TMR4243 - Marine Control Systems II

Homework assignment 1

1 System: Surface ship

The nonlinear surge speed equation of a surface ship can be written[1]

$$M_u \dot{u} + h(u) = \tau_u \tag{1}$$

where τ_u is the surge force control input, $M_{\dot{u}} = m - X_{\dot{u}} > 0$, and $h(u) = -X_u u - X_{|u|u} |u|$ u is monotonically increasing, $h(0) = 0$, and $h(u) u > 0$; $\forall u \neq 0$. Let u_{ref} be a constant reference speed and choose the feedforward control law

$$\tau_u = h(u_{\text{ref}}). \tag{2}$$

1.1 Task: Function properties

Define

$$g(u) := h(u) - h(u_{\text{ref}}). \tag{3}$$

1. Show that $g(u_{\text{ref}}) = 0$.
2. Show that $g(u)(u - u_{\text{ref}}) > 0, \forall u \neq u_{\text{ref}}$.

1.2 Task: Lyapunov analysis

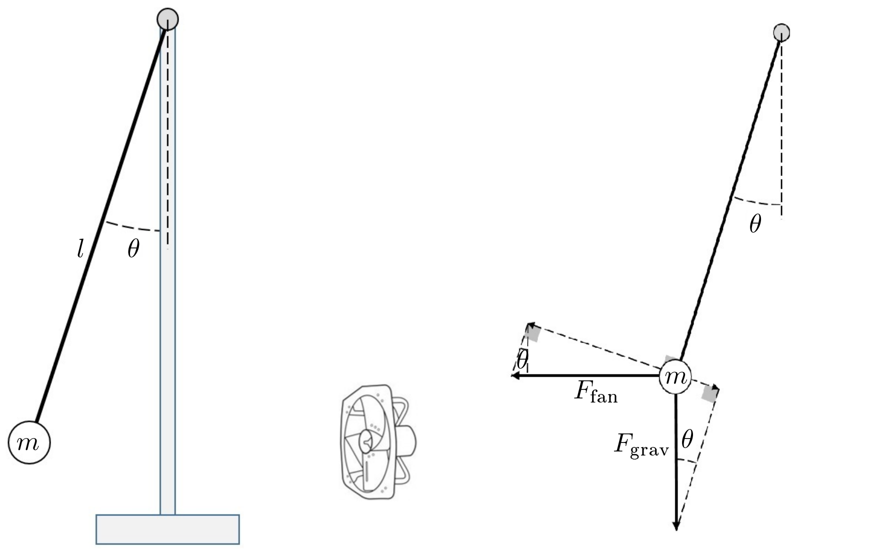
Show by using the Lyapunov function

$$V(u) = \frac{M_{\dot{u}}}{2}(u - u_{\text{ref}})^2 + M_{\dot{u}} \int_{u_{\text{ref}}}^{u} g(y)\, dy \tag{4}$$

that its time derivative along the solutions of the closed-loop system is given by

$$\dot{V}(u) = -(u - u_{\text{ref}}) g(u) - g(u)^2,$$

and that the equilibrium $u - u_{\text{ref}} = 0$ is GAS.



(a) System illustration

(b) Mass free body diagram

Figure 1: Pendulum

m	0.1 **kg**	mass of the bob
l	0.5 **m**	length of the rod attached to the bob
g	$9.81 m/s^2$	
k	0.01	friction coefficient
θ	**rad**	angle between the vertical stand and the bob rod
ω	**rad/s**	angular velocity

Table 1: Parameters and variables

2 System: Pendulum

The system at hand consists of a stand with a bearing. A rod is attached to the outer bearing race. A bob is attached to the other end of the rod. See Figure 1a.

Table 1 summarizes the parameters and variables of the installation.

The control plant model is based on that the angular acceleration $\dot{\omega}$ of the rotating part is proportional to the sum of torques:

$$\dot{\omega} = \frac{1}{J} \sum \tau, \tag{5}$$

where J is the system inertia.

[1] Fossen, Handbook of Marine Craft Hydrodynamics and Motion Control, Eq. 7.32

Considering the bob as a point mass, the inertia is

$$J = ml^2. \tag{6}$$

For simplicity, this is chosen as the inertia of the whole system, thus disregarding the mass of the rod.

