TMR4243 - Marine Control Systems II

Mathematical notations and preliminaries

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1 Mathematical preliminaries

The class of systems considered in this thesis is represented by the ordinary differential equation (ODE)

$$\dot{x} = f(x, u, t) \tag{1}$$

where for each $t \geq 0$ the vector $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is an input vector that can be used to actively manipulate the state, $\dot{x} := \frac{d}{dt}x(t)$ is the time derivative of the state, and $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a nonlinear vector function that can be differentiated sufficiently many times. The state vector x can contain only the original states of the plant, or it can be augmented with some additional dynamic states necessary to solve a control problem.

To this equation belongs an output map

$$y = h(x) \tag{2}$$

where $y(t) \in \mathbb{R}^m$ is the system output that we wish to control.

1.1 Notation

Table 1: Mathematical abbreviations. See e.g. Wikipedia (2016).

Symbol:	Meaning:
$\emptyset$	*empty set*
$\in$	*element of*
$\forall$	*for all*
$\exists$	*there exists*
$\exists!$	*there exists one and only one*
$:=$	*defined as*
$=:$	*defines*
$\Rightarrow$	*implies*
$\Leftrightarrow$	*if and only if*
$\to$	*to* or *converge to*
$\mapsto$	*maps to*
$\nearrow$	*converge to from below*
$\searrow$	*converge to from above*

- Total time derivatives of a function $x(t)$ are denoted $\dot{x}$, $\ddot{x}$, $x^{(3)}, \ldots, x^{(n)}$. A superscript with an argument variable will denote partial differentiation with respect to that argument, that is,

$$\alpha^t(x, \theta, t) := \frac{\partial \alpha(x, \theta, t)}{\partial t}, \quad \alpha^{x^2}(x, \theta, t) := \frac{\partial^2 \alpha(x, \theta, t)}{\partial x^2}, \quad \alpha^{\theta^n}(x, \theta, t) := \frac{\partial^n \alpha(x, \theta, t)}{\partial \theta^n}, \quad \text{etc.}$$

  The gradient $\alpha^x(x, \theta, t)$ with $x \in \mathbb{R}^n$ is defined as a row vector, by convention.

- For a function $f : X \to Y$ we say that f is of class $\mathcal{C}^r$, and write $f \in \mathcal{C}^r$, if $f^{x^k}(x)$, $k \in \{0, 1, \ldots, r\}$, is defined and continuous for all $x \in X$. f is *continuous* if $f \in \mathcal{C}^0$, f is *continuously differentiable* if $f \in \mathcal{C}^1$, and f is *smooth* if $f \in \mathcal{C}^\infty$.

Table 2: Other abbreviations

Symbol:	Meaning:
US	*Uniformly Stable*
UGS	*Uniformly Globally Stable*
UGAS	*Uniformly Globally Asymptotically Stable*
UGES	*Uniformly Globally Exponentially Stable*
LAS	*Locally Asymptotically Stable*
LES	*Locally Exponentially Stable*
UA	*Uniformly Attractive*
UGA	*Uniformly Globally Attractive*
a.e.	*almost everywhere*
a.a.	*almost all*
s.t.	*such that*
w.r.t.	*with respect to*
wlog	*without loss of generality*

- The p-norm of a vector is $|x|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$, where the most commonly used is the 2-norm, or the Euclidean vector norm, simply denoted $|x| := |x|_2 = (x^\top x)^{1/2}$. This reduces to the absolute value for a scalar.
- The signals norms are denoted by $||x||_p := \left(\int_{t_0}^{\infty} |x(t)|^p \, dt\right)^{1/p}$, where in particular $\|x\| := \|x\|_\infty = \operatorname{ess\,sup}\{|x(t)| : t \geq 0\}$. If a specific time-interval $[t_0, t_1]$ is of interest, we use the notation $\left\|x_{[t_0,t_1]}\right\| := \operatorname{ess\,sup}\{|x(t)| : t \in [t_0, t_1]\}$
- The *distance-to-the-set* function is denoted $|x|_{\mathcal{A}} := d\,(x; \mathcal{A}) = \inf\{d(x, y) : \; y \in \mathcal{A}\}$ where the point-to-point distance function is normally taken as $d(x, y) = |x - y|$. Note that for an equilibrium, $\mathcal{A} = \{0\}$, the distance-to-the-set function reduces to the norm, $|x|_{\mathcal{A}} = |x|_{\{0\}} = |x|$.
- A column vector is often stated as $\operatorname{col}(x, y, z) := \left[x^\top, y^\top, z^\top\right]^\top$, while a row vector is $\operatorname{row}(x, y, z) := [x^\top, y^\top, z^\top]$. For a vector $x = \operatorname{col}(x_1, x_2, \ldots, x_n) \in \mathbb{R}^{nm}$ we use the compact notation $\bar{x}_i := \operatorname{col}(x_1, x_2, \ldots, x_i)$, $i = 1, \ldots, m$. Whenever convenient (and clear from the context) the notation $|(x, y, z)| = |\operatorname{col}(x, y, z)|$ is used. A vector of only ones is denoted $\mathbf{1} = [1, 1, \ldots, 1]^\top$.
- The induced norm of a matrix A is denoted $||A||_p := \max_{|x|_p=1} |Ax|_p$, where in particular $\|A\| := \|A\|_2$.
- The minimum and maximum eigenvalues of a matrix A are given by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.
- A diagonal matrix is often written as $\operatorname{diag}(a, b, c, \ldots)$, and the identity matrix is simply written I where its dimension should be clear from the context.
- In control design, the subscript 'd' as in $x_d(t)$ or $y_d(\theta)$ means 'desired.' It will always be used for a varying desired function. For constant set-points the subscript 'ref' is used as in x_{ref} or u_{ref}.

