TMR4243 - Marine Control Systems II

Homework assignment 6 - Observer design

1 Observability

1.1 Observability and detectability

1. For the following system, check if the system is *observable* and/or *detectable*.
   Dynamics:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
   Outputs:
   (a) $y = x_1$
   (b) $y = x_2$

2. For the above system, calculate the observability Gramian matrix Q_∞ for $t = \infty$, calculate its rank, and conclude again on observability for the two cases:
   (a) $y = x_1$
   (b) $y = x_2$

3. For the above system with $y = x_1$, design a Luenberger observer, and:
   (a) Show how to use the `place.m` command to design the injection gain L of the observer, that places the closed-loop observer poles at $p_{obs} = \{-10, -15\}$.
   (b) Show how to use the `place.m` command to design a state feedback control gain K, that renders $x = 0$ UGES and places the closed-loop control feedback poles at $p_{ctrl} = \{-1, -3\}$.
   (c) Show that the *separation principle* holds.

4. For the following system, check if the system is *observable* and/or *detectable*.
   Dynamics:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
   Outputs:
   (a) $y = x_1$
   (b) $y = x_2$
   (c) $y = x_3$
   (d) $y = x_1 + x_2$
   (e) $y_1 = x_1$, $y_2 = x_3$

5. For the above system with $y = x_1 + x_2$, design a Luenberger observer, and:
   (a) Show how to use the `place.m` command to design the injection gain L of the observer, that places the closed-loop observer poles at $p_{obs} = \{-10, -15, -12\}$.

(b) Show how to use the `place.m` command to design a state feedback control gain K, that renders $x = 0$ UGES and places the control feedback poles so that the closed-loop characteristic polynomial becomes $(s + 2)\left(s^2 + 2\xi\omega s + \omega^2\right)$ with damping $\xi = 0.7$ and natural frequency $\omega = 2$.

(c) Show that the *separation principle* holds.

6. For the time-varying system:

$$\begin{aligned}\dot{x} &= A(t)x + u \\ y &= C(t)x\end{aligned}$$

with

$$C(t) := \begin{bmatrix} \cos t & \sin 2t \end{bmatrix}, \qquad A(t) := w(t)C(t)^\top C(t),$$

show that the pair $[A(t), C(t)]$ is UCO.

7. Consider a vessel that has a GPS antenna and receiver. Considering only the horizontal positions, we assume we know the lever arm $l_1 \in \mathbb{R}^2$ from the vessel origin (VO) to GPS antenna 1. Then the vessel owner decides to buy an additional GPS for redundancy reasons, and installs the new antenna at a new location. But we do not know exactly the lever arm $l_2 \in \mathbb{R}^2$ for this new GPS 2 antenna. We assume we have a good measurement of the heading $\psi(t)$ of the vessel from the gyrocompass.

(a) Show that the two position measurements from the two GPS, given in terms of the VO position $p(t)$ in NED and the respective lever arms l_1 and l_2 in the body-frame, can be expressed by:

$$\begin{aligned}p_1(t) &= p(t) + R(\psi(t))l_1 \\ p_2(t) &= p(t) + R(\psi(t))l_2,\end{aligned}$$

where $R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$.

(b) Show that $R \in SO(2)$, that is, R is a 2×2 rotation matrix.

(c) Let the state vector be $x := l_2 \in \mathbb{R}^2$ and output $y(t) := p_2(t) - p_1(t) + R(\psi(t))l_1$, and set up the resulting time-varying linear state-space model.

(d) Show that this system is UCO.

(e) Propose an observer that based on real-time measurements of $p_1(t)$, $p_2(t)$, and $\psi(t)$ online estimates the lever arm l_2.

8. Suppose in MC-Lab we will do a towing test with C/S Enterprise I in pure surge direction, to identify the drag in surge. To this end we have connected the model ship to the towing carriage with a tension sensor at the towing line. We are then able to tow the model ship at various speeds and measure the reulting forces.
The drag force is given by the model

$$\tau_u = X_{|u|u}\,|u|\,u + X_u u$$

where u is the surge velocity, $X_{|u|u}$ and X_u are the nonlinear and linear drag coefficients, and τ_u is the measured towing force. We now consider an observer design to estimate the drag parameters, and we parameterize the system as

$$\begin{aligned}\dot{x} &= 0, & x &= \begin{bmatrix} X_{|u|u} & X_u \end{bmatrix}^\top \\ y &= C(t)x, & y &= \tau_u, \quad C(t) = \begin{bmatrix} |u(t)|\,u(t) & u(t) \end{bmatrix}.\end{aligned}$$

The surge speed $u(t)$ is given by the measured towing speed of the carriage. The above system is then a time-varying linear system.

