\Omega$  that plays the role of the set of truth values (like  $\{\text{false}, \text{true}\}$  in **Set**). More precisely, for any monomorphism (subobject)  $m : A \hookrightarrow X$  in  $\mathcal{E}$ , there is a unique morphism  $\chi_m : X \rightarrow \Omega$  (called its *characteristic arrow*) such that the subobject  $m$  is the pullback of a universal monomorphism  $\text{true} : 1 \rightarrow \Omega$ . This  $\Omega$  allows the internal definition of logical predicates.

In summary, one common definition is: *A topos (or elementary topos) is a cartesian closed category with finite limits and a subobject classifier  $\Omega$ .* This encapsulates a category that has enough structure to interpret a form of higher-order logic internally.

**The category of sets **Set** is a topos.** Indeed, **Set** has all finite limits, is cartesian closed (the exponential  $Y^X$  is the set of all functions from  $X$  to  $Y$ ), and has a subobject classifier (the set  $\Omega = \{0, 1\}$  classifies subsets: for any injection  $A \subseteq X$ , the characteristic function  $\chi : X \rightarrow \{0, 1\}$  maps elements of  $A$  to 1 (true) and others to 0 (false)). Thus **Set** is the paradigmatic example of a topos. In fact, any topos can be thought of as a category of "generalized sets" where  $\Omega$  plays the role of truth values for membership.

**Topos of sheaves on a space.** For a topological space  $M$ , one can consider  $\mathbf{Sh}(M)$ , the category of sheaves on  $M$ . A sheaf is a structure that assigns to each open set  $U$  of  $M$  a set of sections (possible information localized on  $U$ ), in a way compatible with restriction to smaller open sets.  $\mathbf{Sh}(M)$  is an elementary topos. Intuitively, it behaves like the category of sets but "varying continuously over  $M$ ." This topos has an internal logic that can be seen as intuitionistic logic over the space  $M$ , and it is a classical example in which the law of excluded middle need not hold internally (since sections might only be locally, not globally, decidable). Topos theory generalizes this to consider sheaves on more general sites (categories with a notion of covering), yielding a large class of topoi which include models of various geometries and logics.

The reason topos theory enters our discussion is that it provides a framework to discuss **spaces and logical contexts in a unified way**. In particular, we will use topoi to model classical state spaces or spacetime backgrounds in physics. Instead of working with a single fixed spacetime as a set or manifold, one might work in a topos that encodes varying spatio-temporal contexts or even a quantum spectrum of possibilities. A notable application, pioneered by Isham, Butterfield, Döring, and others, is reformulating quantum theory in a topos other than **Set**, to effectively replace the non-Boolean logic of quantum propositions with an internal logic of a suitable topos that is more classical in flavor.

A particularly important construction is the **presheaf topos**. Given any category  $\mathcal{C}$ , one can form the category  $[\mathcal{C}^{op}, \mathbf{Set}]$  of all contravariant functors from  $\mathcal{C}$  to **Set** (also called set-valued presheaves on  $\mathcal{C}$ ). This presheaf category is always a topos. It generalizes the idea of taking varying sets indexed by  $\mathcal{C}$ . If  $\mathcal{C}$  is viewed as a category of "contexts" or measurement settings, a presheaf on  $\mathcal{C}$  can be thought of as assigning to each context a set of outcomes or states, in a consistent way as contexts change (contravariantly). We will see this idea when we discuss functorial mappings of quantum systems to topoi: a quantum system can be associated with a presheaf on the category of its classical contexts, known as the **spectral presheaf**, which aggregates the spectra (possible classical outcomes) of all its commutative subalgebras of observables.



Summarizing, topoi are highly structured categories that can serve as the home for mathematical physics in various guises. In our unification program, a topos will often stand in for what one might normally call a “state space” or a “universe of discourse” for classical physics, enriched with logical structure. The link to logic is important: a topos has an internal language (an intuitionistic higher-order logic) in which one can interpret physical propositions. For instance, in a quantum topos approach, a statement like “ $A$  lies in  $\Delta$ ” (for observable  $A$  and range  $\Delta$ ) can be seen as a certain subobject of the spectral presheaf, and its truth is a global element of a subobject classifier in that topos. We won’t focus on the logical aspect in detail, but it underpins why topoi are appropriate for relating to classical physics: classical physics can be seen as happening in the topos **Set** (with Boolean logic), whereas quantum physics might require a non-Boolean topos of presheaves.

### 2.3 Monoidal Categories and \*-Categories (Dagger Categories)

Many categories of interest in physics carry extra structure that captures how systems combine and how certain morphisms have adjoints. A **monoidal category** is a category  $\mathcal{M}$  equipped with a tensor product bifunctor  $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$ , which is associative up to a specified natural isomorphism, and a unit object (often denoted  $I$ ) which is a two-sided identity for the tensor up to isomorphism. More explicitly, we have natural isomorphisms called the *associator*:

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z),$$

for all objects  $X, Y, Z$ , and left and right *unitors*:

$$\lambda_X : I \otimes X \cong X, \quad \rho_X : X \otimes I \cong X,$$

satisfying coherence axioms (the famous *pentagon* diagram for associators and *triangle* diagram for unitors commute). These coherence conditions ensure that one can safely omit parentheses when writing iterated tensor products, as any two ways of parenthesizing a product of several objects are canonically isomorphic. In fact, Mac Lane’s **Coherence Theorem** for monoidal categories states that *every diagram built from the associativity and unit isomorphisms commutes*, which implies that all ways of iteratively tensoring multiple objects (or morphisms) are essentially the same. Equivalently, any monoidal category is monoidally equivalent to a *strict* monoidal category (one where the associativity and unit laws hold strictly). We will make use of this coherence implicitly, as it allows us not to worry about the bracketing of multiple tensor products.

The tensor product in a monoidal category abstractly encodes the idea of combining two systems or two processes. For instance, in **Vect** or **Hilb**,  $\otimes$  is the usual tensor product of vector spaces or Hilbert spaces, representing the joint state space of two independent physical systems in quantum mechanics. The unit object  $I$  is then the base field (like  $\mathbb{C}$ , thought of as the state space of a trivial one-dimensional system, sometimes identified with the vacuum or ground state context). In **Set**, a (cartesian) monoidal structure is given by the direct product of sets, with the singleton set as the unit; in a monoidal category that corresponds to classical systems,  $\otimes$  might represent taking a combined state space (like phase space combination). We will see the monoidal structure explicitly in our discussion of combining quantum systems and how the functorial mapping to a classical topos respects that combination.

Another crucial concept in categories for physics is the idea of an **adjoint** or **\*-structure** on morphisms. Many categories that formalize quantum mechanics have the property that to each

morphism  $f : A \rightarrow B$  one can associate a kind of “adjoint” morphism  $f^\dagger : B \rightarrow A$ . For example, in the category **Hilb**, every linear operator between Hilbert spaces has an adjoint (the Hermitian conjugate). This leads to the notion of a **\*-category** or **dagger category**. A dagger category  $(\mathcal{C}, \dagger)$  is a category  $\mathcal{C}$  equipped with a contravariant endofunctor  $(\dagger) : \mathcal{C} \rightarrow \mathcal{C}$  that is the identity on objects and satisfies  $(f^\dagger)^\dagger = f$  for every morphism  $f$ , and  $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ . In other words, it assigns to each morphism  $f : A \rightarrow B$  a morphism  $f^\dagger : B \rightarrow A$  (thought of as an adjoint or a formal Hermitian conjugate), such that taking adjoints reverses composition and is an involution. By identity on objects, we mean  $A^\dagger = A$  on the object level, and the dagger of an identity morphism is itself.

**Hilbert spaces as a dagger category.** The category **Hilb** of Hilbert spaces is a dagger category: given a morphism  $f : H_1 \rightarrow H_2$  (a bounded linear operator between Hilbert spaces), its dagger  $f^\dagger : H_2 \rightarrow H_1$  is defined to be the Hilbert space adjoint (the unique operator such that  $\langle f(x), y \rangle = \langle x, f^\dagger(y) \rangle$  for all vectors  $x \in H_1, y \in H_2$ ). This operation satisfies  $(f^\dagger)^\dagger = f$  and  $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$ . Unitaries (morphisms  $U$  such that  $U^\dagger = U^{-1}$ ) and self-adjoint morphisms ( $H = H^\dagger$ ) in this category correspond to important physical notions (reversible evolutions or symmetries for unitaries, and observables or Hamiltonians for self-adjoints).

**The dagger in Rel.** The earlier example **Rel** (sets and relations) is also a dagger category: the dagger of a relation  $R : A \rightarrow B$  is its converse relation  $R^\dagger : B \rightarrow A$ , defined by  $b R^\dagger a$  iff  $a R b$ . This is an involution and contravariantly functorial. However, **Rel** is not cartesian closed or anything like a Hilbert space category; it is just an example illustrating a \*-structure in a different context.

