Marine Control Systems II

Lecture 3: Nonlinear control

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Goals of lecture

- Learn to specify *control problems*, *control objectives*, and the *"Problem Formulation"* in your reports.
- Understand the stabilization problem of a nonlinear plant.
- Understand how the *control design model* is related to or motivates a *control design method*.
- Be able to explain the difference between *local*, *regional*, *global*, and *semiglobal stabilization*.
- Be able to explain *practical stabilization* vs. *stabilization*.
- Explain the main types of control objectives: *regulation*, *tracking*, *path-following*, and *maneuvering*.
- Understand *constructive control*, and the concept of a *Control Lyapunov Function (CLF)*
    - Learn to apply *Sontag's formula* as a feedback control law.

Literature

- Note on “Mathematical Notations and Preliminaries”
- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 8, 9.1-9.2, 9.7, and intro of 10.
- Lecture presentation.

Control specifications

Important to specify and evaluate the system design:

- Control to a setpoint reference or follow reference trajectories.
- Reduce load disturbances.
- Do not inject too much measurement noise.
- Sensitivity and robustness to modeling errors.
- Limitations and constraints.
- Quantitative descriptions:
  - Time and frequency domains.
  - Many classical specifications were geared towards response to reference signals.
  - Important to consider response to disturbances.

Limitations and constraints

Many factors limit the achievable performance:

- Nonlinear effects such as magnitude and rate saturations.
- Measurement noise.
- Disturbances - e.g. sudden environmental loads.
- Dynamics with nonminimum phase characteristics.
- Time delays.

In the **Problem Formulation**, one must design a philosophy that:

- Respects the limitations and constraints.
- Proposes a modification to the process, if possible.
- Prepares the problem so that it is ready for a *Control* and/or *Observer design*.

In all cases one should not formulate unrealistic specifications.

What should be specified in a Problem Formulation?

The *Problem Formulation* prepares the control problem for design. It should as minimum describe:

- The *system setup* and the *design model(s),* incl. simplifying assumptions:
  - Derive the design model if necessary, or state it from a reference.
  - Clarify specifically the *states*, the *control input(s)*, the *output(s) to control*, and the *measured states* (measurements).
  - Note that the **control design model** and **observer design model** could be different.
- Limitations and constraints to be respected.
- The *control objective* in textual and mathematical terms, as relevant.
- Any performance and robustness specifications.
- Guidance setup, assumptions, feasibility, and details related to the:
  - reference point,
  - desired trajectory,
  - desired path and speed along the path, etc.
- Any autonomy considerations?

What should be specified in a Problem Formulation?

For the design model it is important to specify:

- The states $x \in \mathbb{R}^n$.
- The control input(s) $u \in \mathbb{R}^m$.
- The controlled output(s) $y = h(x)$, $h : \mathbb{R}^n \to \mathbb{R}^p$, e.g. position and/or heading of a vessel (ref. *work space* vs. *configuration space*).
- The measured states $z = k(x)$, $k : \mathbb{R}^n \to \mathbb{R}^q$, e.g., position and heading of a vessel, possibly accelerations and angular rates?

In addition, the *Problem Statement* should clarify other aspects:

- Available measurements to the control law, that is,
  - is it a *full-state feedback* design or
  - *output feedback* with an observer?
- Any *system constraints*.
- Possibly a *cost function* for optimization (in optimal control) or as a key performance indicator (KPI).
- Specific care taken to *internal state stability* (minimum vs. nonminimum phase systems).

What should be specified in a Problem Formulation?

When all aspect of the control problem has been described and specified, the *Problem Statement* comes as the conclusion of the Problem Formulation.

This should as minimum describe:

- Reference the control system with inputs and outputs.
- Reference the control objective related to a guidance system.
- Repeat the need for observer design for filtering and state estimation.
- In mathematical terms, state the control task, e.g. "*The control objective is to design a control law for u such that*

$$\lim_{t \to \infty} |y(t) - y_d(t)| = 0,$$

while keeping all system states stable."

