TMR4243 - Marine Control Systems II

Homework assignment 7

1 Maneuvering a scalar plant

Consider the plant

$$\dot{p} = u, \qquad p = col(x, y) \in \mathbb{R}^2,$$

and let the control objective be maneuvering Skjetne (2005), with geometric task to control p to and along a path $p_d(s)$ parameterized by a path variable s, and a dynamic task to satisfy a speed assignment $\upsilon_s(s,t)$ for $\dot{s}$.

Task 1 *Propose a parametrization for $p_d(s)$ that gives a straight-line path.*

Task 2 *Propose a parametrization for $p_d(s)$ that gives an ellipsoidal path, centered in (x_c, y_c) and with radius r_x in the x-direction and radius r_y in the y-direction.*

Let υ_0 be a desired constant speed along the path for $\dot{p}(t)$.

Task 3 *Propose a speed assignment $\upsilon_s(t,s)$ for $\dot{s}(t)$, that ensures path-following at constant speed υ_0* [m/s]*, along:*

1. *The straight-line path.*
2. *The ellipsoidal path.*

Hint: This shall ensure that $|\dot{p}| = |\dot{p}_d| = \upsilon_0$ when p is tracking $p_d(s)$.

Task 4 *Show that with the control law and CLF*

$$u = -K\left(p - p_d(s)\right) + p_d^s(s)\upsilon_s(t,s), \qquad K = K^\top > 0$$
$$V(p,s) = \frac{1}{2}\left(p - p_d(s)\right)^\top \left(p - p_d(s)\right)$$

we get

$$\dot{V} = -\left(p - p_d(s)\right)^\top K \left(p - p_d(s)\right) - V^s(p,s)\left(\upsilon_s(t,s) - \dot{s}\right).$$

What is the expression for $V^s(p,s)$?

Task 5 *Show that the update law for $\dot{s}$ given by*

$$\dot{s} = \upsilon_s(t,s)$$

solves the Maneuvering Problem. Why do we call this a ***Tracking update law?***

Task 6 *Show that the update law for $\dot{s}$ given by*

$$\dot{s} = \upsilon_s(t,s) - \mu V^s(p,s), \qquad \mu \geq 0$$

solves the Maneuvering Problem. Why do we call this a ***Gradient update law?***

Note that the vector $p_d^s(s)$ is for any value s the tangent vector along the path at $p_d(s)$. A modified version of the gradient update law is to ensure that the tangent vector $p_d^s(s)$ is normalized. This will avoid a varying gain from $V^s(p,s)$ along the path according to the parametrization.

Task 7 *Show that the modified gradient update law for $\dot{s}$ given by*

$$\dot{s} = \upsilon_s(t,s) - \frac{\mu}{|p_d^s(s)|} V^s(p,s), \qquad \mu \geq 0$$

solves the Maneuvering Problem. Why do we call this a ***Unit-tangent gradient update law?***

Task 8 *Let the path be the straight-line, and determine path coefficients so that the path becomes the x-axis. Consider $V(p, s)$ to be a cost function, where you fix the position p to be constant and let s be a free optimization variable.*

1. *Let p be located at $p = col(5, 0)$. What is the value of s that minimizes $s \mapsto V(p, s)$?*
2. *Let p be located at $p = col(15, 7)$. What is the value of s that minimizes $s \mapsto V(p, s)$?*

Task 9 *Let $\omega_s = \upsilon_s(t, s) - \dot{s}$ be the path-variable speed error, and*

$$V_2(p, s, \omega_s) := V(p, s) + \frac{1}{2\lambda\mu}\omega_s^2$$

(considering ω_s as an additional state in the system). Show that the update law for $\dot{s}$ given by

$$\begin{aligned} \dot{s} &= \upsilon_s(t, s) - \omega_s \\ \dot{\omega}_s &= -\lambda\left(\omega_s - \mu V^s(p, s)\right), \qquad \mu \geq 0 \end{aligned}$$

gives

$$\dot{V}_2 = -\left(p - p_d(s)\right)^\top K\left(p - p_d(s)\right) - \frac{1}{\mu}\omega_s^2,$$

and solves the Maneuvering Problem. Why do we call this a ***Filtered gradient update law?***

You shall now implement the closed-loop system in Matlab/Simulink with the unit-tangent gradient update law, and the two alternative paths and speed assignments:

- The straight-line path, going through the points $(2, 0)$ and $(10, 4)$.
- The ellipsoidal path, centered at $(x_c, y_c) = (6, 0)$ with radii $(r_x, r_y) = (5, 3)$.

Set the other variables to:

$$\begin{aligned} K &= 1.0 \cdot I \\ \upsilon_0 &= 1.0 \text{ [m/s]} \end{aligned}$$

Test different initial conditions for $p(0) = (x_0, y_0)$ and $s(0)$ for the two paths and speed assignments, and observe the fast transient response for $s(t)$ the first second of simulation and thereby the overall behavior of the trajectories.

