Marine Control Systems II

Lecture 8: DP Control System

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Goals of lecture

- Learn about industrial DP control systems and requirements to system topology, faults and failures, and redundancy and segregation.
- Learn how to model the thruster configuration, formulate the thrust allocation problem, and design a simple thrust allocation algorithm.
- Design a simple state observer for DP.
- Design DP control algorithm(s).
- Implement a maneuvering control strategy for DP.

Literature

- Lecture notes and presentation.

Notation

We use the following notation for the different motion parameters:

- Vectors $x \in \mathbb{R}^3$ are referenced to coordinate reference frames $\{a\}$, $\{b\}$, etc., through sub- and superscripts, where the superscript typically indicates along which coordinate axes the rectangular coordinates are decomposed.
- Position vector $p \in \mathbb{R}^3$ is denoted p^c_{ab} where the scripts means that the position vector is from the origin of frame a to origin frame b, decomposed along the coordinate axes of frame c. All position vectors are generally assumed to be in rectangular coordinates, unless otherwise stated.
- Linear velocity and acceleration vectors $v \in \mathbb{R}^3$ and $a \in \mathbb{R}^3$ are denoted v^c_{ab} and a^c_{ab} meaning the velocity and acceleration of frame b relative to frame a, decomposed in frame c.
- Euler angle orientation vector $\Theta \in \mathcal{S}^3$ is denoted Θ_{ab} meaning the angles from a to b.
- Angular velocity and acceleration vectors $\omega \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}^3$ are denoted ω^c_{ab} and α^c_{ab} meaning the angular velocity and acceleration of frame b with respect to a, decomposed in c.
- Any vector x^c, decomposed in coordinate frame $\{c\}$, can be transformed into another coordinate frame $\{d\}$ by using a rotation matrix $R^d_c \in \mathcal{SO}(3)$, that is, $x^d = R^d_c x^c$.
- The derivative of a rotation matrix is $\dot{R}^d_c = R^d_c S(\omega^c_{dc})$ where $S(\omega^c_{dc}) = -S(\omega^c_{dc})^\top$ is the skew-symmetric matrix constructed by the angular velocity ω^c_{dc}, that is, for $\omega := \mathrm{col}(\omega_x, \omega_y, \omega_z)$ we get

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

6DOF kinematics

We consider two reference frames, the Earth-fixed North-East-Down (NED) frame $\{n\}$ and the body-fixed (BODY) frame $\{b\}$.

- NED is a local frame with origin nearby the operation, e.g. at some intialization point of the position reference systems or at the setpoint of the DP. NED is assumed to be an inertial frame.
- BODY is a fixed coordinate frame in the vessel, typically set at midship and waterline, with x-axis forward, y-axis to starboard, and z-axis down.

We let the 6DOF position and orientation vector be $\eta = \mathrm{col}(p, \Theta) \in \mathbb{R}^3 \times \mathcal{S}^3$, where $p = \mathrm{col}(x, y, z) = p_{bn}^n$ is the position of $\{b\}$ in $\{n\}$, decomposed in $\{n\}$, and $\Theta = \mathrm{col}(\phi, \theta, \psi) = \Theta_{nb}$ is the orientation of $\{b\}$ in $\{n\}$.

The corresponding body-fixed velocity vector $\nu = \mathrm{col}(u, v, w, p, q, r) \in \mathbb{R}^6$, where (u, v, w) is the linear velocity in surge, sway, and heave, and (p, q, r) is the angular rate vector in roll, pitch, and yaw.

6DOF kinematics

The vessel kinematic model is now written (Fossen, 2011)

$$\dot{\eta} = J(\Theta)\nu,$$

where the transformation matrix is

$$J(\Theta) := \begin{bmatrix} R_{3D}(\Theta) & 0 \\ 0 & T_\Theta(\Theta) \end{bmatrix}$$

with $R_{3D}(\Theta) = R_b^n(\Theta_{nb}) = R_{z,\psi} R_{y,\theta} R_{x,\phi}$ being the rotation matrix between $\{b\}$ and $\{n\}$,

$$R_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}, \quad R_{y,\theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}, \quad R_{z,\psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{3D}(\Theta) = \begin{bmatrix} \cos\theta\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

and

$$T_\Theta(\Theta) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \end{bmatrix}.$$

6DOF kinetics

Let an irrotational, constant, and horizontal surface current velocity vector $c \in \mathbb{R}^6$ with speed V_c and direction β_c in NED, be given by

$$c = \begin{bmatrix} c_x \\ c_y \\ 0_{4\times 1} \end{bmatrix} = \begin{bmatrix} V_c \cos\beta_c \\ V_c \sin\beta_c \\ 0_{4\times 1} \end{bmatrix}.$$

The relative body-fixed velocity ν_r between the hull and fluid becomes

$$\nu_r := \nu - \nu_c = \nu - J(\Theta)^{-1} c.$$

The kinetic model is given by

$$M_{rb}\dot{\nu} + C_{rb}(\nu)\nu + M_a\dot{\nu}_r + C_a(\nu_r)\nu_r + D(\nu_r)\nu_r + \mu + g(\eta) = \tau + \tau_{wind} + \tau_{wave},$$

where:

- $M_{rb}\dot{\nu} + C_{rb}(\nu)\nu + M_a\dot{\nu}_r + C_a(\nu_r)\nu_r$ are inertial loads,
- $D(\nu_r)\nu_r + \mu$ are damping loads,
- $g(\eta)$ is restoring load,
- $\tau_{wind} + \tau_{wave}$ are environmental loads, and
- τ are propulsion loads; see Fossen (2011) for details.

6DOF kinetics

Note:

- Hydrodynamic loads depend on relative velocity ν_r between hull and fluid.
- According to Fossen (2011, Property 8.1), $C_{rb}(\nu)$ can, for a constant irrotational current, be parameterized such that

  $$M_{rb}\dot{\nu} + C_{rb}(\nu)\nu = M_{rb}\dot{\nu}_r + C_{rb}(\nu_r)\nu_r.$$

  Hence, letting $M = M_{rb} + M_a$ and $C(\nu_r) = C_{rb}(\nu_r) + C_a(\nu_r)$ we get

  $$M\dot{\nu}_r + C(\nu_r)\nu_r + D(\nu_r)\nu_r + \mu + g(\eta) = \tau + \tau_{wind} + \tau_{wave}$$

For six degrees of freedom (6DOF) control, the thrust load vector is $\tau \in \mathbb{R}^6$. For m thrusters, this is defined as

$$\tau = \tau_1 + \tau_2 + \cdots + \tau_m$$

where for $i \in \{1, \ldots, m\}$,

$$\tau_i := \begin{bmatrix} \mathcal{F}_i \\ \mathcal{M}_i \end{bmatrix}, \qquad \mathcal{F}_i := \begin{bmatrix} X_i \\ Y_i \\ Z_i \end{bmatrix} [\mathrm{N}], \qquad \mathcal{M}_i := \begin{bmatrix} K_i \\ M_i \\ N_i \end{bmatrix} [\mathrm{N\,m}].$$

3DOF kinematics

For 3 DOF horizontal motion of a floating vessel, we assume due to roll-pitch stability and zero heave that $\phi \approx 0$, $\theta \approx 0$, and $z \approx 0$.