(a) Suppose in the interval $[t_0, t_1]$ we tow the model at a constant speed $u(t) = u_0$. Show that the LTV observability Gramian matrix (see Definition UCO in lecture)

$$Q_{[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(\tau,t)^\top C(\tau)^\top C(\tau)\Phi(\tau,t)d\tau$$

is rank deficient in this case.

(b) Suppose in the interval $[t_1, t_2]$ we stepped up the speed to $u(t) = u_1 > u_0$. Calculate again the observability Gramian matrix $Q_{[t_0,t_2]}$ over to total interval $[t_0, t_2]$, by splitting the integral between the two subintervals $[t_0, t_1]$ and $[t_1, t_2]$. Is the Gramian $Q_{[t_0,t_2]}$ still rank deficient?

(c) If it still is rank deficient, repeat again for a third speed $u(t) = u_2 > u_1$ for the period $[t_2, t_3]$. Eventually you will find that the rank is built up over time with sufficient variation in speed u.

(d) Propose an observer that can be implemented to online estimate the drag coefficients while running the experiment.

1.2 Nonlinear observer designs

1.2.1 Slow-speed surface vessel

Consider the low-speed vessel model

$$\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ \dot{b} &= 0 \\ M\dot{\nu} + D\nu &= \tau + b \end{aligned}$$

where $\eta = col(x, y, \psi)$, $\nu = col(u, v, r)$, $D = D^\top > 0$ is the damping matrix, $M = M^\top > 0$ is the mass matrix, $b \in \mathbb{R}^3$ is a bias load vector, τ is the thruster loads, and

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with properties

$$\begin{aligned} R(\psi)^\top R(\psi) &= R(\psi)R(\psi)^\top = I \\ \det(R(\psi)) &= 1 \\ R(\psi)^{-1} &= R(\psi)^\top \\ \dot{R} &= R(\psi)S(r), \qquad S(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad r = \dot{\psi}. \end{aligned}$$

Let $\eta_b := R^\top\eta$, $x := col(\eta_b, b, \nu)$, and $u := \tau$, and assume we have the measurements $y_1 = \eta_b$ and $y_2 = r$.

1. Assume D is a constant matrix.

   (a) Show that the slow speed dynamics can be written

   $$\begin{aligned} \dot{x} &= Ax + \rho(u, y_1, y_2) \\ y_1 &= C_1 x \\ y_2 &= C_2 x. \end{aligned}$$

   (b) Show that the pair (A, C_1) is observable.

(c) Propose a nonlinear observer with state estimate $\hat{x}$ that estimates the state vector x. Show that the estimation error $\tilde{x} := x - \hat{x}$ is rendered UGES.

2. Assume now that $D = D(\nu)$ is indeed a nonlinear damping matrix, and let $d(\nu) := D(\nu)\nu$ be the corresponding monotonically nondecreasing damping load as function of ν.

   (a) Show that the system then can be written on the form

$$\dot{x} = Ax + G\gamma(Hx) + \rho(u, y_1, y_2), \qquad y_1 = C_1 x$$

   (b) Propose a new nonlinear observer for this plant, and give design conditions on the injection gains for the observer error dynamics to be GES.

1.2.2 Example in lecture

In the lecture we looked at the example system

$$\begin{aligned} \dot{x}_1 &= x_2 - k \sin x_1 \\ \dot{x}_2 &= -k \cos x_2 + u \\ y &= x_1, \end{aligned}$$

and we showed that for $k = 0.1$ we could do the "Observer with global Lipschitz condition" design to render the observer error $\tilde{x} = x - \hat{x} = 0$ GES.

Redo this example and see if you can design the injection gain L that renders $\tilde{x} = 0$ GES for the cases $k = 0.175$ and $k = 0.25$.

1.2.3 DP observer with bias

Work through the proof of the theorem on the "DP observer with bias" at the end of the lecture presentation; see (Værnø and Skjetne, 2017).

1 Solution: Observability

1.1 Observability and detectability

1. For the following system, check if the system is *observable* and/or *detectable*.
   Dynamics:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad A = \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   Outputs:

   (a) $y = x_1$. We get

$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{1}$$

$$O_1 = \begin{bmatrix} C_1 \\ C_1 A \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \tag{2}$$

   which clearly is full rank. Hence, the pair (A, C_1) is observable.