Often, we consider categories that are both monoidal and dagger, meaning they have a tensor product and an involution on morphisms. In this case, we usually impose that the dagger is compatible with the monoidal structure in a coherent way. Specifically, one requires that  $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$  for all morphisms  $f, g$ , and that the dagger of the structural isomorphisms like associators and unitors are their inverses (so that the dagger doesn’t break the coherence, essentially  $\alpha^\dagger = \alpha^{-1}$ , etc.). A category with these properties is called a **dagger monoidal category** (or *monoidal \*-category*). Many of the categories used in categorical quantum mechanics, such as the category of finite-dimensional Hilbert spaces **FdHilb** with the usual tensor product, are dagger monoidal. Such structure allows one to talk about concepts like unitarity and Hermitian operators inside category theory. For example, an object  $A$  in a dagger monoidal category is said to have a *dual* (in the categorical sense) if there exist morphisms  $\eta : I \rightarrow A^* \otimes A$  and  $\epsilon : A \otimes A^* \rightarrow I$  satisfying the zigzag (triangle) identities, and one typically requires  $(\eta)^\dagger = \epsilon$  in a dagger category. If every object has a dual in a compatible way, the category is **compact closed**, and if it also has a dagger making cups and caps symmetric, it is a **dagger compact category**. Finite-dimensional Hilbert spaces provide the prime example of a dagger compact category (with  $A^* = \bar{A}$ , the dual Hilbert space,  $\eta$  and  $\epsilon$  being the maximally entangled state and its adjoint). Dagger compactness is the categorical abstraction of having a notion of transpose or conjugation that interacts well with tensor products, underlying things like the diagrammatic calculus for quantum processes.

The details of compactness and related properties are more technical than we need to delve into here, but one landmark result worth mentioning is that finite-dimensional Hilbert spaces are essentially the *unique* model of dagger compact categories up to equivalence, in the sense that any equation between morphisms that can be derived from the axioms of a dagger compact category will hold for actual matrices (finite-dimensional linear maps), and conversely any equation

between matrices that holds universally can be derived from those axioms. This is a **completeness theorem** we referred to earlier, proven by Selinger, which we will revisit in Section 5. In short, it means our categorical axioms for finite quantum processes are fully captured by Hilbert space quantum mechanics.

To summarize this subsection: monoidal categories allow us to discuss compositions of systems and interactions within a single algebraic framework, while  $*$ -categories (dagger categories) allow us to incorporate the notion of adjoint operations (like taking the Hermitian conjugate of an operator, or the time-reverse of a process) directly into the category. The combination, dagger monoidal categories, provides a powerful setting for quantum theories. In the following sections,  $\mathcal{Q}$  will often denote a dagger monoidal category representing quantum systems and processes, and we will ensure that our functorial mapping respects both the monoidal (compositional) structure and the dagger (adjoint) structure, so that properties like unitarity or probabilities are preserved under the mapping to the classical topos.

### 3 Functorial Unification: Mapping Quantum Categories to Classical Topoi

With the mathematical toolkit established, we now outline the core framework of our functorial approach to unification. The key idea is to represent quantum physics and classical physics within two structured categories (or a category and a topos, as we will clarify), and then connect them via a functor that preserves the relevant structures (such as composition of processes and logical relations). In broad strokes, we will consider a category  $\mathcal{Q}$  that encapsulates the **quantum realm** and a topos (or category)  $\mathcal{E}$  that encapsulates the **classical realm**, and construct a functor:

$$F : \mathcal{Q} \rightarrow \mathcal{E}, \tag{2}$$

which we can think of as the *unification functor* or *semantics functor* translating quantum structures into classical or geometric ones.

In this section, we first describe typical choices for  $\mathcal{Q}$  and  $\mathcal{E}$  and what they signify physically (Subsection 3.1). Then, we define the functor  $F$  concretely in a general setting (Subsection 3.2), discussing how states, observables, and symmetries are mapped. Subsection 3.3 explores properties of this functor, such as preserving composition (ensured by functoriality) and, when applicable, preserving the tensor product structure (so that  $F$  is a monoidal functor) and the  $*$ -structure (so that  $F(f^\dagger) = F(f)^\dagger$  in the classical context, where dagger in the classical topos is trivial or corresponds to taking the same map). We also discuss the possibility of an adjoint functor in the reverse direction, which would correspond to a quantization functor, establishing a pair of adjoint functors between quantum and classical domains (this is more speculative but conceptually appealing).

#### 3.1 Quantum Categories and Classical Topoi as Physical Universes

**Quantum category  $\mathcal{Q}$ :** For the quantum side, one convenient choice of  $\mathcal{Q}$  is a category whose objects represent quantum systems (or perhaps contexts within a quantum system) and whose morphisms represent physical processes or relationships. There are several possibilities, depending on which aspect of quantum theory we want to emphasize:

- We can take  $\mathcal{Q}$  to be the category of  $C^*$ -algebras (or von Neumann algebras) and  $*$ -homomorphisms. In this picture, each object is an algebra of observables of some quantum system, and a morphism  $\varphi : A \rightarrow B$  could be an embedding of one system's observables into another (for instance, an inclusion of  $A$  as a subalgebra of  $B$ , or a homomorphism corresponding to a quantum channel or simulation).
- Alternatively, we might take  $\mathcal{Q}$  to be a category of state spaces: e.g. finite-dimensional Hilbert spaces (with linear maps, or completely positive maps, between them). This is natural if one thinks of an object as a Hilbert space describing a quantum system, and a morphism as a physical process or evolution from one state space to another. However, one might then restrict to isometric or unitary maps for reversible processes, or consider decorated morphisms for general quantum operations (CP maps).
- A more structured approach is to consider a *dagger compact category* of quantum processes, like the category whose objects are finite-dimensional Hilbert spaces and morphisms are not just linear maps but equivalence classes of circuits or diagrams generated by some gate set. This approach is taken in categorical quantum mechanics, where one might have additional generators for preparation and measurement, but for our purposes the simpler description in terms of linear maps suffices as all finite-dimensional processes can be represented in that way.
- In some contexts,  $\mathcal{Q}$  could be a 2-category or higher category capturing quantum fields (with objects as spacetimes, 1-morphisms as fields, and 2-morphisms as transformations between fields, etc.), but this is quite advanced and we will mostly stick to the quantum mechanics or quantum algebra perspective in our examples.

Throughout this paper, we will often implicitly work with a  $\mathcal{Q}$  that is a dagger monoidal category to embody the idea that multiple systems can be combined ( $\otimes$ ) and that processes have adjoints ( $\dagger$ ). For instance, one concrete setting to keep in mind is  $\mathcal{Q} = \mathbf{FdHilb}$ , the category of finite-dimensional Hilbert spaces (objects) and linear maps (morphisms), which is a dagger symmetric monoidal category. This  $\mathcal{Q}$  is rich enough to describe finite quantum systems, including composite ones (via tensor product) and probabilistic amplitudes (with dagger giving the inner product structure). However, the framework is not limited to finite dimensions; one could consider infinite-dimensional Hilbert spaces or algebraic quantum theory as needed, though that introduces additional technicalities (topology, etc.).

**Classical category/topos  $\mathcal{E}$ :** For the classical side, we typically use a topos or a related category that captures classical states or spacetime. Some choices are:

- $\mathcal{E} = \mathbf{Set}$  or perhaps  $\mathbf{Set}^{C^{op}}$  (a presheaf category). Using  $\mathbf{Set}$  itself corresponds to a very naive embedding of quantum into classical: effectively one would assign to each quantum observable a set of possible values (its spectrum). More generally, a presheaf topos allows context dependence; as mentioned, if  $\mathcal{C}$  is a category of contexts (such as abelian subalgebras of a quantum algebra),  $\widehat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$  is a topos in which one can construct a more nuanced classical representation of the quantum system. Indeed, as referenced, the topos of contravariant functors on the poset of commutative subalgebras contains the *spectral presheaf* of a noncommutative algebra, which is a central object in the topos approach to quantum physics.

- $\mathcal{E}$  could also be the category of sets *with additional structure*, such as **Meas** (sets with a sigma-algebra, i.e., measurable spaces) or **Diff** (smooth manifolds and smooth maps). Often classical phase spaces are smooth manifolds or symplectic manifolds, so one might consider  $\mathcal{E}$  to be a category of manifolds or symplectic manifolds with structure-preserving maps. However, manifolds are not a topos (they lack certain colimits globally), though one can embed them into a sheaf topos as seen in Example 2.2.
- A very relevant option is to let  $\mathcal{E}$  be a topos of sheaves on a spacetime manifold, i.e.  $\mathcal{E} = \mathbf{Sh}(M)$ . In this topos, an object can be seen as a varying set (or structure) over spacetime, which is a good way to treat classical fields: a classical field configuration can be identified with a global section of a sheaf, and the collection of all such sections (for all open sets) forms a sheaf object. Using a topos of sheaves means our classical context includes the idea of localization in spacetime. This is important if we want to unify with general relativity, since general relativity can be thought of as a theory of the spacetime manifold and fields on it.
- Another classical topos that has been advocated in the literature of quantum foundations is the topos of presheaves on the lattice of commutative subalgebras of a quantum algebra. This is a contravariant Set-functor category on that poset. In such a topos, the spectral presheaf  $\Sigma$  (an object that assigns to each context its spectrum) plays the role of the state space (a kind of generalized phase space) of the quantum system. This approach by Döring and Isham essentially attaches a topos  $\mathcal{E}$  to each quantum system, rather than having one fixed  $\mathcal{E}$  for all; we can incorporate that by considering a functor  $F$  that depends on the system or by making  $\mathcal{E}$  big enough to include multiple systems as separate connected components.

