Marine Control Systems II

Lecture 5: Backstepping

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Goals of lecture

- Carry out the *backstepping* control design method:
  - Integrator backstepping
  - General vectorial backstepping
  - LgV-backtsepping
- To apply integral action in backstepping:
  - applied to a DP control design

Literature

- Inequalities section in note on "Mathematical Notations and Preliminaries": Especially Young's inequality.
- Khalil, H. K. (2015). Nonlinear Control.
  - Chapter: 9.5.
- Skjetne, R. (2005). The Maneuvering Problem.
  - Ch. 4.1 (for presentation of systematic ISS backstepping design).
- Lecture presentation.

Chain of integrators plant

Consider the nonlinear system

$$\begin{aligned} \dot{x}_i &= x_{i+1}, && i = 1, 2, \dots n-1 \\ \dot{x}_n &= f(x) + g(x)u, && g(x) \neq 0,\ \forall x, \\ y &= x_1 \end{aligned}$$

and a regulation control objective of $y = 0$. (Note: $x := \mathrm{col}(x_1, \dots, x_n)$.)
What is the relative degree?

This system correspond to a feedback linearizable system, since it can be written in the controller form

$$\dot{x} = Ax + B\gamma(x)\left[u - \rho(x)\right], \qquad \gamma(x) \neq 0,\ \forall x.$$

Now we will learn **Backstepping**.

Integrator backstepping

Consider the chain of 3 integrators,

$$\dddot{\xi} = F(\xi, \dot{\xi}, \ddot{\xi}) + G(\xi, \dot{\xi}, \ddot{\xi})u$$
$$y = \xi$$

What is the relative degree?

Relative degree 3 implies 3 steps of backstepping. Letting $x_1 := \xi$, $x_2 := \dot{\xi}$, $x_3 := \ddot{\xi}$ and $x = \mathrm{col}(x_1, x_2, x_3)$, we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= f(x) + g(x)u \end{aligned}$$

with $f(x) := F(x_1, x_2, x_3)$ and $g(x) := G(\xi, \dot{\xi}, \ddot{\xi})$.

Control objective: Regulation $y = x_1 \to y_{ref}$

Integrator backstepping

Step 1: Let $z_1 := x_1 - y_{ref}$ and $z_2 := x_2 - \alpha_1(x_1)$ where α_1 takes the role as a virtual control for x_2 to control the $\dot{x}_1$-dynamics.

We also define the CLF $V_1(z_1) := \frac{1}{2}z_1^2$. Differentiating gives

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = x_2 = z_2 + \alpha_1 \\ \dot{V}_1 &= z_1\dot{z}_1 = z_1\alpha_1 + z_1z_2 \end{aligned}$$

The term z_1z_2 we disregard, since we postpone dealing with this until the next step. Choosing

$$\alpha_1(x_1) = -c_1z_1 = -c_1\left(x_1 - y_{ref}\right)$$

gives

$$\begin{aligned} \dot{z}_1 &= -c_1z_1 + z_2 \\ \dot{V}_1 &= -c_1z_1^2 + z_1z_2 \end{aligned}$$

Assuming $z_2 = 0$ then we have that $\dot{V}_1$ is negative definite in z_1 so that $z_1 \to 0$.

However, the term z_1z_2 is there, and we deal with that in next step.
To aid next step, we calculate the derivative of $\alpha_1(x_1)$,

$$\dot{\alpha}_1 = -c_1\dot{x}_1 = -c_1x_2 =: \sigma_1(x_1, x_2)$$

Integrator backstepping

Step 2: We have $z_2 := x_2 - \alpha_1(x_1)$ and define $z_3 := x_3 - \alpha_2(x_1, x_2)$ where α_2 is the second virtual control, now for x_3 to control the $(\dot{x}_1, \dot{x}_2)$-dynamics.

