Marine Control Systems II

Lecture 11: Adaptive backstepping

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Goals of lecture

- Perform an adaptive backstepping design for DP.
- Be able to carry out a scalar adaptive backstepping design.
- Be able to carry out a vectorial adaptive backstepping design.

Literature

Adaptive control:

- Lecture presentation.
- Skjetne (2005). Ch. 4.2

Adaptive DP control

Consider the Dynamic Positioning system

$$\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= -D\nu + R(\psi)^\top b + \tau \end{aligned}$$

$\eta = \mathrm{col}(x, y, \psi)$ the position/heading;
$\nu = \mathrm{col}(u, v, r)$ the velocities;
b is assumed a constant unknown bias load;
Damping matrix $D > 0$; mass matrix $M = M^\top > 0$; τ the control force input.
Notice in particular, for

$$R = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S(r) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that the following properties hold:

$$R(\psi)^\top R(\psi) = R(\psi)R(\psi)^\top = I, \qquad \det R(\psi) = 1$$
$$\dot{R} = R(\psi)S(r), \qquad \dot{R}^\top = -S(r)R(\psi)^\top, \qquad S(r) = -S(r)^\top.$$

Adaptive DP control

Control design model

$$\dot{\eta} = R(\psi)\nu$$
$$M\dot{\nu} = -D\nu + R(\psi)^\top b + \tau$$

We let

$$z_1 := R(\psi)^\top (\eta - \eta_d(t))$$
$$z_2 := \nu - \alpha_1,$$

and $\hat{b}$ an adaptive estimate of b with the error

$$\tilde{b} := b - \hat{b}.$$

Then we do an adaptive backstepping DP control design on the blackboard...
Note; there are no lecture notes on this - we improvise!

Parametric strict feedback form...

... of vector relative degree n:

$$\dot{x}_1 = G_1(x_1)\,x_2 + f_1(x_1) + \Phi_1(x_1)\,\varphi$$
$$\dot{x}_2 = G_2(x_1, x_2)\,x_3 + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\,\varphi$$
$$\vdots$$
$$\dot{x}_n = G_n(x_1, \ldots, x_n)\,u + f_n(x_1, \ldots, x_n) + \Phi_n(x_1, \ldots, x_n)\,\varphi$$
$$y = h(x_1)$$

- $x_i \in \mathbb{R}^m,\ i = 1, \ldots, n$, are the states, $y \in \mathbb{R}^m$ is the output, $u \in \mathbb{R}^m$ is the control, and $\varphi \in \mathbb{R}^p$ is a vector of constant unknown parameters.
- $G_i(x_1, \ldots, x_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible for all x, the map $h(x_1)$ is a diffeomorphism, and G_i, f_i, and Φ_i are smooth.

Example 1: DP

We consider the DP control design model

$$
\begin{aligned}
\dot{\eta} &= R(\psi)\nu \\
M\dot{\nu} &= -D\nu + R(\psi)^{\top} b + \tau.
\end{aligned}
$$

Let $n = 2$ and

$$
\begin{aligned}
x_1 &:= \eta, \qquad x_2 := \nu, \qquad \varphi := b \\
G_1(x_1) &:= R(\psi), \qquad f_1(x_1) = 0, \\
G_2(x_1, x_2) &:= M^{-1}, \qquad f_2(x_1, x_2) = -M^{-1} D\nu
\end{aligned}
$$

and then we need

$$
\Phi_1(x_1) := 0 \qquad \text{and} \qquad \Phi_2(x_1, x_2) := M^{-1} R(\psi)^{\top}
$$

such that

$$
\begin{aligned}
\dot{x}_1 &= G_1(x_1)\, x_2 + f_1(x_1) + \Phi_1(x_1)\, \varphi = R(\psi)\nu \\
\dot{x}_2 &= G_2(x_1, x_2)\, u + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\, \varphi = M^{-1}\tau - M^{-1} D\nu + M^{-1} R(\psi)^{\top} b
\end{aligned}
$$