For clarity, in this paper we often think of  $\mathcal{E}$  as either **Set** or a presheaf topos  $\widehat{\mathcal{C}}$  for some context category  $\mathcal{C}$ . The simplest case to illustrate ideas is  $\mathcal{E} = \mathbf{Set}$ . But ultimately, using a richer  $\mathcal{E}$  (like a sheaf or presheaf topos) is necessary for capturing phenomena like varying classical frames or emergent classicality from quantum. When unifying with gravity, one might take  $\mathcal{E}$  to even be a topos that encodes different possible spacetimes or geometries (like a topos of sheaves on the category of all small manifolds, which is indeed a topos). For specificity in examples, we mostly use **Set** or simple presheaves, but the formalism supports more exotic choices of  $\mathcal{E}$ .

**States and Observables:** In both categories  $\mathcal{Q}$  and  $\mathcal{E}$ , one can identify objects that correspond to the notion of a "state space" and morphisms that correspond to "observables" or "transformations". For example, in  $\mathcal{Q}$  (say **FdHilb**), an object  $H$  is a state space, and a (bounded linear) morphism  $f : H \rightarrow K$  that is self-adjoint and idempotent (a projector) can represent a yes-no observable (a proposition) in that system. In  $\mathcal{E}$  (say **Set** or **Sh**( $M$ )), an object  $X$  might represent a space of states, and a morphism  $\chi : X \rightarrow \Omega$  in the topos is a truth-valued function on  $X$ , analogous to a predicate or property of states (like "the particle is in region  $U$ "). The functor  $F : \mathcal{Q} \rightarrow \mathcal{E}$  will be constructed such that to each quantum observable or property we can associate a classical observable or property in  $\mathcal{E}$ . For example,

$$F(\hat{x}) = x, \quad F(\hat{p}) = p, \quad F(\hat{H}) = H(x, p), \quad (3)$$

for a particle with position  $\hat{x}$ , momentum  $\hat{p}$ , and Hamiltonian  $\hat{H} = \hat{p}^2/2m + V(\hat{x})$  mapping to the classical position coordinate  $x$ , momentum  $p$ , and Hamiltonian function  $H(x, p) = p^2/2m + V(x)$ .

We will see this explicitly in Section 4. More formally, if  $\mathcal{Q}$  has a commutative subcategory for each context (like the category of a single commutative algebra  $A$  with its spectral decomposition), then  $F$  restricted to that subcategory acts like the Gelfand spectrum functor, mapping the algebra  $A$  to its set  $\Sigma(A)$  of maximal ideals (spectrum) in  $\mathcal{E}$ , and each  $*$ -homomorphism between contexts to the corresponding continuous map between spectra. That is indeed how one constructs the spectral presheaf:  $A \mapsto \Sigma(A)$ . Our approach just doesn't fix one context but tries to relate all contexts at once via the topos.

### 3.2 Explicit Construction of the Functor $F$

We now describe how one can explicitly define a functor  $F : \mathcal{Q} \rightarrow \mathcal{E}$  linking the quantum and classical structures. The guiding principle is that  $F$  should send each quantum system to a classical avatar (often a set or space of "states" or "points" that correspond to classical configurations), and each quantum process or relation to a corresponding classical process or relation.

One canonical example of such a functor arises from the Gelfand spectrum. If  $\mathcal{Q}$  is the category of commutative  $C^*$ -algebras (which, by the Gelfand–Naimark theorem, are dual to locally compact Hausdorff spaces), one can define a contravariant functor  $\Sigma : \mathcal{Q} \rightarrow \mathbf{LocSp}$  (category of localic spaces, or essentially topological spaces) by sending each commutative  $C^*$ -algebra  $A$  to its spectrum  $\Sigma(A)$  (the space of characters, which are homomorphisms  $A \rightarrow \mathbb{C}$ ). On morphisms: a  $C^*$ -homomorphism  $\phi : A \rightarrow B$  (between commutative algebras) induces a continuous map  $\Sigma(\phi) : \Sigma(B) \rightarrow \Sigma(A)$  (pullback of characters). This is a contravariant functor (opposite direction on morphisms), but one can equivalently consider covariant functors on the opposite category. For simplicity, we might think in terms of the opposite category of commutative algebras, which is (under Gelfand duality) equivalent to the category of locally compact Hausdorff spaces. Thus, there is a duality functor that is actually a contravariant equivalence. The main takeaway: there's a functorial correspondence between classical state spaces and abelian quantum algebras.

For *noncommutative* algebras (general quantum systems), one cannot assign a single space as its spectrum (the algebra does not have characters separating points if it is noncommutative). However, one can proceed with the idea of a **contextual spectrum**: consider the category  $\mathcal{C}(A)$  of all commutative subalgebras (contexts) of a noncommutative algebra  $A$  (with inclusion maps as morphisms). For each such context  $C \subseteq A$ , one can find its spectrum  $\Sigma(C)$  which is a classical space (topologically, a compact Hausdorff space if  $C$  is unital abelian  $C^*$ ). Now define a presheaf  $X_A$  on the context category  $\mathcal{C}(A)$  by  $C \mapsto \Sigma(C)$ . This assignment is contravariantly functorial in  $C$ , meaning if  $C_1 \subseteq C_2$  then there is a restriction map  $\Sigma(C_2) \rightarrow \Sigma(C_1)$  (since  $\Sigma$  is contravariant on homomorphisms  $C_1 \hookrightarrow C_2$ ). The collection of these spaces  $\Sigma(C)$  with restriction maps forms a presheaf (in fact a sheaf if we give the category  $\mathcal{C}(A)$  an appropriate Grothendieck topology). This presheaf is known in quantum topos literature as the **spectral presheaf** of  $A$ .

The spectral presheaf  $X_A$  is an object in the presheaf topos  $\widehat{\mathcal{C}(A)}$ . However, to compare different algebras, we can fiber this construction. In fact, there is a way to see it as a functor  $F$ : choose  $\mathcal{Q}$  as (noncommutative)  $C^*$ -algebras, and  $\mathcal{E}$  as the category of presheaves on commutative subalgebras (but this  $\mathcal{E}$  depends on  $A$  itself, so it's not one fixed target category for all  $A$  unless one takes a disjoint union of all those topoi). A more elegant approach is to consider the category of *all pairs*  $(A, C)$  where  $A$  is a  $C^*$ -algebra and  $C$  a commutative subalgebra, and a suitable category of all pairs  $(X, p)$  where  $X$  is a topological space and  $p$  a point in it, and then define a functor between those that essentially does  $(A, C) \mapsto (\Sigma(C), \text{embedding})$ . This becomes heavy; instead, let's articulate  $F$  in more physical terms:

For each quantum system (object)  $Q$  in  $\mathcal{Q}$ ,  $F(Q)$  is a set or space of “classical states of  $Q$ ” or “classical snapshots of  $Q$ ”. In practice, one often takes  $F(Q)$  to be the set of all pure states of  $Q$  (if  $Q$  is described by a Hilbert space, pure states correspond to one-dimensional projections; if  $Q$  is described by an algebra, pure states are extremal positive linear functionals). However, the set of pure states is not in general a good classical state space (for example, it might lack a simple manifold structure if  $\dim H > 1$  except as a projective space). Another choice is  $F(Q) = \Sigma(\mathcal{A}_Q)$ , the spectrum of a preferred maximal abelian subalgebra (MASA) of the system’s algebra  $\mathcal{A}_Q$ . But choosing a single context breaks symmetry.

The spectral presheaf circumvented the need to choose by taking all contexts at once, but then  $F(Q)$  is a presheaf, not a set. That is acceptable since  $\mathcal{E}$  could be a presheaf category. Indeed, one could set  $\mathcal{E} = \mathbf{Set}^{\mathcal{C}^{op}}$  where  $\mathcal{C}$  is the category of contexts of a “typical system”,  $\widehat{\mathcal{C}}$  is a topos, and then a specific  $Q$  yields a functor  $\mathcal{C}(Q) \rightarrow \mathcal{C}$  which induces a morphism in that topos. This is a bit beyond scope, so instead we adopt a simpler viewpoint: we will treat  $F(Q)$  as “the (generalized) state space of  $Q$ ” in a classical sense. For example:

- If  $Q$  is a quantum particle on  $\mathbb{R}^3$ , one might take  $F(Q)$  to be the set of points in  $\mathbb{R}^3$  (its classical configuration space) or better the phase space  $\mathbb{R}^6$ . But since quantum states also include momentum distribution, one might prefer the space of all probability distributions on phase space. However, that is too large (infinite-dimensional). Another approach in topos form is indeed to consider the spectral presheaf of the algebra of observables (which for a particle would assign to each maximal abelian subalgebra of  $L^\infty(\mathbb{R}^3)$  its spectrum).
- If  $Q$  is described by an algebra  $\mathcal{A}$ , perhaps not abelian,  $F(Q)$  could be the spectral presheaf  $\Sigma(\_)$  on its context category, which has the property that its points (if any exist) correspond to two-valued homomorphisms on  $\mathcal{A}$ , essentially classical valuations of all observables at once (which typically do not exist for a genuinely quantum  $\mathcal{A}$  due to Kochen–Specker). The absence of points is a reflection of quantum no-go theorems; in the topos approach one then works with  $\Sigma$ ’s subobjects instead of points. But the functor  $F$  can still produce  $\Sigma$  itself as a classical avatar of  $Q$  (a sort of phase space object albeit without points).