In this case we use $\eta = \mathrm{col}(x, y, \psi) \in \mathbb{R}^2 \times \mathcal{S}^1$, where (x, y) is the horizontal position of $\{b\}$ in $\{n\}$, and ψ is the yaw angle - which we refer to as the *ship heading*. The body-fixed velocity vector is $\nu = \mathrm{col}(u, v, r) \in \mathbb{R}^3$ for surge, sway, yaw.

The kinematic model now becomes

$$\dot{\eta} = R(\psi)\nu,$$

where

$$R(\psi) := \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

satisfying:

- $R(\psi)$ is a rotation matrix so that $R(\psi)^\top R(\psi) = R(\psi)R(\psi)^\top = I$ and $\det R(\psi) = 1$.
- $\dot{R} := \frac{d}{dt}R(\psi(t)) = r(t)R(\psi(t))S$ where $r = \dot{\psi}$ and

$$S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -S^\top.$$

3DOF kinetics

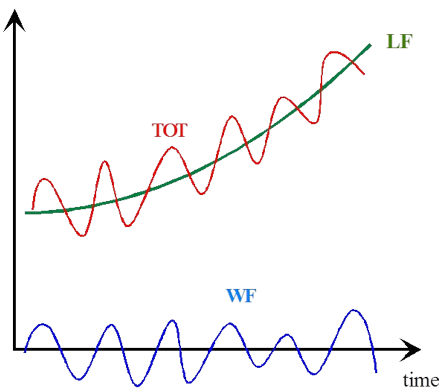
The wave-induced loads τ_{wave} consists of

$$\tau_{wave} = \tau_{wave,1} + \tau_{wave,2}$$

where $\tau_{wave,1}$ and $\tau_{wave,2}$ are the 1$^{\text{st}}$ and 2$^{\text{nd}}$ order wave loads, respectively:

- $\tau_{wave,1}$ produce oscillatory motions with zero mean horizontal loads.
- $\tau_{wave,2}$ is wave-drift loads that cause drift motion.

Hence, the total motion of a slow-speed vessel is assumed to consist of a wave-frequency motion superimposed on the low-frequency dynamics due to propulsion and low-frequency excitation loads, that is, $\eta_{tot} = \eta_{lf} + \eta_{wf}$.



3DOF kinetics

We let $\eta = \eta_{lf}$, $\nu = \nu_{lf}$, and use an *internal model principle*[1] to capture the wave-frequency motion, that is,

$$\begin{aligned} \dot{\xi} &= A_w \xi + E_w \omega_w \\ \eta_{wf} &= C_w \xi. \end{aligned} \qquad \text{Damped harmonic oscillator models WF motion}$$

We also assume slow-speed so that the velocity squared terms and cross-terms become negligible. This justifies that

$$\begin{aligned} C(\nu_r)\nu_r &\approx 0 && \text{No Coriolis} \\ D(\nu_r)\nu_r &\approx D\nu_r && \text{Linear damping} \end{aligned}$$

where D is a constant matrix of damping coefficients. In addition we neglect fluid memory effects and note that $g(\eta) = 0$ for a free-floating vessel in our 3DOF equations.

We then have

$$M\dot{\nu}_r + D\nu_r = \tau + \tau_{wind} + \tau_{wave,2}$$

[1] Internal model principle will be elaborated further in Lecture 9.

3DOF kinetics

In 3DOF, we note that the constant irrotational surface current become $c \in \mathbb{R}^3$ defined as

$$c = \begin{bmatrix} c_x \\ c_y \\ 0 \end{bmatrix} = \begin{bmatrix} V_c \cos\beta_c \\ V_c \sin\beta_c \\ 0 \end{bmatrix}, \qquad \nu_r := \nu - \nu_c = \nu - R(\psi)^\top c$$

Assuming also that $\dot{\nu}_c = \dot{R}^\top c = -rSR(\psi)^\top c \approx 0$, we get

$$M\dot{\nu} + D\nu = \tau + DR(\psi)^\top c + \tau_{wave,2} + \tau_{wind}$$

Let a bias load vector $b \in \mathbb{R}^3$ characterize the slowly-varying loads due to:

- current drag,
- wave drift, and
- unmodeled dynamics in wind, thrusters, and other hydrodynamic loads.

This is typically set the dynamics

$$\begin{aligned} \dot{b} &= -T_b^{-1} b + E_b \omega_b && \text{Markov process} \\ &\text{or} \\ \dot{b} &= \omega_b && \text{Wiener process (random walk).} \end{aligned}$$

As the slowly-varying loads captured by b is as much body-fixed loads as NED-fixed, studies (**?**) have shown that representing b in $\{b\}$ or in $\{n\}$ is equally valid.

3DOF kinetics

The resulting 3DOF stochastic (including noise) **observer design model** for DP (with b in $\{n\}$) now becomes

$$\begin{aligned}
\dot{\xi} &= A_w \xi + E_w \omega_w \\
\dot{\eta} &= R(\psi)\nu \\
\dot{b} &= \omega_b \\
M\dot{\nu} + D\nu &= \tau + R(\psi)^\top b + \tau_{wind} + \omega_\nu \\
y &= \eta + C_w \xi + \upsilon_y
\end{aligned}$$

where ω_w, ω_b, and ω_ν are white process noise, and υ_y are white measueremt noise.

The corresponding 3DOF deterministic (excluding noise) **control design model** for DP becomes

$$\begin{aligned}
\dot{\eta} &= R(\psi)\nu \\
M\dot{\nu} + D\nu &= \tau + R(\psi)^\top b + \tau_{wind}.
\end{aligned}$$

6DOF thruster configuration

The thruster configuration for a 6DOF vessel with m thrusters, is in general given by a set of lever arm vectors $l_i \in \mathbb{R}^3$, $i \in \{1, \ldots, m\}$ from the vessel's coordinate origin (CO) to the location of each individual thruster.

For a single thruster, let the thrust vector it produces in the body frame, and its location in the body frame, be

$$\mathcal{F}_i = \begin{bmatrix} \mathcal{F}_{i,x} \\ \mathcal{F}_{i,y} \\ \mathcal{F}_{i,z} \end{bmatrix}, \qquad l_i = \begin{bmatrix} l_{i,x} \\ l_{i,y} \\ l_{i,z} \end{bmatrix},$$

respectively. Then the corresponding thrust load from this thruster alone becomes

$$\tau_i = \begin{bmatrix} \mathcal{F}_i \\ \mathcal{M}_i \end{bmatrix} = \begin{bmatrix} \mathcal{F}_i \\ l_i \times \mathcal{F}_i \end{bmatrix} = \begin{bmatrix} \mathcal{F}_{i,x} \\ \mathcal{F}_{i,y} \\ \mathcal{F}_{i,z} \\ l_{i,y}\mathcal{F}_{i,z} - l_{i,z}\mathcal{F}_{i,y} \\ l_{i,z}\mathcal{F}_{i,x} - l_{i,x}\mathcal{F}_{i,z} \\ l_{i,x}\mathcal{F}_{i,y} - l_{i,y}\mathcal{F}_{i,x} \end{bmatrix}.$$

6DOF thruster configuration

For a fixed thruster producing thrust F_i in:

- pure surge direction becomes $\mathcal{F}_i = \mathrm{col}(F_i, 0, 0)$,
- pure sway becomes $\mathcal{F}_i = \mathrm{col}(0, F_i, 0)$,
- pure heave becomes $\mathcal{F}_i = \mathrm{col}(0, 0, F_i)$, and
- an azimuth angle $\alpha_i \in \mathcal{S}^1$ in the body-frame, yields

$$\mathcal{F}_i = \begin{bmatrix} F_i \cos\alpha_i \\ F_i \sin\alpha_i \\ 0 \end{bmatrix}.$$