Example 2

Consider the relative degree 3 system

$$
\begin{aligned}
\dot{x}_1 &= x_2 + k\cos(\omega x_1) \\
\dot{x}_2 &= x_3 + a x_1 \\
\dot{x}_3 &= u - b\,|x_3|\, x_3 + d \\
y &= x_1,
\end{aligned}
$$

where (k, a, b, d) are constant unknown parameters.
Then we get

$$
\begin{aligned}
\varphi &:= \begin{bmatrix} k \\ a \\ b \\ d \end{bmatrix} \\
\Phi_1(x_1) &:= \begin{bmatrix} \cos(\omega x_1) & 0 & 0 & 0 \end{bmatrix} \\
\Phi_2(x_1, x_2) &:= \begin{bmatrix} 0 & x_1 & 0 & 0 \end{bmatrix} \\
\Phi_3(x_1, x_2, x_3) &:= \begin{bmatrix} 0 & 0 & -|x_3|\, x_3 & 1 \end{bmatrix}
\end{aligned}
$$

...Example 2

Hence,

$$\varphi := \begin{bmatrix} k \\ a \\ b \\ d \end{bmatrix}$$

$$\Phi_1(x_1) := \begin{bmatrix} \cos(\omega x_1) & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_2(x_1, x_2) := \begin{bmatrix} 0 & x_1 & 0 & 0 \end{bmatrix}$$

$$\Phi_3(x_1, x_2, x_3) := \begin{bmatrix} 0 & 0 & -|x_3|\,x_3 & 1 \end{bmatrix}$$

gives

$$\begin{aligned} \dot{x}_1 &= x_2 + \Phi_1(x_1)\,\varphi \\ \dot{x}_2 &= x_3 + \Phi_2(x_1, x_2)\,\varphi \\ \dot{x}_3 &= u + \Phi_3(x_1, x_2, x_3)\,\varphi \\ y &= x_1, \end{aligned}$$

Adaptive backstepping

Parametric strict feedback form of vector relative degree n:

$$\begin{aligned} \dot{x}_1 &= G_1(x_1)\,x_2 + f_1(x_1) + \Phi_1(x_1)\,\varphi \\ \dot{x}_2 &= G_2(x_1, x_2)\,x_3 + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\,\varphi \\ &\;\;\vdots \\ \dot{x}_n &= G_n(x_1, \ldots, x_n)\,u + f_n(x_1, \ldots, x_n) + \Phi_n(x_1, \ldots, x_n)\,\varphi \\ y &= h(x_1) \end{aligned}$$

- $x_i \in \mathbb{R}^m,\ i = 1, \ldots, n$, are the states, $y \in \mathbb{R}^m$ is the output, $u \in \mathbb{R}^m$ is the control, and $\varphi \in \mathbb{R}^p$ is a vector of constant unknown parameters.
- $G_i(x_1, \ldots, x_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible for all x, the map $h(x_1)$ is a diffeomorphism, and G_i, f_i, and Φ_i are smooth.

Scalar parametric strict feedback form

Parametric strict feedback form of vector relative degree n:

$$\begin{aligned}
\dot{x}_1 &= g_1\left(x_1\right) x_2 + f_1\left(x_1\right) + \Phi_1\left(x_1\right)\varphi \\
\dot{x}_2 &= g_2\left(x_1, x_2\right) x_3 + f_2\left(x_1, x_2\right) + \Phi_2\left(x_1, x_2\right)\varphi \\
&\;\;\vdots \\
\dot{x}_n &= g_n\left(x_1, \ldots, x_n\right) u + f_n(x_1, \ldots, x_n) + \Phi_n\left(x_1, \ldots, x_n\right)\varphi \\
y &= h\left(x_1\right)
\end{aligned}$$

- $x_i \in \mathbb{R},\ i = 1, \ldots, n$, are the states, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control, and $\varphi \in \mathbb{R}^p$ is a vector of constant unknown parameters.
- $g_i(x_1, \ldots, x_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible for all x, the map $h(x_1)$ is a diffeomorphism, and g_i, f_i, and Φ_i are smooth.

Scalar parametric strict feedback form

An example follows on blackboard...