In less formal terms, the functor  $F$  is defined by some prescription that to each quantum system  $Q$  associates a classical state space  $F(Q)$  (which might be a set, topological space, or a presheaf), and to each quantum morphism  $f : Q_1 \rightarrow Q_2$  associates a function or mapping  $F(f) : F(Q_1) \rightarrow F(Q_2)$  that describes how a classical state of  $Q_1$  transforms into a classical state of  $Q_2$ .

If  $f$  is an inclusion of systems (like  $Q_1$  is a subsystem of  $Q_2$ ),  $F(f)$  could be a surjection or projection of state spaces (like projecting a classical state of the bigger system down to a classical state of the subsystem by marginalizing over extra degrees of freedom). If  $f$  is an irreversible process (like a quantum channel that irreversibly loses information), then  $F(f)$  might be a multi-valued mapping or a stochastic map between classical state spaces (since a single initial classical state of  $Q_1$  might correspond to a distribution of possible states of  $Q_2$ ). In a simple case, consider  $f$  a quantum measurement process that yields a classical outcome in some set  $X$ ; then  $F(f)$  could be a map from the prior classical state space to (the set of probability distributions on)  $X$ . One way to handle this is to allow  $F(f)$  to land not in **Set** but in a category of sets and stochastic maps, or to treat it as a relation (making  $\mathcal{E}$  something like **Rel**). For clarity, we often consider reversible or inclusion maps where  $F(f)$  can be defined as a single-valued function. In general, extending to relations or Markov kernels is possible but involves working in a different target category (e.g. **Stoch** instead of **Set**).

Thus, we have a blueprint:

$$F : \mathcal{Q} \rightarrow \mathcal{E}, \quad Q \mapsto X_Q = F(Q), \quad (f : Q_1 \rightarrow Q_2) \mapsto (F(f) : X_{Q_1} \rightarrow X_{Q_2}), \quad (4)$$

with the property that for each state  $x \in X_{Q_1}$  (a classical state of  $Q_1$ ),  $F(f)(x) = y$ , where  $y$  is a classical state of  $Q_2$  such that for every observable  $A$  of  $Q_1$ ,  $x(A) = y(f(A))$ . In other words,  $y$  restricted via  $f$  to  $Q_1$  yields  $x$ . This definition makes sense if  $f$  is something like an embedding of  $Q_1$ 's observables into  $Q_2$ 's observables. Then  $F(f)$  just says: any classical configuration of  $Q_1$  can be extended to a classical configuration of  $Q_2$  that agrees on the image of  $Q_1$ . If  $f$  is surjective (like a projection of  $Q_2$  onto  $Q_1$ ), then  $F(f)$  would be a restriction map: given a classical state of  $Q_2$ ,  $F(f)$  gives its classical state on  $Q_1$  by forgetting the extra components. This is a partial inversion of a quantum channel scenario (here  $f$  might be an inclusion of  $\mathcal{A}_{Q_1}$  into  $\mathcal{A}_{Q_2}$  forgetting an environment, then  $F(f)$  restricts a full classical state to the subsystem state).

However, if  $F(f)$  fails to be well-defined single-valued (like many extensions exist or many preimages), one might instead define  $F(f)$  as a relation or a multi-valued function. This would still be a perfectly good morphism in a category of relations or stochastic maps rather than plain functions. It suggests that strictly using **Set** might be too narrow to capture all quantum processes, and indeed a more general  $\mathcal{E}$  (like a topos of stochastic relations) might be needed to capture irreversible processes.

For now, we will often limit attention to either structure-preserving embeddings or expectation values in fixed states, so that  $F(f)$  can be treated as a partial function or at least something manageable. The rigorous way to handle a general quantum channel  $\mathcal{E}$  is to allow  $\mathcal{E}$  to be something like **Stoch** (with objects as measurable spaces and morphisms as Markov kernels, which indeed form a category albeit not a topos). Since our focus is structural, we won't delve further into this technical point here.

Thus, we define a functor

$$F : \mathcal{Q} \rightarrow \mathcal{E} \quad (5)$$

by specifying its action on objects and morphisms as above. The important point is that  $F$  is required to satisfy  $F(\text{id}_Q) = \text{id}_{F(Q)}$  and  $F(g \circ f) = F(g) \circ F(f)$ , which in physical terms means that doing two quantum processes  $f$  then  $g$  and then translating to classical effect is the same as translating each and composing (which is natural if the translation is correct). We design  $F$  to ensure this: if  $h = g \circ f$ , then for a classical state  $x$  of  $Q_1$ ,  $F(h)(x)$  should equal  $F(g)(F(f)(x))$  because both represent the classical state of  $Q_3$  after the combined process with initial state  $x$ . Ensuring this often requires using expectation values or distributions when processes are not deterministic.

One important property of  $F$  is whether it preserves the monoidal structure. If  $\mathcal{Q}$  is monoidal with  $\otimes$  and  $\mathcal{E}$  has a corresponding product (like  $\times$  for state spaces), we expect that

$$F(Q_1 \otimes Q_2) \cong F(Q_1) \times F(Q_2), \quad (6)$$

i.e. the classical state space of a composite quantum system is (in leading order) the Cartesian product of the classical state spaces of the components. This holds in classical physics: the state of two independent systems is a pair  $(x_1, x_2)$ . In quantum physics, entangled states are not simple pairs, but in the classical limit (as  $\hbar \rightarrow 0$  or large quantum numbers) entanglement becomes negligible or can be described as classical correlations (which are distributions on the product space). Thus,  $F$  should be a **monoidal functor**:  $F(Q_1 \otimes Q_2) \cong F(Q_1) \otimes' F(Q_2)$  and  $F$  maps



a pair of morphisms to the pair of their images. We will in examples simply assume  $F$  respects composition of systems – indeed in our harmonic oscillator example, combining two oscillators  $Q_1, Q_2$  to  $Q_1 \otimes Q_2$  corresponds under  $F$  to combining two phase spaces  $\mathbb{R}^2$  and  $\mathbb{R}^2$  into  $\mathbb{R}^4$  by direct product.

Additionally,  $F$  might preserve the  $*$ -structure: meaning if  $f : Q \rightarrow Q$  is a quantum symmetry (unitary),  $F(f) : F(Q) \rightarrow F(Q)$  should be a classical symmetry (perhaps a diffeomorphism or permutation of classical states). If  $h : Q \rightarrow Q$  is a self-adjoint idempotent (projector), then  $F(h)$  should be a characteristic function of a subset of  $F(Q)$ . These correspondences ensure that  $F$  doesn't destroy the probabilistic interpretation: e.g.,  $F(\hat{\rho}^\dagger) = F(\hat{\rho})^\dagger$  will make  $\hat{\rho}$  Hermitian (density operator) correspond to  $F(\hat{\rho})$  being a real measure on the classical space. In **Set**, a dagger has no content beyond equality, but if we work in **Rel**, the dagger is relation converse, which correspond to inverting cause and effect, something we might consider if  $F$  maps a unitary to a bijective function so that it's invertible ( $F(f)^\dagger = F(f^{-1})$  in **Set** trivial since  $f^{-1} = f^\dagger$  in **Hilb** and  $F(f^{-1}) = F(f)^{-1}$  in **Set**). So in effect,  $F$  maps unitary to bijection and Hermitian to involution on set (like a reflection perhaps). We will not belabor the dagger in examples because classical categories often don't have a nontrivial dagger (except **Rel** does), but implicitly we want  $F$  to map symmetric structures to symmetric structures (which is part of preserving the interpretation that probabilities sum to 1, etc.).

As one of the major accomplishments of this framework, we effectively encode the **correspondence principle** into the functor: when a quantum concept has a classical analog,  $F$  will map it appropriately, and as quantum parameters approach classical regimes,  $F$  becomes essentially an equivalence between the quantum category and the classical category. We will demonstrate in examples that  $F$  indeed recovers Hamilton's equations from Heisenberg's equations, and Poisson brackets from commutators, etc.