1.1.1 A remark on the notation

The partial derivative notation, for instance $\alpha^\theta(x, \theta, t)$, is a convenient compact notation. However, a superscript can be confused with some other mathematical operation, like taking the power of the function. This is solved by always keeping the argument list of the function. Partial derivatives will always come before the argument list, while other mathematical operations are indicated after the list (with partial differentiation as the primary operation). For example, $f^t(x, \theta, t)^2 = \left(\frac{\partial f(x,\theta,t)}{\partial t}\right)^2$ and $\xi^{\theta^2}(\theta, t)^\top = \left(\frac{\partial^2 \xi(\theta,t)}{\partial \theta^2}\right)^\top$. The only exception is $f^{-1}(x)$ which means the inverse map of the function f, while $f(x)^{-1} = 1/f(x)$. If other ambiguous cases are encountered, they are solved using parentheses.

1.2 Inequalities

The following list of inequalities are convenient for proving stability in nonlinear control systems. In all cases, unless otherwise stated, let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be vectors in a normed vector space, $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are scalars, and let $p, q \geq 1$.

- **Triangle inequality**

$$|x+y|_p \leq |x|_p + |y|_p \,. \tag{3}$$

  As a consequence of this follows the **reverse triangle inequality**

$$\Big| |x|_p - |y|_p \Big| \leq |x-y|_p \,. \tag{4}$$

- **Hölder's inequality**

$$\left|x^\top y\right| \leq |x|_p \, |y|_q \,, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{5}$$

  A special case is the **Cauchy-Schwarz inequality** by setting $p = q = 2$ so that

$$\left|x^\top y\right| \leq |x|_2 \, |y|_2 \,. \tag{6}$$

- **Completing the squares** gives

$$x^\top y \leq \frac{1}{2} x^\top x + \frac{1}{2} y^\top y, \tag{7}$$

  which is a vectorial case of **Young's inequality** for a scalar product (setting $p = q = 2$)

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{8}$$

  For any $\kappa > 0$, setting $p = q = 2$ and $a = \sqrt{2\kappa} x_i$, $b = \frac{1}{\sqrt{2\kappa}} y_i$ for $x_i, y_i \in \mathbb{R}$, we get the more common version,

$$\begin{aligned} x_i y_i &\leq \kappa x_i^2 + \frac{y_i^2}{4\kappa}, \qquad \kappa > 0 & (9) \\ x^\top y &\leq \kappa x^\top x + \frac{1}{4\kappa} y^\top y. & (10) \end{aligned}$$

- Some other interesting inequalities are

$$\begin{aligned} |y - \sin y| &\leq \frac{1}{6} |y|^3 \,, \qquad |1 - \cos y| \leq \frac{1}{2} y^2 & (11) \\ |x+y|_2^2 &\leq 2\,|x|_2^2 + 2\,|y|_2^2 & (12) \\ \frac{a}{1+a} &\leq \frac{b}{1+b}, \qquad 0 \leq a \leq b & (13) \\ \frac{a+b}{1+a+b} &\leq \frac{a}{1+a} + \frac{b}{1+b}, \qquad a, b \geq 0. & (14) \end{aligned}$$

- For class-$\mathcal{K}$ function α_1 and $a, b \geq 0$ we get

$$\alpha_1\left(a+b\right) \leq \alpha_1\left(2a\right) + \alpha_1\left(2b\right). \tag{15}$$

1.3 Derivatives

For the functions $a(x) : \mathbb{R}^n \to \mathbb{R}^n$, $b(x) : \mathbb{R}^n \to \mathbb{R}^n$, and $v(x) : \mathbb{R}^n \to \mathbb{R}$, we have the differentiation rules

$$a^x(x) \in \mathbb{R}^{n\times n}, \qquad b^x(x) \in \mathbb{R}^{n\times n}, \qquad v^x(x) \in \mathbb{R}^{1\times n} \tag{16}$$

$$\frac{d}{dx}\left[\frac{a(x)}{v(x)}\right] = \frac{a^x(x)v(x) - a(x)v^x(x)}{v(x)^2} \in \mathbb{R}^{n\times n} \tag{17}$$

$$\frac{d}{dx}\left[\frac{a(x)^\top}{v(x)}\right] = \frac{a^x(x)^\top v(x) - v^x(x)^\top a(x)^\top}{v(x)^2} \in \mathbb{R}^{n\times n} \tag{18}$$

$$\frac{d}{dx}\left[a(x)^\top b(x)\right] = b(x)^\top a^x(x) + a(x)^\top b^x(x) \in \mathbb{R}^{1\times n}. \tag{19}$$