   (b) $y = x_2$. We get

$$C_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{3}$$

$$O_2 = \begin{bmatrix} C_2 \\ C_2 A \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} \tag{4}$$

   which clearly is rank deficient ($\text{rank}(O_2) = 1 < 2$). Hence, the pair (A, C_2) is not observable. However, we note that the eigenvalues of the A matrix is $\{-2, -5\}$ where the observable mode is x_2, and the unobservable mode x_1 is stable with eigenvalue -2. Hence, the system is detectable.

2. For the above system, calculate the observability Gramian matrix Q_∞ for $t = \infty$, calculate its rank, and conclude again on observability for the two cases:

   (a) $y = x_1$. Since the A matrix is Hurwitz, the observability Gramian is given by the Lyapunov equation

$$A^\top Q_1 + Q_1 A = -C_1^\top C_1. \tag{5}$$

   Using `lyap.m` in Matlab, we get

$$Q_1 = \begin{bmatrix} 0.2500 & 0.0357 \\ 0.0357 & 0.0071 \end{bmatrix} \tag{6}$$

   We find that $\text{rank}(Q_1) = 2$, and hence the (A, C_1) is observable.

   (b) $y = x_2$. We calculate the Lyapunov equation

$$A^\top Q_2 + Q_2 A = -C_2^\top C_2 \tag{7}$$

   by using `lyap.m` in Matlab, and get

$$Q_2 = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.1 \end{bmatrix} \tag{8}$$

   We find that $\text{rank}(Q_2) = 1$, and hence the (A, C_2) is not observable.

3. For the above system with $y = x_1$, a Luenberger observer is:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 1 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= \hat{x}_1 \end{aligned}$$

   or

$$\begin{aligned} \dot{\hat{x}} &= (A - LC)\,\hat{x} + Bu + Ly \\ \hat{y} &= C_1 \hat{x} \end{aligned}$$

(a) The Matlab command `place.m` is made to design a feedback gain K to place the poles of $A - BK$ at desited locations. To design L to place the observer poles of $A - LC_1$ at $p_{obs} = \{-10, -15\}$, we note that the transpose gives $A^\top - C_1^\top L^\top$, and thus we call `place.m` with $A^\top$ and $C_1^\top$ to calculate $L^\top$. This gives

$$L = \begin{bmatrix} 18 \\ 50 \end{bmatrix}.$$

(b) We then call the `place.m` command with A, B, and $p_{ctrl} = \{-1, -3\}$ to get the gain

$$K = \begin{bmatrix} -1 & -3 \end{bmatrix}.$$

(c) Letting $\tilde{x} = x - \hat{x}$, we get the closed-loop system

$$\begin{aligned} \dot{\tilde{x}} &= (A - LC_1)\,\tilde{x} \\ \dot{x} &= Ax - BK\hat{x} + BKx - BKx = (A - BK)\,x + BK\tilde{x} \end{aligned}$$

which is a cascade of two closed-loop subsystems, or

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A - LC_1 & 0 \\ BK & A - BK \end{bmatrix} \begin{bmatrix} \tilde{x} \\ x \end{bmatrix}.$$

Hence, since the overall closed-loop system is block-diagonal, the eigenvalues is given by the matrices on the diagonal. These eigenvalues are then given by $A - LC_1$ and $A - BK$, which can be designed separately without affecting stability of the other subsystem.

4. For the following system, check if the system is *observable* and/or *detectable*.
Dynamics:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \\ A &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \qquad A^2 = \begin{bmatrix} 1 & -2 & -4 \\ 0 & 2 & -5 \\ 0 & 10 & 7 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Outputs:

(a) $y = x_1$. We get

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{9}$$

$$O_1 = \begin{bmatrix} C_1 \\ C_1A \\ C_1A^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & -2 & -4 \end{bmatrix} \tag{10}$$

which is full rank. Hence, the pair (A, C_1) is observable.

(b) $y = x_2$. We get

$$C_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{11}$$

$$O_2 = \begin{bmatrix} C_2 \\ C_2A \\ C_2A^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -5 \end{bmatrix} \tag{12}$$

which clearly is rank deficient (has a zero column). Hence, the pair (A, C_2) is not observable.