We also deine the CLF $V_2(z_1, z_2) := V_1(z_1) + \frac{1}{2}z_2^2$. Differentiating gives

$$\begin{aligned}\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = x_3 - \sigma_1(x_1, x_2) = z_3 + \alpha_2 - \sigma_1(x_1, x_2)\\ \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 = -c_1 z_1^2 + z_1 z_2 + z_2\left(\alpha_2 - \sigma_1\right) + z_2 z_3\\ &= -c_1 z_1^2 + z_2\left(z_1 + \alpha_2 - \sigma_1\right) + z_2 z_3\end{aligned}$$

The term $z_2 z_3$ we disregard, since we postpone dealing with this until the next step. Choosing

$$\begin{aligned}\alpha_2(x_1, x_2) &= -z_1 - c_2 z_2 + \sigma_1(x_1, x_2)\\ &= -\left(x_1 - y_{ref}\right) - c_2\left(x_2 - \alpha_1\right) - c_1 x_2\end{aligned}$$

gives

$$\begin{aligned}\dot{z}_2 &= -z_1 - c_2 z_2 + z_3\\ \dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3\end{aligned}$$

Assuming $z_3 = 0$ then we have that $\dot{V}_2$ is negative definite in (z_1, z_2) so that $(z_1, z_2) \to 0$.

However, the term $z_2 z_3$ is there, and we deal with that in next step.
To aid next step, we calculate the derivative of $\alpha_2(x_1, x_2)$,

$$\begin{aligned}\dot{\alpha}_2 &= -\dot{x}_1 - c_2\left(\dot{x}_2 - \dot{\alpha}_1\right) - c_1\dot{x}_2\\ &= -x_2 - c_2\left(x_3 - \sigma_1(x_1, x_2)\right) - c_1 x_3 =: \sigma_2(x_1, x_2, x_3)\end{aligned}$$

Integrator backstepping

Step 3: We have $z_3 := x_3 - \alpha_2(x_1)$ and no more z-states and virtual controls are needed, since we in the 3rd step will hit u to control the $(\dot{x}_1, \dot{x}_2, \dot{x}_3)$-dynamics.

We define the 3rd CLF

$$V_3(z_1, z_2, z_3) := V_2(z_1, z_2) + \frac{1}{2}z_3^2.$$

Differentiating gives

$$\begin{aligned}\dot{z}_3 &= \dot{x}_3 - \dot{\alpha}_2 = f(x) + g(x)u - \sigma_2(x_1, x_2, x_3)\\ \dot{V}_3 &= \dot{V}_2 + z_3\dot{z}_3 = -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_3\left(f(x) + g(x)u - \sigma_2\right)\\ &= -c_1 z_1^2 - c_2 z_2^2 + z_3\left(z_2 + f(x) + g(x)u - \sigma_2\right)\end{aligned}$$

Now there are no more terms to postpone to a next step, and we choose

$$u = \frac{1}{g(x)}\left[-z_2 - f(x) + \sigma_2(x_1, x_2, x_3) - c_3 z_3\right]$$

gives

$$\begin{aligned}\dot{z}_3 &= -z_2 - c_3 z_3\\ \dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3\end{aligned}$$

Integrator backstepping

We now have the closed-loop in the z-dynamics

$$\begin{aligned}\dot{z}_1 &= -c_1 z_1 + z_2 \\ \dot{z}_2 &= -z_1 - c_2 z_2 + z_3 \\ \dot{z}_3 &= -z_2 - c_3 z_3\end{aligned}$$

and

$$\begin{aligned}V_3(z_1, z_2, z_3) &= \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}z_3^2 \\ \dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3\end{aligned}$$

Based on Lyapunov's direct method, what stability do we have of the origin $(z_1, z_2, z_3) = (0, 0, 0)$?

Letting $z := \mathrm{col}(z_1, z_2, z_3)$, we get

$$\dot{z} = \begin{bmatrix} -c_1 & 1 & 0 \\ -1 & -c_2 & 1 \\ 0 & -1 & -c_3 \end{bmatrix} z = -Cz + Sz$$

where $C = \mathrm{diag}(c_1, c_2, c_3)$ and $S = -S^\top$. Then we have the vectorial Lyapunov function

$$\begin{aligned}V &= \frac{1}{2}z^\top z \\ \dot{V} &= z^\top \dot{z} = -z^\top Cz + z^\top Sz = -z^\top Cz\end{aligned}$$

since $z^\top Sz = 0$.

Integrator backstepping

What we have done, is iteratively constructing a CLF $V(z)$ and a state transformation $z = T(x)$ into a quadratically stabilizable system:

$$\begin{aligned}z_1 &= x_1 - y_{ref} \\ z_2 &= x_2 + c_1 x_1 - c_1 y_{ref} \\ z_3 &= (1 + c_1 c_2)\, x_1 + (c_1 + c_2)\, x_2 + x_3 - (1 + c_1 c_2)\, y_{ref}\end{aligned}$$

Note that $(z_1, z_2, z_3) = (0, 0, 0) \Rightarrow (x_1, x_2, x_3) = (y_{ref}, 0, 0)$.

