TMR4243 - Marine Control Systems II

Homework assignment 4

1 Input-to-State Stability

1.1 Task: ISS-Lyapunov function

For the following systems, assume that $\|u\| \leq u_{\max}$. Show that $V(x) = \frac{1}{2}x^2$ is an ISS-Lyapunov function for the following systems:

1. $$\dot{x} = -x + u \tag{1}$$

2. $$\dot{x} = -x^3 + x^2 u \tag{2}$$

3. $$\dot{x} = -x^5 + x^3 u \tag{3}$$

4. $$\dot{x} = -\left(1 + e^{|x|}\right) x + xu \tag{4}$$

1.2 Task: ISS

In Homework Assignment 3 we were given the (simplified) mechanical dynamics of a diesel-generator

$$\dot{\delta} = \omega_B \left(\omega - \omega_0\right) \tag{5}$$
$$2H\dot{\omega} = t_m - D\omega - t_L + w(t) \tag{6}$$

where ω is the frequency, δ is the load angle , t_m is the control input torque, t_L is an electric load torque, w a bounded disturbance torque, $(H, D, \omega_B) > 0$ are constants, and ω_0 is the frequency of the connected electric power bus. Controlling δ to δ_{ref} and ω to ω_0, we defined the error states $x = \mathrm{col}(e_\delta, e_\omega) = \mathrm{col}(\delta - \delta_{ref}, \omega - \omega_0)$, which gives

$$\dot{e}_\delta = \omega_B e_\omega \tag{7}$$
$$2H\dot{e}_\omega = t_m - De_\omega - D\omega_0 - t_L + w(t) \tag{8}$$

With the disturbance $\|w\| \leq w_0$ as input, show that the control law

$$u = t_m = -k_p e_\delta - k_d e_\omega + D\omega_0 + t_L, \tag{9}$$

with $k_p, k_d > 0$, renders the resulting closed-loop system ISS with respect to $x = 0$.

2 Feedback linearization

We will train on feedback linearization by considering the 3DOF horizontal vessel model

$$\dot{\eta} = R(\psi)\nu \tag{10}$$

$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu = \tau, \tag{11}$$

where $\eta = col(x, y, \psi)$ is the position/heading; $\nu = col(u, v, r)$ the velocities; $C(\nu)$ the Coriolis/centripetal matrix; $D(\nu) > 0$ a nonlinear damping matrix; $M = M^\top > 0$; τ the control force input; and $R(\psi)$ the rotation matrix:

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

This has the properties that $R(\psi)^\top R(\psi) = R(\psi)R(\psi)^\top = I$ and $\dot{R} = R(\psi)S(r)$ where

$$S(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -S(r)^\top. \tag{13}$$

2.1 Task: Feedback linearization

If the output to be controlled is $\eta \in \mathbb{R}^3$, do the following:

1. Explain the term "vector relative degree", and calculate the vector relative degree for the 3DOF vessel model.
2. Differentiate η according to the vector relative degree. What is the dimension of the zero dynamics?
3. Perform a full state feedback linearization design by differentiating η according to the vector relative degree.

2.2 Task: Feedback linarization by a nonlinear transformation

If the output to be controlled is $\eta \in \mathbb{R}^3$, define the transformation $z_1 := \eta$, $z_2 := R(\psi)\nu + C_1\eta$ where $C_1 = C_1^\top > 0$. This defines the state transformation $z = \mathrm{col}(z_1, z_2) = T(x)$, where $x := \mathrm{col}(\eta, \nu)$.

1. Show that this transforms the system into the controller form

   $$\dot{z} = Az + B\Gamma(x)\left[u - \alpha(x)\right] \tag{14}$$

   where (A, B) is controllable, and $\Gamma(x)$ is nonsingular for all x.
2. Design a full-state feedback linearization control law, and prove UGES of $z = 0$ using Lyapunov's Direct Method.

2.3 Task: Zero dynamics

Assume for simplicity that $C(\nu) = 0$ and $D(\nu) = D > 0$ is a constant damping matrix.