(c) $y = x_3$. We get

$$C_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{13}$$

$$O_3 = \begin{bmatrix} C_3 \\ C_3A \\ C_3A^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 10 & 7 \end{bmatrix} \tag{14}$$

which clearly is rank deficient (has a zero column). Hence, the pair (A, C_3) is not observable.

(d) $y = x_1 + x_2$. We get

$$C_4 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \tag{15}$$

$$O_4 = \begin{bmatrix} C_4 \\ C_4 A \\ C_4 A^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -2 & 2 \\ 1 & 0 & -9 \end{bmatrix} \tag{16}$$

which has $\text{rank}(O_4) = 3$. Hence, the pair (A, C_4) is observable.

(e) $y_1 = x_1$, $y_2 = x_3$. We get

$$C_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$O_5 = \begin{bmatrix} C_5 \\ C_5 A \\ C_5 A^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & -2 & -3 \\ 1 & -2 & -4 \\ 0 & 10 & 7 \end{bmatrix} \tag{18}$$

which has $\text{rank}(O_5) = 3$. Hence, the pair (A, C_5) is observable.

5. For the above system with $y = x_1 + x_2$, a Luenberger observer is:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} &= \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} (y - \hat{y}) \\ \hat{y} &= C_4 \hat{x} \end{aligned}$$

or

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \qquad \hat{y} = C_4 \hat{x}$$

(a) To design L to place the observer poles of $A - LC_4$ at $p_{obs} = \{-10, -15, -12\}$, we note that the transpose gives $A^\top - C_4^\top L^\top$, and thus we call `place.m` with $A^\top$ and $C_4^\top$ to calculate $L^\top$. This gives

$$L = \begin{bmatrix} 234.8182 \\ -203.8182 \\ 39.0909 \end{bmatrix}.$$

(b) The characteristic polynomial $(s+2)\left(s^2 + 2\xi\omega s + \omega^2\right)$ with damping $\xi = 0.7$ and natural frequency $\omega = 2$ gives poles at $p_{ctrl} = \{-2, -1.4 + j1.4283, -1.4 - j1.4283\}$. We then call the `place.m` command with A, B, and p_{ctrl} to get the gain

$$K = \begin{bmatrix} 2.2 & -2.0 & -1.2 \end{bmatrix}.$$

(c) Letting $\tilde{x} = x - \hat{x}$, we get the closed-loop system

$$\begin{aligned} \dot{\tilde{x}} &= (A - LC_4)\,\tilde{x} \\ \dot{x} &= (A - BK)\,x + BK\tilde{x} \end{aligned}$$

which is a cascade of two closed-loop subsystems, or

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} A - LC_4 & 0 \\ BK & A - BK \end{bmatrix} \begin{bmatrix} \tilde{x} \\ x \end{bmatrix}.$$

Hence, since the overall closed-loop system is block-diagonal, the eigenvalues is given by the matrices on the diagonal. These eigenvalues are then given by $A - LC_4$ and $A - BK$, which can be designed separately without affecting stability of the other subsystem.

6. For the time-varying system:

$$\begin{aligned}\dot{x} &= A(t)x + u\\ y &= C(t)x\end{aligned}$$

with

$$C(t) := \begin{bmatrix} \cos t & \sin 2t \end{bmatrix}, \qquad A(t) := w(t)C(t)^\top C(t),$$

show that the pair $[A(t), C(t)]$ is UCO.
Answer: We invoke the Lemma stating that UCO is invariant under output feedback. This means that we can apply an output feedback control $u = -K(t)y = -K(t)C(t)x$, and then the pair $[A(t), C(t)]$ is UCO iff $[A(t) - K(t)C(t), C(t)]$ is UCO. The trick with the observability Gramian for time-varying systems is to make the transition matrix Φ constant. Since $A(t) = w(t)C(t)^\top C(t)$ we can achieve this by

$$\begin{aligned} A(t) - K(t)C(t) &= w(t)C(t)^\top C(t) - K(t)C(t)\\ &\Downarrow\\ K(t) &= w(t)C(t)^\top\\ &\Downarrow\\ A(t) - K(t)C(t) &= 0\end{aligned}$$