Simulate also with $\mu = 0$ and discuss the differences in the responses.

In particular, test and report the following:

Task 10 *For the straight-line path, start with initial condition $s(0) = 0$ and $p(0) = (6, 5)$.*

1. *Set $\mu = 0$ and simulate for 3 seconds, and store the data.*
2. *Set $\mu = 10$ and simulate for 3 seconds, and store the data.*

For both cases, plot the 2D position responses of $(p(t), p_d(s(t)))$ in the same figure during the two seconds. Plot also the time-plots of $s(t)$ in another figure. Add the figures to your report, using the provided 'plotpdftex.m' script. Discuss the behavior of the tracking update law ($\mu = 0$) versus the gradient update law ($\mu = 10$).

Task 11 *For the ellipsoidal path, start with initial condition $s(0) = 0$ and $p(0) = (7, 2)$.*

1. *Set $\mu = 0$ and simulate for 25 seconds, and store the data.*
2. *Set $\mu = 10$ and simulate for 25 seconds, and store the data.*

For both cases, plot the 2D position responses of $(p(t), p_d(s(t)))$ in the same figure during the five seconds. Plot also the time-plots of $s(t)$ in another figure. Add the figures to your report, using the provided 'plotpdftex.m' script. Discuss the behavior of the tracking update law ($\mu = 0$) versus the gradient update law ($\mu = 10$).

1 Solutions: Maneuvering a scalar plant

Consider the plant

$$\dot{p} = u, \qquad p = col(x, y) \in \mathbb{R}^2,$$

and let the control objective be maneuvering Skjetne (2005), with geometric task to control p to and along a path $p_d(s)$ parameterized by a path variable s, and a dynamic task to satisfy a speed assignment $\upsilon_s(s,t)$ for $\dot{s}$.

Task 1 *Propose a parametrization for $p_d(s)$ that gives a straight-line path.*
Answer: *Let $p_0, p_1 \in \mathbb{R}^2$ be two points on the straight-line path. Then*

$$p_d(s) = (1-s)\, p_0 + s p_1, \qquad s \in \mathbb{R}$$

is a straight-line path such that $p_d(0) = p_0$ and $p_d(1) = p_1$.

Task 2 *Propose a parametrization for $p_d(s)$ that gives an ellipsoidal path, centered in (x_c, y_c) and with radius r_x in the x-direction and radius r_y in the y-direction.*
Answer: *Let $p_0 = (x_c, y_c) \in \mathbb{R}^2$ be the center, $R_{xy} = \mathrm{diag}(r_x, r_y)$, and $\xi(s) = \mathrm{col}\left(\cos(2\pi s), \sin(2\pi s)\right)$. An ellipsoidal path is then given by*

$$p_d(s) = p_0 + R_{xy}\xi(s).$$

This is such that $s : 0 \to 1$ corresponds to tracing the ellipsoid one revolution.

Let υ_0 be a desired constant speed along the path for $\dot{p}(t)$.

Task 3 *Propose a speed assignment $\upsilon_s(t,s)$ for $\dot{s}(t)$, that ensures path-following at constant speed υ_0* [m/s]*, along:*
Answer: *We get*

$$|\dot{p}_d| = |p_d^s(s)\dot{s}| = |p_d^s(s)|\,|\dot{s}| = |\upsilon_0|$$

We want $\mathrm{sgn}(\dot{s}) = \mathrm{sgn}(\upsilon_0)$*. Hence,*

$$\dot{s} = \upsilon_s(t,s) := \frac{\upsilon_0}{|p_d^s(s)|}$$

1. *The straight-line path.*
   Answer: *We get*
   $$\upsilon_s(t,s) = \upsilon_s := \frac{\upsilon_0}{|p_d^s(s)|} = \frac{\upsilon_0}{|p_1 - p_0|}$$

2. *The ellipsoidal path.*
   Answer: *We get*
   $$\upsilon_s(t,s) = \upsilon_s(s) := \frac{\upsilon_0}{|p_d^s(s)|} = \frac{\upsilon_0}{\sqrt{\left(x_d^s(s)\right)^2 + \left(y_d^s(s)\right)^2}} = \frac{\upsilon_0}{\sqrt{\left(-2\pi r_x \sin\left(\frac{s}{2\pi}\right)\right)^2 + \left(2\pi r_y \cos\left(\frac{s}{2\pi}\right)\right)^2}}$$