The recursive construction of the CLF has correspondingly generated:

$$\left.\begin{array}{l} V_1(z_1) = \frac{1}{2}z_1^2 \\ V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}z_2^2 \\ V_3(z_1, z_2, z_3) = V_2(z_1) + \frac{1}{2}z_3^2 \end{array}\right\} \begin{array}{l} V(z) = \frac{1}{2}z^\top z \\ \\ \dot{V} = -z^\top Cz \end{array}$$

Nonlinear plant in strict feedback form

We now consider the nonlinear plant:

$$
\begin{aligned}
\dot{x}_1 &= G_1\left(\bar{x}_1\right) x_2 + f_1\left(\bar{x}_1\right) \\
\dot{x}_2 &= G_2\left(\bar{x}_2\right) x_3 + f_2\left(\bar{x}_2\right) \\
&\vdots \\
\dot{x}_n &= G_n\left(\bar{x}_n\right) u + f_n(\bar{x}_n) \\
y &= h\left(x_1\right)
\end{aligned}
$$

where $\forall t \geq 0$, $x_i(t) \in \mathbb{R}^m$, $i = 1, \ldots, n$, are the states, $y(t) \in \mathbb{R}^m$ is the output, $u(t) \in \mathbb{R}^m$ is the control. Note the compact notation $\bar{x}_j := \mathrm{col}(x_1, \ldots, x_j)$.
The matrices $G_i(x_1, \ldots, x_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible for all x, $h(x_1)$ is a diffeomorphism, and G_i and f_i are smooth.

As an example we consider again $n = 3$.
What is then the dimension of the state space?

Vectorial backstepping

First, we define $\bar{x}_i := \mathrm{col}(x_1, \ldots, x_i)$, each $x_j \in \mathbb{R}^m$, and $x := \bar{x}_3 \in \mathbb{R}^{3m}$. We then consider the vectorial strict feedback form system

$$
\begin{aligned}
\dot{x}_1 &= G_1(\bar{x}_1)x_2 + f_1(\bar{x}_1) \\
\dot{x}_2 &= G_2(\bar{x}_2)x_3 + f_2(\bar{x}_2) \\
\dot{x}_3 &= G_3(\bar{x}_3)u + f_3(\bar{x}_3) \\
y &= h(x_1)
\end{aligned}
$$

The matrices $G_i(\bar{x}_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible for all x, $h(x_1)$ is a bijective mapping, and G_i and f_i are smooth, for $i = 1, 2, 3$.
The system now has vector relative degree $[3, 3, \ldots, 3]$.
The objective is tracking, $y(t) \to y_d(t)$.
We will not write the parameter lists of the functions for simplicity.

Vectorial backstepping

Step 1: Let $z_1 := y - y_d(t)$ and $z_2 := x_2 - \alpha_1$. Differentiating gives

$$\begin{aligned} \dot{z}_1 &= h^{x_1}\dot{x}_1 - \dot{y}_d = h^{x_1}\left[G_1 x_2 + f_1\right] - \dot{y}_d(t) \\ &= h^{x_1}G_1 z_2 + h^{x_1}G_1\alpha_1 + h^{x_1}f_1 - \dot{y}_d(t) \end{aligned}$$

Let A_1 be a desired Hurwitz matrix for the z_1-subsystem and $Q_1 = Q_1^\top > 0$ s.t. $P_1 A_1 + A_1^\top P_1 = -Q_1$.
Define the CLF $V_1(z_1) = z_1^\top P_1 z_1$ s.t.

$$\dot{V}_1 = 2z_1^\top P_1 \dot{z}_1 = 2z_1^\top P_1\left[h^{x_1}G_1\alpha_1 + h^{x_1}f_1 - \dot{y}_d(t)\right] + 2z_1^\top P_1 h^{x_1}G_1 z_2$$

Postponing dealing with z_2 and choosing virtual control

$$\alpha_1(x_1, t) = G_1^{-1}\left(h^{x_1}\right)^{-1}\left[A_1 z_1 - h^{x_1}f_1 + \dot{y}_d(t)\right]$$

gives

$$\begin{aligned} \dot{V}_1 &= 2z_1^\top P_1 A_1 z_1 + 2z_1^\top P_1 h^{x_1}G_1 z_2 \\ &= z_1^\top P_1 A_1 z_1 + z_1^\top A_1^\top P_1 z_1 + 2z_1^\top P_1 h^{x_1}G_1 z_2 \\ &= -z_1^\top Q_1 z_1 + 2z_1^\top P_1 h^{x_1}G_1 z_2 \\ \dot{z}_1 &= A_1 z_1 + h^{x_1}G_1 z_2 \end{aligned}$$