Hence, the pair $[A(t), C(t)]$ is UCO iff $[0, C(t)]$ is UCO, that is, the system

$$\begin{aligned}\dot{x} &= 0\\ y &= C(t)x,\end{aligned}$$

for which the transiton matrix becomes $\Phi = I$. We then get the observability Gramian

$$\begin{aligned} Q(t_0, t_1) &= \int_{t_0}^{t_1} C(\tau)^\top C(\tau) d\tau = \int_{t_0}^{t_1} \begin{bmatrix} \cos^2\tau & \cos\tau\sin 2\tau\\ \cos\tau\sin 2\tau & \sin^2 2\tau \end{bmatrix} d\tau\\ &= \begin{bmatrix} \frac{1}{2}\tau + \frac{1}{4}\sin 2\tau & -\frac{1}{2}\cos\tau - \frac{1}{6}\cos 3\tau\\ -\frac{1}{2}\cos\tau - \frac{1}{6}\cos 3\tau & \frac{1}{2}\tau - \frac{1}{8}\sin 4\tau \end{bmatrix}\Bigg|_{t_0}^{t_1}\\ &= \begin{bmatrix} \frac{1}{2}(t_1 - t_0) + \frac{1}{4}(\sin 2t_1 - \sin 2t_0) & -\frac{1}{2}(\cos t_1 - \cos t_0) - \frac{1}{6}(\cos 3t_1 - \cos 3t_0)\\ -\frac{1}{2}(\cos t_1 - \cos t_0) - \frac{1}{6}(\cos 3t_1 - \cos 3t_0) & \frac{1}{2}(t_1 - t_0) - \frac{1}{8}(\sin 4t_1 - \sin 4t_0) \end{bmatrix}\end{aligned}$$

Letting, for instance, $t_0 = 2k\pi$ and $t_1 = 2k\pi + \frac{\pi}{2}$, for $k = 0, \pm 1, \pm 2, \ldots$ gives

$$Q(0, \frac{\pi}{2}) = \begin{bmatrix} \frac{\pi}{4} & \frac{4}{6}\\ \frac{4}{6} & \frac{\pi}{4}\end{bmatrix},$$

which has the determinant $\det\left(Q(0, \frac{\pi}{2})\right) = \frac{\pi^2}{16} - \frac{4}{9} \approx 0.1724$. Hence, even if the matrix $C(t)^\top C(t)$ is rank deficient for each t, we find that full rank is rapidly built up over time.

7. Consider a vessel that has a GPS antenna and receiver. Considering only the horizontal positions, we assume we know the lever arm $l_1 \in \mathbb{R}^2$ from the vessel origin (VO) to GPS antenna 1. Then the vessel owner decides to buy an additional GPS for redundancy reasons, and installs the new antenna at a new location. But we do not know exactly the lever arm $l_2 \in \mathbb{R}^2$ for this new GPS 2 antenna. We assume we have a good measurement of the heading $\psi(t)$ of the vessel from the gyrocompass.

   (a) For the VO position $p(t)$ and antenna position $p_1(t)$ in NED, the vector $p_{l_1}(t) = p_1(t) - p(t)$ is the arm from the antenna to the vessel origin VO in NED. Rotating to Body-frame, we get

   $$R(\psi(t))^\top p_{l_1}(t) = l_1,$$

   that is, the constant lever arm from VO to Antenna 1 in Body, where $R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi\\ \sin\psi & \cos\psi \end{bmatrix}$

is the rotation matrix from Body to NED. Hence, we get

$$\begin{aligned} R(\psi(t))R(\psi(t))^\top p_{l_1}(t) &= R(\psi(t))l_1 \\ &\Downarrow \\ p_{l_1}(t) &= p_1(t) - p(t) = R(\psi(t))l_1 \\ &\Downarrow \\ p_1(t) &= p(t) + R(\psi(t))l_1 \end{aligned}$$

and likewise for Antenna 2

$$p_2(t) = p(t) + R(\psi(t))l_2.$$

(b) Show that $R \in SO(2)$, that is, R is a 2×2 rotation matrix.
Answer: We have to show that $\det(R) = 1$ and $R^{-1} = R^\top$. We get

$$\begin{aligned} \det R &= \cos^2\psi + \sin^2\psi = 1, \qquad \text{OK} \\ R^\top R &= \begin{bmatrix} \cos^2\psi + \sin^2\psi & 0 \\ 0 & \cos^2\psi + \sin^2\psi \end{bmatrix} = I \\ RR^\top &= \begin{bmatrix} \cos^2\psi + \sin^2\psi & 0 \\ 0 & \cos^2\psi + \sin^2\psi \end{bmatrix} = I \\ &\Downarrow \\ R^\top &= R^{-1} \end{aligned}$$