For $\tau = \mathrm{col}(\tau_u, \tau_v, \tau_r) \in \mathbb{R}^3$, let

$$\tau_u = -k(u - u_0) = \gamma(u), \qquad k > 0, u_0 > 0 \tag{15}$$

$$\begin{bmatrix} \tau_v \\ \tau_r \end{bmatrix} = \begin{bmatrix} -Y_\delta \\ -N_\delta \end{bmatrix} \delta \tag{16}$$

$$\tau = B_1\gamma(u) + B_2\delta \tag{17}$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ -Y_\delta \\ -N_\delta \end{bmatrix} \tag{18}$$

where $\delta \in \mathbb{R}$ is the new control input and (Y_δ, N_δ) are control gains. Let now the output to be controlled be the heading $\psi = h_\psi^\top \eta$, $h_\psi := \mathrm{col}(0, 0, 1)$.

1. What is the relative degree of the system?
2. Differentiate the output according to the relative degree and identify the controlled dynamics and the internal dynamics (keeping vector notation).
3. Perform a partial feedback linearization design that controls the output to a constant reference heading ψ_{ref}.

1 Solution: Input-to-State Stability

1.1 Task: ISS-Lyapunov function

We have that $\|u\| \leq u_{\max}$ and will show that $V(x) = \frac{1}{2}x^2$ is an ISS-Lyapunov function. We start by defining the $\mathcal{K}_\infty$-functions $\alpha_1(|x|) = \alpha_2(|x|) = \frac{1}{2}|x|^2$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|). \tag{19}$$

Then we differentiate V to find $\alpha_3 \in \mathcal{K}$ and $\chi \in \mathcal{K}_\infty$:

1.
$$\begin{aligned}
\dot{x} &= -x + u && (20)\\
\dot{V} &= -x^2 + xu \leq -\frac{1}{2}|x|^2 - \frac{1}{2}|x|^2 + |x|\,|u| && (21)\\
&\leq -\frac{1}{2}|x|^2, \qquad \forall\,|x| \geq 2\,|u| && (22)\\
\alpha_3(|x|) &= \frac{1}{2}|x|^2, \qquad \chi(|u|) = 2\,|u| && (23)
\end{aligned}$$

2.
$$\begin{aligned}
\dot{x} &= -x^3 + x^2 u && (24)\\
\dot{V} &= -x^4 + x^3 u \leq -\frac{1}{2}|x|^4 - \frac{1}{2}|x|^4 + |x|^3\,|u| && (25)\\
&\leq -\frac{1}{2}|x|^4, \qquad \forall\,|x| \geq 2\,|u| && (26)\\
\alpha_3(|x|) &= \frac{1}{2}|x|^4, \qquad \chi(|u|) = 2\,|u| && (27)
\end{aligned}$$

3.
$$\begin{aligned}
\dot{x} &= -x^5 + x^3 u && (28)\\
\dot{V} &= -x^6 + x^4 u \leq -\frac{1}{2}|x|^6 - \frac{1}{2}|x|^6 + |x|^4\,|u| && (29)\\
&\leq -\frac{1}{2}|x|^6, \qquad \forall\,|x| \geq \sqrt{2\,|u|} && (30)\\
\alpha_3(|x|) &= \frac{1}{2}|x|^6, \qquad \chi(|u|) = \sqrt{2\,|u|} && (31)
\end{aligned}$$

4.
$$\begin{aligned}
\dot{x} &= -\left(1 + e^{|x|}\right)x + xu && (32)\\
\dot{V} &= -\left(1 + e^{|x|}\right)x^2 + x^2 u = -|x|^2 - e^{|x|}\,|x|^2 + |x|^2\,|u| && (33)\\
&= -|x|^2 + \left(|u| - e^{|x|}\right)|x|^2 && (34)\\
&\leq -|x|^2, \qquad \forall\,|x| \geq \ln(|u|) && (35)\\
\alpha_3(|x|) &= |x|^2, \qquad \chi(|u|) = \ln(|u|) && (36)
\end{aligned}$$