Aiding next step,

$$\dot{\alpha}_1 = \alpha_1^{x_1}\dot{x}_1 + \alpha_1^t =: \sigma_1(\bar{x}_2, t)$$

Vectorial backstepping

Step 2: We have $z_2 := x_2 - \alpha_1$ and let $z_3 := x_3 - \alpha_2$. Differentiating gives

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = G_2(\bar{x}_2)x_3 + f_2(\bar{x}_2) - \sigma_1(\bar{x}_2, t) \\ &= G_2(\bar{x}_2)z_3 + G_2(\bar{x}_2)\alpha_2 + f_2(\bar{x}_2) - \sigma_1(\bar{x}_2, t) \end{aligned}$$

Let A_2 be a desired Hurwitz matrix for the z_2-subsystem and $Q_2 = Q_2^\top > 0$ s.t. $P_2 A_2 + A_2^\top P_2 = -Q_2$.
Define the CLF $V_2(z_1, z_2) = V_1(z_1) + z_2^\top P_2 z_2$ s.t.

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + 2z_2^\top P_2 \dot{z}_2 = -z_1^\top Q_1 z_1 + 2z_1^\top P_1 h^{x_1}G_1 z_2 + 2z_2^\top P_2\left[G_2 z_3 + G_2\alpha_2 + f_2 - \sigma_1\right] \\ &= -z_1^\top Q_1 z_1 + 2z_2^\top\left\{G_1^\top\left(h^{x_1}\right)^\top P_1 z_1 + P_2\left[G_2\alpha_2 + f_2 - \sigma_1\right]\right\} + 2z_2^\top P_2 G_2 z_3 \end{aligned}$$

Postponing dealing with z_3 and choosing virtual control

$$\alpha_2(x_1, x_2, t) = G_2^{-1}\left[-P_2^{-1}G_1^\top\left(h^{x_1}\right)^\top P_1 z_1 + A_2 z_2 - f_2 + \sigma_1\right]$$

gives

$$\begin{aligned} \dot{V}_2 &= -z_1^\top Q_1 z_1 + 2z_2^\top P_2 A_2 z_2 + 2z_2^\top P_2 G_2 z_3 \\ &= -z_1^\top Q_1 z_1 + z_2^\top\left[P_2 A_2 + A_2^\top P_2\right]z_2 + 2z_2^\top P_2 G_2 z_3 \\ &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + 2z_2^\top P_2 G_2 z_3 \\ \dot{z}_2 &= -P_2^{-1}G_1^\top\left(h^{x_1}\right)^\top P_1 z_1 + A_2 z_2 + G_2(\bar{x}_2)z_3 \end{aligned}$$

Aiding next step,

$$\dot{\alpha}_2 = \alpha_2^{x_1}\dot{x}_1 + \alpha_2^{x_2}\dot{x}_2 + \alpha_2^t =: \sigma_2(\bar{x}_3, t)$$

Vectorial backstepping

Step 3: We have $z_3 := x_3 - \alpha_2$ and now we will hit the control u. Differentiating gives

$$\dot{z}_3 = \dot{x}_3 - \dot{\alpha}_2 = G_3(\bar{x}_3)u + f_3(\bar{x}_3) - \sigma_2(\bar{x}_3, t)$$

Let A_3 be a desired Hurwitz matrix for the z_3-subsystem and $Q_3 = Q_3^\top > 0$ s.t. $P_3 A_3 + A_3^\top P_3 = -Q_3$.
Define the CLF $V_3(z_1, z_2, z_3) = V_2(z_1, z_2) + z_3^\top P_3 z_3$ s.t.