Generally, for a thruster producing thrust F_i at alignment $\alpha_i = \mathrm{col}(\alpha_{i,x}, \alpha_{i,y}, \alpha_{i,z}) \in \mathcal{S}^3$ in the body-frame becomes

$$\mathcal{F}_i = R_{3D}(\alpha_i) \begin{bmatrix} F_i \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} F_i \cos\alpha_{i,y} \cos\alpha_{i,z} \\ F_i \cos\alpha_{i,y} \sin\alpha_{i,z} \\ -F_i \sin\alpha_{i,y} \end{bmatrix}.$$

6DOF thruster configuration

For m thrusters, each with a horizontal orientation $\alpha_i \in \mathcal{S}^1$, we then get the total thrust loads

$$\tau = \sum_{i=1}^{m} \tau_i = \sum_{i=1}^{m} \begin{bmatrix} \mathcal{F}_i \\ l_i \times \mathcal{F}_i \end{bmatrix} = \sum_{i=1}^{m} \begin{bmatrix} \cos\alpha_i \\ \sin\alpha_i \\ 0 \\ -l_{i,z} \sin\alpha_i \\ l_{i,z} \cos\alpha_i \\ l_{i,x} \sin\alpha_i - l_{i,y} \cos\alpha_i \end{bmatrix} F_i$$

Letting $F := \mathrm{col}(F_1, F_2, \cdots, F_m)$ be the vector of thruster forces and $\alpha := \mathrm{col}(\alpha_1, \ldots, \alpha_m)$ the vector of azimuth angles, we can write this as

$$\begin{aligned} \tau &= B(\alpha) F \\ &= \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \cdots & \cos\alpha_m \\ \sin\alpha_1 & \sin\alpha_2 & \cdots & \sin\alpha_m \\ 0 & 0 & \cdots & 0 \\ -l_{1,z}\sin\alpha_1 & -l_{2,z}\sin\alpha_2 & \cdots & -l_{m,z}\sin\alpha_m \\ l_{1,z}\cos\alpha_1 & l_{2,z}\cos\alpha_2 & \cdots & l_{m,z}\cos\alpha_m \\ l_{1,x}\sin\alpha_1 - l_{1,y}\cos\alpha_1 & l_{2,x}\sin\alpha_2 - l_{2,y}\cos\alpha_2 & \cdots & l_{m,x}\sin\alpha_m - l_{m,y}\cos\alpha_m \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix} \end{aligned}$$

The matrix $B(\alpha) \in \mathbb{R}^{6\times m}$ we call the **thruster configuration matrix**.

3DOF thruster configuration

In 3DOF we delete dimensions 3-5, such that

$$\tau = \sum_{i=1}^{m} \begin{bmatrix} \cos\alpha_i \\ \sin\alpha_i \\ l_{i,x}\sin\alpha_i - l_{i,y}\cos\alpha_i \end{bmatrix} F_i$$

and on vector form

$$\tau = B(\alpha)F$$

$$= \begin{bmatrix} \cos\alpha_1 & \cos\alpha_2 & \cdots & \cos\alpha_m \\ \sin\alpha_1 & \sin\alpha_2 & \cdots & \sin\alpha_m \\ l_{1,x}\sin\alpha_1 - l_{1,y}\cos\alpha_1 & l_{2,x}\sin\alpha_2 - l_{2,y}\cos\alpha_2 & \cdots & l_{m,x}\sin\alpha_m - l_{m,y}\cos\alpha_m \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$$

The *thruster configuration matrix* $B(\alpha) \in \mathbb{R}^{3\times m}$ determines to what degree you have full actuation and redundancy, that is:

- If $\operatorname{rank} B(\alpha) = 3$ (or 6 in 6DOF case), then you have a **fully actuated thruster system**.
- If $\operatorname{rank} B(\alpha) < 3$ (or 6 in 6DOF case), then you have an **underactuated thruster system**.
- If you can delete any one column of $B(\alpha)$ and still maintain full rank ($\operatorname{rank} B(\alpha) = 3$), then you have an **overactuated thruster system**.

3DOF thruster configuration in rectangular thrust vectors

Let us represent the thrust force from an azimuth thruster in rectangular coordinates, as

$$\mathcal{F}_i = \begin{bmatrix} \mathcal{F}_{i,x} \\ \mathcal{F}_{i,y} \\ 0 \end{bmatrix} = \begin{bmatrix} F_i\cos\alpha_i \\ F_i\sin\alpha_i \\ 0 \end{bmatrix}, \qquad F_i = \sqrt{\mathcal{F}_{i,x}^2 + \mathcal{F}_{i,y}^2}, \qquad \alpha_i = \operatorname{atan}\left(\frac{\mathcal{F}_{i,y}}{\mathcal{F}_{i,x}}\right)$$

In this case we get

$$\tau = \sum_{i=1}^{m} \begin{bmatrix} \mathcal{F}_{i,x} \\ \mathcal{F}_{i,y} \\ l_{i,x}\mathcal{F}_{i,y} - l_{i,y}\mathcal{F}_{i,x} \end{bmatrix}$$

Considering now the extended thrust vector

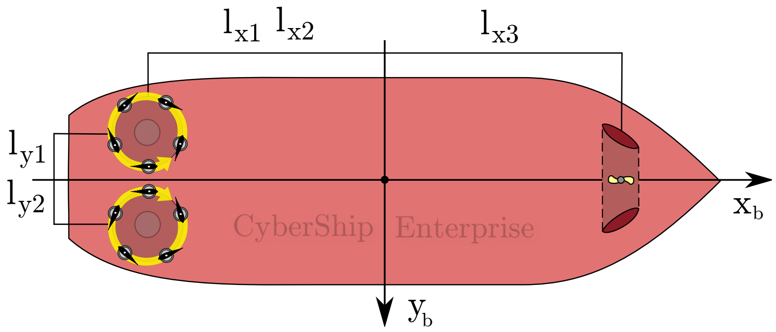
$$F = \begin{bmatrix} \mathcal{F}_{1,x} & \mathcal{F}_{1,y} & \mathcal{F}_{2,x} & \mathcal{F}_{2,y} & \cdots & \mathcal{F}_{m,x} & \mathcal{F}_{m,y} \end{bmatrix}^\top \in \mathbb{R}^{2m},$$

we can write this with a constant *thruster configuration matrix* $B \in \mathbb{R}^{3\times m}$ as

$$\tau = BF := \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \\ -l_{1,y} & l_{1,x} & -l_{2,y} & l_{2,x} & \cdots & -l_{m,y} & l_{m,x} \end{bmatrix} \begin{bmatrix} \mathcal{F}_{1,x} \\ \mathcal{F}_{1,y} \\ \mathcal{F}_{2,x} \\ \mathcal{F}_{2,y} \\ \vdots \\ \mathcal{F}_{m,x} \\ \mathcal{F}_{m,y} \end{bmatrix}.$$

Example: Thruster configuration for CSE

Consider the thruster configuration for C/S Enterprise in the below figure



The thruster configuration is

Table : The C/S Enterprise thruster configuration.