Task 4 ***Answer:*** *With the control law and CLF*

$$u = -K\left(p - p_d(s)\right) + p_d^s(s)\upsilon_s(t,s), \qquad K = K^\top > 0$$
$$V(p,s) = \frac{1}{2}\left(p - p_d(s)\right)^\top \left(p - p_d(s)\right)$$

we get

$$\begin{aligned}
\dot{V} &= \left(p - p_d(s)\right)^\top \left(\dot{p} - \dot{p}_d(s)\right) \\
&= \left(p - p_d(s)\right)^\top \left(-K\left(p - p_d(s)\right) + p_d^s(s)\upsilon_s(t,s) - p_d^s(s)\dot{s}\right) \\
&= -\left(p - p_d(s)\right)^\top K\left(p - p_d(s)\right) + \left(p - p_d(s)\right)^\top p_d^s(s)\left(\upsilon_s(t,s) - \dot{s}\right) \\
&= -\left(p - p_d(s)\right)^\top K\left(p - p_d(s)\right) - V^s(p,s)\left(\upsilon_s(t,s) - \dot{s}\right), \qquad Q.E.D.
\end{aligned}$$

where

$$V^s(p,s) = -\left(p - p_d(s)\right)^\top p_d^s(s).$$

Task 5 ***Answer:*** *Choosing* $\dot{s} = \upsilon_s(t,s)$ *gives*

$$\dot{V} = -\left(p - p_d(s)\right)^\top K \left(p - p_d(s)\right).$$

Letting $e := p - p_d(s)$ *gives*

$$\begin{aligned} \dot{e} &= \dot{p} - p_d^s(s)\dot{s} = -Ke + p_d^s(s)\left(\upsilon_s(t,s) - \dot{s}\right) = -Ke \\ V &= \frac{1}{2}e^\top e, \qquad \dot{V} = -e^\top K e \end{aligned}$$

Hence $e = 0$ *is UGES. We call this a* ***Tracking update law*** *because for*

$$\left.\begin{array}{l} \dot{\xi} = \upsilon_s(t,\xi) \\ s = \xi \end{array}\right\} \qquad \bar{p}_d(t) := p_d(s(t))$$

forms a tracking problem of $p(t) \to \bar{p}_d(t)$ *where* $s(t)$ *just becomes a time signal generated by an exosystem.*

Task 6 ***Answer:*** *For* $\dot{s} = \upsilon_s(t,s) - \mu V^s(p,s)$, $\mu \geq 0$, *and* $e := p - p_d(s)$, *we get*

$$\dot{V} = -e^\top K e - \mu V^s(p,s)^2 \leq -e^\top K e.$$

Hence $e = 0$ *is again UGES. We call this a* ***Gradient update law*** *because* $\dot{s}$ *in addition to being driven by* $\upsilon_s(t,s)$ *takes feedback from the gradient of the Lyapunov function w.r.t.* s. *This mechanism ensures for large* μ *that* s *is rapidly driven to a minimizer of* $s \mapsto V(p,s)$.

Note that the vector $p_d^s(s)$ is for any value s the tangent vector along the path at $p_d(s)$. A modified version of the gradient update law is to ensure that the tangent vector $p_d^s(s)$ is normalized. This will avoid a varying gain from $V^s(p,s)$ along the path according to the parametrization.

Task 7 ***Answer:*** *For* $\dot{s} = \upsilon_s(t,s) - \frac{\mu}{|p_d^s(s)|}V^s(p,s)$, $\mu \geq 0$, *and* $e := p - p_d(s)$, *we get*

$$\dot{V} = -e^\top K e - \frac{\mu}{|p_d^s(s)|}V^s(p,s)^2 \leq -e^\top K e.$$

Hence $e = 0$ *is again UGES. We call this a* ***Unit-tangent gradient update law*** *because now* $\dot{s}$ *takes feedback from*

$$\frac{V^s(p,s)}{|p_d^s(s)|} = -\frac{p_d^s(s)^\top\left(p - p_d(s)\right)}{|p_d^s(s)|} = -\frac{p_d^s(s)^\top}{|p_d^s(s)|}e,$$

that is, the inner product between the unit path tangent vector $\frac{p_d^s(s)}{|p_d^s(s)|}$ *and the error vector* $e = p - p_d(s)$. *Obvioulsy, the minimum is attained when these are perpendicular.*

Task 8 *Let the path be the straight-line, and determine path coefficients so that the path becomes the x-axis. Consider* $V(p,s)$ *to be a cost function, where you fix the position* p *to be constant and let* s *be a free optimization variable.*
Answer: *We let* $p_0 = \mathrm{col}(0,0)$ *and* $p_1 = \mathrm{col}(1,0)$. *Then* s *traces the* x*-axis with*

$$\begin{aligned} p_d(s) &= sp_1 = \mathrm{col}(s,0), \qquad s \in \mathbb{R}. \\ V(p,s) &= \frac{1}{2}|p - sp_1|^2 = \frac{1}{2}(x-s)^2 + \frac{1}{2}y^2 \end{aligned}$$

1. *Let* p *be located at* $p = col(5,0)$. *What is the value of* s *that minimizes* $s \mapsto V(p,s)$?
   Answer: *We get* $V(p,s) = \frac{1}{2}(5-s)^2$ *and* $s = 5$ *will minimize* V.
2. *Let* p *be located at* $p = col(15,7)$. *What is the value of* s *that minimizes* $s \mapsto V(p,s)$?
   Answer: *We get* $V(p,s) = \frac{1}{2}(15-s)^2 + 24.5$ *and* $s = 15$ *will minimize* V.