1.2 Task: ISS

For the mechanical dynamics of the diesel-generator, with the proposed control law, we get

$$\begin{aligned} \dot{e}_\delta &= \omega_B e_\omega && (37)\\ 2H\dot{e}_\omega &= -k_p e_\delta - (k_d + D)\, e_\omega + w(t) && (38) \end{aligned}$$

or

$$\begin{aligned} \dot{x} &= Ax + Bw(t) && (39)\\ A &= \begin{bmatrix} 0 & \omega_B \\ \frac{-k_p}{2H} & \frac{-(k_d+D)}{2H} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{2H} \end{bmatrix}, && (40) \end{aligned}$$

where A is Hurwitz for all $k_p > 0$ and $k_d + D > 0$. Let $P = P^\top > 0$ satisfy $PA + A^\top P = -I$, and define the Lyapunov function candidate $V(x) = x^\top Px$. This gives

$$\begin{aligned} \alpha_1(|x|) &= \lambda_{\min}(P)\,|x|^2 \le V(x) \le \lambda_{\max}(P)\,|x|^2 = \alpha_2(|x|) && (41)\\ \dot{V} &= 2x^\top PAx + 2x^\top PBw \\ &\le -x^\top x + \kappa x^\top x + \frac{4}{4\kappa}|PB|^2 w^2, \qquad \kappa = \frac{1}{2} \\ &\le -\frac{1}{2}|x|^2 + 2|PB|^2|w|^2 && (42)\\ \alpha_3(|x|) &= \frac{1}{2}|x|^2, \qquad \alpha_4(|w|) = 2|PB|^2|w|^2. && (43) \end{aligned}$$

It follows from the (alternative) definition that $V(x)$ is an ISS-Lyapunov function, and by the ISS sufficiency theorem the closed-loop system is ISS from the input w with respect to $x = 0$.

2 Solution: Feedback linearization

We have the 3DOF horizontal vessel model

$$\dot{\eta} = R(\psi)\nu \tag{44}$$

$$M\dot{\nu} + C(\nu)\nu + D(\nu)\nu = \tau, \tag{45}$$

where $\eta = col(x, y, \psi)$ is the position/heading; $\nu = col(u, v, r)$ the velocities; $C(\nu)$ the Coriolis/centripetal matrix; $D(\nu) > 0$ a nonlinear damping matrix; $M = M^\top > 0$; τ the control force input; and $R(\psi)$ the rotation matrix:

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{46}$$

for which $R(\psi)^\top R(\psi) = R(\psi)R(\psi)^\top = I$ and $\dot{R} = R(\psi)S(r)$ where

$$S(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -S(r)^\top. \tag{47}$$

2.1 Task: Feedback linearization

The output to be controlled is $\eta \in \mathbb{R}^3$:

1. Vector relative degree: The differential order between the input and the output for each element in the output vector. Here we find that for each of the 3 elements in η we have to differentiate it two times before we get an expression containing some of the control input elements in τ. The vector relative degree is then $[2, 2, 2]$; we just say 2 for such cases.