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 + 2z_3^\top P_3 \dot{z}_3 = -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + 2z_2^\top P_2 G_2 z_3 + 2z_3^\top P_3 \left[G_3 u + f_3 - \sigma_2\right] \\ &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + 2z_3^\top \left\{G_2^\top P_2 z_2 + P_3\left[G_3 u + f_3 - \sigma_2\right]\right\}\end{aligned}$$

We then assign the control law

$$u = G_3^{-1}\left[-P_3^{-1} G_2^\top P_2 z_2 + A_3 z_3 - f_3 + \sigma_2\right]$$

gives

$$\begin{aligned}\dot{V}_3 &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + 2z_3^\top P_3 A_3 z_3 \\ &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + z_3^\top P_3 A_3 z_3 + z_3^\top P_3 A_3 z_3 \\ &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + z_3^\top P_3 A_3 z_3 + z_3^\top A_3^\top P_3 z_3 \\ &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 + z_3^\top \left(P_3 A_3 + A_3^\top P_3\right) z_3 \\ &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 - z_3^\top Q_3 z_3 \\ \dot{z}_3 &= -P_3^{-1} G_2^\top P_2 z_2 + A_3 z_3\end{aligned}$$

Vectorial backstepping

Again we can summarize what we have done. We got the closed-loop:

$$\begin{aligned}\dot{z}_1 &= A_1 z_1 + h^{x_1}(\bar{x}_1) G_1(\bar{x}_1) z_2 \\ \dot{z}_2 &= -P_2^{-1} G_1(\bar{x}_1)^\top h^{x_1}(\bar{x}_1)^\top P_1 z_1 + A_2 z_2 + G_2(\bar{x}_2) z_3 \\ \dot{z}_3 &= -P_3^{-1} G_2(\bar{x}_2)^\top P_2 z_2 + A_3 z_3\end{aligned}$$

and the CLF

$$\begin{aligned}V_3(z_1, z_2, z_3) &= z_1^\top P_1 z_1 + z_2^\top P_2 z_2 + z_3^\top P_3 z_3 \\ \dot{V}_3 &= -z_1^\top Q_1 z_1 - z_2^\top Q_2 z_2 - z_3^\top Q_3 z_3\end{aligned}$$

In vector form, let $z = \mathrm{col}(z_1, z_2, z_3)$, $P = \mathrm{diag}(P_1, P_2, P_3)$, and $Q = \mathrm{diag}(Q_1, Q_2, Q_3)$. Then the design results in the closed-loop system

$$\dot{z} = \begin{bmatrix} A_1 & h^{x_1}(\bar{x}_1) G_1(\bar{x}_1) & 0 \\ -P_2^{-1} G_1(\bar{x}_1)^\top h^{x_1}(\bar{x}_1)^\top P_1 & A_2 & G_2(\bar{x}_2) \\ 0 & -P_3^{-1} G_2(\bar{x}_2)^\top P_2 & A_3 \end{bmatrix} z = H(\bar{x}_2) z$$

where we note that the closed-loop system is now nonlinear.
The recursively generated CLF is similarly

$$V(z) = z^\top P z, \qquad \dot{V} = -z^\top Q z$$

We also see that the matrix $H(\bar{x}_2)$ can be written

$$H(\bar{x}_2) = A + G(\bar{x}_2)$$

where $A = \mathrm{diag}(A_1, A_2, A_3)$ is Hurwitz, and $G(\bar{x}_2)$ has a skew-symmetry-like structure so that

$$V^z(z) G(\bar{x}_2) = 0.$$

Vectorial LgV-backstepping

This design is based upon the article by Arcak and Kokotović (2001).

We consider the nonlinear plant:

$$\begin{aligned}\dot{x}_1 &= G_1(\bar{x}_1)\,x_2 + f_1(\bar{x}_1)\\ \dot{x}_2 &= G_2(\bar{x}_2)\,x_3 + f_2(\bar{x}_2)\\ &\vdots\\ \dot{x}_n &= G_n(\bar{x}_n)\,u + f_n(\bar{x}_n)\\ y &= x_1\end{aligned}$$

and a tracking objective $y(t) \to y_d(t)$ using LgV backstepping.

Note again the compact notation $\bar{x}_j := \mathrm{col}(x_1, \ldots, x_j)$. Note that we also, for simplicity, assume $y = x_1$ and not $y = h(x_1)$.

Again, let's do 3 steps of *Backstepping* for a system with $n = 3$, but now with the LgV technique.

