Observer for simplified DP model: Design and proof

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1 Control design model

We consider the typical low speed dynamics of a marine vessel of the form

$$\dot{\eta} = R(\psi)\nu \tag{1a}$$

$$M\dot{\nu} = -D\nu + R(\psi)^\top b + \tau \tag{1b}$$

$$\dot{b} = 0, \tag{1c}$$

where $\eta = \mathrm{col}\,(\eta_N, \eta_E, \psi) \in \mathbb{R}^3$ contains the North/East position and heading angle of the vessel, and $\nu = \mathrm{col}\,(u, v, r) \in \mathbb{R}^3$ contains the surge/sway velocity and yaw rate in the body frame of the vessel,

2 Observers

Assuming only measurements of η is available, the classical observer for (1) is designed by copying the plant dynamics and adding injection terms, as follows

$$\dot{\hat{\eta}} = R(\psi)\hat{\nu} + L_1\bar{\eta} \tag{2a}$$

$$M\dot{\hat{\nu}} = -D\hat{\nu} + R(\psi)^\top \hat{b} + \tau + K_2 R(\psi)^\top L_2\bar{\eta} \tag{2b}$$

$$\dot{\hat{b}} = L_3\bar{\eta}, \tag{2c}$$

where $(\hat{\eta}, \hat{\nu}, \hat{b})$ are the state estimates, and $\bar{\eta} := \eta - \hat{\eta}$. The objective of the observer problem is to give conditions on the injection gains L_1, L_2, L_3 and K_2, together with necessary assumptions on the plant, in order to achieve global stability and attractivity of the equilibrium $(\bar{\eta}, \bar{\nu}, \bar{b}) = 0$.

2.1 No bias

First we consider the case where $b = 0$. Let $\bar{\eta} := \eta - \hat{\eta}$ and $\bar{\nu} := \nu - \hat{\nu}$ be the estimation error states. The closed-loop error system is

$$\dot{\bar{\eta}} = R(\psi)\bar{\nu} - L_1\bar{\eta} \tag{3a}$$

$$M\dot{\bar{\nu}} = -D\bar{\nu} - K_2 R(\psi)^\top L_2\bar{\eta}, \tag{3b}$$

for which the following result apply.

Proposition 1 *The equilibrium $(\bar{\eta}, \bar{\nu}) = 0$ of (3) is UGAS under the following conditions:*

- *The damping matrix satisfy $D + D^\top \geq 0$.*
- *The gain $K_2 = I$, and L_1 and L_2 are symmetric positive definite and satisfy $L_1 L_2 + L_2 L_1 > 0$.*

If the damping matrix satisfies $D + D^\top > 0$, then the equilibrium $(\bar{\eta}, \bar{\nu}) = 0$ of (3) is UGES.

Proof 1 *Consider the Lyapunov function*

$$p_m |x|^2 \leq V(x) = x^\top P x \leq p_M |x|^2$$

where $\bar{x} = \begin{bmatrix} \bar{\eta}^\top & \bar{\nu}^\top \end{bmatrix}^\top$ and $P = P^\top = diag\{L_2,\ M\}$. The function $V(x)$ is positive definite if L_2 and M are positive definite. The time derivative of $V(\bar{x})$ along (4a) and (4b) gives

$$\dot{V} = -\bar{\eta}^\top [L_1 L_2 + L_2 L_1] \bar{\eta} - \bar{\nu}^\top (D + D^\top) \bar{\nu}.$$

Clearly, for the conditions $D + D^\top > 0$ and $L_1 L_2 + L_2 L_1 > 0$, $\dot{V}$ is negative definite, and the equilibrium is UGES. However, if we only require $D + D^\top \geq 0$, we need to show UGAS also when $D = 0$. We evoke Matrosov's Thereom in [1] [Theorem 1]. The equilibrium $(\bar{\eta}, \bar{\nu}) = 0$ is UGS since $V(\bar{x})$ is positive definite and $\dot{V} \leq -c_1 \bar{\eta}^\top \bar{\eta}$ where $c_1 = 2\lambda_{min}(L_1 L_2)$. Define the auxillary function as $W = -\bar{\nu}^\top R(\psi)^\top \bar{\eta}$, $Y_1 := -c_1 \bar{\eta}^\top \bar{\eta}$, $Y_2 = -\bar{\nu}^\top \bar{\nu} - \bar{\nu}^\top [\dot{R}(\psi)^\top - D^\top M^{-1} R(\psi)^\top] \bar{\eta} - \bar{\eta}^\top L_2 R(\psi) M^{-1} R(\psi)^\top \bar{\eta}$. Then $\dot{V} \leq Y_1$ and $\dot{W} \leq Y_2$ For bounded states, V and W are bounded, and Y_1, Y_2 are continuous. For $\bar{\nu} \neq 0$, $Y_1 = 0 \Rightarrow Y_2 = -\bar{\nu}^\top \bar{\nu} < 0$, and for $Y_1 = Y_2 = 0 \Rightarrow (\bar{\eta}, \bar{\nu}) = 0$. This proofs that the equilibrium $(\bar{\eta}, \bar{\nu}) = 0$ is UGAS for $D + D^\top \geq 0$.

