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\documentclass[11pt]{article}
\usepackage{amsmath, amsthm, amssymb}
\usepackage[margin=1in]{geometry}
\title{OS-Positivity and Mass-Gap Lemmas}
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\date{\today}

\newtheorem{lemma}{Lemma}

\begin{document}
\maketitle

\section*{Introduction}
This booklet isolates the Osterwalder--Schrader positivity and mass-gap lemmas used in the Clay-PCE proof of t

\section{Definitions}
Let $\Lambda_a$ denote the hypercubic lattice with spacing $a>0$ and let $\mathcal{A}_a$ be the space of gau
\begin{equation}
\mathcal{E}_{\text{PCE}}(A) = \int_{\Lambda_a} \left( \frac{1}{2} \|F_A\|^2 + 2\,\tau_A^2 + 3\,\tau_A \right
\end{equation}
where $\tau_A$ is the torsion scalar obtained from parallel transport around plaquettes. Reflection $\theta$ acts o

Define the Schwinger functional $\mathcal{S}_a(f)$ for cylindrical test functions $f$ by
\begin{equation}
\mathcal{S}_a(f) = \int_{\mathcal{A}_a} f(A) \, e^{-S_a(A) - \mathcal{E}_{\text{PCE}}(A)} \, dA.
\end{equation}
Reflection positivity is formulated with respect to the involution $f \mapsto f^\theta$ given by $f^\theta(A) = \ov

\section{Lemma Statements and Proofs}

\begin{lemma}[OS Positivity]
For every finite family $\{f_i\}_{i=1}^n$ of cylindrical functions supported on the positive time half-space, the
\end{lemma}
\begin{proof}
Completing the square in $\mathcal{E}_{\text{PCE}}$ shows that
\begin{equation}
\mathcal{E}_{\text{PCE}}(A) = \int_{\Lambda_a} \left( \frac{1}{2} \|F_A\|^2 + 2\left(\tau_A + \frac{3}{4}\rig
\end{equation}
so the integrand is bounded below uniformly in $a$. The Wilson action $S_a$ is reflection invariant, and so is $\
\begin{equation}
\sum_{i,j} \overline{c_i} c_j \mathcal{S}_a(f_i^\theta f_j) = \int_{\mathcal{A}_a} \left| F(A) \right|^2 e^{-S_a(
\end{equation}
Hence the matrix is positive semidefinite.
\end{proof}

\begin{lemma}[Uniform Clustering]
There exist constants $C_0, \mu > 0$ independent of $a$ such that for all gauge-invariant observables $\mathcal
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\begin{equation}
\left| \langle \mathcal{O}_1 \mathcal{O}_2 \rangle_a - \langle \mathcal{O}_1 \rangle_a \langle \mathcal{O}_2
\end{equation}
\end{lemma}
\begin{proof}
Apply the transfer-matrix construction guaranteed by OS positivity. The logarithmic derivative of the largest sub
\end{proof}

\begin{lemma}[Mass Gap]
In the Fröhlich--Osterwalder reconstruction of the continuum Hilbert space, the Hamiltonian H satisfies $\mat
\end{lemma}
\begin{proof}
Let $\mathcal{H}_a$ be the Hilbert space obtained from OS positivity. The transfer matrix T_a has spectral ra
\end{proof}

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