LgV vectorial backstepping

Step 1: Let $z_1 := y - y_d(t)$ and $z_2 := x_2 - \alpha_1$. Differentiating gives

$$\dot{z}_1 = \dot{x}_1 - \dot{y}_d = G_1 x_2 + f_1 - \dot{y}_d(t) = G_1 z_2 + G_1\alpha_1 + f_1 - \dot{y}_d(t)$$

Instead of choosing a Hurwitz matrix A_1, we will now just choose $A_1 = -C_1$ where $C_1 = C_1^\top > 0$. Then choosing $P_1 = \frac{1}{2}I$ gives $P_1A_1 + A_1^\top P_1 = -C_1 = -Q_1$.
Define the CLF $V_1(z_1) = \frac{1}{2}z_1^\top z_1$ s.t.
$\dot{V}_1 = z_1^\top \dot{z}_1 = z_1^\top\left[G_1\alpha_1 + f_1 - \dot{y}_d(t)\right] + z_1^\top G_1 z_2$. We now choose the virtual control

$$\alpha_1(x_1, t) = G_1^{-1}\left[-C_1 z_1 - f_1 + \dot{y}_d(t) + \alpha_{10}\right]$$

where α_{10} is yet to be assigned. This gives

$$\dot{V}_1 = -z_1^\top C_1 z_1 + z_1^\top\alpha_{10} + z_1^\top G_1 z_2.$$

Using Young's inequality ($a^\top = z_1^\top G_1$, $b = z_2$), we get

$$\begin{aligned}\dot{V}_1 &\leq -z_1^\top C_1 z_1 + z_1^\top\alpha_{10} + \kappa_1 z_1^\top G_1 G_1^\top z_1 + \frac{1}{4\kappa_1}z_2^\top z_2\\ &= -z_1^\top C_1 z_1 + z_1^\top\left[\alpha_{10} + \kappa_1 G_1 G_1^\top z_1\right] + \frac{1}{4\kappa_1}z_2^\top z_2\end{aligned}$$

and choose

$$\alpha_{10} = -\kappa_1 G_1 G_1^\top z_1 \quad \Rightarrow \quad \begin{cases}\dot{V}_1 \leq -z_1^\top C_1 z_1 + \frac{1}{4\kappa_1}z_2^\top z_2\\ \dot{z}_1 = -\left(C_1 + \kappa_1 G_1 G_1^\top\right)z_1 + G_1 z_2\end{cases}$$

Aiding next step,

$$\dot{\alpha}_1 = \alpha_1^{x_1}\dot{x}_1 + \alpha_1^t =: \sigma_1(\bar{x}_2, t)$$

LgV vectorial backstepping

Step 2: We have $z_2 := x_2 - \alpha_1$ and let $z_3 := x_3 - \alpha_2$. Differentiating gives

$$\begin{aligned}\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 &= G_2(\bar{x}_2)x_3 + f_2(\bar{x}_2) - \sigma_1(\bar{x}_2, t)\\ &= G_2(\bar{x}_2)z_3 + G_2(\bar{x}_2)\alpha_2 + f_2(\bar{x}_2) - \sigma_1(\bar{x}_2, t)\end{aligned}$$

Define the CLF $V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2}z_2^\top z_2$ s.t.

$$\dot{V}_2 = \dot{V}_1 + z_2^\top \dot{z}_2 \le -z_1^\top C_1 z_1 + \frac{1}{4\kappa_1} z_2^\top z_2 + z_2^\top [G_2\alpha_2 + f_2 - \sigma_1] + z_2^\top G_2 z_3$$

Postponing dealing with z_3 and choosing virtual control

$$\alpha_2(x_1, x_2, t) = G_2^{-1}[-C_2 z_2 - f_2 + \sigma_1 + \alpha_{20}]$$

where α_{20} is yet to be assigned. This gives

$$\begin{aligned}\dot{V}_2 &\le -z_1^\top C_1 z_1 + \frac{1}{4\kappa_1} z_2^\top z_2 - z_2^\top C_2 z_2 + z_2^\top \alpha_{20} + z_2^\top G_2 z_3\\ &\le -z_1^\top C_1 z_1 - z_2^\top \left[C_2 - \frac{1}{4\kappa_1} I\right] z_2 + z_2^\top \alpha_{20} + \kappa_2 z_2^\top G_2 G_2^\top z_2 + \frac{1}{4\kappa_2} z_3^\top z_3\\ &= -z_1^\top C_1 z_1 - z_2^\top \left[C_2 - \frac{1}{4\kappa_1} I\right] z_2 + z_2^\top \left[\alpha_{20} + \kappa_2 G_2 G_2^\top z_2\right] + \frac{1}{4\kappa_2} z_3^\top z_3\end{aligned}$$

and choose

$$\alpha_{20} = -\kappa_2 G_2 G_2^\top z_2 \quad \Rightarrow \quad \begin{cases} \dot{V}_2 \le -z_1^\top C_1 z_1 - z_2^\top \left[C_2 - \frac{1}{4\kappa_1} I\right] z_2 + \frac{1}{4\kappa_2} z_3^\top z_3 \\ \dot{z}_2 = -\left[C_2 + \kappa_2 G_2 G_2^\top\right] z_2 + G_2 z_3 \end{cases}$$