Nonlinear plant

Typically the high-fidelity **simulation model**

$$\dot{\xi} = F(\xi, u)$$

is too complex, unnecessarily realistic, and of too high fidelity to base a model-based control design upon.

In other cases, since the state-space representation is not unique, it is desired to transform the model into another more convenient representation.

In any case, we seek justifiable simplifications to transform the process model into a simplified **design model** that control or observer design can be based upon. This is typically achieved by linearization or by some method of model reduction.

Given a high-fidelity state ξ, a reduced state for control design is often determined $x \in \{\xi\}$ as a subset of ξ. E.g., for DP we use $\eta_{3DOF} = \{x, y, \cancel{z, \phi, \theta}, \psi\}$ and $\nu_{3DOF} = \{u, v, \cancel{w, p, q}, r\}$. All state reduction must be justified.

Nonlinear plant

After model reduction, consider the nonlinear control design model

$$\dot{x} = f(x, u), \quad y = h(x)$$

where for each $t \geq 0$:

- $x(t) \in \mathbb{R}^n$ is the state vector,
- $u(t) \in \mathbb{R}^p$ is the control,
- $y(t) \in \mathbb{R}^m$ is the output to control, and
- $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ are smooth functions.

Control design (continuous control) is to construct a, possibly dynamic, control law

$$\begin{aligned} \dot{\xi} &= g(t, x, \xi) \\ u &= \kappa(t, x, \xi) \end{aligned}$$

in order to solve a specified control problem.

Stabilization

We want to stabilize

$$\dot{x} = f(x, u)$$

at an equilibrium $x = x_0$, where f is locally Lipschitz.

The **Steady State Problem** is then to find u_0 s.t.

$$0 = f(x_0, u_0)$$

Let $x_\delta = x - x_0$ and $u_\delta = u - u_0$. Then

$$\dot{x}_\delta = f(x_\delta + x_0, u_\delta + u_0) := f_\delta(x_\delta, u_\delta)$$

for which $f_\delta(0, 0) = 0$.

State feedback stabilization: Design $u_\delta = \alpha(x_\delta)$, α locally Lipschitz, s.t. $x_\delta = 0$ is stable for $\dot{x}_\delta = f_\delta(x_\delta, \alpha(x_\delta))$, which by

$$u = u_0 + \alpha(x_\delta)$$

implies that $x = x_0$ is stable for $\dot{x} = f(x, u_0 + \alpha(x_\delta))$.

Linearization

A common technique for further model simplification is linearization:

- Linearized model:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

  where

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=x_0,\ u=u_0}, \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x=x_0,\ u=u_0}$$

$$C = \left.\frac{\partial h(x)}{\partial x}\right|_{x=x_0}$$

Linear state feedback control

Suppose the entire state vector x is available, i.e., $C = I$.
Assume (A, B) is stabilizable - i.e., controllable or every uncontrollable eigenvalue has a negative real part.
Then linear state feedback control is to find a matrix K such that $(A - BK)$ is Hurwitz.
Control law:

$$u = -Kx$$

How do you find K?

- Trial and Error - e.g., on a sea trials.
- Eigenvalue/Pole placement.
- Eigenvalue-Eigenvector placement.
- Linear Quadratic Regulator (LQR).
- Derivative-free optimization (DFO).
- Artificial intelligence (AI).

Linear state feedback control

The linearized closed-loop system:

$$\dot{x} = (A - BK)x$$

is obviously GES.

The original system,

$$\dot{x} = f(x, -Kx)$$

is, on the other hand, typically only LES. Its region of convergence (ROC) may be difficult to quantify.

Example 1

We consider an inverted pendulum on a cart,

$$L\ddot{\theta} - \ddot{p}\cos\theta = g\sin\theta$$
$$(M+m)\,\ddot{p} - mL\ddot{\theta}\cos\theta + mL\dot{\theta}^2\sin\theta = u$$

where θ is the angle from upright position, p is the position of the cart, u is a control force input on the cart, m is the point mass of the pendulum, M is the mass of the cart, and L is the pendulum length.