1.3.1 Example: Maneuvering guidance signals

As an example, consider a path variable $s \in \mathbb{R}$, a time variable $t \in \mathbb{R}$, and a guidance system providing the following signals:

$$p_d(s) : \mathbb{R} \to \mathbb{R}^2, \qquad p_d^s(s) : \mathbb{R} \to \mathbb{R}^2, \qquad p_d^{s^2}(s) : \mathbb{R} \to \mathbb{R}^2 \tag{20}$$

$$u_d(t) : \mathbb{R} \to \mathbb{R}, \qquad \dot{u}_d(t) : \mathbb{R} \to \mathbb{R} \tag{21}$$

$$\upsilon_d(s,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \qquad \upsilon_d^s(s,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \qquad \upsilon_d^t(s,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \tag{22}$$

where $p_d(s)$ is a point on the surface for each s, $u_d(t)$ is a desired surge velocity as function of time, and $\upsilon_d(s,t)$ is a desired speed assignment along the path as a function of the path and time variables.

For $p_d(s) = col(x_d(s), y_d(s))$, then we have

$$|p_d^s(s)| = \sqrt{p_d^s(s)^\top p_d^s(s)} = \sqrt{x_d^s(s)^2 + y_d^s(s)^2} \in \mathbb{R} \tag{23}$$

$$(|p_d^s(s)|)^s := \frac{\partial\,|p_d^s(s)|}{\partial s} = \frac{p_d^s(s)^\top p_d^{s^2}(s)}{|p_d^s(s)|} = \frac{x_d^s(s)x_d^{s^2}(s) + y_d^s(s)y_d^{s^2}(s)}{\sqrt{x_d^s(s)^2 + y_d^s(s)^2}} \in \mathbb{R}, \tag{24}$$

where $|p_d^s(s)| > 0$ for any regular parametrization of the path (it is zero only if the path is a singleton).

If the desired heading of the ship, $\psi_d(s)$, shall be defined to point in the direction of the tangent vector to the path, then this is given by:

$$\psi_d(s) = \arctan\left(\frac{y_d^s(s)}{x_d^s(s)}\right) \tag{25}$$

$$\psi_d^s(s) = \frac{x_d^s(s)y_d^{s^2}(s) - x_d^{s^2}(s)y_d^s(s)}{x_d^s(s)^2 + y_d^s(s)^2} \tag{26}$$

$$\psi_d^{s^2}(s) = \frac{x_d^s(s)y_d^{s^3}(s) - x_d^{s^3}(s)y_d^s(s)}{x_d^s(s)^2 + y_d^s(s)^2} - 2\frac{\left(x_d^s(s)y_d^{s^2}(s) - x_d^{s^2}(s)y_d^s(s)\right)\left(x_d^s(s)x_d^{s^2}(s) + y_d^s(s)y_d^{s^2}(s)\right)}{\left[x_d^s(s)^2 + y_d^s(s)^2\right]^2} \tag{27}$$

Typically, given your favorite parametrization along a path, $p_d(s)$, the speed assignment υ_d corresponding to a desired surge speed, is

$$\upsilon_d(s,t) := \frac{u_d(t)}{|p_d^s(s)|}. \tag{28}$$

Then we get

$$\upsilon_d^s(s,t) = -\frac{(|p_d^s(s)|)^s}{|p_d^s(s)|^2}u_d(t) = -\frac{p_d^s(s)^\top p_d^{s^2}(s)}{|p_d^s(s)|^3}u_d(t) \in \mathbb{R} \tag{29}$$

$$\upsilon_d^t(s,t) = \frac{\dot{u}_d(t)}{|p_d^s(s)|} \in \mathbb{R}. \tag{30}$$

Note again, that for a regular parametrization of the path, the denominator is never zero.

1.3.2 Differentiation under the integral sign

Derived using the fundamental theorem of calculus, one get to the general form of the Leibniz integral rule; see (Flanders, 1973):

$$I(x) = \int_{u(x)}^{v(x)} f(x,y)dy \tag{31}$$

$$\frac{dI(x)}{dx} = f(x,v(x))\frac{dv}{dx} - f(x,u(x))\frac{du}{dx} + \int_{u(x)}^{v(x)} \frac{\partial f(x,y)}{\partial x}dy \tag{32}$$