Aiding next step,

$$\dot{\alpha}_2 = \alpha_2^{x_1}\dot{x}_1 + \alpha_2^{x_2}\dot{x}_2 + \alpha_2^t =: \sigma_2(\bar{x}_3, t)$$

LgV vectorial backstepping

Step 3: We have $z_3 := x_3 - \alpha_2$ and now we will hit the control u. Differentiating gives

$$\dot{z}_3 = \dot{x}_3 - \dot{\alpha}_2 = G_3(\bar{x}_3)u + f_3(\bar{x}_3) - \sigma_2(\bar{x}_3, t)$$

Define the CLF $V_3(z_1, z_2, z_3) = V_2(z_1, z_2) + \frac{1}{2}z_3^\top z_3$ s.t.

$$\dot{V}_3 = \dot{V}_2 + z_3^\top \dot{z}_3 \le -z_1^\top C_1 z_1 - z_2^\top \left[C_2 - \frac{1}{4\kappa_1} I\right] z_2 + \frac{1}{4\kappa_2} z_3^\top z_3 + z_3^\top [G_3 u + f_3 - \sigma_2]$$

We then assign the control law

$$u = G_3^{-1}[-C_3 z_3 - f_3 + \sigma_2]$$

gives

$$\begin{aligned}\dot{V}_3 &= \dot{V}_2 + z_3^\top \dot{z}_3 \le -z_1^\top C_1 z_1 - z_2^\top \left[C_2 - \frac{1}{4\kappa_1} I\right] z_2 - z_3^\top \left[C_3 + \frac{1}{4\kappa_2} I\right] z_3\\ \dot{z}_3 &= -C_3 z_3\end{aligned}$$

We get that $(z_1, z_2, z_3) = (0, 0, 0)$ is UGES for $C_2 > \frac{1}{4\kappa_1} I$ and $C_3 > \frac{1}{4\kappa_2} I$.

LgV vectorial backstepping

Again we can summarize what we have done. We got the closed-loop:

$$
\begin{aligned}
\dot z_1 &= -\left(C_1 + \kappa_1 G_1 G_1^\top\right) z_1 + G_1 z_2 \\
\dot z_2 &= -\left[C_2 + \kappa_2 G_2 G_2^\top\right] z_2 + G_2 z_3 \\
\dot z_3 &= -C_3 z_3.
\end{aligned}
$$

Do you notice anything peculiar about this closed-loop system?

We also get the CLF

$$
\begin{aligned}
V_3(z_1, z_2, z_3) &= \frac{1}{2} z_1^\top z_1 + \frac{1}{2} z_2^\top z_2 + \frac{1}{2} z_3^\top z_3 \\
\dot V_3 &\leq -z_1^\top C_1 z_1 - z_2^\top \left[C_2 - \frac{1}{4\kappa_1} I\right] z_2 - z_3^\top \left[C_3 + \frac{1}{4\kappa_2} I\right] z_3
\end{aligned}
$$

In vector form, using $z = \mathrm{col}(z_1, z_2, z_3)$, the closed-loop system becomes

$$
\dot z = \begin{bmatrix} -C_1 - \kappa_1 G_1(\bar x_1) G_1(\bar x_1)^\top & G_1(\bar x_1) & 0 \\ 0 & -C_2 - \kappa_2 G_2(\bar x_2) G_2(\bar x_2)^\top & G_2(\bar x_2) \\ 0 & 0 & -C_3 \end{bmatrix} z =: H(\bar x_2) z
$$

where we note that the closed-loop system is now given by an upper triangular matrix $H(\bar x_2)$, indicating a purely cascaded system.
The recursively generated CLF is similarly

$$
V(z) = \frac{1}{2} z^\top z, \qquad \dot V \leq -q z^\top z.
$$

A nonlinear DP plant with a bias

We consider the DP plant:

$$
\left.\begin{aligned} \dot\eta &= R(\psi)\nu \\ M\dot\nu &= \tau - D\nu + b \end{aligned}\right\} \quad \text{Parametric strict feedback form}
$$

where $\eta = \mathrm{col}(x, y, \psi)$, $\nu = \mathrm{col}(u, v, r)$, $M = M^\top > 0$ and $D > 0$ are the mass and damping matrices, respectively, b is a constant bias load, and R is the rotation matrix.

