TMR4243 - MARINE CONTROL SYSTEMS II

Homework assignment 1

1 System: Surface ship

The nonlinear surge speed equation of a surface ship can be written[1]

$$M_u \dot{u} + h(u) = \tau_u \tag{1}$$

where τ_u is the surge force control input, $M_{\dot{u}} = m - X_{\dot{u}} > 0$, and $h(u) = -X_u u - X_{|u|u} |u|$ u is monotonically increasing, $h(0) = 0$, and $h(u)u > 0$; $\forall u \neq 0$. Let u_{ref} be a constant reference speed and choose the feedforward control law

$$\tau_u = h(u_{\mathrm{ref}}). \tag{2}$$

1.1 Task: Function properties

Define

$$g(u) := h(u) - h(u_{\mathrm{ref}}). \tag{3}$$

1. Show that $g(u_{\mathrm{ref}}) = 0$.
2. Show that $g(u)(u - u_{\mathrm{ref}}) > 0, \forall u \neq u_{\mathrm{ref}}$.

1.2 Task: Lyapunov analysis

Show by using the Lyapunov function

$$V(u) = \frac{M_{\dot{u}}}{2}(u - u_{\mathrm{ref}})^2 + M_{\dot{u}} \int_{u_{\mathrm{ref}}}^{u} g(y)\, dy \tag{4}$$

that its time derivative along the solutions of the closed-loop system is given by

$$\dot{V}(u) = -(u - u_{\mathrm{ref}})\, g(u) - g(u)^2,$$

and that the equilibrium $u - u_{\mathrm{ref}} = 0$ is GAS.

2 System: Pendulum

The system at hand consists of a stand with a bearing. A rod is attached to the outer bearing race. A bob is attached to the other end of the rod. See Figure 1a.

Table 1 summarizes the parameters and variables of the installation.

The control plant model is based on that the angular acceleration $\dot{\omega}$ of the rotating part is proportional to the sum of torques:

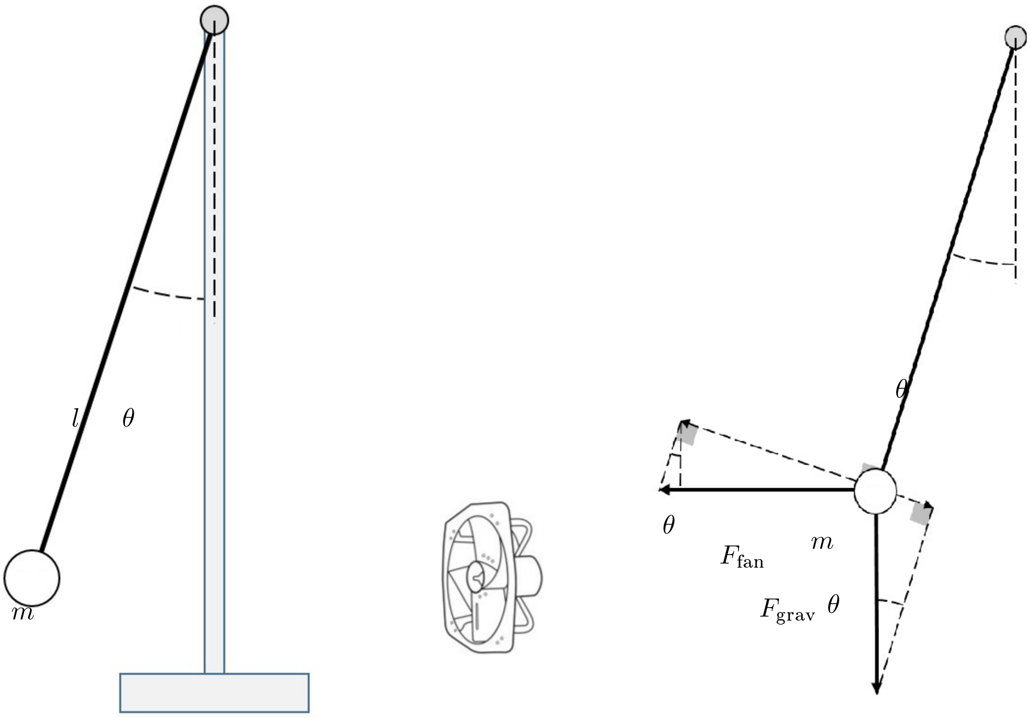
$$\dot{\omega} = \frac{1}{J}\sum \tau, \tag{5}$$

where J is the system inertia.