	No.	Propulsor	Location	Power [W]	Force [N]
Stern	1	Voith	[-0.4574 -0.055]	–	–
	2	Voith	[-0.4574 0.055]	–	–
Bow	3	Tunnel	[0.3875 0.00]	–	–

Example: Thruster configuration for CSE

Using rectangular thrust vectors for the Voith-Schneider propellers, we get $\tau = BF$ with $F = \begin{bmatrix} F_{1,x} & F_{1,y} & F_{2,x} & F_{2,y} & F_3 \end{bmatrix}^\top \in \mathbb{R}^5$ and

$$B := \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0.055 & -0.4574 & -0.055 & -0.4574 & 0.3875 \end{bmatrix}.$$

Suppose now that we have calculated F_{cmd}. Then

$$F_{1,cmd} = \sqrt{F_{1,x,cmd}^2 + F_{1,y,cmd}^2}, \qquad \alpha_{1,cmd} = \operatorname{atan}\left(\frac{F_{1,y,cmd}}{F_{1,x,cmd}}\right)$$

$$F_{2,cmd} = \sqrt{F_{2,x,cmd}^2 + F_{2,y,cmd}^2}, \qquad \alpha_{2,cmd} = \operatorname{atan}\left(\frac{F_{2,y,cmd}}{F_{2,x,cmd}}\right)$$

$$F_{3,cmd} = F_{3,cmd}$$

Possibly, each thrust is scaled so that

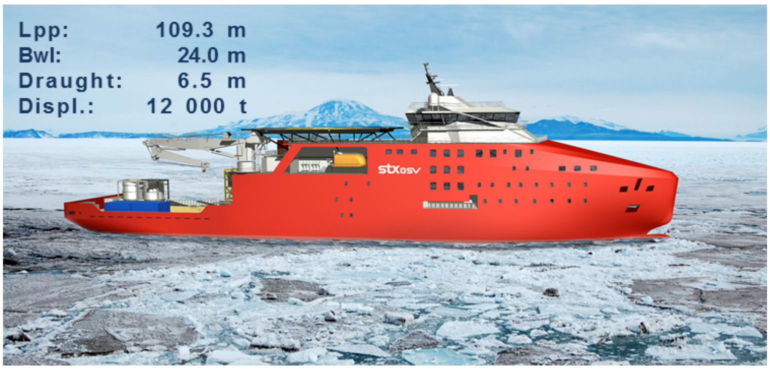
$$F_i = k_i u_i$$

where k_i is a scaling gain and u_i is a normalized signal, say $u_i \in [-1, 1]$. Then we find

$$u_{i,cmd} = \frac{1}{k_i} F_{i,cmd}, \qquad i = 1, 2, 3.$$

Example: Thruster configuration for CIV Arctic vessel

Consider the thruster configuration for CIV Arctic vessel in the below figure



The thruster configuration is

Table : The CIV Arctic thruster configuration.

	No.	Propulsor	Location	Power [MW]	Force [kN]
Stern	1	Azimuth	[-49.9 6.0]	5	720
	2	Azimuth	[-49.9 -6.0]	5	720
Bow	3	Azimuth	[44.6 0]	1.5	216
	4	Azimuth	[43.5 0]	1.5	216
	5	Tunnel	[49,1 0]	1.5	200
	6	Tunnel	[31.9 0]	1.5	200

Example: Thruster configuration for CIV Arctic vessel

Using rectangular thrust vectors for the Voith-Schneider propellers, we get $\tau = BF$ with

$$F = \begin{bmatrix} F_{1,x} & F_{1,y} & F_{2,x} & F_{2,y} & F_{3,x} & F_{3,y} & F_{4,x} & F_{4,y} & F_5 & F_6 \end{bmatrix}^\top \in \mathbb{R}^{10}$$

and

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ -l_{1,y} & l_{1,x} & -l_{2,y} & l_{2,x} & -l_{3,y} & l_{3,x} & -l_{4,y} & l_{4,x} & l_{5,x} & l_{6,x} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ -6 & -49.9 & 6 & -49.9 & 0 & 44.6 & 0 & 43.5 & 49.1 & 31.9 \end{bmatrix}$$

We must then map

$$F_i = \sqrt{F_{i,x}^2 + F_{i,y}^2}, \qquad \alpha_i = \operatorname{atan}\left(\frac{F_{i,y}}{F_{i,x}}\right), \qquad i = \{1, 2, 3, 4\}.$$

3DOF thruster configuration in rectangular thrust vectors

A few considerations on the rectangular thruster configuration:

- Only azimuthing thrusters will be decomposed into its $x - y$ components.
- A tunnel thruster only gives pure sway force. Hence, $\mathcal{F}_{i,x} = 0$ and the corresponding element of F and column of B should be deleted.
- An aft propeller (without rudder) only gives pure surge force. Hence, $\mathcal{F}_{i,y} = 0$ and the corresponding element of F and column of B should be deleted.
- A thruster fixed at an azimuth α_i should keep its column in B as

$$\begin{bmatrix} \cos\alpha_i \\ \sin\alpha_i \\ l_{i,x}\sin\alpha_i - l_{i,y}\cos\alpha_i \end{bmatrix}$$

  (the column will still be constant) and F_i in the corresponding element of F.
- Note that commanding a step in the pair $(\mathcal{F}_{i,x}, \mathcal{F}_{i,y})$ from one sample to the next implies that we can step the thrust magnitude $F_i = \sqrt{\mathcal{F}_{i,x}^2 + \mathcal{F}_{i,y}^2}$ and azimuth $\alpha_i = \operatorname{atan}\left(\frac{\mathcal{F}_{i,y}}{\mathcal{F}_{i,x}}\right)$ equally fast. However, azimuth in particular has a rate limit not allowing large changes in a short time ($360°$ in $1-2$ minutes).

Thrust allocation

For our dynamic DP model

$$\dot{\eta} = R(\psi)\nu$$
$$M\dot{\nu} + D\nu = \tau + R(\psi)^\top b + \tau_{wind}.$$

we consider τ as our control input.

In reality there is a local thruster controller, for each thruster, that controls its speed, propeller pitch (for CPP), and azimuth (for azimuth thrusters). This has a **Direct Thruster Control** mode where the thruster is controlled by a lever, either remotely from the bridge or from the local thuster room (required).

Several functions will command the τ vector (surge and sway forces and yaw moment):

- **Joystick control:** *Direct body-relative motion control* – to provide thrust loads in surge,sway, yaw as directly commanded by a joystick.
- **DP control:** *3DOF automatic stationkeeping control* – where DP controller automatically commands the τ-vector as feedback control to keep station for the vessel.
- **AutoPilot control:** *2DOF voyaging control* – where the Autopilot controller determines particularly the yaw moment and surge force to keep the vessel at a desired speed and steer it at desired heading.
- ...

Thrust allocation

For our dynamic DP model

$$\dot{\eta} = R(\psi)\nu$$
$$M\dot{\nu} + D\nu = \tau + R(\psi)^\top b + \tau_{wind}.$$

we consider τ as our control input.