Notice in particular, for

$$
R = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

that the following properties hold:

$$
\begin{aligned}
R(\psi)^\top R(\psi) = R(\psi) R(\psi)^\top &= I, \qquad \det R(\psi) = 1 \\
\dot R = R(\psi) S(r), \qquad \dot R^\top &= -S(r) R(\psi)^\top, \qquad S(r) = -S(r)^\top.
\end{aligned}
$$

Method A - Integral action on output

Integral action: With $z_1 = R(\psi)^\top (\eta - \eta_d(t))$ and $z_2 = \nu - \alpha_1$, use

$$\xi(t) = \int_0^t z_1(\tau) d\tau \quad \Rightarrow \quad \dot{\xi} = z_1 = R(\psi)^\top (\eta - \eta_d(t))$$

Step 1: Assuming $z_2 = 0$, design $\alpha_1(\xi, \eta, t)$ to stabilize:

$$\begin{aligned}
\dot{\xi} &= z_1 \\
\dot{z}_1 &= \dot{R}^\top (\eta - \eta_d) + R^\top (\dot{\eta} - \dot{\eta}_d) \\
&= -SR^\top (\eta - \eta_d) + R^\top (R\nu - \dot{\eta}_d) \\
&= -Sz_1 + z_2 + \alpha_1 - R^\top \dot{\eta}_d
\end{aligned}$$

Step 2: Assume (for now) $b = 0$ and design τ to stabilize $(\dot{\xi}, \dot{z}_1)$ and

$$M\dot{z}_2 = \tau - D\nu - M\dot{\alpha}_1$$

...Method A

Resulting closed loop system

$$\begin{aligned}
\dot{\xi} &= z_1 \\
\dot{z}_1 &= -C_0\xi - C_1 z_1 - Sz_1 + z_2 \\
M\dot{z}_2 &= -z_1 - (C_2 + D) z_2 + b
\end{aligned}$$

- What is the equilibrium, if the reference $\eta_d(t) = \eta_{ref}$ is constant and:
  - the bias $b = 0$?
  - the bias $b \neq 0$?
- What is the equilibrium, if the reference $\eta_d(t)$ is time-varying and:
  - the bias $b = 0$?
  - the bias $b \neq 0$?

- Is the bias matched with the integral action?

Method B - Integral action matched to bias

Integral action: With $z_1 = R(\psi)^\top (\eta - \eta_d(t))$ and $z_2 = \nu - \alpha_1$, use

$$\xi(t) = \int_0^t z_2(\tau) d\tau \quad \Rightarrow \quad \dot{\xi} = z_2 = \nu - \alpha_1(\eta, t)$$

Step 1: Assuming $z_2 = 0$, design α_1 to stabilize:

$$\dot{z}_1 = -S z_1 + z_2 + \alpha_1 - R^\top \dot{\eta}_d$$

Step 2: Assume (for now) $b = 0$ and design τ to stabilize $\dot{z}_1$ and

$$\begin{aligned} \dot{\xi} &= z_2 \\ M \dot{z}_2 &= \tau - D\nu - M\dot{\alpha}_1 \end{aligned}$$

...Method B

Resulting closed loop system

$$\begin{aligned} \dot{z}_1 &= -C_1 z_1 - S z_1 + z_2 \\ \dot{\xi} &= z_2 \\ M \dot{z}_2 &= -C_0 \xi - z_1 - (C_2 + D) z_2 + b \end{aligned}$$

- What is the equilibrium, if:
  - the bias $b = 0$?
  - the bias $b \neq 0$?

- Let $\tilde{\xi} := \xi - C_0^{-1} b$ and differentiate $\tilde{\xi}$. What is then:
  - closed-loop?
  - equilibrium?

Preparations for next lecture

Observer designs:

- Khalil, H. K. (2015). Nonlinear Control:
  - Chapter 11
- Lecture note “Observer for simplified DP model: Design and proof”.
- Lecture presentation.

Bibliography

Arcak, M. and Kokotović, P. (2001). Redesign of backstepping for robustness against unmodelled dynamics. *Int. J. Robust Nonlinear Contr.*, 11(7):633–643. Robustness in identification and control.