Considering the bob as a point mass, the inertia is

$$J = ml^2. \tag{6}$$

[1]Fossen, Handbook of Marine Craft Hydrodynamics and Motion Control, Eq. 7.32



(a) System illustration

(b) Mass free body diagram

Figure 1: Pendulum

m	0.1 **kg**	mass of the bob
l	0.5 **m**	length of the rod attached to the bob
g	$9.81 m/s^2$	
k	0.01	friction coefficient
θ	**rad**	angle between the vertical stand and the bob rod
ω	**rad/s**	angular velocity

Table 1: Parameters and variables

For simplicity, this is chosen as the inertia of the whole system, thus disregarding the mass of the rod.

Figure 1b holds a diagram of the forces acting on the system. Since the forces parallel to the rod are counteracted by the latter, only the tangent components are of interest here:

- The torque τ_{grav} due to gravity is

$$\begin{aligned} \tau_{\text{grav}} &= -lF_{\text{grav}} \sin(\theta) \\ &= -lmg \sin(\theta). \end{aligned} \tag{7}$$

- The torque τ_{fan} due to the fan pressure on the rod is

$$\begin{aligned} \tau_{\text{fan}} &= lF_{\text{fan}} \sin\left(\frac{\pi}{4} - \theta\right), \\ &= lF_{\text{fan}} \cos(\theta) \end{aligned} \tag{8}$$

  where F_{fan} is the force from the fan.

Additionally, the friction in the bearing is modeled by a torque τ_{fric} proportional to the velocity:

$$\tau_{\text{fric}} = -k\omega. \tag{9}$$

Inserting (7)-(9) and substituting (6) in (5) yields

$$\begin{aligned} \dot{\omega} &= \frac{1}{ml^2} \left(-lmg \sin(\theta) - k\omega + lF_{\text{fan}} \cos(\theta)\right) \\ &= -\frac{g}{l} \sin(\theta) - \frac{k}{ml^2}\omega + \frac{F_{\text{fan}}}{ml} \cos(\theta) \end{aligned} \tag{10}$$

2.1 Task: State equations

1. Write the state equation $\dot{x} = f(x, u)$ using the state vector $x = \begin{bmatrix} \theta \\ \omega \end{bmatrix}$ and $u = F_{\text{fan}}$.
2. Program a corresponding Simulink model.

2.2 Task: System properties

Assuming $|F_{\text{fan}}|$ is bounded, explain why the system is or isn't:

1. Forward complete.
2. Backward complete.
3. Complete
4. Locally Lipschitz.
5. Globally Lipschitz.

2.3 Task: Simple pendulum equilibrium point

Assume the fan is off, i.e. $F_{\text{fan}} = 0$ N.

1. Show that $x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an equilibrium point of the unforced system.
2. Explain why x^* is or isn't
   (a) a unique equilibrium point,
   (b) an isolated equilibrium point.

3. Describe the physical situation(s) the equilibrium point(s) correspond(s) to.
4. Simulate the system with initial condition $x(0) = x^*$ to confirm the behavior at the equilibrium point.

2.4 Task: Linearized simple pendulum model

The linearized state equations are

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{ml^2} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}.$$

1. Show that x^* is also an equilibrium point for the linearized system.
2. Explain why x^* is or isn't:
   (a) Locally stable.
   (b) Globally stable.

2.5 Task: Equilibrium point with fan

Assume that the fan is again running, with $F_{\text{fan}} = 0.56638$ N.

1. Calculate the angle at which the bob now stabilizes.
2. Confirm through simulation.

2.6 Task: Angle control

In order to set the fan to stabilize the bob at $\theta = 60°$,

1. choose a change of variables such that the equilibrium is shifted to this angle,
2. write the state equations using the new states, and
3. determine F_{fan} necessary to the new equilibrium.
4. Confirm through simulation.