(c) With the state vector be $x := l_2 \in \mathbb{R}^2$ and output $y(t) := p_2(t) - p_1(t) + R(\psi(t))l_1$, we get

$$\begin{aligned} \dot{x} &= 0 \\ y &= p_2(t) - p_1(t) + R(\psi(t))l_1 \\ &= p(t) + R(\psi(t))l_2 - p(t) - R(\psi(t))l_1 + R(\psi(t))l_1 \\ &= R(\psi(t))x. \end{aligned}$$

Letting $A = 0$ and $C(t) = R(\psi(t))$ such that

$$\dot{x} = Ax, \qquad y = C(t)x.$$

(d) To show that this system is UCO, we notice that the transition matrix is $\Phi = I$, and thus the Gramian becomes

$$Q(t_0, t_1) = \int_{t_0}^{t_1} C(\tau)^\top C(\tau) d\tau = \int_{t_0}^{t_1} R(\psi(t))^\top R(\psi(t)) d\tau = \int_{t_0}^{t_1} I d\tau = I\,(t_1 - t_0)\,.$$

According to the definition of UCO for time-varying systems, it is clear that this is satisfied, and the system is UCO.

(e) As an observer based on real-time measurements of $p_1(t)$, $p_2(t)$, and $\psi(t)$, and $A = 0$, we could use a Kalman-Bucy filter

$$\begin{aligned} \dot{\hat{x}} &= -L(t)R(\psi(t))\hat{x} + L(t)y \\ \dot{P} &= -L(t)\mathcal{R}L(t)^\top + \varepsilon I \\ L(t) &= PR(\psi(t))^\top \mathcal{R}^{-1} \end{aligned}$$

where $\varepsilon > 0$ is a small parameter accounting for small uncertainty (no process noise) in the $\dot{x}$ dynamics, and $\mathcal{R}$ is the covariance of the measurement noise in y.

8. Suppose in MC-Lab we will do a towing test with C/S Enterprise I in pure surge direction, to identify the drag in surge. To this end we have connected the model ship to the towing carriage with a tension sensor at the towing line. We are then able to tow the model ship at various speeds

and measure the reulting forces.
The drag force is given by the model

$$\tau_u = X_{|u|u} \, |u| \, u + X_u u$$

where u is the surge velocity, $X_{|u|u}$ and X_u are the nonlinear and linear drag coefficients, and τ_u is the measured towing force. We now consider an observer design to estimate the drag parameters, and we parameterize the system as

$$\begin{aligned} \dot{x} &= 0, & x &= \left[\begin{array}{cc} X_{|u|u} & X_u \end{array} \right]^\top \\ y &= C(t)x, & y &= \tau_u, \quad C(t) = \left[\begin{array}{cc} |u(t)| \, u(t) & u(t) \end{array} \right]. \end{aligned}$$

The surge speed $u(t)$ is given by the measured towing speed of the carriage. The above system is then a time-varying linear system.

(a) Suppose in the interval $[t_0, t_1]$ we tow the model at a constant speed $u(t) = u_0$. Show that the LTV observability Gramian matrix (see Definition UCO in lecture)

$$Q_{[t_0,t_1]} = \int_{t_0}^{t_1} \Phi(\tau, t)^\top C(\tau)^\top C(\tau) \Phi(\tau, t) d\tau$$

is rank deficient in this case. Explain why.
Answer: We get $\Phi(\tau, t) = I$ and

$$\begin{aligned} Q_{[t_0,t_1]} &= \int_{t_0}^{t_1} C(\tau)^\top C(\tau) d\tau = \int_{t_0}^{t_1} \left[\begin{array}{c} u_0 \, |u_0| \\ u_0 \end{array} \right] \left[\begin{array}{cc} |u_0| \, u_0 & u_0 \end{array} \right] d\tau \\ &= \int_{t_0}^{t_1} \left[\begin{array}{cc} u_0^2 \, |u_0|^2 & u_0^2 \, |u_0| \\ u_0^2 \, |u_0| & u_0^2 \end{array} \right] d\tau \\ &= \left[\begin{array}{cc} u_0^2 \, |u_0|^2 & u_0^2 \, |u_0| \\ u_0^2 \, |u_0| & u_0^2 \end{array} \right] (t_1 - t_0), \end{aligned}$$

for which we get $\det Q_{[t_0,t_1]} = 0$. Hence, this is rank deficient. The reason is that one specific speed is not enough to identify the nonlinear drag curve and its two parameters.