If the control objective is to control $(x, \theta) \to (x_0, 0)$, then the steady state problem is given by $u_0 = 0$.

...Example 1

Linearizing these equations, we get

$$L\ddot{\theta} - \ddot{p} = g\theta$$
$$(M+m)\,\ddot{p} - mL\ddot{\theta} = u.$$

Let $x_1 = \operatorname{col}(\theta, p)$ and $x_2 = \operatorname{col}\left(\dot{\theta}, \dot{p}\right)$, then we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \begin{bmatrix} \frac{g}{L} + \frac{gm}{LM} & 0 \\ \frac{gm}{M} & 0 \end{bmatrix} x_1 + \begin{bmatrix} \frac{1}{LM} \\ \frac{1}{M} \end{bmatrix} u \end{aligned}$$

Other typical classes suitable for feedback control

Various model-based nonlinear design techniques exist for equally various classes of control models:

- *Feedback linearizable systems.*
- *Strict feedback form.*
- *Parametric strict feedback form.*
- *Feedforward systems.*
- *Cascaded systems.*
- *Interconnections of passive systems.*
- etc.

Strict feedback form

- *Strict feedback form* of vector relative degree n :

$$\begin{aligned}
\dot{x}_1 &= G_1(x_1)\,x_2 + f_1(x_1) + W_1(x_1)\,\delta_1(t) \\
\dot{x}_2 &= G_2(x_1, x_2)\,x_3 + f_2(x_1, x_2) + W_2(x_1, x_2)\,\delta_2(t) \\
&\;\;\vdots \\
\dot{x}_n &= G_n(x_1, \ldots, x_n)\,u + f_n(x_1, \ldots, x_n) + W_n(x_1, \ldots, x_n)\,\delta_n(t) \\
y &= h(x_1)
\end{aligned}$$

  - $x_i(t) \in \mathbb{R}^m$, $i = 1, \ldots, n$, are the states, $y(t) \in \mathbb{R}^m$ is the output, $u(t) \in \mathbb{R}^m$ is the control.
  - $\delta_i(\,\cdot\,)$ are unknown bounded disturbances.
  - $G_i(x_1, \ldots, x_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible, $h(x_1)$ is a diffeomorphism, and G_i, f_i, and W_i are smooth.

- This system is prepared for a *backstepping* design or *feedback linearization* (for $\delta_i = 0$).

Parametric strict feedback form

- *Parametric strict feedback form* of vector relative degree n :

$$
\begin{aligned}
\dot{x}_1 &= G_1\left(x_1\right) x_2 + f_1\left(x_1\right) + \Phi_1\left(x_1\right) \varphi \\
\dot{x}_2 &= G_2\left(x_1, x_2\right) x_3 + f_2\left(x_1, x_2\right) + \Phi_2\left(x_1, x_2\right) \varphi \\
&\ \vdots \\
\dot{x}_n &= G_n\left(x_1, \ldots, x_n\right) u + f_n(x_1, \ldots, x_n) + \Phi_n\left(x_1, \ldots, x_n\right) \varphi \\
y &= h\left(x_1\right)
\end{aligned}
$$

  - $x_i \in \mathbb{R}^m,\ i = 1, \ldots, n$, are the states, $y \in \mathbb{R}^m$ is the output, $u \in \mathbb{R}^m$ is the control, and $\varphi \in \mathbb{R}^p$ is a vector of constant unknown parameters.
  - $G_i(x_1, \ldots, x_i)$ and $h^{x_1}(x_1) := \frac{\partial h}{\partial x_1}(x_1)$ are invertible for all $\bar{x}_i$, the map $h(x_1)$ is a diffeomorphism, and G_i, f_i, and Φ_i are smooth.
- This system is prepared for an *adaptive backstepping* design.