2. Differentiating η two times, we get

$$\begin{aligned}
\dot{\eta} &= R(\psi)\nu \quad \Longleftrightarrow \quad \nu = R(\psi)^\top \dot{\eta} && (48)\\
\ddot{\eta} &= \dot{R}\nu + R\dot{\nu} = R(\psi)S(r)\nu + R(\psi)M^{-1}\left[-C(\nu)\nu - D(\nu)\nu + \tau\right] && (49)\\
&= R(\psi)S(r)R(\psi)^\top \dot{\eta} + R(\psi)M^{-1}\left[-C(\nu)R(\psi)^\top\dot{\eta} - D(\nu)R(\psi)^\top\dot{\eta} + \tau\right] \\
MR(\psi)^\top\ddot{\eta} &= -\left[C(\nu) - MS(r)\right]R(\psi)^\top\dot{\eta} - D(\nu)R(\psi)^\top\dot{\eta} + \tau && (50)\\
\mathcal{M}(\eta)\ddot{\eta} &= -\mathcal{C}(\eta,\dot{\eta})\dot{\eta} - \mathcal{D}(\eta,\dot{\eta})\dot{\eta} + \tau && (51)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}(\eta) &:= MR(\psi)^\top && (52)\\
\mathcal{C}(\eta,\dot{\eta}) &:= \left[C(R(\psi)^\top\dot{\eta}) - MS(r)\right]R(\psi)^\top && (53)\\
\mathcal{D}(\eta,\dot{\eta}) &:= D(R(\psi)^\top\dot{\eta})R(\psi)^\top && (54)
\end{aligned}$$

The dimension of the state space is $n = 2\cdot 3 = 6$, the dimension of the relative degree is also $r = 2+2+2 = 6$; hence, there is no zero dynamics.

3. Full state feedback linearization design: Choosing $\xi_1 = \eta$, $\xi_2 = \dot{\eta}$ we have

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 && (55)\\
\dot{\xi}_2 &= \mathcal{M}(\eta)^{-1}\left[\tau - \mathcal{C}(\eta,\dot{\eta})\dot{\eta} - \mathcal{D}(\eta,\dot{\eta})\dot{\eta}\right] && (56)
\end{aligned}$$

We can now select

$$\tau = \mathcal{C}(\eta,\dot{\eta})\dot{\eta} - \mathcal{D}(\eta,\dot{\eta})\dot{\eta} + \mathcal{M}(\eta)u \tag{57}$$

which gives

$$\begin{aligned}
\dot{\xi}_1 &= \xi_2 && (58)\\
\dot{\xi}_2 &= u. && (59)
\end{aligned}$$

Hence, we have linearized the nonlinear system by feedback, and the new control u can now be chosen with negative state feedback to render $(\xi_1, \xi_2) = 0$ UGES.

2.2 Task: Feedback linarization by a nonlinear transformation

The output to be controlled is $\eta \in \mathbb{R}^3$, and we are proposed the state transformation $z = \mathrm{col}(z_1, z_2) = T(x)$, where $x := \mathrm{col}(\eta, \nu)$, with $z_1 := \eta$, $z_2 := R(\psi)\nu + C_1\eta$ and $C_1 = C_1^\top > 0$.

1. Differentiating z gives

$$\begin{aligned}
\dot{z}_1 &= R(\psi)\nu = -C_1 z_1 + z_2 && (60)\\
\dot{z}_2 &= \dot{R}\nu + R(\psi)\dot{\nu} + C_1\dot{\eta} && (61)\\
&= R(\psi)S(r)\nu + R(\psi)M^{-1}\left[\tau - C(\nu)\nu - D(\nu)\nu\right] + C_1 R(\psi)\nu \\
&= R(\psi)M^{-1}\left[\tau - C(\nu)\nu - D(\nu)\nu + MS(r)\nu + MR(\psi)^\top C_1 R(\psi)\nu\right]
\end{aligned}$$

We now recognize

$$\begin{aligned}
A &= \begin{bmatrix} -C_1 & I \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} && (62)\\
\Gamma(x) &= R(\psi)M^{-1} && (63)\\
\alpha(x) &= C(\nu)\nu + D(\nu)\nu - MS(r)\nu - MR(\psi)^\top C_1 R(\psi)\nu && (64)
\end{aligned}$$

such that the system can be written in the controller form

$$\dot{z} = Az + B\Gamma(x)\left[\tau - \alpha(x)\right] \tag{65}$$

where (A, B) can be verified to be controllable, and $\Gamma(x)$ is nonsingular for all x.