(b) Suppose in the interval $[t_1, t_2]$ we stepped up the speed to $u(t) = u_1 > u_0$. Calculate again the observability Gramian matrix $Q_{[t_0,t_2]}$ over to total interval $[t_0, t_2]$, by splitting the integral between the two subintervals $[t_0, t_1]$ and $[t_1, t_2]$. Is the Gramian $Q_{[t_0,t_2]}$ still rank deficient?
Answer: We get

$$\begin{aligned} Q_{[t_0,t_2]} &= \int_{t_0}^{t_2} C(\tau)^\top C(\tau) d\tau = \int_{t_0}^{t_1} C(\tau)^\top C(\tau) d\tau + \int_{t_1}^{t_2} C(\tau)^\top C(\tau) d\tau \\ &= \left[\begin{array}{cc} u_0^2 \, |u_0|^2 & u_0^2 \, |u_0| \\ u_0^2 \, |u_0| & u_0^2 \end{array} \right] (t_1 - t_0) + \left[\begin{array}{cc} u_1^2 \, |u_1|^2 & u_1^2 \, |u_1| \\ u_1^2 \, |u_1| & u_1^2 \end{array} \right] (t_2 - t_1) \end{aligned}$$

To check if full rank is achieved, it is easiest to test with some numbers. Let $u_0 = 0.1$m/s, $u_1 = 0.5$m/s, $t_0 = 0$s, $t_1 = 30$s, and $t_2 = 60$s. This gives

$$Q_{[0,60]} = \left[\begin{array}{cc} 1.878 & 3.780 \\ 3.780 & 7.800 \end{array} \right]$$

which is full rank. Hence, we have shown that two towing speeds are enough to achieve observability of the parameters in x. However, testing over a larger speed range will improve the fit of the drag force function to the real drag.

(c) If it still is rank deficient, repeat again for a third speed $u(t) = u_2 > u_1$ for the period $[t_2, t_3]$. Eventually you will find that the rank is built up over time with sufficient variation in speed u. **Answer:** No need for this.

(d) Propose an observer that can be implemented to online estimate the drag coefficients while running the experiment. **Answer:** As in previous problem, a Kalman filter for time-varying systems is recommended. This is not repeated here, since the filter is essentially the same as in previous problem, only with a new $C(t)$ matrix.

1.2 Nonlinear observer designs

1.2.1 Slow-speed surface vessel

Consider the low-speed vessel model

$$\begin{aligned}\dot{\eta} &= R(\psi)\nu \\ \dot{b} &= 0 \\ M\dot{\nu} + D\nu &= \tau + b\end{aligned}$$

where $\eta = col(x, y, \psi)$, $\nu = col(u, v, r)$, $D = D^\top > 0$ is the damping matrix, $M = M^\top > 0$ is the mass matrix, $b \in \mathbb{R}^3$ is a bias load vector, τ is the thruster loads, and

$$R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with properties

$$\begin{aligned}R(\psi)^\top R(\psi) &= R(\psi)R(\psi)^\top = I \\ \det(R(\psi)) &= 1 \\ R(\psi)^{-1} &= R(\psi)^\top \\ \dot{R} &= R(\psi)S(r), \qquad S(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad r = \dot{\psi}.\end{aligned}$$

Let $\eta_b := R^\top \eta$, $x := col\,(\eta_b, b, \nu)$, and $u := \tau$, and assume we have the measurements $y_1 = \eta_b$ and $y_2 = r$.

1. Assume D is a constant matrix.

   (a) Show that the slow speed dynamics can be written

   $$\begin{aligned}\dot{x} &= Ax + \rho(u, y_1, y_2) \\ y_1 &= C_1 x \\ y_2 &= C_2 x.\end{aligned}$$

   Answer: We differentiate η_b to get

   $$\begin{aligned}\dot{\eta}_b &= \dot{R}^\top \eta + R^\top \dot{\eta} = -S(r)R^\top \eta + R^\top R(\psi)\nu = -S(r)\eta_b + \nu \\ \dot{b} &= 0 \\ \dot{\nu} &= M^{-1}b - M^{-1}D\nu + M^{-1}\tau.\end{aligned}$$