The Leibniz integral rule is particularly interesting if we have a system $\dot{x} = -g(x)$ where $g(0) = 0$ and $g(\cdot)$ is monotonically increasing, in which case we can use

$$V(x) = \int_0^x g(y)dy \tag{33}$$

$$V^x(x)\left(-g(x)\right) = -g(x)^2 < 0, \quad \forall x \neq 0. \tag{34}$$

2 Stability of sets

Consider the *ordinary differential equation*

$$\dot{x} = f(x) \tag{35}$$

frequently referred to as the 'system,' where for each $t \geq 0$ the vector $x(t) \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth vector function. Let $x(t, x_0)$ denote the solution of (35) at time t with initial state $x(0) = x_0$. If there is no ambiguity from the context, the solution is simply written as $x(t)$. It is defined on some maximal interval of existence $(T_{\min}(x_0), T_{\max}(x_0))$ where $T_{\min}(x_0) < 0 < T_{\max}(x_0)$. The system (35) is said to be *forward complete* if $T_{\max}(x_0) = +\infty$ for all x_0, *backward complete* if $T_{\min}(x_0) = -\infty$ for all x_0, and *complete* if it is both forward and backward complete (Lin et al., 1996).

A convenient but crude way to ensure forward completeness is:

Proposition 1 *(Teel, 2002) Suppose the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies conditions for existence and uniqueness of solutions. Suppose also that $f(\cdot)$ satisfies a global sector bound, that is, $\exists L \geq 0$ and $c \geq 0$ such that $\forall x$*

$$|f(x)| \leq L\,|x| + c. \tag{36}$$

Then all solutions are defined for all $t \geq t_0$.

In this thesis, stability for (35) will frequently be analyzed with respect to a closed, not necessarily bounded, set $\mathcal{A} \subset \mathbb{R}^n$. Such a set is said to be *forward invariant* for a forward complete system (35) if $\forall x_0 \in \mathcal{A}$ the solution $x(t, x_0) \in \mathcal{A}$, $\forall t \geq 0$. In order to measure the distance away from the set, the "distance to the set $\mathcal{A}$ function" is defined as

$$|x|_{\mathcal{A}} := d\left(x; \mathcal{A}\right) = \inf\left\{d(x,y) : \; y \in \mathcal{A}\right\} \tag{37}$$

where the point-to-point distance function is in this thesis simply taken as the Euclidean distance $d(x,y) = |x - y|$.

An example is given by an equilibrium. An equilibrium point $x_e \in \mathbb{R}^n$ of (35) is a point such that $f(x_e) = 0$. It can be represented by the compact set

$$\mathcal{A} := \left\{x \in \mathbb{R}^n : \; x = x_e\right\}.$$

The distance function is in this case $|x|_{\mathcal{A}} = \inf\left\{|x - y| : \; y = x_e\right\} = |x - x_e|$ showing that the distance function reduces to the traditional norm function.

Another example is the ε-ball given by the compact set

$$\mathcal{A}_\varepsilon = \left\{x \in \mathbb{R}^n : \; |x| \leq \varepsilon\right\},$$

for which the distance function becomes $|x|_{\mathcal{A}_\varepsilon} = \max\{0,\ |x| - \varepsilon\}$.

In the case when $\mathcal{A}$ is a compact set (closed and bounded) and it can be established for a solution to (35) that the distance $|x(t, x_0)|_{\mathcal{A}}$ is bounded on the maximal interval of existence, then the trajectory itself must necessarily be bounded away from infinity on the maximal interval of existence. By a contradiction argument it follows that the system must be forward complete. In the case when $\mathcal{A}$ is not bounded (non-compact) this is not necessarily true, and other means must be used to established forward completeness. In stability definitions for such sets, *forward completeness* is therefore a prerequisite that must hold for the system.

Definition 2 *If the system (35) is forward complete, then for this system a closed, forward invariant set $\mathcal{A}$ is:*

- Uniformly Stable *(US) if there exists $\delta(\cdot) \in \mathcal{K}_\infty$ such that for any $\varepsilon > 0$,*

$$|x_0|_{\mathcal{A}} \le \delta(\varepsilon),\ t \ge 0 \quad \Rightarrow \quad |x(t, x_0)|_{\mathcal{A}} \le \varepsilon. \tag{38}$$

- Uniformly Globally Asymptotically Stable *(UGAS) if it is US and* Uniformly Attractive *(UA), that is, for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that*

$$|x_0|_{\mathcal{A}} \le r,\ t \ge T \quad \Rightarrow \quad |x(t, x_0)|_{\mathcal{A}} \le \varepsilon. \tag{39}$$

Some convenient classes of functions are next defined. These are instrumental in nonlinear control theory.

Definition 3

- *A function $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\alpha(0) = 0$ is positive semi-definite if $\alpha(s) \ge 0$ for $s > 0$ and positive definite if $\alpha(s) > 0$ for $s > 0$.*
- *A function $\alpha : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ with $\alpha(0) = 0$ belongs to class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, $\alpha(0) = 0$, and $\alpha(s_2) > \alpha(s_1)$, $\forall s_2 > s_1$. It belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.*
- *A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ belongs to class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if for each fixed $t \ge 0$, $\beta(\cdot, t) \in \mathcal{K}$, and for each fixed $s \ge 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.*

The above stability definitions using $\varepsilon - \delta$ bounds, is as shown by Lin et al. (1996) and Khalil (2002) equivalent to using class-$\mathcal{K}$ and -$\mathcal{KL}$ estimates.