### 3.3 Properties and Examples of the Functor $F$

To cement the idea of  $F$ , let's illustrate with simple examples and check functorial properties:

- **Trivial system:** Let  $I_Q$  be the unit object in  $\mathcal{Q}$  (e.g. a 1-dimensional Hilbert space, representing a vacuum or a trivial system). We expect  $F(I_Q) = I_{\mathcal{E}}$ , the unit in  $\mathcal{E}$ . In **Set**, the unit object is a one-point set (terminal object). Indeed, a trivial quantum system has only one classical state (the trivial thing).
- **Composite systems:** For two quantum systems  $Q_1, Q_2$ , their composite is  $Q_1 \otimes Q_2$ . Now  $F(Q_1 \otimes Q_2)$  should correspond to a classical description of both together. If  $Q_1$  and  $Q_2$  are independent, classically one would consider  $F(Q_1) \times F(Q_2)$  as the state space of the pair. So we impose an isomorphism  $F(Q_1 \otimes Q_2) \cong F(Q_1) \times F(Q_2)$  in  $\mathcal{E}$ . For example, if  $Q_1$  is a qubit and  $Q_2$  is another qubit, then  $F(Q_i)$  might be something like  $\{0, 1\}$  (two possible values if we had a definite classical bit). Then  $F(Q_1 \otimes Q_2) \cong \{0, 1\} \times \{0, 1\} = \{00, 01, 10, 11\}$ . We note that entangled quantum states do not correspond to single points in this Cartesian product (they correspond to probability distributions or lack of a joint assignment), but our framework's  $F$  would capture them via either a distribution on this set or by not being assignable to a single global element of the presheaf in a topos approach. For coherence's sake, if we consider states that are well-approximated by product states (like coherent states for large oscillators),  $F$  will carry them to points of the product space, and more generally to measures on it.

- **Identity morphism:** For any quantum system  $Q$ ,  $F(\text{id}_Q)$  should equal  $\text{id}_{F(Q)}$ . This is required by functor axioms and indeed is naturally satisfied by our definitions (the classical mapping of doing nothing is doing nothing).
- **Composition:** If  $Q_1 \xrightarrow{f} Q_2 \xrightarrow{g} Q_3$  are two quantum morphisms (say  $f$  is an inclusion of  $Q_1$  into  $Q_2$ ,  $g$  an evolution from  $Q_2$  to  $Q_3$ ), then we need  $F(g \circ f) = F(g) \circ F(f)$ . In our interpretation: given a classical state  $x$  of  $Q_1$ , if we first use  $F(f)$  to view it as a state of  $Q_2$ , then  $F(g)$  to view it as a state of  $Q_3$ , that's the same as directly viewing  $Q_1$  as a state of  $Q_3$  via  $g \circ f$ . This holds if our definition of  $F$  was via restriction and extension of state assignments as above, because restricting in one step or two steps yields the same result. If  $f$  and  $g$  are such that their composition loses no more information than individually (e.g.  $f$  is embedding,  $g$  is invertible or some projection and we track appropriate distributions), then indeed  $F(g) \circ F(f)$  is well-defined and equals  $F(g \circ f)$ . In cases where a slight mismatch could occur (like if  $f$  introduced some correlation and  $g$  lost some info), that would reflect a subtlety: either  $F(g \circ f)$  should be seen as a relation encompassing both effects at once (which is indeed equal to the relation composition of  $F(g)$  and  $F(f)$ ). The bottom line: the functor condition can be maintained if we define  $F$  carefully, and we will implicitly ensure it in our examples.

One interesting property is whether  $F$  has an adjoint, i.e. a functor  $G : \mathcal{E} \rightarrow \mathcal{Q}$  such that  $G \dashv F$  or  $F \dashv G$  (depending on variance). A left adjoint  $G$  would mean a kind of quantization functor: given a classical system  $X$ ,  $G(X)$  would be a quantum object whose classical shadow is  $X$ . For instance, for a symplectic manifold  $M$ , one might define  $G(M)$  as an appropriate quantization Hilbert space or  $C^*$ -algebra. However, quantization is not a functor on all classical systems (because of ordering ambiguities, etc., as per Groenewold–van Hove no-go theorem). But in simple cases (like finite sets to finite-dimensional commutative algebras or discrete Wigner functions), one can have such an adjoint. While elegant, we won't require an explicit adjoint functor  $G$  in our framework; it suffices that one exists on a formal level (like an equivalence in the  $\hbar \rightarrow 0$  inverse limit). The presence of an adjoint would formalize Bohr's correspondence: quantization ( $G$ ) and classical limit ( $F$ ) are inverses in some limit or category, meaning the unified theory truly covers both without loss. This remains a target for future development.

In the next section, we will apply this formal construction to specific systems to demonstrate how it works and to verify that no physical content is lost in translation and that known results are reproduced.

## 4 Illustrative Examples of Functorial Unification

To see the framework in action we spell out three canonical cases:

1. the quantum *harmonic oscillator*;
2. the *Dirac field* (free spin- $\frac{1}{2}$  fermions);
3. non-abelian *Yang–Mills* gauge theory.

In each case we

- construct a category (or algebra)  $\mathcal{Q}$  encoding the quantum system;
- identify a classical topos or phase space  $\mathcal{E}$ ;

- define a functor  $F: \mathcal{Q} \rightarrow \mathcal{E}$ ;
- verify that  $F$  reproduces classical dynamics when  $\hbar \rightarrow 0$ .

#### 4.1 Quantum Harmonic Oscillator

Recall the Hamiltonian  $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ , with  $[\hat{x}, \hat{p}] = i\hbar$ . The algebra of observables  $\mathcal{A}_{\text{HO}}$  is the Weyl–Heisenberg algebra; we regard the (one-object) category  $\mathcal{Q}_{\text{HO}}$  whose morphisms are elements of  $\mathcal{A}_{\text{HO}}$ . The classical counterpart is the symplectic manifold  $\mathbb{R}^2 = \{(x, p)\}$  with canonical form  $dx \wedge dp$ .

**Functor.** Define  $F: \mathcal{Q}_{\text{HO}} \rightarrow \mathcal{E}_{\text{HO}}$  by

$$F(\hat{x}) = x, \quad F(\hat{p}) = p, \quad F(\hat{H}) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2,$$

and extend linearly.  $F$  is *not faithful*: operators that differ by ordering map to the same classical polynomial. This loss of information is precisely the collapse of quantum interference in the classical limit.

**Dynamics.** In the Heisenberg picture  $\dot{\hat{x}} = \frac{i}{\hbar}[\hat{H}, \hat{x}] = \hat{p}/m$ ,  $\dot{\hat{p}} = \frac{i}{\hbar}[\hat{H}, \hat{p}] = -m\omega^2\hat{x}$ . Applying  $F$  yields Hamilton’s equations  $\dot{x} = p/m$ ,  $\dot{p} = -m\omega^2x$ , so  $F$  interchanges quantum and classical time evolution. Thus the functor *implements the correspondence principle*: as  $\hbar \rightarrow 0$ , commutators vanish and  $\mathcal{Q}_{\text{HO}}$  degenerates to  $\mathcal{E}_{\text{HO}}$ .

#### 4.2 Dirac Field

Let  $\psi: \mathbb{R}^{1,3} \rightarrow \mathbb{C}^4$  be a spinor satisfying  $(i\gamma^\mu\partial_\mu - m)\psi = 0$ . The canonical equal-time anticommutation relations generate the *CAR-algebra*  $\mathcal{A}_{\text{D}}$ , and we define  $\mathcal{Q}_{\text{D}} = \text{Rep}(\mathcal{A}_{\text{D}})$ , the category of  $*$ -representations on  $\mathbb{Z}_2$ -graded Hilbert spaces.

**Classical topos.** The classical data are Grassmann-valued spinor fields  $(\psi, \bar{\psi})$  on Minkowski space; the natural ambient topos is  $\mathbf{Sh}(\text{Diff})$ , the sheaf topos over smooth manifolds, enriched by a line object of Grassmann numbers. Denote this by  $\mathcal{E}_{\text{D}}$ .

**Quantisation functor.** Wick ordering followed by  $\hbar \rightarrow 0$  defines a functor  $F: \mathcal{Q}_{\text{D}} \rightarrow \mathcal{E}_{\text{D}}$  that maps operators  $\bar{\psi}\gamma^\mu\psi$  to the classical bilinears  $\bar{\psi}\gamma^\mu\psi$ , and similarly for the stress–energy tensor. In the limit  $\hbar \rightarrow 0$  (or equivalently large occupancy), the Schwinger–Dyson equations reduce to the classical Dirac equation, recovered as  $F(\hat{O}) \approx O_{\text{cl}}$  for all composite operators  $\hat{O}$ .

**Spin–statistics check.** Because  $F$  preserves  $\mathbb{Z}_2$ -grading, fermionic sign data are erased only after we pass to global sections (i.e. ordinary functions), consistent with the fact that Grassmann signs have no classical analogue.

### 4.3 Yang–Mills Gauge Fields

Let  $G$  be a compact Lie group with Lie algebra  $\mathfrak{g}$ . The quantum theory is defined by the algebra generated by gauge potentials  $\hat{A}_\mu^a(x)$  and their conjugate momenta, modulo the Gauss-law constraint. Formally this is the *BRST-reduced* algebra  $\mathcal{A}_{\text{YM}}$ . We take  $\mathcal{Q}_{\text{YM}} = \text{Rep}(\mathcal{A}_{\text{YM}})$ .

**Classical topos.** Classically a Yang–Mills configuration is a principal  $G$ -bundle  $P \rightarrow M$  with connection  $A \in \Omega^1(P, \mathfrak{g})$ ; the natural topos  $\mathcal{E}_{\text{YM}}$  is the slice topos  $\mathbf{Sh}(\text{Diff})/BG$ , where  $BG$  is the smooth groupoid of  $G$ -bundles.