11.11.2011

(8)

Adaptive backstepping

Example: Simple adaptive system.

$$\dot{x} = ax - b|x|x + k\cos(\omega x) + d + u$$

Nominal control law: PI

$$\dot{z} = x$$

$$u = -k_p x - k\cos(\omega x) - k_i z \qquad k_p > a, \quad k_i > 0$$

$$\Rightarrow \quad \begin{aligned} \dot{z} &= x \\ \dot{x} &= -(k_p - a)x - b|x|x - k_i z + d \end{aligned}$$

Eq. $x = 0, \quad z = \frac{d}{k_i}$

$$e_1 = z - \frac{d}{k_i} \qquad e_2 = x$$

$$\boxed{\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= -(k_p - a)e_2 - b|e_2|e_2 - k_i e_1 \end{aligned}} \qquad \text{UGES.} \quad \underline{\text{OK}}$$

Assume a, b, k, d are unknown and rewrite system:

$$\dot{x} = u + \phi(x)^T c \qquad c = [a \ b \ k \ d]^T$$

$$\phi(x)^T = [x \ \ -|x|x \ \ \cos(\omega x) \ \ 1]$$

Let $\hat{c}$ be an estimate of c and $\tilde{c} = c - \hat{c}$.

$\Rightarrow$

... Example:

CLF:

$$V = \frac{1}{2}x^2 + \frac{1}{2}\tilde{c}^T \Gamma^{-1} \tilde{c} \qquad \Gamma = \Gamma^T > 0$$

$$\dot{V} = x[u + \phi(x)^T \hat{c}] + x\phi(x)^T \tilde{c} + \tilde{c}^T \Gamma^{-1} \dot{\tilde{c}}$$

$$\boxed{u = -k_p x - \phi(x)^T \hat{c}}$$

$$\dot{V} = -k_p |x|^2 + x\phi(x)^T \tilde{c} - \tilde{c}^T \Gamma^{-1} \dot{\hat{c}}$$

$$= -k_p |x|^2 + \tilde{c}^T [\phi(x)x - \Gamma^{-1}\dot{\hat{c}}]$$

$$\boxed{\dot{\hat{c}} = \Gamma \phi(x) x}$$

$$\Rightarrow \dot{V} = -k_p |x|^2 \leq 0$$

By LaSalle-Yoshizawa $x(t) \to 0$.

Let $\Gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_4)$

$$\Rightarrow \left.\begin{aligned} \dot{\hat{a}} &= \gamma_1 x^2 \\ \dot{\hat{b}} &= -\gamma_2 |x| x^2 \\ \dot{\hat{k}} &= \gamma_3 x \cos(\omega x) \\ \dot{\hat{d}} &= \gamma_4 x \end{aligned}\right\} \quad \dot{\tilde{c}} = -\Gamma \phi(x) x$$

$$\dot{x} = -k_p x + \phi(x)^T \tilde{c}$$

Eq. $x = 0$

$$\phi(x)^T \tilde{c} = 0$$

$$\Rightarrow x\tilde{a} - |x|x\tilde{b} + \cos(\omega x)\tilde{k} + \tilde{d} = 0$$

Can then show that

$$\left.\begin{aligned} &\tilde{a}, \tilde{b} \not\to 0 \\ &\tilde{k}\cos(\omega x) + \tilde{d} \to 0 \end{aligned}\right\} \text{ Generally need PE.}$$

-after note 9 from 11.11.2011

$\boxed{15.3.17}$

Suppose now that

$$\dot{x} = ax - b|x|x + k\cos(\omega x) + d + u$$

$$\dot{u} = v \implies v \text{ is our control input}$$

Is this in parametric strict feedback form?