Different mathematical tools can be applied to establish stability of a system of the form (35). By far the most important in this thesis is Lyapunov's direct method. Additionally, other tools like Barbalat's Lemma (Barbălat, 1959) and Matrosov's Theorem (Matrosov, 1962) will be used.

2.1 Set-stability for time-varying systems

Consider the time-varying system

$$\dot{z} = g(z, t) \tag{40}$$

where $z(t, t_0, z_0) \in \mathbb{R}^n$ is the solution, evolving from z_0 at time $t = t_0 \ge 0$. Within the framework of set-stability of noncompact sets it is possible to analyze such a time-varying system as if it is time-invariant. Using a letter p to represent the explicit time-variation t in (40), having dynamics $\dot{p} = 1$, $p(0) = t_0$ so that $p(t) = t + t_0$, $\forall t \ge 0$, and defining $x := \mathrm{col}(z, p)$, then we can define the equivalent and seemingly time-invariant system

$$\dot{x} = \begin{bmatrix} \dot{z} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} g(z, p) \\ 1 \end{bmatrix} =: f(x), \qquad x_0 = \begin{bmatrix} z_0 \\ t_0 \end{bmatrix}, \tag{41}$$

for which $x(t, x_0)$ is the solution, evolving from x_0 at time $t = 0$. According to Lin (1992, Lemma 5.1.1) it follows that $z(t, t_0, z_0)$ is a solution of (40) for $t \ge t_0 \ge 0$ if and only if $x(t, x_0) := \mathrm{col}\,(z(t + t_0, t_0, z_0), t + t_0)$ is a solution of (41) for $t \ge 0$.

If stability for the original system (40) is analyzed with respect to the origin $z = 0$, then for the new system this is analyzed as stability of the noncompact set

$$\mathcal{A} = \{(z, p) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : \ z = 0\},$$

for which the distance function is $|(z, p)|_{\mathcal{A}} = |z|$.

This 'trick' will frequently be applied in this thesis for analysis of closed-loop maneuvering systems. The advantage is to avoid analyzing stability of sets that in the state-space vary with time. One can instead apply the rather extensive theory developed for stability of noncompact sets; see e.g. Lin (1992); Lin et al. (1995); Sontag and Wang (1995a); Lin et al. (1996); Teel and Praly (2000); Teel (2002). A disadvantage is, according to Teel and Praly (2000), that it usually imposes stronger than necessary conditions on the time-dependence of the right-hand side of (40), e.g. continuity when only measurability is needed.

It should be noted, that in this thesis we will usually not go to the step of introducing the variable p but rather just use t as is with $\dot{t} = 1$ and $t(0) = 0$. One should therefore be careful to distinguish this explicit time variation from the implicit time, e.g. $t \mapsto x(t)$ where x is a state, in equations of this thesis (though they physically are the same).

Remark 1 *Since the explicit time-variation often only enters through the designed reference system, e.g. $t \mapsto x_d(t)$ or $t \mapsto \upsilon_s(\theta, t)$, we could in principle define a new state p, running inside the control computer with dynamics $\dot{p} = 1$, $p(0) = t_0$, in order to implement t in these functions. This would indeed make the closed-loop system time-invariant. There is, however, no quantitative difference in this as compared to just using time t, for $t \geq t_0$, directly.*

2.2 Systems with inputs

In some cases we consider systems with input

$$\dot{x} = f(x, u, t) \tag{42}$$

where $x(t) \in \mathbb{R}^n$, $\forall t \geq t_0 \geq 0$, is the state, and $u(\cdot)$ is a measurable, locally essentially bounded input function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. The space of such input functions is denoted $\mathcal{L}_\infty^m$ with the norm $||u_{[t_0,\infty)}|| := \operatorname{ess\,sup}\{u(t) : \ t \geq t_0\}$. For each initial time and state $x(t_0) = x_0 \in \mathbb{R}^n$ and each $u \in \mathcal{L}_\infty^m$, let $x(t, t_0, x_0, u)$ denote the solution of (42) at time t. More general stability concepts for (42) are given in terms of *input-to-state stability* (ISS); see Sontag and Wang (1995b); Lin (1992); Edwards et al. (2000):

Definition 4 *The system (42) is* input-to-state stable *if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for each input $u \in \mathcal{L}_\infty^m$ and each $x_0 \in \mathbb{R}^n$, it holds that*

$$|x(t, t_0, x_0, u)| \leq \beta\left(|x_0|, \ t - t_0\right) + \gamma\left(\left|\left|u_{[t_0,\infty)}\right|\right|\right) \tag{43}$$

for each $t \geq t_0 \geq 0$.

By causality the same definition holds if we replace $||u_{[t_0,\infty)}||$ with $||u_{[t_0,t]}||$.