3 Lyapunov function

Consider the differential equations

$$\begin{aligned} \dot{x}_1 &= u_1 && \text{(11a)} \\ \dot{x}_2 &= u_2, && \text{(11b)} \end{aligned}$$

and a function

$$V(x_1, x_2) = c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2, \quad (12)$$

$$V = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (13)$$

where c_1, c_2, c_3 are positive scalars.

3.1 Task: Differentiation

Do the following:

1. Differentiate $V(x_1, x_2)$ with respect to time (we say, "along the solutions of (11)") and set up the resulting expression in terms of the states (x_1, x_2) and the inputs (u_1, u_2).
2. Let $x := \mathrm{col}(x_1, x_2)$ and $u := \mathrm{col}(u_1, u_2)$. Show that V can be written as $V(x) = x^\top P x$ where $P = P^\top$ (symmetric).
3. Give conditions on (c_1, c_2, c_3) for P to be a positive definite matrix ($P = P^\top > 0$).
4. Show that taking the vector differentiation of $V(x)$ gives $\dot{V} = 2x^\top P\dot{x}$ and that this equals the answer in the Subtask 1.
5. Let $u_1 = -x_1 + x_2$ and $u_2 = -x_2$ such that the closed-loop system becomes

$$\dot{x} = Ax \tag{14}$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}. \tag{15}$$

   Choose values for (c_1, c_2, c_3) such that $P = P^\top > 0$ and $PA + A^\top P = -Q$ where $Q > 0$ is a diagonal positive matrix.

6. Differentiate again $V(x)$ along the soolutions of (14) and show that $\dot{V} = -x^\top Qx$ for your chosen values of (c_1, c_2, c_3).

	$h(u) - h(u_{\text{ref}})$	$u - u_{\text{ref}}$
$u > u_{\text{ref}}$	+	+
$u < u_{\text{ref}}$	-	-

Table 2: Sign of terms given ratio of u and u_{ref}

1 Solution: Surface ship

1.1 Task: Function properties

Inserting $u = u_{ref}$ in (3) yields

$$\begin{aligned} g(u_{\text{ref}}) &= h(u_{\text{ref}}) - h(u_{\text{ref}}) \\ &= 0. \end{aligned} \tag{16}$$

Also,

$$g(u)(u - u_{\text{ref}}) = (h(u) - h(u_{\text{ref}}))(u - u_{\text{ref}}). \tag{17}$$

Since $h(\cdot)$ is monotonically increasing,

$$\begin{aligned} u > u_{\text{ref}} &\Leftrightarrow h(u) > h(u_{\text{ref}}), \text{ and} \\ u < u_{\text{ref}} &\Leftrightarrow h(u) < h(u_{\text{ref}}) \end{aligned}$$

by definition. By inspection, summarized in Table 2, the two terms of (17) have the same sign when $u \neq u_{ref}$ and the product is thus positive.

1.2 Task: Lyapunov analysis

Differentiation of (4) yields

$$\dot{V}(u) = M_{\dot{u}}(u - u_{\text{ref}})(\dot{u} - \dot{u}_{\text{ref}}) + M_{\dot{u}}g(u)\dot{u}. \tag{18}$$

Rearranging (1), then substituting (2), then (3) gives

$$\begin{aligned} \dot{u} &= \frac{\tau_u - h(u)}{M_u}. \\ &= \frac{h(u_{\text{ref}}) - h(u)}{M_u}. \\ &= -\frac{g(u)}{M_u}. \end{aligned} \tag{19}$$

Also, since u_{ref} is constant,

$$\dot{u}_{\text{ref}} = 0. \tag{20}$$

Applying (19), (20) and (16) to (18) yields

$$\begin{aligned} \dot{V}(u) &= M_{\dot{u}}(u - u_{\text{ref}})\left(-\frac{g(u)}{M_u} - 0\right) + M_{\dot{u}}g(u)\left(-\frac{g(u)}{M_u}\right). \\ &= -(u - u_{\text{ref}})g(u) - g(u)^2 =: -\alpha(|u|) < 0, \quad \forall |u| \neq 0 \end{aligned} \tag{21}$$

The speed equilibrium is at $\dot{u} = 0$. From (19) it follows that this requires $g(u) = 0$. (16) reveals $g(u_{\text{ref}})$ as one equilibrium. The Lyapunov function is

- $\dot{V}(u_{\text{ref}}) = 0$, by (21) with (16), and
- $\dot{V}(u) < 0 \ \forall u \neq u_{\text{ref}}$, by the inspection of (17) and since $g(u)^2 \geq 0$,

thus the equilibrium is globally asymptotically stable.