After Step 1 we got, with α_1 as virtual control and $z_2 = u - \alpha_1$

$$V = \tfrac{1}{2}x^2 + \tfrac{1}{2}\tilde{c}^T\Gamma^{-1}\tilde{c}$$

$$\alpha_1 = -k_p x - \phi(x)^T\hat{c}$$

Now, it would be nice to assign the adaptive update law, but...

$$\dot{V} = -k_p x^2 + \tilde{c}^T\left[\phi(x)x - \Gamma^{-1}\dot{\hat{c}}\right] + xz_2$$

$$\dot{\alpha}_1 = -k_p\dot{x} - \dot{\phi}^T\hat{c} - \phi^T\dot{\hat{c}}$$

$$= -k_p\dot{x} - \left[\dot{x} \quad -2\,\mathrm{sgn}(x)x\dot{x} \quad \omega\sin(\omega x)\dot{x} \quad 0\right]\hat{c} - \phi^T\dot{\hat{c}}$$

but...

$$= -\left(k_p + \hat{a} - 2\,\mathrm{sgn}(x)x\hat{b} + \omega\sin(\omega x)\hat{k}\right)\left(u + \phi^T(\tilde{c} + \hat{c})\right) - \phi^T\dot{\hat{c}}$$

Would in next step have to cancel $\tilde{c}$!

$$= \sigma_1(x,u,\hat{c}) + \chi_{11}(x,u,\hat{c})\tilde{c} + \chi_{12}(x)\dot{\hat{c}}$$

Let also: $\boxed{\Delta_1 := \phi(x)^T}$ $\boxed{\rho_1 := \Delta_1^T x = \phi x}$

$$\implies \dot{V}_1 = -k_p x^2 + \tilde{c}^T\left[\rho_1 - \Gamma^{-1}\dot{\hat{c}}\right] + xz_2$$

$$\dot{x} = -k_p x + \Delta_1^T\tilde{c}$$

Must postpone assignment of update law until the end.

step 2 $\implies$

Step 2:

$$\dot{z}_2 = \dot{u} - \dot{\alpha}_1 = v - \sigma_1 - \chi_{11}\tilde{c} - \chi_{12}\dot{\hat{c}}$$

$$V_2 = V + \tfrac{1}{2} z_2^2$$

$$\dot{V}_2 = -k_p x^2 + \tilde{c}^T [\rho_1 - \Gamma^{-1}\dot{\hat{c}}]$$
$$+ z_2 [x + v - \sigma_1 - \chi_{12}\dot{\hat{c}}] - z_2 \chi_{11} \tilde{c}$$

Let $\boxed{\Delta_2 := -\chi_{11} \quad \rho_2 := \rho_1 + \Delta_2^T z_2}$

$$\Rightarrow \dot{V}_2 = -k_p x^2 + \tilde{c}^T [\rho_2 - \Gamma^{-1}\dot{\hat{c}}] + z_2 [x + v - \sigma_1 - \chi_{12}\dot{\hat{c}}]$$

Now we assign v and $\dot{\hat{c}}$:

$$\boxed{\dot{\hat{c}} = \Gamma \rho_2}$$

$$\Rightarrow \dot{V}_2 = -k_p x^2 + z_2 [x + v - \sigma_1 - \chi_{12}\Gamma\rho_2]$$

$$\boxed{v = -x - k_d z_2 + \sigma_1 + \chi_{12}\Gamma\rho_2}$$

$$\Rightarrow \dot{V}_2 = -k_p x^2 - k_d z_2^2 \leq 0$$

Stability and convergence?

2-step Adaptive Backstepping - Model 1

Two models in *parametric strict feedback form* of vector relative degree n:

$$\begin{aligned}\dot{x}_1 &= G_1(x_1)\,x_2 + f_1(x_1) + \Phi_1(x_1)\,\varphi \\ \dot{x}_2 &= G_2(x_1, x_2)\,u + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\,\varphi \\ y &= h(x_1)\end{aligned}$$

$\Phi_1\varphi$ is an *unmatched* uncertainty while $\Phi_2\varphi$ is a *matched* uncertainty.
This is different to...

2-step Adaptive Backstepping - Model 2

Two models in *parametric strict feedback form* of vector relative degree n:

$$\begin{aligned}\dot{x}_1 &= G_1(x_1)\,x_2 + f_1(x_1) \\ \dot{x}_2 &= G_2(x_1, x_2)\,u + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\,\varphi \\ y &= h(x_1)\end{aligned}$$

Note the difference.
Matched versus unmatched uncertainty vector makes a big difference on the complexity of the design.