ISS is a robust stability concept that guarantees bounded state for all bounded inputs and UGAS of the origin when the input vanishes. ISS also generalizes to set-stability, and a treatment is given in Appendix **??**.

An application of the set-stability and ISS tools is illustrated by the following example.

Example 1 ***Claim:*** *The noncompact set*

$$\mathcal{A} = \{(x, t) : \ x = x_d(t)\}$$

is UGAS with respect to the scalar system

$$\dot{x} = -\left(x^3 - x_d(t)^3\right) + \dot{x}_d(t) =: f(x, t)$$

where the desired state $x_d(t)$ is bounded and absolutely continuous, and $|\dot{x}_d(t)| \leq M$, a.a. $t \geq 0$.

Proof: *Forward completeness is established by the auxiliary function* $W := \frac{1}{2}x^2$ *having a derivative* $\dot{W} = -x^4 + x\delta(t) \leq -\varepsilon |x|^4, \forall |x| \geq \sqrt[3]{\frac{\delta_0}{1-\varepsilon}}$ *where* δ_0 *is a bound on* $\delta(t) := x_d(t)^3 + \dot{x}_d(t)$ *and* $\varepsilon \in (0,1)$. *This shows input-to-state stability (ISS) of the system with* δ *as input (see Appendix* **??***), and consequently that* $x(t)$ *and* $f(x(t),t)$ *are bounded for all* $t \geq 0$.
For the distance function we have that

$$|(x,t)|_{\mathcal{A}} = \inf_{(y,\tau)\in\mathcal{A}} \left\| \begin{bmatrix} x-y \\ t-\tau \end{bmatrix} \right\| = \inf_{\tau} \left\| \begin{bmatrix} x-x_d(\tau) \\ t-\tau \end{bmatrix} \right\| \leq |x - x_d(t)| \,.$$

The absolute continuity of $x_d(t)$ *together with boundedness of* $\dot{x}_d(t)$ *implies that* $x_d(t)$ *is globally Lipschitz such that* $|x_d(t) - x_d(\tau)| \leq M\,|t-\tau|$ *holds. Let* τ^* *be the (optimal) value that satisfies the above infimum. Then*

$$\begin{aligned} |x - x_d(t)| &= |x - x_d(\tau^*) + x_d(\tau^*) - x_d(t)| \\ &\leq |x - x_d(\tau^*)| + |x_d(\tau^*) - x_d(t)| \\ &\leq |x - x_d(\tau^*)| + M\,|t-\tau^*| \leq \max\{1,M\} \left\| \begin{bmatrix} x-x_d(\tau^*) \\ t-\tau^* \end{bmatrix} \right\|_1 \\ &\leq \sqrt{2}\max\{1,M\}\,|(x,t)|_{\mathcal{A}} \,. \end{aligned}$$

Defining $k := \sqrt{2}\max\{1,M\}$ *the result is the equivalence relation*

$$\frac{1}{k}\,|x - x_d(t)| \leq |(x,t)|_{\mathcal{A}} \leq |x - x_d(t)| \,.$$

Let a smooth Lyapunov function be $V(x,t) := \frac{1}{2}\left(x - x_d(t)\right)^2$. *This has the bounding functions, according to (*??*) and (*??*), defined as:*

$$\begin{aligned} \alpha_1\left(|(x,t)|_{\mathcal{A}}\right) := \frac{1}{2}\,|(x,t)|_{\mathcal{A}}^2 \leq V(x,t) \leq \frac{k^2}{2}\,|(x,t)|_{\mathcal{A}}^2 =: \alpha_2\left(|(x,t)|_{\mathcal{A}}\right) \\ V^x(x,t)f(x,t) + V^t(x,t) = -\left(x - x_d(t)\right)\left(x^3 - x_d(t)^3\right) =: -\alpha_3(|(x,t)|_{\mathcal{A}}). \end{aligned}$$

Recall the property $(x-y)(d(x)-d(y)) > 0,\ \forall x \neq y,$ *of a monotonically strictly increasing function* $d(x)$. *Using this with* $d(x) = x^3$ *shows that* α_3 *is a positive definite function, and* $\mathcal{A}$ *is therefore UGAS.*

3 Geometric relationships

3.1 Vectors and reference frames

A *vector* $\vec{x}$ is a quantity describing a magnitude and a direction. When not related to any reference frame[1], the vector is said to be *coordinate-free*. However, in this thesis a vector will be related to a Euclidean space $\mathbb{R}^n$ (sometimes called a Cartesian space) spanned by a set of orthogonal unit vectors $\{\vec{\epsilon}_1, \vec{\epsilon}_2, \ldots, \vec{\epsilon}_n\}$ so that

$$\vec{x} = x_1\vec{\epsilon}_1 + x_2\vec{\epsilon}_2 + \cdots + x_n\vec{\epsilon}_n \tag{44}$$

where $x_i = \vec{x}\cdot\vec{\epsilon}_i$, $i \in \{1,2,\ldots,n\}$, are the *Cartesian coordinates* of $\vec{x}$ in $\mathbb{R}^n$. The vector $\vec{x}$ can then be conveniently described as a *coordinate vector*

$$x = \mathrm{col}\left(x_1, x_2, \ldots, x_n\right),$$

for which we drop the 'arrow' notation. This gives the orthogonal coordinate vectors $\epsilon_1 = \mathrm{col}(1,0,0,\ldots,0)$, $\epsilon_2 = \mathrm{col}(0,1,0,\ldots,0)$, …, $\epsilon_n = \mathrm{col}(0,0,0,\ldots,1)$ representing the 'axes' of $\mathbb{R}^n$.