2 Solution: Pendulum

2.1 Task: State equations

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} &= \begin{bmatrix} \omega \\ -\frac{g}{l}\sin(\theta) - \frac{k}{ml^2}\omega + \frac{1}{ml}\cos(\theta) F_{\text{fan}} \end{bmatrix}
\end{aligned}
$$

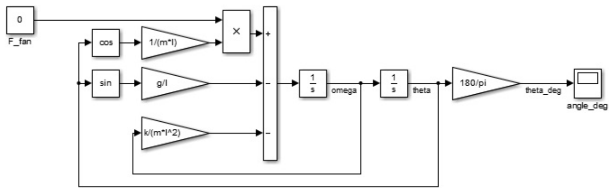


Figure 2: Simulink diagram

2.2 Task: System properties

Assuming $|F_{\text{fan}}|$ is bounded, explain why the system is or isn't:

1. Forward complete: This means that the solutions exists for all forward time. We get for the system that

$$
\begin{aligned}
\left|\dot{\theta}\right| &\leq |\omega| \leq |x| \\
|\dot{\omega}| &\leq \left| -\frac{g}{l}\sin(\theta) - \frac{k}{ml^2}\omega + \frac{1}{ml}\cos(\theta) F_{\text{fan}} \right| \\
&\leq \frac{g}{l} + \frac{k}{ml^2}|\omega| + \frac{1}{ml}|F_{\text{fan}}| \leq a + b|x|
\end{aligned}
$$

   We see that the derivatives have linear sector bounds in the states, and it follows that the solutions can only grow to infinity by an exponential rate - which takes infinite time[2]. Hence, the solutions must be forward complete.

2. Backward complete: Assume for simplicity that $F_{\text{fan}}(t) = 0$ for $t < 0$. Substitute $s = -t$ as a new free variable running backwards in time, and $z = x(s(t))$ is the corresponding solution of x backwards in time. Then

$$
\dot{z} = \frac{dx}{ds}\frac{ds}{dt} = -f(x(s), 0) = -f(z, 0).
$$

   By the same argument as above, this system is forward complete. Since the solutions of $z(t)$ forward in time is the same as the solutions of $x(t)$ backwards in time, it follows that the pendulum system is backwards complete.

3. Complete: Since the system is both forward complete and backwards complete, it is complete.

4. Locally Lipschitz: Sine all p–notrms are equivalent, it is easiest to use the 1-norm (note: $|x|_2 \leq$

[2] This is formally proved in Proposition A.1 in Skjetne et al. (2005).

$|x|_1 \leq \sqrt{2}\,|x|_2$). Assume $u_{\max} := \|F_{\text{fan}}\|_\infty$. We check

$$
\begin{aligned}
& |f(x,u) - f(y,u)|_2 \leq |f(x,u) - f(y,u)|_1 \\
= & \left\| \begin{bmatrix} \omega_x - \omega_y \\ -\frac{g}{l}\left(\sin(\theta_x) - \sin(\theta_y)\right) - \frac{k}{ml^2}\left(\omega_x - \omega_y\right) + \frac{1}{ml}\left(\cos(\theta_x) - \cos(\theta_y)\right) F_{\text{fan}} \end{bmatrix} \right\|_1 \\
= & \left\| \begin{bmatrix} \omega_x - \omega_y \\ -2\frac{g}{l}\cos\left(\frac{\theta_x+\theta_y}{2}\right)\sin\left(\frac{\theta_x-\theta_y}{2}\right) - \frac{k}{ml^2}\left(\omega_x - \omega_y\right) - 2\frac{1}{ml}\sin\left(\frac{\theta_x+\theta_y}{2}\right)\sin\left(\frac{\theta_x-\theta_y}{2}\right) F_{\text{fan}} \end{bmatrix} \right\|_1 \\
= & |\omega_x - \omega_y| \\
& + \left| -2\frac{g}{l}\cos\left(\frac{\theta_x+\theta_y}{2}\right)\sin\left(\frac{\theta_x-\theta_y}{2}\right) - \frac{k}{ml^2}\left(\omega_x - \omega_y\right) - 2\frac{1}{ml}\sin\left(\frac{\theta_x+\theta_y}{2}\right)\sin\left(\frac{\theta_x-\theta_y}{2}\right) F_{\text{fan}} \right| \\
\leq & |\omega_x - \omega_y| + \frac{g}{l}|\theta_x - \theta_y| + \frac{k}{ml^2}|\omega_x - \omega_y| + \frac{u_{\max}}{ml}|\theta_x - \theta_y| \\
\leq & \max\left\{1, \frac{g}{l}, \frac{k}{ml^2}, \frac{u_{\max}}{ml}\right\}|x - y|_1 \leq L\,|x - y|_2 \\
& L = \sqrt{2}\max\left\{1, \frac{g}{l}, \frac{k}{ml^2}, \frac{u_{\max}}{ml}\right\}
\end{aligned}
$$