Let τ_{cmd} be the commanded thrust load vector. **Thrust allocation** is then to calculate the commanded forces and azimuths to be generated by each thruster, that is, find $F_{cmd} \in \mathbb{R}^m$ and $\alpha_{cmd} \in \mathbb{R}^m$ such that

$$B(\alpha_{cmd})F_{cmd} = \tau_{cmd}.$$

Suppose we are successful in this allocation of the thrust. In DP, it is then assumed that the local thruster control systems control the thrusters fast enough (much faster than the vessel time constant) so that the real thrust τ matches τ_{cmd}, that is

$$F(t) \to F_{cmd}, \qquad \alpha(t) \to \alpha_{cmd}$$
$$\Downarrow$$
$$\tau(t) = B(\alpha(t))F(t) \to \tau_{cmd}$$

Thrust allocation

Due to overactuation in DP, there is an infinite number of solutions to the thrust allocation problem. Hence, a solution must be found that considers:

- Minimizing an objective function, e.g. amount of thrust, energy, fuel, etc.
- Respecting constraints such as magnitude and rate saturation of thrusters, forbidden sectors.

Assuming the "fixed-B" configuration, the thrust allocation problem is: Given τ calculate u according to[2]

$$\tau = BF, \qquad F = Ku$$

where $K = \mathrm{diag}(k_1, k_2, \cdots, k_m)$ is a gain matrix or possible the *thrust coefficient matrix* (Fossen, 2011, Chapter 12.3.5). This is ensured by an optimization problem:

$$\boxed{\begin{array}{l} \min F^\top W F + s^\top Q s \\ \text{subject to} \\ \qquad BF = \tau + s \\ \qquad F_{\min} \preceq F \preceq F_{\max} \\ \qquad \Delta F_{\min} \preceq \Delta F \preceq \Delta F_{\max} \end{array}} \quad \text{or} \quad \boxed{\begin{array}{l} \min u^\top W u + s^\top Q s \\ \text{subject to} \\ \qquad BKu = \tau + s \\ \qquad u_{\min} \preceq u \preceq u_{\max} \\ \qquad \Delta u_{\min} \preceq \Delta u \preceq \Delta u_{\max} \end{array}}$$

[2] u means here a control signal vector, not surge velocity.

Unconstrained minimum norm control allocation

A linear set of equations

$$Ax = b, \qquad A \in \mathbb{R}^{n \times m}, \tag{1}$$

with A full rank, is underdetermined when $n < m$ (fat), and there exist infinitely many solutions for x characterized by

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^m : \ Ax = b\} \\ &= \{x \in \mathbb{R}^m : \ x = x_p + x_0, \ Ax_p = b, \ x_0 \in \mathcal{N}(A)\} \end{aligned}$$

where x_p is a particular solution and x_0 is any other vector lying in the nullspace (kernel) of A.

A popular particular solution is the *weighted minimum norm solution*, given by the **weighted Moore-Penrose pseudoinverse**:

Lemma

The weighted minimum norm solution of (1), solving the optimization problem

$$\begin{aligned} &\text{minimize } x^\top W x, \qquad W = W^\top > 0 \\ &\text{subject to } Ax = b, \end{aligned}$$

is

$$x = A_W^\dagger b, \qquad A_W^\dagger := W^{-1} A^\top \left[A W^{-1} A^\top \right]^{-1}.$$

Unconstrained minimum norm control allocation

Lemma

The weighted minimum norm solution of (1), solving the optimization problem

$$\begin{aligned} &\text{minimize } x^\top W x, \qquad W = W^\top > 0 \\ &\text{subject to } Ax = b, \end{aligned}$$

is

$$x = A_W^\dagger b, \qquad A_W^\dagger := W^{-1} A^\top \left[A W^{-1} A^\top \right]^{-1}.$$

Proof.

Applying Lagrange multipliers $L(x, \lambda) = x^\top W x + \lambda^\top (Ax - b)$ *for which the optimality conditions require* $L^x(x, \lambda)^\top = 2Wx + A^\top \lambda = 0$ *and* $L^\lambda(x, \lambda)^\top = (Ax - b) = 0$, *giving* $x = -\frac{1}{2} W^{-1} A^\top \lambda$ *and* $Ax = -\frac{1}{2} A W^{-1} A^\top \lambda = b$. *Thus* $\lambda = -2 \left[A W^{-1} A^\top \right]^{-1} b$, *and the solution is* $x = W^{-1} A^\top \left[A W^{-1} A^\top \right]^{-1} b =: A_W^\dagger b$.

For a reference, see e.g. (Fossen & Sagatun, 1991; Virnig & Bodden, 1994). □

Unconstrained minimum norm control allocation

Defining $Q_\Gamma := I - A_\Gamma^\dagger A$, where $\Gamma = \Gamma^\top > 0$, then $Q_\Gamma \in \mathbb{R}^{m\times m}$ is the Γ-weighted orthogonal projection onto the kernel of A, that is, $AQ_\Gamma = A - AA_\Gamma^\dagger A = A - A = 0$, where $\operatorname{rank}(Q_\Gamma) = m - n$ since the nullity of A is $m - n$.

This has the symmetry property $\Gamma Q_\Gamma = Q_\Gamma^\top \Gamma$ and projection property $Q_\Gamma^2 = Q_\Gamma$.

It follows that all solutions to (1) are given by

$$\mathcal{X} = \left\{ x \in \mathbb{R}^m : \ x = A_W^\dagger b + Q_\Gamma z, \ z \in \mathbb{R}^m \right\}.$$

The unconstrained control allocation problem using a linear effector model (Johansen & Fossen, 2013) is the solution to

$$\text{minimize} \ \ (u - u_d)^\top W (u - u_d) \tag{2a}$$
$$\text{subject to} \ \tau = BKu, \tag{2b}$$

where $W = W^\top > 0$, $\tau \in \mathbb{R}^n$ is the control vector to satisfy, $u \in \mathbb{R}^m$ is the control effectors to be allocated, u_d is a desired value, $B \in \mathbb{R}^{n\times m}$ is the effector configuration matrix, and $K \in \mathbb{R}^{m\times m}$ is a diagonal gain matrix.

For $m > n$, meaning an overactuated allocation problem, then (2b) is an *underdetermined linear set of equations* to be solved for u.

Unconstrained minimum norm control allocation

Corollary

The weighted minimum norm solution of (2) is $u = u^* = H_W^\dagger \tau + Q_W u_d$ *where* $H := BK$ *and* $Q_W := I - H_W^\dagger H$ *and* $H_W^\dagger := W^{-1}H^\top \left[HW^{-1}H^\top\right]^{-1}$.

Proof.

Setting $x = u - u_d$, $A = H = BK$, *and* $\tau = b$, *such that* $BKu = Hu = H(x + u_d) = b \Rightarrow Hx = \tau - Hu_d = \tilde{b}$, *then from the Lemma above the explicit solution is given by* $x = H_W^\dagger \tilde{b} = H_W^\dagger (\tau - Hu_d) = u - u_d$ *and* $u = \left(I - H_W^\dagger H\right) u_d + H_W^\dagger \tau = H_W^\dagger \tau + Q_W u_d$. □

Corollary

Let $u_d = 0$. *Then the weighted minimum norm solution of (2) is* $u = u^* = H_W^\dagger \tau$ *where* $H := BK$ *and* $H_W^\dagger := W^{-1}H^\top \left[HW^{-1}H^\top\right]^{-1}$.

Proof.