In rigid-body dynamics the 3-dimensional space $\mathbb{R}^3$ is of particular interest. The dynamic equations of motion involve *kinematics,* which is "the study of motion without reference to the forces which cause motion," and *kinetics* which is "the study of the relationships between the motion and the forces that

[1]The names 'coordinate frame' and 'reference frame' means the same thing and will be used interchangeably.

cause or accompany the motion" (Meriam and Kraige, 1993). Several Cartesian *reference frames* are important in this context. The absolute motion of a rigid body must be measured in a fixed coordinate frame $\mathcal{E}$, called an *inertial reference frame*, in $\mathbb{R}^3$ for Newton's laws of motion to apply. An inertial frame is not unique. Any fixed coordinate frame in $\mathbb{R}^3$ (our universe) can be used, and two such equivalent frames are related by a *translation* and *rotation*.

For terrestrial navigation the interesting reference frames are:

ECI: The Earth-Centered Inertial reference frame. This is approximately an inertial frame in which Newton's laws of motion apply. It is a nonrotating frame with origin at the center of mass of the Earth, z-axis along the Earth's spin axis and directed towards north, x-axis directed towards the vernal equinox, and y-axis directed to make out a right-hand triad.

ECEF: The Earth-Centered Earth-Fixed reference frame. This frame rotates with the Earth. Its origin and z-axis coincide with the ECI-frame, while the x-axis intersects the Greenwich meridian (0° longitude) and the y-axis is directed to make out a right-hand triad. For slow speed vehicles, navigating close to the Earth's surface, this frame is usually assumed inertial since the Earth's angular rate of rotation ($\omega_e = 7.2921 \cdot 10^{-5} rad/s$) is small.

NED: The North-East-Down reference frame. The origin of this frame is located at the surface of the Earth with coordinates determined by two angles (l, μ) denoting the *longitude* and *latitude*. Its x-axis is pointing towards true North, y-axis towards East, and z-axis pointing downwards and normal to the Earth's surface. For local navigation of vehicles, close to the surface, it is common to assume that this frame is inertial, and that the coordinates of the vehicle is given in the xy-plane (tangential plane) of the NED frame (flat Earth navigation). It will frequently be referred to as the $\mathcal{E}$-frame.

In addition, there are some particularly interesting local frames:

Body: A reference frame fixed to the body of the vehicle. For a marine vessel the origin of this frame is usually chosen in the *principal plane of symmetry* (The Society of Naval Architects and Marine Engineers, 1950) with x-axis – the longitudinal axis – directed from the aft to the bow, y-axis – the transverse axis – directed from port to starboard, and z-axis – the normal axis – directed from top to bottom. It will frequently be referred to as the $\mathcal{B}$-frame.

Path: A reference frame with origin at a point along a path. Its axes are given by the x-axis directed along the *unit tangent vector*, the y-axis directed along the *unit principal normal vector*, and the z-axis directed along the *unit binormal vector* (Lipschutz, 1969). For a continuously parametrized path it is often called the *Serret-Frenet frame*. It will be referred to as the $\mathcal{R}$-frame.

See Fossen (2002); Skjetne and Fossen (2001) for more details on these frames and the transformations between them.

3.2 Rotations

It is pertinent in control applications for robotics, vehicles, aerospace, marine systems, and navigation systems to represent a vector $\vec{x}$ with respect to several Cartesian frames. Let $\mathcal{A}$ and $\mathcal{B}$ be two such frames with orthogonal unit vectors $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ and $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$, respectively. Let $x_a \in \mathcal{A}$ and $x_b \in \mathcal{B}$ be[2] the corresponding coordinate vectors of $\vec{x}$. It can then be shown (Egeland and Gravdahl, 2002) that these are related as

$$x_a = R_b^a x_b, \quad R_b^a := \left\{ \vec{a}_i \cdot \vec{b}_j \right\} \tag{45}$$

where R_b^a is the rotation matrix from $\mathcal{B}$ to $\mathcal{A}$. Since $R_a^b := \left\{ \vec{b}_i \cdot \vec{a}_j \right\}$ we get that $x_b = R_a^b x_a = R_a^b R_b^a x_b$ which implies that

$$R_a^b R_b^a = I \quad \text{and} \quad R_b^a = \left(R_a^b\right)^{-1}. \tag{46}$$

[2]The notation $x \in \mathcal{A}$ means in this context that the elements of x are the coordinates of $\vec{x}$ along the axes of $\mathcal{A}$. In other words, x is decomposed in $\mathcal{A}$.