$\diamond$

2.2 Including bias

We now turn our attention to the case where the bias is included. In this case, with reference to (1) and (2), the observer error dynamics becomes

$$\dot{\bar{\eta}} = R(\psi)\bar{\nu} - L_1 \bar{\eta} \tag{4a}$$

$$M\dot{\bar{\nu}} = -D\bar{\nu} + R(\psi)^\top \bar{b} - K_2 R(\psi)^\top L_2 \bar{\eta} \tag{4b}$$

$$\dot{\bar{b}} = -L_3 \bar{\eta}, \tag{4c}$$

for which the following result apply.

Proposition 2 *The equilibrium $(\bar{\eta}, \bar{\nu}, \bar{b}) = 0$ of (4) is UGAS under the following conditions:*

- *The damping matrix satisfy $D + D^\top \geq 0$.*
- *The gain $K_2 = I$, and the matrices L_1, L_2, L_3 are symmetric positive definite and satisfy that L_3 and L_1 are commutative, and that the symmetric matrices $L_1 L_2 + L_2 L_1 - 2L_3$ and $L_3^{-1} L_1 - L_2^{-1}$ are positive definite.*

Proof 2 *Consider the state vector* $\bar{x} = \begin{bmatrix} \bar{\eta}^\top & \bar{\nu}^\top & \bar{b}^\top \end{bmatrix}^\top$*, and a quadratic Lyapunov function*

$$p_m|x|^2 \leq V(\bar{x}) = \bar{x}^\top P \bar{x} \leq p_M|x|^2,$$

where

$$P = \begin{bmatrix} L_2 & 0 & -I \\ 0 & M & 0 \\ -I & 0 & L_3^{-1}L_1 \end{bmatrix}$$

It can be verified that sufficient conditions for for P *positive definite are*

$$L_2 = L_2^\top > 0, \quad M = M^\top > 0, \quad L_3^{-1}L_1 - L_2^{-1} > 0,$$

and note that the last matrix is symmetric due to the commuting property of L_3 *and* L_1*.*

Using the Schur complement: Schur complement states that for

$$P = \begin{bmatrix} L_2 & B \\ B^\top & C \end{bmatrix}$$

P *is positive definite if* $L_2 > 0$ *and* $C - B^\top L_2^{-1} B > 0$*. For our case this gives*

$$C - B^\top L_2^{-1} B = \begin{bmatrix} M & 0 \\ 0 & L_3^{-1}L_1 \end{bmatrix} - \begin{bmatrix} 0 \\ -I \end{bmatrix} L_2^{-1} \begin{bmatrix} 0 & -I \end{bmatrix}$$

$$= \begin{bmatrix} M & 0 \\ 0 & L_3^{-1}L_1 - L_2^{-1} \end{bmatrix} > 0$$

and this is satisfied if $M > 0$ *and* $L_3^{-1}L_1 - L_2^{-1} > 0$

The time derivative of $V(x)$ *along the trajectories (4) becomes*

$$\dot{V} = -\bar{\eta}^\top [L_1 L_2 + L_2 L_1 - 2L_3] \bar{\eta} - \bar{\nu}^\top (D + D^\top) \bar{\nu}. \tag{5}$$