Follows directly from above Corollary. □

Low-speed passive observer

The following *Observer Design Model* and observer algorithm encompass the DP passive observer:

ODM:

$$
\begin{aligned}
\dot{\xi} &= A_w \xi + E_w \omega_w \\
\dot{\eta} &= R(y_3)\nu \\
\dot{b} &= \omega_b \\
M\dot{\nu} &= -D\nu + R(y_3)^\top b + \tau + \tau_{\text{wind}} \\
y &= \eta + C_w \xi + \upsilon_y
\end{aligned}
$$

Observer:

$$
\begin{aligned}
\dot{\hat{\xi}} &= A_w \hat{\xi} + K_1(\omega_0)\tilde{y} \\
\dot{\hat{\eta}} &= R(y_3)\hat{\nu} + K_2 \tilde{y} \\
\dot{\hat{b}} &= K_3 \tilde{y} \\
M\dot{\hat{\nu}} &= -D\hat{\nu} + R(y_3)^\top \hat{b} + \tau + \tau_{\text{wind}} + R(y_3)^\top K_4 \tilde{y} \\
\hat{y} &= \hat{\eta} + C_w \hat{\xi}
\end{aligned}
$$

where:

- $y_3 = \psi + \xi_\psi$ is the measured yaw angle,
- $\tilde{y} := y - \hat{y} \in \mathbb{R}^2 \times \mathcal{S}_1$,
- $K_1(\omega_0) \in \mathbb{R}^{6\times 3}$, and $K_2, K_3, K_4 \in \mathbb{R}^{3\times 3}$.
- ω_0 is the wave frequency, and the wave motion model is given by

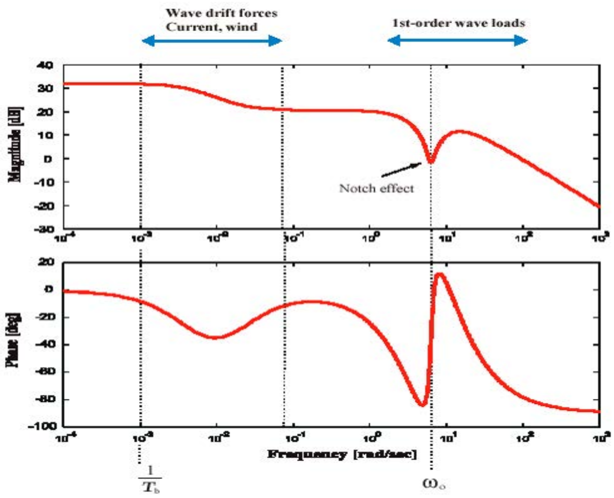
$$
C_w = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \end{bmatrix}, \quad A_w = \begin{bmatrix} 0_{3\times 3} & I_{3\times 3} \\ -\Omega^2 & -2\Lambda\Omega \end{bmatrix} \in \mathbb{R}^{6\times 6}, \quad E_w = \begin{bmatrix} 0_{3\times 3} \\ K_w \end{bmatrix}
$$

$$
\Omega = \mathrm{diag}(\omega_1, \omega_2, \omega_3), \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3), \quad K_w = \mathrm{diag}(k_{w1}, k_{w2}, k_{w3})
$$

See Fossen (2011, Chapter 11.4.1) for details on stability and tuning rules of the injection gain matrices K_i.

Low-speed passive observer

Note the wave filtering property of the observer through the notch effect in the frequency response:



We also see that the phase $< 90°$ which means the filter is passive.

Time-varying DP observer

Time-varying gains in the DP observer is shown by Værnø & Skjetne (2017) to be important when conditions are rapidly changing. The following *Observer Design Model* and observer algorithm are proposed:

ODM:

$$\boxed{\begin{aligned}\dot{\xi} &= A_w \xi + E_w \omega_w \\ \dot{\eta} &= R(y_3)\nu \\ \dot{b} &= -T_b^{-1} b + \omega_b \\ M\dot{\nu} &= -D\nu + R(y_3)^\top b + \tau + \tau_{\text{wind}} \\ y &= \eta + C_w \xi + \upsilon_y\end{aligned}}$$

Observer:

$$\boxed{\begin{aligned}\dot{\hat{\xi}} &= A_w \hat{\xi} + K_1(\omega_0)\tilde{y} \\ \dot{\hat{\eta}} &= R(y_3)\hat{\nu} + K_2 \tilde{y} \\ \dot{\hat{b}} &= -T_b^{-1}\hat{b} + K_3(\sigma(t))\tilde{y} \\ M\dot{\hat{\nu}} &= -D\hat{\nu} + R(y_3)^\top \hat{b} + \tau + \tau_{\text{wind}} + K_4(\sigma(t))R(y_3)^\top \tilde{y} \\ \hat{y} &= \hat{\eta} + C_w \hat{\xi}\end{aligned}}$$

where:

- $y_3 = \psi + \xi_\psi \approx \psi$, $\tilde{y} = y - \hat{y} \in \mathbb{R}^2 \times \mathcal{S}_1$, $C_w = [0,\ I]$, and ω_0 is the wave frequency.
- $K_1(\omega_0) = \mathrm{col}(K_{11}(\omega_0), K_{12}(\omega_0)) \in \mathbb{R}^{6\times 3}$, $K_2 \in \mathbb{R}^{3\times 3}$, $K_3(t) \in \mathbb{R}^{3\times 3}$, and $K_4(t) \in \mathbb{R}^{3\times 3}$.
- The time-varying gains are parameterized by the gain-scheduling signal $\sigma\colon \mathbb{R}_{\geq 0} \to [0, 1]$ according to:

$$\begin{aligned}K_i(\sigma(t)) &= \sigma(t) K_{i,\max} + (1 - \sigma(t))\, K_{i,\min}, \qquad i = 3, 4 \\ \sigma(t) &= \max\{0,\ \min\{\varepsilon_1 |r_d(t)| + \varepsilon_2 |\tilde{y}_f(t)|, 2\} - 1\}, \qquad \varepsilon_1, \varepsilon_2 \geq 0 \\ \dot{\tilde{y}}_f &= -T_f^{-1}(\tilde{y}_f - \tilde{y})\end{aligned}$$

- $K_{i,\max}$ and $K_{i,\min}$ are max/min gain matrices, that is, aggressive (high) and relaxed (low) observer gains.
- $\varepsilon_1, \varepsilon_2$ are tuning constants,
- r_d is desired yaw rate from reference filter, and
- $\tilde{y}_f$ is a lowpass filter state of $\tilde{y}$.

Time-varying DP observer

Letting the estimation error states be $\tilde{\xi} := \xi - \hat{\xi}$, $\tilde{\eta} := \eta - \hat{\eta}$, $\tilde{b} := b - \hat{b}$, $\tilde{\nu} := \nu - \hat{\nu}$, and $\tilde{y} = y - \hat{y}$, the closed-loop error system becomes

$$\boxed{\begin{aligned}\dot{\tilde{\xi}} &= A_w \tilde{\xi} - K_1\left(\tilde{\eta} + C_w \tilde{\xi}\right) \\ \dot{\tilde{\eta}} &= R(y_3)\tilde{\nu} - K_2\left(\tilde{\eta} + C_w \tilde{\xi}\right) \\ \dot{\tilde{b}} &= -T_b^{-1}\tilde{b} - K_3(\sigma(t))\left(\tilde{\eta} + C_w \tilde{\xi}\right) \\ \dot{\tilde{\nu}} &= -M^{-1}D\tilde{\nu} + M^{-1}R(y_3)^\top \tilde{b} - M^{-1}K_4(\sigma(t))R(y_3)^\top\left(\tilde{\eta} + C_w \tilde{\xi}\right) \\ \tilde{y} &= \tilde{\eta} + C_w \tilde{\xi}\end{aligned}}$$