Two other important properties hold for a rotation matrix:

$$R^a_b = \left(R^b_a\right)^\top \quad (47)$$
$$\det R^a_b = 1. \quad (48)$$

showing that R_{ab} is orthogonal with a unitary determinant. Such matrices belongs to the set

$$\mathcal{SO}(3) := \left\{R \in \mathbb{R}^{3\times 3} : \ R^\top R = I \text{ and } \det R = 1\right\}$$

called the *Special Orthogonal group of order 3.* A matrix R is a rotation matrix iff $R \in \mathcal{SO}(3)$.

A rotation about a fixed axis is called a *simple rotation.* Euler's Theorem states that the relative orientation between two reference frames $\mathcal{A}$ and $\mathcal{B}$ can be produced by a simple rotation of $\mathcal{B}$ about some line in $\mathcal{A}$. Any simple rotation can again be produced by three rotations, called the *principal rotations*, about the axes of $\mathcal{A}$. For the nine elements in a rotation matrix, $R \in \mathbb{R}^{3\times 3}$, six constraints due to orthogonality implies that a minimum of three variables are necessary to parametrize it. These are conveniently given by the Euler angles: *roll* ϕ – rotation about the x-axis, *pitch* θ – rotation about the y-axis[3], and *yaw* ψ – rotation about the z-axis. The principal rotations of angles ϕ, θ, and ψ about the axes x, y, and z are then given by the rotation matrices

$$R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad (49)$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad (50)$$

$$R_{z,\psi} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (51)$$

Let $\Theta := \mathrm{col}(\phi, \theta, \psi)$ give the orientation of frame $\mathcal{B}$ with respect to frame $\mathcal{A}$. The corresponding rotation matrix from $\mathcal{B}$ to $\mathcal{A}$ becomes

$$\begin{aligned} R(\Theta)^a_b :=& R_{z,\psi}R_{y,\theta}R_{x,\phi} \quad (52) \\ =& \begin{bmatrix} c\theta c\psi & -c\phi s\psi + s\theta c\psi s\phi & s\phi s\psi + c\phi s\theta c\psi \\ c\theta s\psi & c\phi c\psi + s\theta s\phi s\psi & -c\psi s\phi + c\phi s\theta s\psi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \end{aligned}$$

where $c\cdot = \cos(\cdot)$ and $s\cdot = \sin(\cdot)$ and the commonly used zyx-convention have been applied.

The translational part of the motion of a rigid body is now given by the coordinate vector $p_e = \mathrm{col}(x, y, z)$ in the inertial frame $\mathcal{E}$. The rotational part is given by Θ which describes the orientation of the body-fixed frame $\mathcal{B}$ in $\mathcal{E}$. Let $v_b = \mathrm{col}(u, v, w)$ be the velocity vector of the rigid-body, decomposed in $\mathcal{B}$. This gives the kinematic relationship

$$\dot{p}_e = R(\Theta)^e_b v_b \quad (53)$$

for a rigid body. Likewise, let $\omega_{eb} = \mathrm{col}(p, q, r)$ be the angular velocity vector of $\mathcal{B}$ in $\mathcal{E}$. This can be related to the Euler rate vector $\dot{\Theta}$ as

$$\dot{\Theta} = T_\Theta(\Theta)\omega_b \quad (54)$$

where the transformation matrix $T_\Theta(\cdot)$ is

$$T_\Theta(\Theta) := \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}. \quad (55)$$

It is singular for $\theta = \pm 90^\circ$ which can pose a problem in some applications. This is resolved by using for instance *unit quaternions* instead of the Euler angles to parametrize R_{eb} and T_Θ. See Fossen (2002) for more details.

[3]Note that since θ is used as the path variable in this thesis, we will frequently use σ as the pitch angle in later chapters.

In some applications only the two-dimensional plane is important. In this case the coordinate vector of the origin of $\mathcal{B}$ in $\mathcal{E}$ is $p_e = \mathrm{col}(x, y)$, whereas the orientation of $\mathcal{B}$ in $\mathcal{E}$ is simply given by the yaw angle ψ. Setting $\phi = \theta = 0$ in (52) and (55) and eliminating the z-dimension gives the rotation matrix from $\mathcal{B}$ to $\mathcal{E}$ in $\mathbb{R}^2$ as

$$R_2(\psi)_b^e := \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}. \tag{56}$$

Letting $v_b = \mathrm{col}(u, v)$ be the body-fixed velocity vector we get $\dot{p}_e = R_2(\psi)_b^e v_b$ and $\dot{\psi} = r$. These two equations are commonly put together by defining $\eta := \mathrm{col}(p_e, \psi)$, $\nu := \mathrm{col}(v_b, r)$, and $R(\psi) := \mathrm{diag}(R_2(\psi)_b^e, 1)$ to get the kinematic relationship

$$\dot{\eta} = R(\psi)\nu, \tag{57}$$

commonly used in 3 degrees-of-freedom control applications for vehicles.

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