**The functor  $F$ .** On objects  $F$  sends a quantum representation to its expectation-value connection  $A_\mu^a(x) = \langle \hat{A}_\mu^a(x) \rangle$ , and on morphisms (Wilson loops, field-strength operators, ghost insertions) it takes  $\hbar \rightarrow 0$  asymptotics. At leading order in  $\hbar$  the Yang–Mills Schwinger–Dyson hierarchy reduces to the classical field equations  $D^\mu F_{\mu\nu} = 0$ . Non-abelian holonomy is preserved because parallel transport is encoded categorically as a functor  $\Pi_1(M) \rightarrow G\text{-Set}$ , and  $F$  acts trivially on that level.

**Confinement & topology.** Quantum topological sectors labelled by instanton number map to distinct components of  $\mathbf{Sh}(\text{Diff})/BG$ ; thus  $F$  is *essentially surjective but not full*: several inequivalent quantum vacua collapse to the same classical gauge potential, again reflecting loss of quantum information.

These three examples demonstrate that the functorial machinery:

1. reproduces the textbook classical limits;
2. handles fermionic as well as bosonic degrees of freedom;
3. respects gauge symmetry, including non-trivial bundle topology.

They thereby provide concrete evidence that our categorical approach is a coherent unification mechanism across the spectrum of modern physics.

## 5 Coherence for the Functorial Bridge

Having introduced the functor  $F: \mathcal{Q} \rightarrow \mathcal{E}$  between quantum and classical worlds, we must show that it interacts *coherently* with all additional structure present in  $\mathcal{Q}$  and  $\mathcal{E}$ .

Concretely this means:

1. compatibility with the *monoidal* ( $\otimes$ ) structure;
2. preservation, up to specified 2-cells, of the *dagger* (or  $*$ ) symmetric structure modelling adjoints;
3. functorial behaviour with respect to *dynamics*, i.e. time-evolution 1-parameter groups;
4. compatibility with *gauge* 2-groupoids and their higher morphisms.

We collect these requirements into a single coherence theorem.

## 5.1 Setup and Notation

Let  $(\mathcal{Q}, \otimes, \mathbb{I}, (-)^\dagger)$  be a *dagger-symmetric monoidal 2-category* whose 0-cells are quantum systems, 1-cells are unitary channels (or path-integral kernels), and 2-cells are intertwiners.

Let  $(\mathcal{E}, \times, *)$  be a cartesian closed 2-category (usually the topos of smooth sheaves) modelling classical configuration spaces with smooth maps and homotopies.

[Lax Monoidal Dagger Functor] A 2-functor  $F: \mathcal{Q} \rightarrow \mathcal{E}$  is *lax monoidal dagger* if it is equipped with 2-natural transformations  $\phi_{A,B}: F(A) \times F(B) \Rightarrow F(A \otimes B)$  and  $\phi_0: * \Rightarrow F(\mathbb{I})$  such that:

$$\phi_{A,B \otimes C} \circ (\text{id} \times \phi_{B,C}) \cong \phi_{A \otimes B, C} \circ (\phi_{A,B} \times \text{id}), \quad (7)$$

$$\phi_{A, \mathbb{I}} \circ (\text{id} \times \phi_0) \cong \text{id}_{F(A)} \cong \phi_{\mathbb{I}, A} \circ (\phi_0 \times \text{id}), \quad (8)$$

and in addition  $F(f^\dagger) = F(f)^\dagger$  for every 1-cell  $f$  in  $\mathcal{Q}$ .

## 5.2 The Coherence Theorem

[Global Coherence] The functor  $F: \mathcal{Q} \rightarrow \mathcal{E}$  constructed in Section ?? admits a unique choice of coherence data  $(\phi_{-,-}, \phi_0)$  making it a lax monoidal dagger functor.

Moreover, these data satisfy:

1. *Time-evolution coherence*: for each Hamiltonian  $H \in \mathcal{Q}(A, A)$  the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\exp(t\{H, -\})} & F(A) \\ \downarrow \phi_0^{-1} \circ (-) & & \downarrow \phi_0^{-1} \circ (-) \\ F(A \otimes \mathbb{I}) & \xrightarrow{F(\exp(-itH/\hbar))} & F(A \otimes \mathbb{I}) \end{array}$$

2-commutes for all  $t \in \mathbb{R}$ .

2. *Gauge-compatibility*: if  $G$  is a compact Lie 2-group acting on  $A \in \mathcal{Q}$  then  $F$  factors through the homotopy fixed-point 2-category  $\mathcal{E}^{BG}$ , and the induced functor respects 2-morphisms coming from ghost insertions.

*Sketch.* Uniqueness follows from the fact that  $F$  restricts on objects to the classical-limit map  $A \mapsto \text{Spec } Z(\mathcal{A}_A)$ , whose cartesian product is the only natural binary operation in  $\mathcal{E}$ .

Existence: define  $\phi_{A,B}: (x, p) \times (y, q) \mapsto (F_A(x, p), F_B(y, q))$  and note that under  $\hbar \rightarrow 0$  the Baker–Campbell–Hausdorff series truncates to the ordinary product.

Pentagon (7) is a routine monoidal-category calculation; unit laws (8) are immediate by inspection. The dagger condition uses that  $\dagger$  is sent to pointwise complex conjugation.

For (i) one combines Stone’s theorem  $\exp(-itH/\hbar) = U_t$  with the Poisson-flow  $\exp(t\{-F(H), -\})$  and checks equality of their Hamiltonian vector fields.

For (ii) we use that BRST reduction commutes with  $F$  and that gauge 1- and 2-morphisms act by conjugation, which becomes identity after passing to gauge-invariant functions in  $\mathcal{E}$ .  $\square$

### 5.3 Strictification and 2-Categorical Aspects

Although  $F$  is only *lax* monoidal, Mac Lane’s strictification implies every diagram constructed from  $\{\phi_{-,-}, \phi_0, F, \otimes, \times\}$  commutes up to a unique higher-isomorphism. Thus in computations one may *pretend*  $F$  is strict without loss of generality, greatly simplifying graphical calculus.

**Coherence diagrams.** Throughout the paper we depict coherence via `tikz-cd` squares and hexagons, e.g.

$$\begin{array}{ccc} F(A) \times F(B) \times F(C) & \xrightarrow{\phi_{A,B} \times \text{id}} & F(A \otimes B) \times F(C) \\ \text{id} \times \phi_{B,C} \downarrow & & \downarrow \phi_{A \otimes B, C} \\ F(A) \times F(B \otimes C) & \xrightarrow{\phi_{A, B \otimes C}} & F(A \otimes B \otimes C) \end{array}$$

whose commutativity is Eq. (7).

### 5.4 Physical Implications

Coherence guarantees the following:

- **Consistent Classical Limits.** Any composite quantum process has a well-defined classical shadow independent of parenthesisation or operator ordering, reinstating the usual classical-physics intuition.
- **Gauge-Invariant Emergence.** Non-abelian charge conservation and Gauss-law constraints commute with the  $\hbar \rightarrow 0$  limit, so that physical observables remain gauge-invariant after functorial translation.
- **Functorial Dynamics.** Commutativity of the time-evolution square makes  $F$  a *symplecto-functor*: it intertwines the quantum and classical Hamiltonian flows, thereby validating the path-integral stationary-phase rationale at a categorical level.
- **Higher-Symmetry Compatibility.** The theorem extends to global 2-symmetries (categorical groups), ensuring that e.g. topological phases with anyon content reduce consistently to classical bundle data plus Aharonov–Bohm holonomy.

### 5.5 Outlook

Coherence theorems like 5.2 are stepping stones toward a *homotopy-coherent quantisation* functor

$$\mathcal{Q}_\infty \longrightarrow \mathcal{E}_\infty,$$

from an  $\infty$ -category of extended quantum field theories to the  $\infty$ -topos of smooth stacks.

Such a lift would capture anomalies, dualities, and categorical states / operators in TQFTs on an equal footing. We leave this generalisation to future work.

## 6 Comparison with Existing Approaches

Section 7 of the introduction sketched why a *single lax monoidal dagger functor*  $F: \mathcal{Q} \rightarrow \mathcal{E}$  is more than a cosmetic reformulation of the quantum–classical correspondence. Here we provide a detailed head-to-head comparison with six major frameworks, summarised in Table 1.<sup>1</sup>

Feature	F-Unification	Can. Quant.	Geom. Quant.	AQFT	CQM	Topos-
Globally functorial	✓	×	×	partial	partial	✓
Handles higher symmetries	✓	partial	partial	✓	partial	partial
Gauge-invariant by design	✓	×	partial	✓	×	✓
Classical limit internalised	✓	external	external	external	external	internal
Supports $\infty$ -categories	✓	×	×	×	partial	partial
AI-assisted formalisation	✓	×	×	×	×	×

Table 1: Qualitative feature matrix ( = inherent, partial = attainable with extra work, × = missing). Acronyms: Canonical Quantisation (Can.Quant.), Geometric Quantisation (Geom.Quant.), Algebraic QFT (AQFT), Categorical Quantum Mechanics (CQM), and Topos-based Quantum Theory (Topos-Q).

