Inverse Theory

* Linear Problems
* Non-linear problems
* Parametric inversion
* Linear inverse problems

# Geophysical Inversion

Basic geophysical experiment:

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| ./images/basic\_gephys\_expt.jpg |

Input: geophysical survey

Output: data

Earth model:

* physical property values, e.g. density, magnetic susceptibility, conductivity, elastic properties, etc.
* parameters (e.g. magnetics , are the strength and orientation of a dipole)

General notation:

: data

: model (e.g. )

: forward modeling operator

e.g. gravity:

To solve this problem numerically we divided the earth into cells, each with a constant , and wrote the data as

where

Writing the datum as we have

We also wrote magnetic data in the same way:

after meshing, we wrote:

where:

In both (1) and (2) we can write the relationship between the data and model as

$$& \vec{d} = G \vec{m} \ \
& \text{where:} \
& \vec{d} \in \mathbb{R}^N \qquad (d\_1, ..., d\_N) \
& \vec{m} \in \mathbb{R}^M \qquad (m\_1, ..., m\_M) \
& G \quad \text{is an} \quad N \times M \quad \text{matrix}$$

In other cases, the general notation can t be written as a simple vector-matrix product. As we shall see in DC resistivity, the governing equation is:

where is the electrical conductivity. In the notation

* : measured voltages
* : electrical conductivity
* : involves the solution of (3) done through finite element or finite volume

Lastly, we can have situations where the earth model is a set of parameters.

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| ./images/cube.jpg |

Suppose we wanted to find a best fit dipole that generated surfaces magnetic response.

$$& \vec{B} = \frac{\mu\_0 (m)}{4 \pi r^3} (3 (\hat{m} \cdot \hat{r}) \hat{r} - \hat{m}) \ \
& \vec{m} = |m| \hat{m}$$

The parameters are where the first three define location and the last three are amplitude and orientation of the dipole. So general representation , the model would be the set of parameters.

Remark: The ability to generate data, , given a "model" is called "forward modeling."

# Inverse Problem

Given observed data with and an estimate of the data errors, and the forward modeling relationship what is the model that produced the data?

There are two solution strategies with as the number of data and the number of model parameters.

**Case 1:** when , that is, when there is more data than unknowns, (e.g., finding the magnetic dipole parameters) We can post the inverse problem as one of the finding the best fitting model

**Case 2:** when , that is, when there are more unknowns than data, (e.g. 3D inversion of gravity or magnetics). This is a non-unique problem. We need to incorporate additional information about the model into the inversion.

Remark: Case 1, finding a few parameters is simpler, so we will deal with that first.

# Inverse Problem 1: Find a few parameters

Let from be the observed data.

Let be an estimated standard deviation for the ith datum. Assume that data errors are Gaussian with zero mean and a standard deviation of

Define the misfit (this is an appropriate misfit function for Gaussian statistics):

Goal: find the model that minimizes .

Remark: In general, the solution of the problem requires:

1. A starting model $\m^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}, ...)$
2. Finding how each datum changes when a parameter changes (sensitivity):
3. Computing a perturbation
4. Forming a new model estimate
5. Continue steps 2 to 4 until a minimum has been found.

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| ./images/phi\_d\_min.jpg |

Remarks: final results can depend upon the starting model. Consider minimizing a function of a single variable. In the figure below is a better solution.

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| ./images/sing\_var\_min.jpg |

# Inverse problem II: finding a function

Consider the simple case where we write:

with as an matrix. There are infinitely many solutions. For example, consider the case where there are two unknowns and and one datum:

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| ./images/line.jpg |

and the solution is

Any point along the line is a viable solution. To find a particular solution we need to have additional information (a priori knowledge) about the solution. For example, suppose we knew that the earth model was one where was small. Then let

For our toy example this gives . Question: what is the that minimize and still fits the data constraint?

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| ./images/toy\_solution.jpg |

$$& \text{min } \phi\_m = m\_1^2 + m\_2^2 \
& \text{subject to } m\_1 + m\_2 = 1$$

The answer is

So this is a way to get a single solution that also includes additional information about the earth model.

For a general problem in 3D, we introduce a model objective function

The first term makes a solution close to a reference model (which can be zero); the second term minimizes structure in the x-direction, the third term in the y-direction, and the fourth term in the z-direction. The parameters are constants that control the relative wighting of the different penalty terms.

Again, to solve the problem numerically we need to divide the earth into cells (as done in the forward modeling)

$$\phi\_m & = \alpha\_s \|W\_s (m-m\_{ref}) \|^2 + \alpha\_x \|W\_x (m-m\_{ref}) \|^2 + \alpha\_y \|W\_y (m-m\_{ref}) \|^2 +\alpha\_z \|W\_z (m-m\_{ref}) \|^2 \
& = (m-m\_{ref}) \left( \alpha\_s W\_s^T W\_s + x W\_x^T W\_x \alpha\_y W\_y^T W\_y + \alpha\_z W\_z^T W\_z \right) (m-m\_{ref}) \
\phi\_m & = \|W\_m(m-m\_{ref})\|^2$$

where is an matrix ( are alos matrices).

So our prior knowledge about what kind of solution we want is encoded into . This is extremely important. If we followed the ideas in the toy example we would be led to formulate our problem as

But the observation have errors. We don't want to find a solution that fits the inaccurate data exactly (then we would be guaranteed to have the wrong model). Rather, we define the misfit as

If the data are contaminated with Gaussian error then, if are at the solution ,

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| ./images/obs\_pred.jpg |

Solve the inverse problem by

where is a regularization parameter. As changes from zero to infinity

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| ./images/phi\_d\_phi\_m.jpg |

As , min small misfit (), large model norm ().

As , min small model norm (), large misfit ().

Putting these together yields the Tikhonov curve

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| ./images/tikhonov.jpg |

When we minimize every value of gives a difference solution. We can experiment and find that value of for which the misfit is equal to some desired target level .