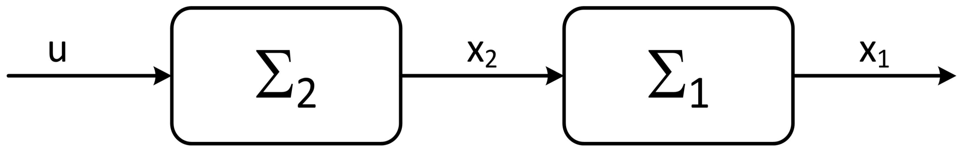
Strict feedforward form

- Nonlinear plant in *strict feedforward form:*

$$
\begin{aligned}
\dot{x}_1 &= f_1\left(x_1, \ldots, x_n\right) + G_1\left(x_1, \ldots, x_n\right) u \\
\dot{x}_2 &= f_2\left(x_2, \ldots, x_n\right) + G_2\left(x_2, \ldots, x_n\right) u \\
&\ \vdots \\
\dot{x}_n &= f_n\left(x_n\right) + G_n\left(x_n\right) u
\end{aligned}
$$

  where $x_i \in \mathbb{R}^m,\ i = 1, \ldots, n$ are the states and $u \in \mathbb{R}^m$ is the control. The matrices $G_i(\,\cdot\,)$ are invertible for all x, and G_i and f_i are smooth.
- This system is prepared for a *feedforwarding control* design.

Time-invariant cascaded systems



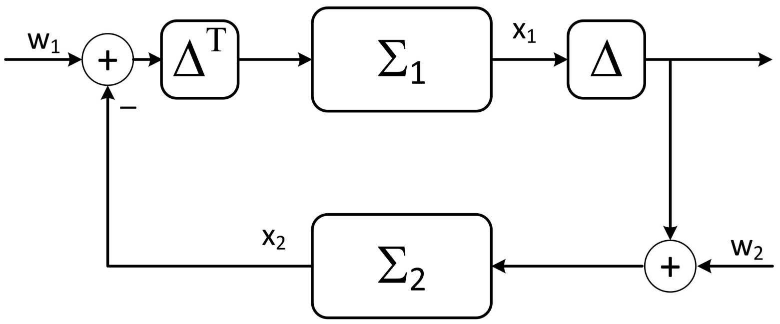
- Nonlinear plant in *cascaded form:*

$$\begin{aligned} \Sigma_1 : \quad & \dot{x}_1 = f_1(x_1, x_2) \\ \Sigma_2 : \quad & \dot{x}_2 = f_2(x_2, u) \end{aligned}$$

  where $x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m$, and $u \in \mathbb{R}^p$ is the control. The function f_1 is continuously differentiable in (x_1, x_2).

- The control objective is now to find a control $u = \alpha(x_1, x_2)$ or $u = \alpha(x_2)$ such that the cascaded interconnection is GAS or GS.

Time-invariant passive systems



- Nonlinear plant in *interconnected form:*

$$\begin{aligned} \Sigma_1 : \quad & \dot{x}_1 = f_1(x_1, x_2, u_1) \\ \Sigma_2 : \quad & \dot{x}_2 = f_2(x_2, x_1, u_2) \end{aligned}$$

  where $x_1 \in \mathbb{R}^n,\ x_2 \in \mathbb{R}^m$, and (u_1, u_2) are the controls.

- The control objective is now to find a controls $u_1 = \alpha_1(x_1, x_2, w_1)$ and $u_2 = \alpha_2(x_2, x_1, w_2)$ such that the interconnection is passive.