Figure 1b holds a diagram of the forces acting on the system. Since the forces parallel to the rod are counteracted by the latter, only the tangent components are of interest here:

- The torque τ_{grav} due to gravity is

$$\begin{aligned} \tau_{\text{grav}} &= -lF_{\text{grav}} \sin(\theta) \\ &= -lmg \sin(\theta). \end{aligned} \tag{7}$$

- The torque τ_{fan} due to the fan pressure on the rod is

$$\begin{aligned} \tau_{\text{fan}} &= lF_{\text{fan}} \sin\left(\frac{\pi}{4} - \theta\right), \\ &= lF_{\text{fan}} \cos(\theta) \end{aligned} \tag{8}$$

  where F_{fan} is the force from the fan.

Additionally, the friction in the bearing is modeled by a torque τ_{fric} proportional to the velocity:

$$\tau_{\text{fric}} = -k\omega. \tag{9}$$

Inserting (7)-(9) and substituting (6) in (5) yields

$$\begin{aligned} \dot{\omega} &= \frac{1}{ml^2}\left(-lmg \sin(\theta) - k\omega + lF_{\text{fan}} \cos(\theta)\right) \\ &= -\frac{g}{l} \sin(\theta) - \frac{k}{ml^2}\omega + \frac{F_{\text{fan}}}{ml} \cos(\theta) \end{aligned} \tag{10}$$

2.1 Task: State equations

1. Write the state equation $\dot{x} = f(x, u)$ using the state vector $x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$ and $u = F_{\text{fan}}$.
2. Program a corresponding Simulink model.

2.2 Task: System properties

Assuming $|F_{\text{fan}}|$ is bounded, explain why the system is or isn't:

1. Forward complete.

2. Backward complete.
3. Complete
4. Locally Lipschitz.
5. Globally Lipschitz.

2.3 Task: Simple pendulum equilibrium point

Assume the fan is off, i.e. $F_{\text{fan}} = 0$ N.

1. Show that $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point of the unforced system.
2. Explain why x^* is or isn't
   (a) a unique equilibrium point,
   (b) an isolated equilibrium point.
3. Describe the physical situation(s) the equilibrium point(s) correspond(s) to.
4. Simulate the system with initial condition $x(0) = x^*$ to confirm the behavior at the equilibrium point.

2.4 Task: Linearized simple pendulum model

The linearized state equations are

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{ml^2} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}.$$

1. Show that x^* is also an equilibrium point for the linearized system.
2. Explain why x^* is or isn't:
   (a) Locally stable.
   (b) Globally stable.

2.5 Task: Equilibrium point with fan

Assume that the fan is again running, with $F_{\text{fan}} = 0.56638$ N.

1. Calculate the angle at which the bob now stabilizes.
2. Confirm through simulation.

2.6 Task: Angle control

In order to set the fan to stabilize the bob at $\theta = 60°$,

1. choose a change of variables such that the equilibrium is shifted to this angle,
2. write the state equations using the new states, and
3. determine F_{fan} necessary to the new equilibrium.
4. Confirm through simulation.

3 Lyapunov function

Consider the differential equations

$$\dot{x}_1 = u_1 \tag{11a}$$

$$\dot{x}_2 = u_2, \tag{11b}$$

and a function

$$V(x_1, x_2) = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2, \tag{12}$$

$$V = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{13}$$

where c_1, c_2, c_3 are positive scalars.

3.1 Task: Differentiation

Do the following:

1. Differentiate $V(x_1, x_2)$ with respect to time (we say, "along the solutions of (11)") and set up the resulting expression in terms of the states (x_1, x_2) and the inputs (u_1, u_2).
2. Let $x := \text{col}(x_1, x_2)$ and $u := \text{col}(u_1, u_2)$. Show that V can be written as $V(x) = x^\top P x$ where $P = P^\top$ (symmetric).
3. Give conditions on (c_1, c_2, c_3) for P to be a positive definite matrix ($P = P^\top > 0$).
4. Show that taking the vector differentiation of $V(x)$ gives $\dot{V} = 2x^\top P \dot{x}$ and that this equals the answer in the Subtask 1.

5. Let $u_1 = -x_1 + x_2$ and $u_2 = -x_2$ such that the closed-loop system becomes

$$\dot{x} = Ax \tag{14}$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}. \tag{15}$$

   Choose values for (c_1, c_2, c_3) such that $P = P^\top > 0$ and $PA + A^\top P = -Q$ where $Q > 0$ is a diagonal positive matrix.

6. Differentiate again $V(x)$ along the soolutions of (14) and show that $\dot{V} = -x^\top Qx$ for your chosen values of (c_1, c_2, c_3).

References