   Thus,

   $$\begin{aligned}A &= \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & M^{-1} & -M^{-1}D \end{bmatrix}, \qquad \rho(u, y_1, y_2) = \begin{bmatrix} -S(y_2)R(y_{1,3})^\top y_1 \\ 0 \\ M^{-1}u \end{bmatrix} \\ C_1 &= \begin{bmatrix} I & 0 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0 & 0 & [0, 0, 1] \end{bmatrix}\end{aligned}$$

   (b) Show that the pair (A, C_1) is observable.
   Answer: We get

   $$A^2 = \begin{bmatrix} 0 & M^{-1} & -M^{-1}D \\ 0 & 0 & 0 \\ 0 & -M^{-1}DM^{-1} & M^{-1}DM^{-1}D \end{bmatrix},$$

   and the observability matrix becomes

   $$O = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & M^{-1} & -M^{-1}D \end{bmatrix}.$$

   Since $M = M^\top > 0$ it is clear that O is full rank, and (A, C_1) is observable.

(c) Propose a nonlinear observer with state estimate $\hat{x}$ that estimates the state vector x. Show that the estimation error $\tilde{x} := x - \hat{x}$ is rendered UGES.
Answer: Since the nonlinearity only depends on the input and outputs, we choose the nonlinear observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \rho(u, y_1, y_2) + L(y_1 - \hat{y}) \\ \hat{y} &= C_1 \hat{x}, \end{aligned}$$

for which the eroor dynamics becomes

$$\dot{\tilde{x}} = (A - LC_1)\,\tilde{x}.$$

Since the pair (A, C_1) is observable, we can place the poles of $A - LC_1$ so that this is Hurwitz and $\tilde{x} = 0$ is UGES.

2. Assume now that $D = D(\nu)$ is indeed a nonlinear damping matrix, and let $d(\nu) := D(\nu)\nu$ be the corresponding monotonically nondecreasing damping load as function of ν.

(a) Show that the system then can be written on the form

$$\dot{x} = Ax + G\gamma\,(Hx) + \rho(u, y_1, y_2), \qquad y_1 = C_1 x$$

Answer: We now get

$$\begin{aligned} A &= \begin{bmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & M^{-1} & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ -I \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \\ \gamma(Hx) &= M^{-1}D(\nu)\nu = M^{-1}D(Hx)Hx \\ \rho(u, y_1, y_2) &= \begin{bmatrix} -S(y_2)R(y_{1,3})^\top y_1 \\ 0 \\ M^{-1}u \end{bmatrix}, \qquad C_1 = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \end{aligned}$$

(b) Propose a new nonlinear observer for this plant, and give design conditions on the injection gains for the observer error dynamics to be GES.
Answer: Since $M = M^\top > 0$ and $D(\nu)$ is positive definite, we get that γ is monotonically increasing and satisfy

$$(v - w)\,(\gamma(v) - \gamma(w)) > 0, \quad \forall v \neq w.$$

We then apply the "Observer based on circle criterion", that is,

$$\dot{\hat{x}} = A\hat{x} + G\gamma(H\hat{x} + K\,(y_1 - C_1\hat{x})) + \rho(u, y) + L\,(y_1 - C_1\hat{x})$$

where K and L are injection gains to be determined to achieve stability. This is done by solving the LMI

$$\begin{bmatrix} (A - LC_1)^\top P + P\,(A - LC) & PG + (H - KC_1)^\top \\ G^\top P + (H - KC_1) & 0 \end{bmatrix} \leq 0,$$

for which a Matlab LMI solver can be used to generate L and K.

1.2.2 Example in lecture

In the lecture we looked at the example system

$$\begin{aligned} \dot{x}_1 &= x_2 - k \sin x_1 \\ \dot{x}_2 &= -k \cos x_2 + u \\ y &= x_1, \end{aligned}$$

and we showed that for $k = 0.1$ we could do the "Observer with global Lipschitz condition" design to render the observer error $\tilde{x} = x - \hat{x} = 0$ GES.

Redo this example and see if you can design the injection gain L that renders $\tilde{x} = 0$ GES for the cases $k = 0.175$ and $k = 0.25$.

Answer: The solution is given in the lecture. Good luck.

1.2.3 DP observer with bias

Work through the proof of the theorem on the “DP observer with bias” at the end of the lecture presentation; see (Værnø and Skjetne, 2017).

References

Værnø, S. A. and Skjetne, R. (2017). Observer for simplified dp model: Design and proof. Lecture note, Norwegian Univ. Sci. & Tech., Trondheim, Norway.