2-step Adaptive Backstepping - Model 1

We do the general design for:

$$\begin{aligned}
\dot{x}_1 &= G_1(x_1)\,x_2 + f_1(x_1) + \Phi_1(x_1)\,\varphi \\
\dot{x}_2 &= G_2(x_1, x_2)\,u + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\,\varphi \\
y &= h(x_1)
\end{aligned}$$

3-step Adaptive Backstepping

We now do the general design for:

$$\begin{aligned}
\dot{x}_1 &= G_1(x_1)\,x_2 + f_1(x_1) + \Phi_1(x_1)\,\varphi \\
\dot{x}_2 &= G_2(x_1, x_2)\,x_3 + f_2(x_1, x_2) + \Phi_2(x_1, x_2)\,\varphi \\
\dot{x}_3 &= G_3(x_1, x_2, x_3)\,u + f_3(x_1, x_2, x_3) + \Phi_3(x_1, x_2, x_3)\,\varphi \\
y &= h(x_1)
\end{aligned}$$

3+ step adaptive backstepping is the difficult design...

Adaptive backstepping:

$$\begin{aligned}\dot{x}_1 &= G_1(x_1)x_2 + f_1(x_1) + \Phi_1(x_1)\varphi \\ \dot{x}_2 &= G_2(x_{12})u + f_2(x_{12}) + \Phi_2(x_{12})\varphi\end{aligned}$$

$\varphi \in \mathbb{R}^p$ vector of unknown param.

Tracking: $x_1 \to x_d(t)$

Step 1: $z_1 = x_1 - x_d(t)$ $\quad z_2 = x_2 - \alpha_1(x_1, t)$ $\quad \tilde{\varphi} = \varphi - \hat{\varphi}$

$$\dot{z}_1 = G_1 z_2 + G_1\alpha_1 + f_1 + \Phi_1\varphi - \dot{x}_d$$

$$V_1 = z_1^T P_1 z_1 + \frac{1}{2}\tilde{\varphi}^T\Gamma^{-1}\tilde{\varphi} \qquad P_1A_1 + A_1^T P_1 = -Q_1$$

$$\begin{aligned}\dot{V}_1 = {} & 2z_1^T P_1\left[G_1\alpha_1 + f_1 - \dot{x}_d + \Phi_1\hat{\varphi}\right] + 2z_1^T P_1 G_1 z_2 \\ & + 2z_1^T P_1 \Phi_1\tilde{\varphi} - \tilde{\varphi}^T\Gamma^{-1}\dot{\hat{\varphi}}\end{aligned}$$

$$\boxed{\alpha_1 = G_1^{-1}\left[A_1 z_1 - f_1 + \dot{x}_d - \Phi_1\hat{\varphi}\right] = \alpha_1(x_1, \hat{\varphi}, t)}$$

$$\boxed{\rho_1 = 2\Phi_1^T P_1 z_1 = \rho_1(x_1, t)}$$

$$\Rightarrow \dot{V}_1 = -z_1^T Q_1 z_1 + 2z_1^T P_1 G_1 z_2 + \tilde{\varphi}^T\left[\rho_1 - \Gamma^{-1}\dot{\hat{\varphi}}\right]$$

$$\dot{z}_1 = A_1 z_1 + G_1 z_2 + \Phi_1\tilde{\varphi}$$

$$\begin{aligned}\dot{\alpha}_1 = \alpha_1^{x_1}\dot{x}_1 + \alpha_1^{\hat{\varphi}}\dot{\hat{\varphi}} + \alpha_1^t = {} & \sigma_1(x_{12}, \hat{\varphi}, t) \\ & + \chi_{11}(x_1, \hat{\varphi}, t)\tilde{\varphi} \\ & + \chi_{12}(x_1, \hat{\varphi}, t)\dot{\hat{\varphi}}\end{aligned}$$