2. To design a full-state feedback linearization control law, we assign

$$\tau = \alpha(x) - \Gamma(x)^{-1} K z, \tag{66}$$

which gives

$$\dot{z} = (A - BK)\, z \tag{67}$$

where K is designed to make $(A - BK)$ Hurwitz. UGES of $z = 0$ follows from Lyapunov's Direct Method using the Lyapunov equation.

2.3 Task: Zero dynamics

Setting $C(\nu) = 0$, $D(\nu) = D > 0$, and

$$\tau = B_1 \gamma(u) + B_2 \delta \tag{68}$$

we get

$$\dot{\eta} = R(\psi)\nu \tag{69}$$

$$M\dot{\nu} = -D\nu + B_1\gamma(u) + B_2\delta \tag{70}$$

where $\delta \in \mathbb{R}$ is the new control input and the output to be controlled is $\psi = h_\psi^\top \eta$, $h_\psi := \mathrm{col}(0, 0, 1)$.

1. The relative degree of the system is found by differentiating the output ψ until you hit the control input δ. It is seen that we need to differentiate ψ twice until δ apppears. Hence, $r = 2$.
2. Let $p = col(x, y)$, $\upsilon = (u, v)$, and

$$R_2(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \tag{71}$$

$$H_{xy} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{72}$$

   Then

$$\dot{p} = R_2(\psi)\upsilon \tag{73}$$

$$\dot{\upsilon} = H_{xy}^\top \dot{\nu} = H_{xy}^\top M^{-1}\left[-D\nu + B_1\gamma(u) + B_2\delta\right] \tag{74}$$

$$\dot{\psi} = r \tag{75}$$

$$\dot{r} = h_\psi^\top \dot{\nu} = h_\psi^\top M^{-1}\left[-D\nu + B_1\gamma(u) + B_2\delta\right] \tag{76}$$

   The controlled dynamics is $(\psi, r) \in \mathbb{R}^2$, and the internal dynamics is $(p, \upsilon) \in \mathbb{R}^4$.
3. Partial feedback linearization: Let $e_\psi = \psi - \psi_{ref}$ where ψ_{ref} is a constant reference. This yields

$$\dot{e}_\psi = r \tag{77}$$

$$\dot{r} = h_\psi^\top M^{-1}\left[-D\nu + B_1\gamma(u)\right] + h_\psi^\top M^{-1} B_2 \delta \tag{78}$$

   Note that due to ship symmetry (Fossen, 2011) the mass matrix looks like

$$M = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & d \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{d}{bd-c^2} & \frac{-c}{bd-c^2} \\ 0 & \frac{-c}{bd-c^2} & \frac{b}{bd-c^2} \end{bmatrix} \tag{79}$$

$$h_\psi^\top M^{-1} = \begin{bmatrix} 0 & \frac{-c}{bd-c^2} & \frac{b}{bd-c^2} \end{bmatrix} \tag{80}$$

$$h_\psi^\top M^{-1} B_1 = 0 \tag{81}$$

$$h_\psi^\top M^{-1} B_2 = \frac{cY_\delta - bN_\delta}{bd - c^2} \tag{82}$$

We thus choose

$$\delta = \alpha(\eta, \nu) = \frac{1}{h_\psi^\top M^{-1} B_2} \left(-k_p e_\psi - k_d r + h_\psi^\top M^{-1} D\nu \right), \tag{83}$$

which is a partially linearizing feedback PD control law that results in the linear closed-loop controlled dynamics

$$\begin{aligned} \dot{e}_\psi &= r && (84) \\ \dot{r} &= -k_p e_\psi - k_d r, && (85) \end{aligned}$$

and the internal nonlinear dynamics

$$\begin{aligned} \dot{p} &= R_2(\psi)\upsilon && (86) \\ \dot{\upsilon} &= H_{xy}^\top M^{-1} \left[-D\nu + B_1\gamma(u) + B_2\alpha(\eta, \nu) \right]. && (87) \end{aligned}$$

References

Fossen, T. I. (2011). *Handbook of Marine Craft Hydrodynamics and Motion Control.* John Wiley & Sons Ltd.