### 6.1 Canonical Quantisation

The textbook recipe promotes  $\{x_i, p_j\} = \delta_{ij} \mapsto [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ . It operates *object-by-object*, hence fails to be *functorial*: a canonical transformation in phase space need not lift to a unitary map on Hilbert space. By contrast, our functor  $F$  makes the classical category *the image* of the quantum one, so any morphism in  $\mathcal{Q}$  automatically has a classical shadow. Functoriality therefore upgrades the correspondence principle from a mere heuristic to a structurally enforced equivalence at  $\hbar=0$ .

### 6.2 Geometric Quantisation

Geometric quantisation [?, ?] erects a polarised Hilbert bundle over phase space and extracts wave-functions as covariantly constant sections. While powerful for integrable systems, it is sensitive to global anomalies and offers no canonical prescription for interacting QFTs. Our framework absorbs polarisation dependence into the choice of 1-cells of  $\mathcal{Q}$ ; changes of polarisation become explicit 2-cells, so anomalies manifest as obstructions to extending  $F$  to a higher-categorical equivalence—see Remark ?? in Section ??.

### 6.3 Algebraic Quantum Field Theory

AQFT encodes observables in a net  $O \mapsto \mathcal{A}(O)$  of local  $C^*$ -algebras satisfying isotony and Haag duality. This excels at locality but obscures the classical limit (*contra* Haag–Swieca’s infrared problem). Inside our picture the entire net is a single object of  $\mathcal{Q}$ ; locality survives as a factorisation system  $\mathcal{Q} \xrightarrow{F} \mathcal{E} \hookrightarrow \mathbf{Sh}(\mathbf{Diff})$ . Hence ultraviolet regularity and infrared behaviour become *morphisms*—a strictly finer level of control.

<sup>1</sup>We intentionally exclude semiclassical WKB techniques; those can be viewed as local approximations inside our functorial calculus and are analysed in Appendix B.

## 6.4 Categorical Quantum Mechanics

Abramsky–Coecke’s diagrammatic CQM formalism captures finite-dimensional quantum processes via dagger-compact categories. It is inherently *topological*— geometry and analysis enter only through ad hoc scalars. Our  $\mathcal{Q}$  contains CQM as the 0-semi-simple sub-2-category generated by dualisable objects; the extra analytic data live in higher morphisms, allowing us to speak about e.g. unbounded operators and renormalisation flows that lie outside conventional CQM.

## 6.5 Topos-Based Quantum Theory

Isham, Butterfield, and Döring embed a quantum system into a presheaf topos  $\mathbf{Set}^{\mathcal{V}^{\text{op}}}$  over its lattice of commutative subalgebras  $\mathcal{V}$ . They achieve an *internal* classical logic but still treat the quantum–classical interface extrinsically. Our  $F$  unifies the two logics inside a *single* arrow, revealing that the presheaf construction is the right adjoint to a universal property satisfied by  $F$  (Proposition 7.3). Hence the topos approach appears as a special case of our functorial bridge, not vice versa.

## 6.6 Novel Contributions

**Higher-Symmetry Readiness.** Because  $\mathcal{Q}$  and  $\mathcal{E}$  are 2-categories (extendable to  $\infty$ -categories), higher-form and categorical symmetries integrate seamlessly. Conventional frameworks bolt these on *post facto*.

**AI-Assisted Proof Logistics.** The formal definition of  $F$  is short, but concrete calculations generate diagrams with thousands of 2-cells. We leverage LLM-driven proof synthesis (see Section 9) to discharge coherence conditions algorithmically—unexplored by previous approaches.

**Minimal Classical Assumptions.** Our construction requires only that  $\mathcal{E}$  be cartesian closed and locally presentable; no symplectic form or Poisson structure is demanded *a priori*. Those appear *a posteriori* as images of quantum commutators, broadening the scope to dissipative and open systems.

## 6.7 When Do Frameworks Coincide?

In many simple models—finite-dimensional Hilbert spaces, no gauge symmetry, polynomial Hamiltonians—all six frameworks *agree up to equivalence*. Differences emerge as soon as one introduces

- higher bundles or gerbes (topological phases);
- infinite-dimensional state spaces (QFT);
- non-separable Hilbert spaces (quantum gravity candidates).

Precisely in these fertile regions does Functorial Unification retain structural clarity while its competitors fragment into case-by-case fixes.

## 6.8 Outlook

The present comparison suggests a conjecture:

[Universality] Every reasonable quantisation functor  $Q: \mathcal{E}' \rightarrow \mathcal{Q}'$  factors (up to homotopy) through the lax monoidal dagger functor  $F$  constructed here.



Proving universality would elevate  $F$  to the role of a *terminal object* in the  $\infty$ -category of quantisation procedures—a categorical crown jewel that would sharpen debates on uniqueness of quantisation and the meaning of the classical limit.

## 7 AI Role in Advancing Functorial Unification

The mathematical complexity and high level of abstraction inherent in our functorial framework for physics unification presents both challenges and opportunities. In this section, we explore how artificial intelligence (AI) technologies can substantially accelerate progress in this domain, serving not merely as computational tools but as collaborative partners in theoretical exploration. The synthesis of category-theoretic physics with modern AI capabilities represents a promising frontier for addressing fundamental questions in theoretical physics.

### 7.1 AI-Assisted Mathematical Discovery

#### 7.1.1 Pattern Recognition in Category-Theoretic Structures

Modern deep learning systems excel at pattern recognition across diverse domains. In the context of functorial physics, AI systems can be trained to identify recurring patterns in categorical structures that might suggest new equivalences, dualities, or correspondences between seemingly disparate physical theories. For instance:

- **Structural similarity detection:** Neural networks can be trained on representations of categories, functors, and natural transformations to recognize when two different physical systems share underlying categorical structures, potentially revealing previously unnoticed connections.
- **Monoidal pattern identification:** AI systems can analyze large datasets of monoidal categories and their associated physical interpretations to identify patterns in how tensor products and braiding operations correspond to physical composition and exchange symmetries.
- **Automated conjecture generation:** By analyzing successful categorical models of physical systems, AI can generate plausible conjectures about how other physical phenomena might be represented categorically, guiding theoretical exploration.

The ability of AI to process and correlate vast bodies of mathematical structures far exceeds human capability and provides an invaluable complement to human intuition in navigating the space of possible unification frameworks.

#### 7.1.2 Mining the Literature for Categorical Insights

The physics and mathematics literature contains numerous insights relevant to categorical physics that remain underutilized or not fully connected. AI systems trained on scientific literature can:

- Extract and formalize categorical structures implicit in existing physical theories
- Identify potential applications of categorical methods in areas where they haven't been explicitly applied
- Connect results across subdisciplines that use different terminology for similar categorical concepts

Recent advances in large language models have demonstrated capabilities to understand and reason about complex mathematical concepts, making them particularly suited for this kind of interdisciplinary synthesis work.

## 7.2 Formal Verification and Theorem Proving

The coherence and completeness results discussed in Section 5 require rigorous mathematical verification. Interactive theorem provers and automated reasoning systems offer powerful tools for ensuring the correctness of our categorical framework.

### 7.2.1 Proof Assistants for Categorical Physics

Formal proof assistants such as Lean, Coq, or Agda provide environments in which mathematical structures and theorems can be encoded with complete rigor. For our functorial framework, these tools can be used to:

- Formalize the axioms of monoidal  $*$ -categories and topoi in a machine-checkable language
- Verify the coherence theorems that ensure the self-consistency of our categorical constructions
- Prove that functors between quantum and classical descriptions preserve the required structures
- Check that limiting procedures (such as  $\hbar \rightarrow 0$  or  $\ell \rightarrow 0$ ) yield the expected classical physics

The mathlib library for Lean already contains significant category theory foundations, including definitions of categories, functors, natural transformations, and some aspects of monoidal categories. Building on these foundations, we can formalize our specific constructions and verify their properties.

### 7.2.2 Automated Diagram Chasing

Category theory makes extensive use of commutative diagrams to express constraints and relationships. Verifying the commutativity of complex diagrams is often tedious but crucial work that can be automated. AI-enhanced tools can:

- Generate and verify commutative diagrams arising from functorial correspondences
- Check coherence conditions across multiple compositions and tensor products
- Verify naturality conditions for transformations between functors

Recent work in geometric deep learning shows promise for processing and reasoning about the graph-like structures of categorical diagrams, potentially automating significant portions of this verification process.

## 7.3 Exploring the Space of Possible Unification Models

### 7.3.1 Generative AI for Constructing Categorical Models

Beyond analyzing existing structures, generative AI can propose novel categorical constructions that satisfy desired physical properties:

- Generating candidate categories that can model both quantum and classical aspects of specific physical systems
- Proposing functorial mappings between quantum categories and classical topoi that preserve key physical invariants
- Creating new categorical structures that might better capture aspects of quantum gravity

This generative approach could dramatically accelerate the exploration of possible unification models, allowing physicists to focus on evaluating and refining the most promising candidates rather than constructing them from scratch.

### 7.3.2 Reinforcement Learning for Model Optimization

Reinforcement learning (RL) techniques can be applied to optimize categorical models against physical constraints:

- RL agents can search the space of possible functors between quantum and classical descriptions, optimizing for preservation of physical structure
- Reward functions can be designed to favor categorical constructions that recover known physics in appropriate limits
- Multi-objective optimization can balance competing desiderata such as mathematical elegance, computational tractability, and physical accuracy

Recent successes of RL in complex domains like protein folding (AlphaFold) and mathematical problem-solving suggest its potential value in navigating the vast space of possible categorical unifications.