5. Globally Lipschitz: Since the above derived L is valid for all $x \in \mathbb{R}^2$, the above derivation in fact shows that

2.3 Task: Simple pendulum equilibrium point

The equilibrium requires, i.e.

$$0 = \omega \tag{22}$$

$$0 = -\frac{g}{l}\sin(\theta) - \frac{k}{ml^2}\omega + \frac{0}{ml}\cos(\theta). \tag{23}$$

From (22) it follows that $0 = \omega$. Substitution in (23) gives $0 = -\frac{g}{l}\sin(\theta)$, thus $0 = \sin(\theta)$ since g and l are constants. Equilibrium points are thus at

$$(\theta, \omega) = (\pm n\pi, 0) \quad \forall n \in \mathbb{Z}.$$

For even n (including $n = 0$), the bob stands still in the lower position with the rod counteracting gravity and no friction effect. For odd n, the bob stands still upright. The infinite amount of equilibrium points corresponds to the bob assuming the equilibrium angles after any number of rotations.

2.4 Task: Linearized simple pendulum model

Setting $\dot{\theta} = \dot{\omega} = 0$ for the linearized equations shows that $x^* = col(0,0)$ is an equilibrium point also for the linearized system.

1. Explain why x^* is or isn't:
   (a) Locally stable: We find that the poles of the linearized A-matrix becomes

$$
\begin{aligned}
\det(sI - A) &= \det\begin{bmatrix} s & -1 \\ \frac{g}{l} & s + \frac{k}{ml^2} \end{bmatrix} = s^2 + \frac{k}{ml^2}s + \frac{g}{l} \\
s_0 &= -\frac{k}{2ml^2} \pm \frac{1}{2}\sqrt{\left(\frac{k}{ml^2}\right)^2 - 4\frac{g}{l}}
\end{aligned}
$$

   We see that the real part of the poles will always be negative for $k, m, l > 0$, and hence the linerarized system is exponentially stable. For the nonlinear system we can then conclude that $x^* = col(0,0)$ is locally stable.

(b) Globally stable: The linearized system, by itself, is globally stable. However, since the non-linear system has multiple equilibria, for $\theta^* = k\pi$, $k \in \mathbb{Z}$, it cannot be globally stable. In fact we will find that every equilibria $(\theta^*, \omega^*) = (2k\pi, 0)$, $k \in \mathbb{Z}$ (pendulum hanging downright), is locally stable, and $(\theta^*, \omega^*) = ((2k+1)\pi, 0)$, $k \in \mathbb{Z}$ (pendulum staying upright), is locally unstable.

2.5 Task: Equilibrium point with fan

(22)-(23) yields

$$\frac{g}{l}\sin(\theta) = \frac{F_{\text{fan}}}{ml}\cos(\theta)$$

which in turn gives

$$\begin{aligned}
\frac{\sin(\theta)}{\cos(\theta)} &= \frac{F_{\text{fan}}}{mg} \\
&\Downarrow \\
\theta &= \tan^{-1}\left(\frac{F_{\text{fan}}}{mg}\right) \qquad (24)
\end{aligned}$$

which corresponds to $\theta = 0.5236$ rad $= 30°$.