Notions of stabilization

From [Khalil, 2015, Ch. 9.1], [Khalil, 2002a, Lecture 25], we have for

$$\dot{x} = f(x, u), \qquad u = \alpha(x)$$

- **Local stabilization:** The origin of $\dot{x} = f(x, \alpha(x))$ is LAS (e.g., by linearization).
- **Regional stabilization:** The origin of $\dot{x} = f(x, \alpha(x))$ is LAS, and a given region $\mathcal{G}$ is a subset of the ROC ($\forall x(0) \in \mathcal{G}$, $\lim_{t\to\infty} x(t) = 0$). E.g., $\mathcal{G} \subset \Omega_c := \{V(x) \le c\}$ where Ω_c is an estimate of the ROC.
- **Global stabilization:** The origin of $\dot{x} = f(x, \alpha(x))$ is GAS.
- **Semiglobal stabilization:** The origin of $\dot{x} = f(x, \alpha(x))$ is LAS, and α can be designed so that any specified compact set can be included in the ROC.
  - Typically, $u = \alpha(p, x)$ such that for any compact set $\mathcal{G}$, the parameter p can be set to ensure $\mathcal{G}$ $\subset$ROC.

Notions of stabilization

What is the difference between *Global stabilization* and *Semiglobal stabilization*?

Practical stabilization

Consider

$$\dot{x} = f(x,u) + \delta(t,x,u)$$
$$f(0,0) = 0, \qquad \delta(t,0,0) \neq 0$$
$$|\delta(t,x,u)| \leq \delta_0 < \infty, \quad \forall x \in D_x,\ u \in D_u,\ t \geq 0$$

There is no control $u = \alpha(x)$ with $\alpha(0) = 0$ that renders the origin of

$$\dot{x} = f(x, \alpha(x)) + \delta(t, x, \alpha(x))$$

ULAS, since the origin is not an equilibrium point.

Practical stabilization

Definition

[Khalil, 2015, Def. 10.1] The system

$$\dot{x} = f(x,u) + \delta(t,x,u)$$

is practically stabilizable if for any $\varepsilon > 0$, $\exists u = \alpha(x)$ such that the solutions of

$$\dot{x} = f(x, \alpha(x)) + \delta(t, x, \alpha(x))$$

are uniformly ultimately bounded by ε, that is,

$$|x(t)| \leq \varepsilon, \qquad \forall t \geq T$$

Typically, $u = \alpha(p, x)$ such that for any $\varepsilon > 0$, the parameter p can be set to ensure that ε is an ultimate bound.

Practical stabilization

With practical stabilization, one can again have

- **Local practical stabilization**,
- **Regional practical stabilization**,
- **Global practical stabilization**, or
- **Semiglobal practical stabilization**

depending on allowable region of the initial state.

Regulation

For your system

$$\dot{x} = f(x, u), \qquad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$$
$$y = h(x), \qquad h : \mathbb{R}^n \to \mathbb{R}^m$$

let the control objective be to regulate the output $y(t)$ to a constant output *reference* y_{ref} or to regulate the state $x(t)$ to a constant state reference x_{ref} (i.e., $h = I$).
In other words, to asymptotically stabilize a designed equilibrium

$$y(t) = y_{ref} \quad \text{or} \quad x(t) = x_{ref}$$

Note in regulation that the reference y_{ref} may typically be piecewise constant and reset to new values intermittently.

Often a *reference filter* is used to change the reference point smoothly to avoid step responses.

Tracking

For your system

$$\begin{aligned} \dot{x} &= f(x,u), \qquad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \\ y &= h(x), \qquad h : \mathbb{R}^n \to \mathbb{R}^m \end{aligned}$$

let the control objective be for the output $y(t)$ to track a desired output $y_d(t)$, that is, to asymptotically stabilize

$$y = y_d(t) = h_d(x_d(t)).$$

Note in tracking that both the time evolution $y_d(t)$ and its dynamic motion $\dot{y}_d(t)$, $\ddot{y}_d(t)$, etc. is specified in one combined package, called the *desired trajectory*.

A *guidance system* or a *reference filter* typically generates the desired trajectory and its necessary derivatives.