Step 2:

$$\dot{z}_2 = G_2 u + f_2 + \Phi_2 \phi - \sigma_1 - \chi_{11} \tilde{\phi} - \chi_{12} \dot{\hat{\phi}}$$

$$V_2 = V_1 + z_2^T P_2 z_2 \qquad P_2 A_2 + A_2^T P_2 = -Q_2$$

$$\dot{V}_2 = -z_1^T Q_1 z_1 + \tilde{\phi}^T [\rho_1 - \Gamma^{-1} \dot{\hat{\phi}}]$$

$$+ 2 z_2^T \{ G_1^T P_1 z_1 + P_2 [G_2 u + f_2 + \Phi_2 \hat{\phi} - \sigma_1 - \chi_{12} \dot{\hat{\phi}}] \}$$

$$+ 2 z_2^T P_2 [\Phi_2 - \chi_{11}] \tilde{\phi}$$

$$\boxed{u = G_2^{-1} [A_2 z_2 - P_2^{-1} G_1^T P_1 z_1 - f_2 - \Phi_2 \hat{\phi} + \sigma_1 + \chi_{12} \Gamma \rho_2]}$$

Define: $\rho_2 = \rho_1 + 2(\Phi_2 - \chi_{11})^T P_2 z_2$

$$\Rightarrow \dot{V}_2 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 + \tilde{\phi}^T [\rho_1 - \Gamma^{-1} \dot{\hat{\phi}}]$$

$$+ 2 z_2^T P_2 \chi_{12} [\Gamma \rho_2 - \dot{\hat{\phi}}] + 2 \tilde{\phi}^T (\Phi_2 - \chi_{11})^T P_2 z_2$$

$$= -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 + \tilde{\phi}^T [\rho_2 - \Gamma^{-1} \dot{\hat{\phi}}]$$

$$+ 2 z_2^T P_2 \chi_{12} [\Gamma \rho_2 - \dot{\hat{\phi}}]$$

We now choose the adaptive update law:

$$\boxed{\dot{\hat{\phi}} = \Gamma \rho_2 = 2 \Gamma \Phi_1^T P_1 z_1 + 2 \Gamma (\Phi_2 - \chi_{11})^T P_2 z_2}$$

$$\Rightarrow \dot{V}_2 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 \leq 0$$

$$\dot{z}_2 = -P_2^{-1} G_1^T P_1 z_1 + A_2 z_2 + [\Phi_2 - \chi_{11}] \tilde{\phi}$$

Using LaSalle-Yoshizawa $\Rightarrow$ $z(t) \to 0$

Moreover: $\Phi_1 \tilde{\phi} \to 0$ and $(\Phi_2 - \chi_{11}) \tilde{\phi} \to 0$

Adaptive backstepping

$$\boxed{\begin{aligned}\dot{x}_1 &= G_1 x_2 + f_1 + \phi_1^T \theta \\ \dot{x}_2 &= G_2 x_3 + f_2 + \phi_2^T \theta \\ \dot{x}_3 &= G_3 u + f_3 + \phi_3^T \theta\end{aligned}}$$

$$z_1 = x_1 - x_d$$
$$z_2 = x_2 - \alpha_1$$
$$z_3 = x_3 - \alpha_2$$
$$\tilde{\theta} = \theta - \hat{\theta}$$

Step 1: $\dot{z}_1 = G_1 z_2 + G_1 \alpha_1 + f_1 + \phi_1^T \theta - \dot{x}_d$

$$V_1 = z_1^T P_1 z_1 + \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} \qquad P_1 A_1 + A_1^T P_1 = -Q_1$$

$$\dot{V}_1 = 2 z_1^T P_1 \left[G_1 \alpha_1 + f_1 - \dot{x}_d + \phi_1^T \hat{\theta} \right] + 2 z_1^T P_1 G_1 z_2 + 2 z_1^T P_1 \phi_1^T \tilde{\theta} - \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}}$$

$$\boxed{\begin{aligned}&\alpha_1 = G_1^{-1} \left[A_1 z_1 - f_1 + \dot{x}_d - \phi_1^T \hat{\theta} \right] = \alpha_1(x_1, \hat{\theta}, t) \\ &\rho_1 = 2 \phi_1 P_1 z_1 = \rho_1(x_1, t) \qquad \Delta_1 = \phi_1^T\end{aligned}}$$