Task 9 *Let* $\omega_s = \upsilon_s(t,s) - \dot{s}$ *be the path-variable speed error, and*

$$V_2(p,s,\omega_s) := V(p,s) + \frac{1}{2\lambda\mu}\omega_s^2$$

(considering ω_s *as an additional state in the system). Show that the update law for* $\dot{s}$ *given by*

$$\begin{aligned} \dot{s} &= \upsilon_s(t,s) - \omega_s \\ \dot{\omega}_s &= -\lambda\left(\omega_s - \mu V^s(p,s)\right), \qquad \mu \geq 0 \end{aligned}$$

gives

$$\dot{V}_2 = -\left(p - p_d(s)\right)^\top K\left(p - p_d(s)\right) - \frac{1}{\mu}\omega_s^2,$$

and solves the Maneuvering Problem.
Answer: *We get*

$$\begin{aligned} \dot{V}_2 &= \dot{V} + \frac{1}{\lambda\mu}\omega_s\dot{\omega}_s = -e^\top Ke - \omega_s\left(V^s(p,s) - \frac{1}{\lambda\mu}\dot{\omega}_s\right) \\ &= -e^\top Ke - \omega_s\left(V^s(p,s) + \frac{\omega_s - \mu V^s(p,s)}{\mu}\right) = -e^\top Ke - \frac{\omega_s^2}{\mu} < 0 \end{aligned}$$

so that $(e,\omega_s) = (0,0)$ *is UGES.*
We call this a ***Filtered gradient update law*** *because the gradient term* $u = V^s(p,s)$ *is input to and filtered by the 1st-order lowpass filter* $\dot{\omega}_s = -\lambda\left(\omega_s - \mu u\right)$ *before entering the path speed dynamics* $\dot{s}$*.*

You shall now implement the closed-loop system in Matlab/Simulink with the modified gradient update law, and the two alternative paths and speed assignments:

- The straight-line path, going through the points $(2,0)$ and $(10,4)$.
- The ellipsoidal path, centered at $(x_c, y_c) = (6,0)$ with radii $(r_x, r_y) = (5,3)$.

Set the other variables to:

$$\begin{aligned} K &= 1.0 \cdot I \\ \upsilon_0 &= 1.0 \text{ [m/s]} \end{aligned}$$

Test different initial conditions for $p(0) = (x_0, y_0)$ and $s(0)$ for the two paths and speed assignments, and observe the fast transient response for $s(t)$ the first second of simulation and thereby the overall behavior of the trajectories.

Simulate also with $\mu = 0$ and discuss the differences in the responses.

In particular, test and report the following:

Task 10 *For the straight-line path, start with initial condition* $s(0) = 0$ *and* $p(0) = (6,5)$*.*

1. *Set* $\mu = 0$ *and simulate for* 3 *seconds, and store the data.*
2. *Set* $\mu = 10$ *and simulate for* 3 *seconds, and store the data.*

For both cases, plot the 2D position responses of $(p(t), p_d(s(t)))$ *in the same figure during the two seconds. Plot also the time-plots of* $s(t)$ *in another figure. Add the figures to your report, using vector graphics. Discuss the behavior of the tracking update law (*$\mu = 0$*) versus the gradient update law (*$\mu = 10$*).*

Task 11 *For the ellipsoidal path, start with initial condition* $s(0) = 0$ *and* $p(0) = (7,2)$*.*

1. *Set* $\mu = 0$ *and simulate for* 25 *seconds, and store the data.*
2. *Set* $\mu = 10$ *and simulate for* 25 *seconds, and store the data.*

For both cases, plot the 2D position responses of $(p(t), p_d(s(t)))$ *in the same figure during the five seconds. Plot also the time-plots of* $s(t)$ *in another figure. Add the figures to your report, using the provided 'plotpdftex.m' script. Discuss the behavior of the tracking update law (*$\mu = 0$*) versus the gradient update law (*$\mu = 10$*).*

Answer: See and run the Matlab script "`Exc7_ManeuveringTasks.m`" and run the different tasks by setting the "`Task_mode`" flag.

References

Skjetne, R. (2005). *The Maneuvering Problem.* PhD thesis, Norwegian Univ. Sci. & Tech., Trondheim, Norway.