Path-following

For your system

$$\begin{aligned} \dot{x} &= f(x,u), \qquad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \\ y &= h(x), \qquad h : \mathbb{R}^n \to \mathbb{R}^m \end{aligned}$$

let a *desired path* for the output y be specified in some way. The method for specifying the path is important because it affects the control design.

In path-following, the important task is to stay on and follow the path. The dynamic motion along the path (time scheduling, speed, acceleration, etc.) is less important.

...Path-following

Let the path be specified by the set of points

$$\mathcal{P} = \{ y \in \mathbb{R}^m : \ |h_p(y)| \leq \varepsilon \} .$$

Then the path-following control objective is for the output $y(t)$ to enter and stay within $\mathcal{P}$ at the same time as the speed satisfies $|\dot{y}(t)| \geq U_0 > 0$.

A *guidance system* is used to generate/specify the path.

Here the path was specified by a maximum and minimum constraint for deviating from the curve $h_p(y) = 0$, as sort of following a road. Typically, however, the path is specified by a continuous curve as elaborated next . . .

...Path-following

Typically, the path-following objective is to stay on and follow a curve perfectly, i.e., the set

$$\mathcal{P} = \{ y \in \mathbb{R}^m : \ h_p(y) = 0 \} .$$

This curve can be parametrized by discrete points (waypoints), by straight lines and circular arcs, or continuously. The latter implies a continuous parametrization such as

$$\mathcal{P} = \{ y \in \mathbb{R}^m : \ \exists s \in \mathbb{R} \text{ such that } y = y_d(s) \} .$$

In this case one can typically generate motion for $s(t)$ such that the path-following control objective becomes a tracking task $y(t) \to y_d(s(t))$. However, this eliminates some of the flexibility in the path-following control problem.

Maneuvering

For your system

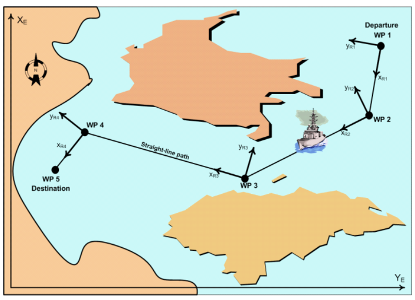
$$\begin{aligned} \dot{x} &= f(x,u), \qquad f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \\ y &= h(x), \qquad h : \mathbb{R}^n \to \mathbb{R}^m \end{aligned}$$

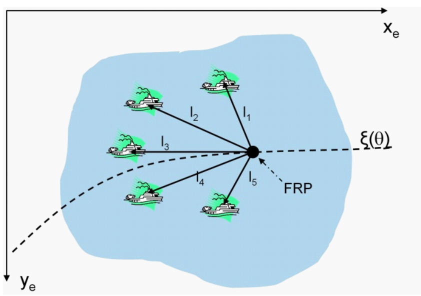
let the control objective be to follow a desired path continuously parametrized by the curve $s \mapsto y_d(s)$.

Moreover, let the speed (or dynamic behavior) along the path be specified by some function $U(t)$ such that $|\dot{y}(t)| \to U(t)$.

This problem can be translated into the following problem statement …

...Maneuvering





The Maneuvering Problem is comprised of the two tasks, in prioritized order:

1. **Geometric Task:** -for some absolutely continuous function $s(t)$, force the output y to converge to the desired parametrized path $y_d(s)$, i.e.,
$$\lim_{t\to\infty} |y(t) - y_d(s(t))| = 0.$$

...Maneuvering

2. **Dynamic Task:** Satisfy one or more of the assignments:
   0.1 *Time Assignment:* -force s to converge to a desired time assignment $\tau(t)$,
   $$\lim_{t\to\infty} |s(t) - \tau(t)| = 0.$$
   0.2 *Speed Assignment:* -force $\dot{s}$ to converge to a desired speed assignment $\upsilon(s,t)$,
   $$\lim_{t\to\infty} |\dot{s}(t) - \upsilon(s(t),t)| = 0.$$
   0.3 *Acceleration Assignment:* -force $\ddot{s}$ to converge to a desired acceleration assignment $\alpha(\dot{s}(t), s(t), t)$,
   $$\lim_{t\to\infty} |\ddot{s}(t) - \alpha(\dot{s}(t), s(t), t)| = 0.$$