For the derivation of the above, we get:

$$PA_0(\psi) = \begin{bmatrix} L_2 & 0 & -I \\ 0 & M & 0 \\ -I & 0 & L_3^{-1}L_1 \end{bmatrix} \begin{bmatrix} -L_1 & R(\psi) & 0 \\ -M^{-1}R(\psi)^\top L_2 & -M^{-1}D & M^{-1}R(\psi)^\top \\ -L_3 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -L_2 L_1 + L_3 & L_2 R(\psi) & 0 \\ -R(\psi)^\top L_2 & -D & R(\psi)^\top \\ L_1 - L_1 & -R(\psi) & 0 \end{bmatrix}, \tag{6}$$

$$A_0(\psi)^\top P = \begin{bmatrix} -L_1 & -L_2 R(\psi) M^{-1} & -L_3 \\ R(\psi)^\top & -D^\top M^{-1} & 0 \\ 0 & R(\psi) M^{-1} & 0 \end{bmatrix} \begin{bmatrix} L_2 & 0 & -I \\ 0 & M & 0 \\ -I & 0 & L_3^{-1}L_1 \end{bmatrix}$$

$$= \begin{bmatrix} -L_1 L_2 + L_3 & -L_2 R(\psi) & L_1 - L_1 \\ R(\psi)^\top L_2 & -D^\top & -R(\psi)^\top \\ 0 & R(\psi) & 0 \end{bmatrix} \tag{7}$$

and putting this together gives

$$
\begin{aligned}
&PA_0(\psi) + A(\psi)^\top P \\
&= \begin{bmatrix} -L_2L_1 - L_1L_2 + 2L_3 & 0 & 0 \\ 0 & -(D+D^\top) & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{8}
$$

Define $G := L_1L_2 + L_2L_1 - 2L_3$. *The time derivative of* $V(x)$ *is then upper bounded by*

$$\dot{V} \leq -\varepsilon g_m|\bar{\eta}|^2 - 2d_m|\bar{\nu}|^2,$$

where d_m *is the smallest eigenvalue of* D, *and* g_m *is the smallest eigenvalue of* G. *To show UGAS we only require* $D + D^\top \geq 0$, *so* $d_m \geq 0$. *To show UGAS two steps of the nested version of Matrosov's Theorem in [1] [Theorem 1] is used. The equilibrium* $(\bar{\eta}, \bar{\nu}, \bar{b}) = 0$ *is UGS by the fact that* $V(\bar{x})$ *is positive definite, and* $\dot{V} \leq -\varepsilon l_{3m}|\bar{\eta}|^2$. *We define the first auxiliary function as* $W_1 = -\bar{\nu}^\top R(\psi)^\top \bar{\eta}$ *and the second as* $W_2 = -\bar{b}^\top R(\psi)M\bar{\nu}$, *and define* $Y_1 := -\varepsilon l_{3m}\|\bar{\eta}\|^2$, $Y_2 := -\bar{\nu}^\top\bar{\nu} - \bar{\nu}^\top[\dot{R}^\top - D^\top M^{-1}R^\top]\bar{\eta} - \bar{b}^\top RM^{-1}R^\top\bar{\eta} - \bar{\eta}^\top L_2RM^{-1}R^\top\bar{\eta}$, *and* $Y_3 := -\bar{b}^\top b - \bar{b}^\top[\dot{R}M - RD]\bar{\nu} - \bar{b}^\top L_2\bar{\eta} + \bar{\eta}^\top L_3RM\bar{\nu}$. *Then* $\dot{V} \leq Y_1$, $\dot{W}_1 \leq Y_2$, *and* $\dot{W}_2 \leq Y_3$. *For bounded states* V, W_1, *and* W_2 *are bounded, and* Y_1, Y_2, *and* Y_3 *are continuous. For* $\bar{\nu} \neq 0$, $\bar{b} \neq 0$, $Y_1 = 0 \Rightarrow Y_2 = -\bar{\nu}^\top\bar{\nu} < 0$. *When* $\bar{b} \neq 0$, $Y_1 = Y_2 = 0 \Rightarrow Y_3 \leq -\bar{b}^\top\bar{b} < 0$. *Furthermore, when* $Y_1 = Y_2 = Y_3 = 0 \Rightarrow (\bar{\eta}, \bar{\nu}, \bar{b}) = 0$. *This proves that the equilibrium* $(\bar{\eta}, \bar{\nu}, \bar{b}) = 0$ *is UGAS.*

References

[1] Antonio Loría, Elena Panteley, Dobrivoje Popović, and Andrew R Teel. A nested matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems. *Automatic Control, IEEE Transactions on*, 50(2):183–198, 2005.