$$\dot{V}_1 = -z_1^T Q_1 z_1 + 2 z_1^T P_1 G_1 z_2 + \tilde{\theta}^T \left[\rho_1 - \Gamma^{-1} \dot{\hat{\theta}} \right]$$

$$\dot{z}_1 = A_1 z_1 + G_1 z_2 + \phi_1^T \tilde{\theta}$$

$$\begin{aligned}\dot{\alpha}_1 = \alpha_1^{x_1} \dot{x}_1 + \alpha_1^{\hat{\theta}} \dot{\hat{\theta}} + \alpha_1^t &= \sigma_1(x_1, x_2, \hat{\theta}, t) \\ &+ \chi_{11}(x_1, \hat{\theta}, t) \tilde{\theta} \\ &+ \chi_{12}(x_1, \hat{\theta}, t) \dot{\hat{\theta}}\end{aligned}$$

Step 2:

$$\dot{z}_2 = G_{12} z_3 + G_{12}\alpha_2 + f_2 + \phi_2^T \theta - \sigma_1 - \chi_{11}\tilde{\theta} - \chi_{12}\dot{\hat{\theta}}$$

$$V_2 = V_1 + z_2^T P_2 z_2 \qquad P_2 A_2 + A_2^T P_2 = -Q_2$$

$$\dot{V}_2 = -z_1^T Q_1 z_1 + \tilde{\theta}^T \left[\rho_1 - \Gamma^{-1}\dot{\hat{\theta}}\right]$$

$$+ 2 z_2^T \left\{ G_1^T P_1 z_1 + P_2 \left[G_{12}\alpha_2 + f_2 + \phi_2^T \hat{\theta} - \sigma_1 - \chi_{12}\dot{\hat{\theta}} \right] \right\}$$

$$+ 2 z_2^T P_2 G_{12} z_3 \qquad + 2 z_2^T P_2 \left[\phi_2^T - \chi_{11}\right]\tilde{\theta}$$

$$\boxed{\begin{array}{l} \alpha_2 = G_{12}^{-1}\left[-P_2^{-1} G_1^T P_1 z_1 + A_2 z_2 - f_2 - \phi_2^T \hat{\theta} + \sigma_1 + \chi_{12}\Gamma\rho_2 \right] \\ \Delta_2 = \phi_2^T - \chi_{11} \qquad \Big| \; \omega_2^T = 2 z_2^T P_2 \chi_{12} \\ \rho_2 = \rho_1 + 2\Delta_2^T P_2 z_2 \end{array}}$$

$$\dot{V}_2 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 + 2 z_2^T P_2 G_{12} z_3$$

$$+ 2 z_2^T P_2 \chi_{12}\left[\Gamma\rho_2 - \dot{\hat{\theta}}\right] + \tilde{\theta}^T \left[\rho_1 - \Gamma^{-1}\dot{\hat{\theta}} + 2\Delta_2^T P_2 z_2\right]$$

$$= -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 + 2 z_2^T P_2 G_{12} z_3$$

$$+ \omega_2^T \left[\Gamma\rho_2 - \dot{\hat{\theta}}\right] + \tilde{\theta}^T \left[\rho_2 - \Gamma^{-1}\dot{\hat{\theta}}\right]$$

$$\dot{z}_2 = -P_2^{-1} G_1^T P_1 z_1 + A_2 z_2 + G_{12} z_3 + \Delta_2 \tilde{\theta} + \chi_{12}\left[\Gamma\rho_2 - \dot{\hat{\theta}}\right]$$

$$\dot{\alpha}_2 = \sigma_2 + \chi_{21}\tilde{\theta} + \chi_{22}\dot{\hat{\theta}}$$