...Maneuvering

It follows for a speed assignment that The Maneuvering Problem is to construct a dynamic control law

$$\begin{aligned}\dot{s} &= \omega(t,x,s)\\ u &= \kappa(t,x,s)\end{aligned}$$

to render the set

$$\mathcal{A} = \{(\tau, x, s) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R} : \quad h(x) = y_d(s), \;\; \omega(\tau, x, s) = \upsilon(s, \tau)\}$$

UGAS for the closed-loop system

$$\begin{aligned}\dot{s} &= \omega(t,x,s)\\ \dot{x} &= f(x, \kappa(t,x,s)).\end{aligned}$$

Control Lyapunov Function (CLF)

See [Khalil, 2002a], Lecture 30, on Control Lyapunov Functions. Consider

$$\dot{x} = f(x) + G(x)u, \qquad f(0) = 0,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $G : \mathbb{R}^n \to \mathbb{R}^{n \times m}$.

Suppose there exist a continuous state feedback law $\psi(x) : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $x = 0$ for $\dot{x} = f(x) + G(x)\psi(x)$ is LAS.

Then by the converse Lyapunov theorem, there is a $V(x)$ s.t.

$$V^x(x)\left[f(x) + G(x)\psi(x)\right] < 0, \qquad \forall x \in D \backslash \{0\}.$$

If $u = \psi(x)$ is globally stabilizing, $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded.

...Control Lyapunov Function (CLF)

$$\begin{array}{l} V^x(x)\left[f(x) + G(x)\psi(x)\right] < 0, \qquad \forall x \in D \backslash \{0\} \\ \qquad\qquad\qquad \Downarrow \\ V^x(x)G(x) = 0 \text{ for } x \in D \backslash \{0\} \quad \Rightarrow \quad V^x(x)f(x) < 0 \end{array}$$

This implies the **Small Control Property:**

Since $\psi(x)$ is continuous and $\psi(0) = 0$, then for any $\varepsilon > 0$, $\exists \delta > 0$ so that for $|x| < \delta$, $x \neq 0$, there is u with $|u| < \varepsilon$ such that

$$V^x(x)\left[f(x) + G(x)u\right] < 0$$

...Control Lyapunov Function (CLF)

See [Khalil, 2002b, Khalil, 2015].

Definition

A continuously differentiable positive definite function $V(x)$ is called a **Control Lyapunov Function** (CLF) for $\dot{x} = f(x) + G(x)u$ if

1. $V^x(x)G(x) = 0$ for $x \in D \backslash \{0\}$ implies $V^x(x)f(x) < 0$.
2. $V(x)$ satisfies the *Small Control Property*.

If $V(x)$ is radially unbounded and satisfies 1. with $D = \mathbb{R}^n$, then it is a Global CLF.

The system $\dot{x} = f(x) + G(x)u$ is stabilizable by a continuous state feedback control ONLY IF it has a CLF.

Sontag's formula

See [Khalil, 2002b, Khalil, 2015].

Theorem

Let $V(x)$ be a CLF for $\dot{x} = f(x) + G(x)u$. Then the origin $x = 0$ is stabilizable by $u = \psi(x)$, where

$$\psi = \begin{cases} -\frac{\left[V^x f + \sqrt{(V^x f)^2 + ((V^x G)(V^x G)^\top)^2}\right]}{(V^x G)(V^x G)^\top} (V^x G)^\top, & \text{if } V^x G \neq 0 \\ 0, & \text{if } V^x G = 0 \end{cases}$$

*This is called **Sontag's formula**.*

The control law $\psi(x)$ is continuous for all $x \in D$ including $x = 0$.

If f and G are smooth, then ψ is smooth for $x \neq 0$.

