TMR4243 - Marine Control Systems II

Homework assignment 2

1 A time-varying Lyapunov function

Consider the Lyapunov function

$$V(t, x_1, x_2) = \phi(t)x_1^2 + \frac{1}{2}x_2^2, \tag{1}$$

where $\phi(t) > 0$ is a continuously differentiable function (we say, $\phi \in \mathcal{C}^1$, i.e. the set of all functions for which the derivative exists and is continuous), and (x_1, x_2) are the states driven by $\dot{x}_1 = f_1$ and $\dot{x}_2 = f_2$. Show that

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \tag{2}$$

and calculate this as an expression of (t, x_1, x_2, f_1, f_2).

2 A nonautonomous system

Consider the time-varying linear system

$$\dot{x}_1 = g(t)x_2 \tag{3a}$$

$$\dot{x}_2 = -cg(t)x_1 - x_2, \qquad c > 0, \quad 0 < g_0 \leq |g(t)| \leq g_1, \forall t \geq 0. \tag{3b}$$

1. Show by Lyapunov's Direct Method that $(x_1, x_2) = (0, 0)$ is UGS, using

$$V_1(x) := \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2. \tag{4}$$

2. Verify by Barbalat's Lemma that you can prove convergence of $x_2(t) \to 0$.
3. Verify by the LaSalle-Yoshizawa theorem that you can prove UGS and convergence of $x_2(t) \to 0$.
4. Verify by the Nested Matrosov theorem that you can prove that $(x_1, x_2) = (0, 0)$ is UGES.
5. Explain why Krasovskii-LaSalle's Invariance Principle is not applicable to this system.
6. Assume that $g(t) = 3$ and $c = 2$ so that the system becomes time-invariant. Show then by Krasovskii-LaSalle that $(x_1, x_2) = 0$ is indeed UGES.

3 An autonomous system

Consider the time-invariant linear system

$$\dot{x}_1 = x_2 + x_3 \tag{5a}$$

$$\dot{x}_2 = -x_1 + x_3 \tag{5b}$$

$$\dot{x}_3 = -x_1 - x_2 - x_3 \tag{5c}$$

1. Verify that $(x_1, x_2, x_3) = 0$ is the single equilibrium for this system.

2. Verify that the origin is GES by linear system methods.
3. What stability property are you able to prove by Lyapunov's Direct Method by the Lyapunov function
$$V_1(x) := \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \tag{6}$$
4. Based on V_1 and $\dot{V}_1$ above, use the Krasovskii-LaSalle's Invariance Principle to show that the origin is GES.
5. Using V_1 from above and
$$\begin{aligned} V_2(x) &: = x_1x_3 + \frac{1}{2}x_1^2 \qquad (7) \\ V_3(x) &: = x_2x_3, \qquad (8) \end{aligned}$$
verify by the Nested Matrosov theorem that you can prove that the origin is UGAS.
6. Conjecture: If this system is UGAS, then it must be GES. Is this correct? Explain.

1 Solution: A time-varying Lyapunov function

Differentiating, we get

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \\
\frac{\partial V}{\partial t} &= \frac{\partial \phi}{\partial t} x_1^2 = \dot{\phi}(t) x_1^2 \\
\frac{\partial V}{\partial x_1} &= 2\phi(t) x_1 \\
\frac{\partial V}{\partial x_2} &= x_2
\end{aligned}$$

Hence, we get

$$\dot{V} = \dot{\phi}(t) x_1^2 + 2\phi(t) x_1 f_1 + x_2 f_2$$

2 Solution: A nonautonomous system

1. For $V_1(x) = \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2$, we get

$$\alpha_1(|x|) := \frac{1}{2}\min\{1,c\}\,|x|^2 \le V(x) \le \frac{1}{2}\max\{1,c\}\,|x|^2 =: \alpha_2(|x|)$$

and

$$\begin{aligned}
\dot{V}_1 &= cx_1\dot{x}_1 + x_2\dot{x}_2 = cx_1 g(t) x_2 + x_2\left(-cg(t)x_1 - x_2\right) \\
&= -x_2^2 =: -\alpha_3(|x|) \le 0
\end{aligned}$$

Since α_1 and α_2 are class-$\mathcal{K}_\infty$ functions and α_3 is **positive semidefinite**, then UGS of $x = 0$ follows from Lyapunov's direct method.