Step 3:

$$\dot z_3 = G_{13} u + f_3 + \phi_3^T \theta - \sigma_2 - \chi_{21} \tilde\theta - \chi_{22} \dot{\hat\theta}$$

$$V_3 = V_2 + z_3^T P_3 z_3 \qquad P_3 A_3 + A_3^T P_3 = -Q_3$$

$$\dot V_3 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2$$
$$+ 2 z_3^T \left\{ G_{12}^T P_2 z_2 + P_3 \left[G_{13} u + f_3 + \phi_3^T \hat\theta - \sigma_2 - \chi_{22} \dot{\hat\theta} \right] \right\}$$
$$+ 2 z_3^T P_3 \left[\phi_3^T - \chi_{21} \right] \tilde\theta + w_2^T \left[\Gamma \rho_2 - \dot{\hat\theta} \right] + \tilde\theta^T \left[\rho_2 - \Gamma^{-1} \dot{\hat\theta} \right]$$

$$\boxed{\Delta_3 = \phi_3^T - \chi_{21}} \qquad \boxed{w_3^T = w_2^T + 2 z_3^T P_3 \chi_{22}}$$

$$\boxed{\rho_3 = \rho_2 + 2 \Delta_3^T P_3 z_3}$$

$$\boxed{u = G_{13}^{-1} \left[-P_3^{-1} G_{12}^T P_2 z_2 + A_3 z_3 - f_3 - \phi_3^T \hat\theta + \sigma_2 + \chi_{22} \Gamma \rho_3 + u_0 \right]}$$

$$\dot V_3 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 - z_3^T Q_3 z_3$$
$$+ 2 z_3^T P_3 \chi_{22} \left[\Gamma \rho_3 - \dot{\hat\theta} \right] + 2 z_3^T P_3 u_0$$
$$+ 2 z_3^T P_3 \Delta_3 \tilde\theta + w_2^T \left[\Gamma \rho_2 - \dot{\hat\theta} \right] + \tilde\theta^T \left[\rho_2 - \Gamma^{-1} \dot{\hat\theta} \right]$$

$$= -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 - z_3^T Q_3 z_3$$
$$+ \tilde\theta^T \left[\rho_3 - \Gamma^{-1} \dot{\hat\theta} \right] + w_3^T \left[\Gamma \rho_3 - \dot{\hat\theta} \right]$$
$$+ \underbrace{w_2^T \left[\Gamma \rho_2 - \cancel{\dot{\hat\theta}} - \Gamma \rho_3 + \cancel{\dot{\hat\theta}} \right]}_{w_2^T \Gamma (\rho_2 - \rho_3) = w_2^T \Gamma (-2 \Delta_3^T P_3 z_3) = -2 z_3^T P_3 \Delta_3 \Gamma w_2} + 2 z_3^T P_3 u_0$$

$$\Longrightarrow \boxed{u_0 = \Delta_3 \Gamma w_2}$$

$$\dot z_3 = -P_3^{-1} G_{12}^T P_2 z_2 + A_3 z_3 + \Delta_3 \tilde\theta + \Delta_3 \Gamma w_2 \quad \Longrightarrow$$

$$\Rightarrow \dot{V}_3 = -z_1^T Q_1 z_1 - z_2^T Q_2 z_2 - z_3^T Q_3 z_3$$
$$+ \tilde{\theta}^T \left[\rho_3 - \Gamma^{-1} \dot{\hat{\theta}} \right] + w_3^T \left[\Gamma \rho_3 - \dot{\hat{\theta}} \right]$$

Adaptive update law:

$$\dot{\hat{\theta}} = \Gamma \rho_3 \quad \Rightarrow \quad \dot{V}_3 = -z^T Q z$$

This design is called a "tuning function design"

where

$$\rho_1 = 2 \Delta_1^T P_1 z_1$$
$$\rho_2 = \rho_1 + 2 \Delta_2^T P_2 z_2$$
$$\rho_3 = \rho_2 + 2 \Delta_3^T P_3 z_3$$

are called "tuning functions".

The equilibrium $(z, \tilde{\theta}) = 0$ is UGS and by LaSalle-Yoshizawa we get that $\lim_{t \to \infty} |z(t)| = 0$.

Preparations for next lecture

Summary lecture:

- The students ask questions.

Bibliography

Skjetne, R. (2005). *The Maneuvering Problem*. PhD thesis, Norwegian Univ. Sci. & Tech., Trondheim, Norway.