Let $\tilde{x} = \mathrm{col}(\tilde{\xi}, \tilde{\eta}, \tilde{b}, \tilde{\nu}) \in \mathbb{R}^{15}$. Then we can write

$$\begin{aligned}\dot{\tilde{x}} &= A(y_3, \sigma(t))\tilde{x} \\ \tilde{y} &= C\tilde{x}\end{aligned}$$

where

$$A(y_3, \sigma) := \begin{bmatrix} A_w - K_1 C_w & -K_1 & 0 & 0 \\ -K_2 C_w & -K_2 & 0 & R(y_3) \\ -K_3(\sigma) C_w & -K_3(\sigma) & -T_b^{-1} & 0 \\ -M^{-1}K_4(\sigma)R(y_3)^\top C_w & -M^{-1}K_4(\sigma)R(y_3)^\top & M^{-1}R(y_3)^\top & -M^{-1}D \end{bmatrix}$$

$$C := \begin{bmatrix} C_w & I & 0 & 0 \end{bmatrix}$$

Time-varying DP observer

We make the following assumptions:

- **Assumption:** The following matrices commute with the rotation matrix R:

$$RK_2 = K_2 R, \quad RK_3(\sigma) = K_3(\sigma)R, \quad RT_b^{-1} = T_b^{-1}R$$
$$\mathrm{diag}(R,R)K_1 = K_1 R, \quad RC_w = C_w\,\mathrm{diag}(R,R), \quad \mathrm{diag}(R,R)A_w = A_w\,\mathrm{diag}(R,R)$$

- This is achived if $K_2, K_3, T_b, K_{11}(\omega_0), K_{12}(\omega_0), \Omega, \Lambda$ are on the form $\mathrm{diag}(a,a,b)$.
- In this case, we can write

$$A(y_3,\sigma) = T(R(y_3))A(0,\sigma)T(R(y_3))^\top, \qquad T(R) := \mathrm{diag}(R,R,R,R,I)$$
$$A(0,\sigma) = \sigma A_{\max} + (1-\sigma)\,A_{\min}, \qquad A_{\max} = A(0,1), \quad A_{\min} = A(0,0)$$

Theorem

Suppose the above assumption is satisfied. The equilibrium $\tilde{x} = 0$ is then UGES under the following conditions:

- *There exist $Q = Q^\top > 0$ and $P = P^\top > 0$ such that*

$$PA_{\min} + A_{\min}^\top P \le -Q \qquad \text{and} \qquad PA_{\max} + A_{\max}^\top P \le -Q$$

- *For the skew symmetric matrix $S = -S^\top$, such that $\dot{R} = \dot{y}_3 R(y_3)S$, let $\Sigma(S) := \mathrm{diag}(S,S,S,S,0_{3\times 3}) \in \mathbb{R}^{15\times 15}$. Then P satisfy:*

$$P\Sigma = \Sigma P$$

Time-varying DP observer

Proof: Let $R = R(y_3)$ and

$$S = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -S^\top$$

be such that $\dot{R} = \dot{y}_3 RS$ and $\dot{T} = \dot{y}_3 T(R)\Sigma(S)$, and note $\Sigma(S) = -\Sigma(S)^\top$. Considering the transformation $z = T(R)^\top \tilde{x}$ (with inverse map $\tilde{x} = T(R)z$), we get

$$\begin{aligned}
\dot{z} &= \dot{T}^\top \tilde{x} + T^\top \dot{\tilde{x}} = \dot{y}_3 \Sigma(S)^\top T(R)^\top \tilde{x} + T^\top A(y_3,\sigma)\tilde{x} \\
&= \dot{y}_3 \Sigma(S)^\top T(R)^\top T(R) z + T(R)^\top T(R) A(0,\sigma) T(R)^\top T(R) z \\
&= \dot{y}_3 \Sigma(S)^\top z + A(0,\sigma) z
\end{aligned}$$

Let $V = z^\top P z$ for which the total derivative becomes

$$\begin{aligned}
\dot{V} &= 2\dot{y}_3 z^\top P\Sigma(S)^\top z + 2z^\top PA(0,\sigma)z \\
&= \dot{y}_3 z^\top \left[P\Sigma(S)^\top + \Sigma(S)P\right] z + z^\top \left[PA(0,\sigma) + A(0,\sigma)^\top P\right] z \\
&= \dot{y}_3 z^\top P \left[\Sigma(S)^\top - \Sigma(S)^\top\right] z + z^\top \left[\sigma PA_{\max} + (1-\sigma)\,PA_{\min} + \sigma A_{\max}^\top P + (1-\sigma)\,A_{\min}^\top P\right] z \\
&= \sigma z^\top \left[P\left(A_{\max} - A_{\min}\right) + \left(A_{\max} - A_{\min}\right)^\top P\right] z + z^\top \left[PA_{\min} + A_{\min}^\top P\right] z \\
&\le -\sigma z^\top Q z - (1-\sigma)\, z^\top Q z \\
&= -z^\top Q z
\end{aligned}$$

Since $|\tilde{x}| = \sqrt{z^\top T(R) T(R)^\top z} = |z|$, it follows that $\exists k, \lambda > 0$ such that

$$|\tilde{x}(t)| = |z(t)| \le k\,|z(t_0)|\,e^{-\lambda(t-t_0)} = k\,|\tilde{x}(t_0)|\,e^{-\lambda(t-t_0)}.$$

Q.E.D.

Time-varying DP observer

Tuning the gains, we note that at min/max we get

$$A_{\min} = A(0,0) = \begin{bmatrix} A_w & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -T_b^{-1} & 0 \\ 0 & 0 & M^{-1} & -M^{-1}D \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \\ K_{3,\min} \\ M^{-1}K_{4,\min} \end{bmatrix} [C_w \ I \ 0 \ 0] = A_0 - K_{\min}C$$

$$A_{\max} = A(0,1) = \begin{bmatrix} A_w & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -T_b^{-1} & 0 \\ 0 & 0 & M^{-1} & -M^{-1}D \end{bmatrix} - \begin{bmatrix} K_1 \\ K_2 \\ K_{3,\max} \\ M^{-1}K_{4,\max} \end{bmatrix} [C_w \ I \ 0 \ 0] = A_0 - K_{\max}C$$

Hence, we first choose $K_{\min}$ with relaxed gains and $K_{\max}$ with aggressive gains so that in each case $A_{\min}$ and $A_{\max}$ are Hurwitz.

Then, we set up the *Linear Matrix Inequality* with $P \in \mathbb{R}^{15\times 15}$ as variable:

$$\begin{bmatrix} PA_{\min} + A_{\min}^\top P + Q \\ PA_{\max} + A_{\max}^\top P + Q \\ P\Sigma - \Sigma P \\ \Sigma P - P\Sigma \end{bmatrix} \leq 0$$

This can be solved in Matlab, with, say, `lmiedit.m`.