If V is a global CLF, then $u = \psi(x)$ is globally stabilizing.

...Sontag's formula

Sketch of proof:

We have $V^x(x)\,[f(x) + G(x)\psi(x)]$.

If $V^x G = 0$ then $\dot{V} = V^x(x) f(x) < 0$ for $x \neq 0$ by definition of the CLF.

If $V^x G \neq 0$ then

$$\begin{aligned}
\dot{V} &= V^x f - \left[V^x f + \sqrt{(V^x f)^2 + ((V^x G)(V^x G)^\top)^2} \right] \frac{(V^x G)(V^x G)^\top}{(V^x G)(V^x G)^\top} \\
&= V^x f - \left[V^x f + \sqrt{(V^x f)^2 + ((V^x G)(V^x G)^\top)^2} \right] \\
&= -\sqrt{(V^x f)^2 + ((V^x G)(V^x G)^\top)^2} < 0, \qquad \forall x \neq 0.
\end{aligned}$$

How to find a CLF?

- If you know of any control law $u = \psi(x)$ with an associated Lyapunov function $V(x)$, then this is a CLF.
- Systematic design methods:
  - Feedback linearization.
  - Backstepping.

Example 1

Consider

$$\dot{x} = ax - bx^3 + u, \qquad a, b > 0$$

such that $f(x) = ax - bx^3$ and $G(x) = 1$. By feedback linearization we would choose

$$u = \psi_{FL} = -ax + bx^3 - kx, \qquad k > 0$$

such that $\dot{x} = -kx$.
Now $V(x) = \frac{1}{2}x^2$ is a CLF:

$$V^x G = x \text{ and } V^x f = ax^2 - bx^4$$

We get that $V^x G = 0$ only for $x = 0$.

Moreover, for any $\varepsilon > 0$ there should exist $\delta > 0$ where $|x| < \delta \Rightarrow \exists\, |u| < \varepsilon$ s.t.

$$V^x(x)\left[f(x) + G(x)u\right] = ax^2 - bx^4 + xu < 0.$$

...Example 1

Let, for instance, $u = -ax$ and $\delta = \frac{\varepsilon}{a}$. Then $|x| < \delta$ gives

$$|u| = |-ax| = a\,|x| < a\delta = \varepsilon.$$

Hence, the small control property is satisfied.

Sontag's formula gives for $V^x G = x \neq 0$

$$\begin{aligned} u = \psi_{SF} &= -\frac{\left[ax^2 - bx^4 + \sqrt{(ax^2 - bx^4)^2 + x^4}\right]}{x} \\ &= -\frac{x^2\left(a - bx^2\right)}{x} - \frac{\sqrt{x^4(a - bx^2)^2 + x^4}}{x} \\ &= -ax + bx^3 - x\sqrt{(a - bx^2)^2 + 1} \end{aligned}$$

...Example 1

We analyze:

Method	Control law	Closed-loop
FL	$u_{FL} = -ax + bx^3 - kx$	$\dot{x} = -kx$
CLF	$u_{SF} = -ax + bx^3 - \sqrt{(a-bx^2)^2+1}x$	$\dot{x} = -\sqrt{(a-bx^2)^2+1}x$
FL: $\lvert x\rvert \ll 1$	$u_{FL} \approx -(a+k)x$	$\dot{x} \approx -kx$
CLF: $\lvert x\rvert \ll 1$	$u_{SF} \approx -(a+\sqrt{a^2+1})x$	$\dot{x} \approx -\sqrt{a^2+1}x$
FL: $\lvert x\rvert \gg 1$	$u_{FL} \approx bx^3$	$\dot{x} \approx -kx$
CLF: $\lvert x\rvert \gg 1$	$u_{SF} \approx -ax$	$\dot{x} \approx -bx^3$

Preparations for next lecture

ISS and Feedback linearization:

- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 4.2-4.4 and 9.1-9.4.
- Lecture presentation.

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