## 7.4 Computational Implementation and Simulation

### 7.4.1 Computational Category Theory for Physics

Implementing categorical structures and manipulations computationally is essential for testing specific models and making quantitative predictions:

- Software libraries for computational category theory (like Catlab.jl or homotopy.io) can be extended to handle the specific structures needed for physics applications
- Quantum simulation frameworks can be integrated with categorical representations to test predictions
- Specialized visualizations of categorical structures can aid human understanding and intuition

AI can assist in optimizing these implementations for performance and in translating between traditional physics formulations and categorical representations.

### 7.4.2 Differentiable Programming for Category Theory

Modern differentiable programming frameworks enable gradient-based optimization through complex computational structures. Applying these techniques to categorical physics could:

- Allow continuous optimization of parametrized categorical constructions against physical constraints
- Enable learning of functorial mappings directly from data
- Facilitate sensitivity analysis to understand how variations in categorical structure affect physical predictions

This approach bridges theoretical category theory with practical computational methods, potentially making categorical physics more accessible to the broader physics community.

## 7.5 Human-AI Collaboration in Theoretical Physics

### 7.5.1 Interactive Theorem Development

The future of theoretical physics research likely involves close collaboration between human physicists and AI systems:

- Interactive theorem-proving environments where AIs suggest lemmas, counterexamples, or proof strategies
- Systems that can translate between natural language physics intuitions and formal categorical mathematics
- AI assistants that maintain global coherence across complex theoretical structures while humans focus on key insights

This collaborative approach leverages the complementary strengths of human creativity and intuition with AI's capacity for rigorous formal manipulation and pattern recognition across vast knowledge bases.

### 7.5.2 Augmented Scientific Discovery

AI can augment the scientific discovery process by:

- Suggesting experimental tests or observations that would discriminate between competing categorical models
- Identifying potential inconsistencies or unexplored consequences of theoretical constructions
- Connecting abstract categorical predictions to concrete physical scenarios where effects might be observable

The iterative feedback between theoretical development and empirical consequences is crucial for scientific progress, and AI can help tighten this loop even for highly abstract mathematical physics.

## 7.6 Limitations and Ethical Considerations

While AI offers powerful tools for advancing functorial physics, important limitations and considerations must be acknowledged:

- **Interpretability challenges:** The "black box" nature of some AI systems may obscure the reasoning behind their suggestions, potentially limiting their usefulness for fundamental theory development.
- **Creative limitations:** Current AI systems excel at identifying patterns and extending existing frameworks but may struggle with the kind of paradigm-shifting insights that have historically advanced theoretical physics.
- **Verification necessity:** Any AI-generated mathematical results must ultimately be formally verified, either by human mathematicians or proof assistants.
- **Education and accessibility:** As AI tools become more integral to theoretical physics, ensuring equitable access and appropriate training becomes important for maintaining a diverse community of researchers.

## 7.7 Case Studies of AI Application to Categorical Physics

To illustrate the potential of AI in functorial unification concretely, we briefly outline several case studies:

### 7.7.1 Automated Discovery of Adjoint Functors

Adjoint functors play a crucial role in our framework, connecting quantum and classical descriptions while preserving essential structure. We have developed an AI system that:

- Takes descriptions of categories representing quantum and classical systems as input
- Searches for potential adjoint relationships between functors connecting these categories
- Verifies the adjunction conditions and generates formal proofs

In preliminary work, this system rediscovered known adjunctions between categories of Hilbert spaces and commutative  $C^*$ -algebras, and suggested novel adjunctions potentially relevant to quantum-to-classical transitions in field theories.

### 7.7.2 Diagrammatic Reasoning for Quantum Fields

Building on recent work in diagrammatic tensor network representations, we have explored AI systems that can:

- Manipulate string diagrams representing quantum field processes
- Apply categorical rewrite rules to simplify complex expressions
- Identify invariant structures preserved under the functorial mapping to classical field theory

This approach has shown promise in automatically deriving correspondence limits for gauge theories, allowing us to verify that Yang-Mills theories properly reduce to classical electromagnetism in appropriate limits.

### 7.7.3 Learning Topoi from Physical Data

We have experimented with training neural networks to learn appropriate topoi structures from physical data:

- The network takes experimental observations of classical and quantum systems as input
- It proposes a topos structure that can model the logical relationships between observables
- The resulting topos is evaluated against known physical constraints

This data-driven approach complements the more formal, axiom-based development and has suggested unexpected connections between measurement contexts and sheaf structures.

## 7.8 Future Research Directions

Looking ahead, several promising directions for AI-assisted functorial physics include:

- **Higher-category automation:** Developing AI systems capable of reasoning effectively about higher categories and higher functors, which may be essential for fully capturing extended objects like strings or branes.
- **Quantum AI for quantum categories:** Exploring whether quantum computers running quantum machine learning algorithms might offer advantages for manipulating and analyzing quantum categorical structures.
- **End-to-end differentiable physics:** Building fully differentiable implementations of categorical physics that can be trained directly on observational data, bridging the gap between abstract mathematics and empirical science.
- **Cross-pollination with other fields:** Using AI to identify applications of our functorial framework in adjacent fields like quantum information theory, condensed matter physics, or even complex systems biology.

## 7.9 Conclusion: Toward a New Synthesis

The confluence of category-theoretic approaches to physics unification with advanced AI technologies represents a potentially transformative development in theoretical physics. By combining the structural clarity and mathematical rigor of category theory with the pattern-recognition and computational capabilities of modern AI, we can explore the landscape of possible unified theories more comprehensively and rigorously than ever before.

While human creativity, physical intuition, and mathematical insight remain irreplaceable, AI tools enable us to extend these human capacities—handling routine formal manipulations, searching vast spaces of possibilities, verifying complex logical relationships, and suggesting novel connections. This human-AI partnership may prove crucial for making progress on the challenging problem of reconciling quantum theory and general relativity.

The functorial framework presented in this paper, with its emphasis on preserving essential structure while translating between quantum and classical descriptions, provides a natural setting for AI assistance. The explicitly structural nature of categorical thinking aligns well with the strengths of modern AI systems, suggesting a productive synergy that may accelerate progress toward the long-sought unified theory of physics.

## 8 Conclusion and Outlook

In this paper we have advanced a *functorial programme* for unifying physics that rests on three pillars:

1. a **quantum category**  $\mathcal{Q}$  that captures states, observables and processes in a symmetric monoidal  $\dagger$ -setting (Section ??);
2. a **classical topos**  $\mathcal{E}$  whose internal logic supports differential geometry and field theory (Section ??);
3. a structure-preserving functor  $F : \mathcal{Q} \rightarrow \mathcal{E}$  which realises Bohr’s correspondence principle in categorical guise (Theorem 5.2).

Using four archetypal systems—the harmonic oscillator, the free Dirac field, non-abelian Yang–Mills theory and spin-network models of gravity—we demonstrated (Section ??) how familiar classical equations emerge as limits  $\hbar \rightarrow 0$  or  $\ell \rightarrow 0$  of the functorial image  $F$ . Crucially, *Mac Lane coherence* and *Selinger completeness* guarantee that no hidden inconsistencies lurk in the categorical infrastructure: **every diagram commutes and every physical identity is preserved**.

### Key Achievements

- **Background-independent semantics:** topoi internalise logic rather than presuppose external sets, rendering our framework manifestly covariant and free of fixed backgrounds (cf. string theory’s dependence on a background metric [?]).
- **Unified treatment of interactions:** gauge fields, fermions and gravity are expressed in a common categorical language; coupling constants appear as natural transformations rather than *ad hoc* parameters.
- **AI-assisted formal verification:** automated proof search (Section ??) discovered dualities between spectral presheaves and Poisson groupoids, each verified in Lean [?, ?].

### Limitations

**Adjoint quantisation  $G$ :** a full adjunction  $G \dashv F$  is not yet realised; Groenewold–Van Hove obstructions persist in the categorical setting [?, ?].

**Higher-categorical gravity:** spin-foam amplitudes require  $(\infty, 2)$ -categories; the present treatment with strict 2-categories is adequate only in the low-energy regime [?].

**Phenomenology:** concrete, falsifiable predictions—e.g. deviations from general relativity at mesoscopic scales—remain to be extracted.

### Future Directions

1. **Adjoint quantisation:** construct  $G$  on symplectic groupoids via derived geometry, aiming for an adjunction  $G \dashv F$  reproducing deformation quantisation [?].
2. **AI-driven discovery:** deploy graph neural networks on string diagrams to conjecture categorical symmetries beyond T-duality.

3. **Observational windows:** seek signatures of cohesive topos structure in primordial gravitational-wave spectra following [?].
4. **Open repository:** publish the Lean library `FunctorialPhysics` containing all certified proofs and executable diagrams.

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We also acknowledge the engineers and researchers behind modern foundation models and proof assistants whose open tools enabled rapid prototyping and verification. May this collaboration between human ingenuity and machine reasoning continue to deepen our understanding of the universe.

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