2. Let $\phi(t) = \dot{V}(t)$. This function is uniformly continuous since the solution trajectory of $x_2(t)$ by definition is an absolutely continuous function of t. Then

$$\int_0^t \phi(\tau)d\tau = \int_0^t \dot{V}(\tau)d\tau = V(t) - V(0).$$

Since $V(t) \ge 0$, $\forall t \ge 0$, and monotonically nonincreasing, it will converge. Hence, $\lim_{t\to\infty}\int_0^t \phi(\tau)d\tau$ exists and is finite. This proves that $\phi(t) = \dot{V}(t) \to 0 \implies x_2(t) \to 0$ as $t \to \infty$.

3. LaSalle-Yoshizawa proves directly from the result of Lyapunov's Direct method in 1. above that $x = 0$ is UGS and that $\alpha_3(|x(t)|) = x_2(t)^2 \to 0$ as $t \to \infty$.

4. Nested Matrosov Theorem:
   (a) From 1. above we get that $x = 0$ is UGS, using $V_1(x)$.
   (b) Select $\Delta > 0$ and let

$$\begin{aligned}
j &: = 2 \\
V_2(x) &: = \mathrm{sgn}(g(t)) x_1 x_2 \\
\phi(t) &: = |g(t)|
\end{aligned}$$

   It follows that $\exists \mu > g_1$ such that $\max\{|V_1(x)|, V_2(x), \phi(t)\} \le \mu$ for all $(x,t) \in \mathcal{B}^2(\Delta) \times \mathbb{R}_{\ge 0}$ (all x bounded and t arbitrary). We also get

$$\begin{aligned}
\dot{V}_1 &= -x_2^2 =: Y_1(x) \\
\dot{V}_2 &= |g(t)|\,x_2^2 - c\,|g(t)|\,x_1^2 - x_1x_2 =: Y_2(x, \phi(t))
\end{aligned}$$

(c) We find then that $Y_1(x) = 0 \Rightarrow x_2 = 0 \Rightarrow Y_2(x, \phi(t)) = -c\,|g(t)|\, x_1^2 \leq 0$.

(d) We get that $Y_1(x) = Y_2(x, \phi(t)) = 0 \Rightarrow x = 0$. Hence, it follows from the Nested Matrosov theorem that $x = 0$ is UGAS - and UGES since the system is linear.

5. The Krasovskii-LaSalle's Invariance Principle is not applicable since the closed-loop system generally is time-varying (nonautonomous) due to the time-varying gain $g(t)$.

6. If $g(t) = 3$ and $c = 2$ then the system becomes

$$\dot{x}_1 = 3x_2 \tag{9a}$$
$$\dot{x}_2 = -6x_1 - x_2. \tag{9b}$$

Using $V(x) = x_1^2 + \frac{1}{2}x_2^2$ gives

$$\dot{V} = 6x_1x_2 - 6x_1x_2 - x_2^2 = -x_2^2 \leq 0$$

which proves UGS. We look for the largest invariant set $\mathcal{M}$ inside the set

$$\Omega = \left\{ x \in \mathbb{R}^2 : x_2 = 0 \right\}.$$

Setting $x_2 \equiv 0$ implies $\dot{x}_2 = 0$ and gives

$$\begin{aligned} \dot{x}_1 &= 0 \\ 0 &= \dot{x}_2 = -6x_1 - 0 \Rightarrow x_1 = 0, \end{aligned}$$

which shows that $\mathcal{M} = \{x \in \Omega : x = 0\}$ is indeed the origin. Hence, $x = 0$ is (U)GAS - and correspondingly (U)GES.

3 Solution: An autonomous system

Consider the time-invariant linear system

$$\dot{x}_1 = x_2 + x_3 \tag{10a}$$
$$\dot{x}_2 = -x_1 + x_3 \tag{10b}$$
$$\dot{x}_3 = -x_1 - x_2 - x_3 \tag{10c}$$

1. We get the linear system

$$\dot{x} = Ax, \qquad A := \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \tag{11}$$

and it is straightforward to verify that A is nonsingular. Hence, $(x_1, x_2, x_3) = 0$ is the single equilibrium for the system.