Low-speed fixed gain observer without wave filter

We can now consider a simpler DP observer not having "the mandatory" wave filter.
We use the following Observer Design Model and fixed gain observer algorithm:

ODM:

$$\boxed{\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ \dot{b} &= -T_b^{-1}b + \omega_b \\ M\dot{\nu} &= -D\nu + R(\psi)^\top b + \tau + \tau_{\text{wind}} \\ y &= \eta + v_y \end{aligned}}$$

Observer:

$$\boxed{\begin{aligned} \dot{\hat{\eta}} &= R(\psi)\hat{\nu} + L_1\tilde{y} \\ \dot{\hat{b}} &= -T_b^{-1}\hat{b} + L_2\tilde{y} \\ M\dot{\hat{\nu}} &= -D\hat{\nu} + R(\psi)^\top\hat{b} + \tau + \tau_{\text{wind}} + L_3R(\psi)^\top\tilde{y} \\ \hat{y} &= \hat{\eta} \end{aligned}}$$

where:

- $\tilde{y} = y - \hat{y} = \tilde{\eta}$ (assuming $v_y = 0$).
- $L_1 = L_1^\top > 0$
- $L_2 = L_2^\top > 0$
- $L_3 = L_3^\top > 0$

Low-speed fixed gain observer without wave filter

The closed-loop error system becomes

$$\boxed{\begin{aligned}\dot{\tilde{\eta}} &= R(\psi)\tilde{\nu} - L_1\tilde{\eta} \\ \dot{\tilde{b}} &= -T_b^{-1}\tilde{b} - L_2\tilde{\eta} \\ \dot{\tilde{\nu}} &= -M^{-1}D\tilde{\nu} + M^{-1}R(\psi)^\top\tilde{b} - M^{-1}L_3R(\psi)^\top\tilde{\eta} \\ \tilde{y} &= \tilde{\eta}\end{aligned}}$$

Letting $\tilde{x} = \mathrm{col}(\tilde{\eta}, \tilde{b}, \tilde{\nu}) \in \mathbb{R}^9$, yields

$$\begin{aligned}\dot{\tilde{x}} &= A(\psi)\tilde{x} \\ \tilde{y} &= C\tilde{x}\end{aligned}$$

where

$$A(\psi) := \begin{bmatrix} -L_1 & 0 & R(\psi) \\ -L_2 & -T_b^{-1} & 0 \\ -M^{-1}L_3R(\psi)^\top & M^{-1}R(\psi)^\top & -M^{-1}D \end{bmatrix}$$

$$C := \begin{bmatrix} I & 0 & 0 \end{bmatrix}$$

We make the following assumptions:

- **Assumption:** The following matrices commute with the rotation matrix R:
$$RL_1 = L_1R, \quad RL_2 = L_2R, \quad RT_b^{-1} = T_b^{-1}R$$
- This is achived if L_1, L_2, T_b are on the form $\mathrm{diag}(a, a, b)$.
- In this case, we can write
$$A(\psi) = T(R(\psi))A(0)T(R(\psi))^\top, \qquad T(R) := \mathrm{diag}(R, R, I) \in \mathbb{R}^{9\times 9}$$

Low-speed fixed gain observer without wave filter

Theorem

Suppose the above assumption is satisfied. The equilibrium $\tilde{x} = 0$ is then UGES under the following conditions:

- *There exist $Q = Q^\top > 0$ and $P = P^\top > 0$ such that*
$$PA(0) + A(0)^\top P \leq -Q$$
- *For the skew symmetric matrix $S = -S^\top$, such that $\dot{R} = rR(\psi)S$, let $\Sigma(S) := \mathrm{diag}(S, S, 0_{3\times 3}) \in \mathbb{R}^{9\times 9}$. Then P satisfy:*
$$P\Sigma = \Sigma P$$

Proof.

The proof is identical as for the time-varying observer. For completeness, it is summarized on next page. □

Note that the last LMI constraint requires P to be on the form

$$P = \begin{bmatrix} P_1 & P_{12} & P_{13} \\ P_{12}^\top & P_2 & P_{23} \\ P_{13}^\top & P_{23}^\top & P_3 \end{bmatrix}, \quad P_1, P_2, P_{12} \sim \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, \quad P_{13}, P_{23} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{bmatrix},$$

$$P_3 = P_3^\top > 0$$

Low-speed fixed gain observer without wave filter

Proof: Considering the transformation $z = T(R)^\top \tilde{x}$ (with inverse map $\tilde{x} = T(R)z$), we get

$$\begin{aligned}
\dot{z} &= \dot{T}^\top \tilde{x} + T^\top \dot{\tilde{x}} = r\Sigma(S)^\top T(R)^\top \tilde{x} + T^\top A(\psi)\tilde{x} \\
&= r\Sigma(S)^\top T(R)^\top T(R)z + T(R)^\top T(R)A(0)T(R)^\top T(R)z \\
&= r\Sigma(S)^\top z + A(0)z
\end{aligned}$$

Note that $\Sigma(S) = -\Sigma(S)^\top$. Let $V = z^\top P z$ for which the total derivative becomes

$$\begin{aligned}
\dot{V} &= 2rz^\top P\Sigma(S)^\top z + 2z^\top PA(0)z \\
&= rz^\top \left[P\Sigma(S)^\top + \Sigma(S)P\right] z + z^\top \left[PA(0) + A(0)^\top P\right] z \\
&\leq -z^\top Q z
\end{aligned}$$

Since $|\tilde{x}| = \sqrt{z^\top T(R)T(R)^\top z} = |z|$, it follows that $\exists k, \lambda > 0$ such that

$$|\tilde{x}(t)| = |z(t)| \leq k\,|z(t_0)|\, e^{-\lambda(t-t_0)} = k\,|\tilde{x}(t_0)|\, e^{-\lambda(t-t_0)}.$$

Q.E.D.

PID control

The **control design model** for DP is

$$\begin{aligned}
\dot{\eta} &= R(\psi)\nu \\
M\dot{\nu} + D\nu &= \tau + b + \tau_{wind}.
\end{aligned}$$

The simplest control law is the PID control law - a model-free control. For DP we need to think about the corresponding coordinate frames.
Hence, a PID controller is

$$\begin{aligned}
\dot{\xi} &= \hat{\eta} - \eta_d \\
\tau &= \tau_{pid} - \tau_{wind} \\
&= -K_i R(y_3)^\top \xi - K_p R(y_3)^\top (\hat{\eta} - \eta_d) - K_d (\hat{\nu} - \nu_d) - \tau_{wind}
\end{aligned}$$

where $K_i = K_i^\top > 0$, $K_p = K_p^\top > 0$, $K_d = K_d^\top > 0$, and $\nu_d = R(\psi_d)^\top \dot{\eta}_d$.

Now, maybe the most common way of including integral action in the DP controller, is to compensate the bias b by its estimate $\hat{b}$ from the observer.
In this case the control law becomes

$$\begin{aligned}
\tau &= \tau_{pd} - \hat{b} - \tau_{wind} \\
&= -K_p R(y_3)^\top (\hat{\eta} - \eta_d) - K_d (\hat{\nu} - \nu_d) - \hat{b} - \tau_{wind}
\end{aligned}$$

that is, a PD control law with wind feedforward and bias compensation.

Backstepping DP control

The **control design model** for DP is

$$\dot{\eta} = R(\psi)\nu$$
$$M\dot{\nu} + D\nu = \tau + b + \tau_{wind}.$$

We will do a few backstepping designs on the blackboard...

Manuvering DP control law

We do this on blackboard...

Preparations for next lecture

Disturbance rejection and integral action:

- Khalil (2015).
  - Chapters: 13.1-13.4
- Lecture presentation.

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