2.6 Task: Angle control

Introducing a new variable

$$\tilde{\theta} = \theta - \frac{\pi}{3}$$

gives

$$\begin{aligned}
\theta &= \tilde{\theta} + \frac{\pi}{3}, \text{ and} \\
\dot{\tilde{\theta}} &= \dot{\theta}.
\end{aligned}$$

The state equations are then

$$\begin{aligned}
\dot{\tilde{\theta}} &= \omega \\
\dot{\omega} &= -\frac{g}{l}\sin\left(\tilde{\theta} + \frac{\pi}{3}\right) - \frac{k}{ml^2}\omega + \frac{F_{\text{fan}}}{ml}\cos\left(\tilde{\theta} + \frac{\pi}{3}\right)
\end{aligned}$$

The equilibrium requires $\dot{\tilde{\theta}} = \dot{\omega} = 0$, also $\tilde{\theta} = 0$ here, i.e.

$$\begin{aligned}
0 &= \omega \\
0 &= -\frac{g}{l}\sin\left(0 + \frac{\pi}{3}\right) - \frac{k}{ml^2}\omega + \frac{F_{\text{fan}}}{ml}\cos\left(0 + \frac{\pi}{3}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{g}{l}\sin\left(\frac{\pi}{3}\right) + \frac{k}{ml^2}0 &= \frac{F_{\text{fan}}}{ml}\cos\left(\frac{\pi}{3}\right) \\
&\Downarrow \\
F_{\text{fan}} &= mg\frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} \\
&= mg\tan\left(\frac{\pi}{3}\right)
\end{aligned}$$

Note: The task could also have been solved by solving (24) for F_{fan}, however revising changes of variables was part of the intention here.

3 Solution: Lyapunov function

3.1 Task: Differentiation

1. Differentiating $V(x_1, x_2)$ along the solutions of (11)results in

$$\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = (2c_1x_1 + c_2x_2)\,u_1 + (c_2x_1 + 2c_3x_2)\,u_2.$$

2. We get

$$\begin{aligned} V(x_1,x_2) &= c_1x_1^2 + c_2x_1x_2 + c_3x_2^2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ P &= \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \end{aligned}$$

3. For P to be a positive definite its leading principal minors must all be positive. This means that $c_1 > 0$ and $c_1c_3 - \frac{1}{4}c_2^2 > 0$. It follows that $c_3 > 0$ and $c_2 < 2\sqrt{c_1c_3}$.
4. Differentiating $V(x)$ gives

$$\begin{aligned} \dot{V} &= \dot{x}^\top Px + x^\top P\dot{x} = 2x^\top P\dot{x} = 2x^\top Pu \\ &= 2\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= (2c_1x_1 + c_2x_2)\,u_1 + (c_2x_1 + 2c_3x_2)\,u_2 \qquad \text{Q.E.D.} \end{aligned}$$

5. We have from above that $c_1 > 0$, $c_3 > 0$, and $c_2 < 2\sqrt{c_1c_3}$. Calculating $PA + A^\top P$ gives

$$\begin{aligned} & PA + A^\top P \\ = & \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & \frac{c_2}{2} \\ \frac{c_2}{2} & c_3 \end{bmatrix} \\ = & -\begin{bmatrix} 2c_1 & -c_1 + c_2 \\ -c_1 + c_2 & 2c_3 - c_2 \end{bmatrix} =: -Q. \end{aligned}$$

Hence, for Q to be diagonal we need $c_1 = c_2 > \dot{0}$. Moreover, for it to be positive we need $c_1 > 0$ and $\det Q > 0$. This gives

$$\det Q = -c_1^2 + 4c_3c_1 - c_2^2 = 4c_3c_1 - 2c_1^2 = 2c_1\,(2c_3 - c_1) > 0.$$

Choosing for instance $c_3 = c_1 = c_2 > 0$ satisfies this, e.g. $c_1 = c_2 = c_3 = 1$.

6. Differentiating $V(x)$ gives

$$\dot{V} = \dot{x}^\top Px + x^\top P\dot{x} = \left(x^\top A^\top\right) Px + x^\top P(Ax) = x^\top \left(PA + A^\top P\right) x = -x^\top Qx$$

References

Skjetne, R., Fossen, T. I., and Kokotović, P. V. (2005). Adaptive maneuvering, with experiments, for a model ship in a marine control laboratory. *Automatica*, 41(2):289–298.