2. Calculating the eigenvalues of A gives $\lambda_{1,2} = -0.3194 \pm j1.6332$ and $\lambda_3 = -0.3611$. All real values are negative and, thus, the origin mus be GES.

3. Using $V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2$ we get $\alpha_1(|x|) = \alpha_2(|x|) = \frac{1}{2}x^\top x = \frac{1}{2}\,|x|^2$ and

$$\begin{aligned} \dot{V}_1 &= x_1x_2 + x_1x_3 - x_1x_2 + x_2x_3 - x_1x_3 - x_2x_3 - x_3^2 \\ &= -x_3^2 := -\alpha_3(|x|) \leq 0. \end{aligned} \tag{12}$$

Since α_3 is only positive semidefinite, the origin can only be proven GS by this Lyapunov function.

4. Use the Krasovskii-LaSalle's Invariance Principle we look for the largest invariant set in

$$\Omega = \left\{ x \in \mathbb{R}^3 : x_3 = 0 \right\}. \tag{13}$$

Making Ω invariant implies that $x_3 = 0$ and

$$\dot{x}_3 = 0 = -x_1 - x_2 \implies x_1 = -x_2 \tag{14}$$

This gives

$$\begin{aligned} \dot{x}_1 &= x_2 = -x_1 && (15)\\ \dot{x}_2 &= -x_1 = x_2 && (16) \end{aligned}$$

We want to prove that $x_1 = -x_2 = 0$ is the only invariant condition for this constraint. Assume conversely that $x_1 = -x_2 = c \neq 0$ is an invariant solution for $x \in \Omega$. Then the solution of the two equations, while satisfying the constraint $x_1 = -x_2$ is

$$x_1(t) = ce^{-t} \quad \text{and} \quad x_2(t) = -ce^{t} \tag{17}$$

It follows that $x_1 = -x_2$ holds only for $t = 0$, while for $t > 0$ the constraint fails implying that the solution must then leave Ω. By contradiction, the only state that satisfies the constraint $x_1 = -x_2$ is, indeed, $x_1 = x_2 = 0$. Hence, the largest invariant set $\mathcal{M}$ in Ω must be the origin itself, which then is GAS.

5. We already have that the origin is UGS by V_1 as a Lyapunov function. Since the system is autonomous, we get $\phi(t) = 0$, which makes the analysis easier. Differentiating V_1, V_2, and V_3 gives

$$\begin{aligned} \dot{V}_1 &= -x_3^2 =: Y_1(x) && (18)\\ \dot{V}_2 &= \dot{x}_1 x_3 + x_1\dot{x}_3 + x_1\dot{x}_1 = x_2x_3 + x_3^2 - x_1^2 - x_1x_2 - x_1x_3 + x_1x_2 + x_1x_3 \\ &= x_2x_3 + x_3^2 - x_1^2 =: Y_2(x) && (19)\\ \dot{V}_3 &= \dot{x}_2x_3 + x_2\dot{x}_3 = -x_1x_3 + x_3^2 - x_1x_2 - x_2^2 - x_2x_3 := Y_3(x) && (20) \end{aligned}$$

These polynomial functions are obviously all bounded when the state $|x|$ is bounded. We get that $Y_1(x) = 0 \implies x_3 = 0 \implies Y_2(x) = -x_1^2 \leq 0$ and $Y_1(x) = Y_2(x) = 0 \implies x_3 = x_1 = 0 \implies Y_3(x) = -x_2^2 \leq 0$, and finally $Y_1(x) = Y_2(x) = Y_3(x) = 0 \implies x = 0$. Hence, by the Nested Matrosov Theorem, the origin is UGAS.

6. Conjecture: If this system is UGAS, then it must be GES.
Since the system is autonomous, convergence and stability must be uniform, that is, for a time-invariant system UGS/UGAS/UGES = GS/GAS/GES. Moreover, since convergence for a linear system must be exponential, the UGAS=GAS becomes GES.

References