TMR4243 - Marine Control Systems II

Exam

Spring 2017

Notation: Throughout this exam $|x|$ means the vector 2-norm, i.e. $|x| = \sqrt{x^\top x}$. For a scalar x, this corresponds to the absolute value.

States and variables are scalars unless these are specifically defined as vectors, e.g., x_1 is a scalar while $x_2 \in \mathbb{R}^n$ is an n-dimensional vector.

1 Properties of nonlinear systems (20 pts)

1. Consider the three ordinary differential equations (ODEs):

$$\dot{z} = g_1(z) = -cz \quad (1)$$
$$\dot{z} = g_2(z) = -cz^3 \quad (2)$$
$$\dot{z} = g_3(z) = -c\sin(z) \quad (3)$$

where $z \in \mathbb{R}$, $z_0 = z(0)$, and c is a positive constant.

(a) For each of the three ODEs, explain if these are *Locally Lipschitz* and/or *Globally Lipschitz* or not Lipschitz at all.

(b) What can you say about *existence*, *uniqueness*, and *forward completeness* of the solutions for the three ODEs?

2. For each of the three systems

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - 2x_2 \end{aligned}} \tag{4}$$

$$\boxed{\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \cos x_1 \\ \dot{x}_2 &= x_1 \cos x_1 - x_2 \left(\cos x_1\right)^2 + x_3 \\ \dot{x}_3 &= -x_2^3 \end{aligned}} \tag{5}$$

$$\boxed{\begin{aligned} \dot{x}_1 &= 3 \sin x_2 \\ \dot{x}_2 &= -\sin x_2 \end{aligned}} \tag{6}$$

Find the equilibria of these systems, and explain what type of equilibria these are.

3. The scalar ODE

$$\dot{x} = -x^2, \qquad x_0 = -1$$

admits the solution

$$x(t) = \frac{1}{t-1}, \qquad t \geq 0$$

(a) Discuss the Lipschitz properties of this ODE and the corresponding existence, uniqueness, and forward completeness of its solutions.

(b) Is it stable?

2 Lyapunov stability (27 pts)

1. For the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -\left(x_1 + x_1^3\right) - 2x_1x_2^2 \\ \dot{x}_2 &= x_1^2x_2 - \left(x_2 + x_2^5\right)\end{aligned}$$

let a Lyapunov function candidate be

$$V(x_1, x_2) = c_1x_1^2 + c_2x_2^2$$

where $c_1 > 0$ and $c_2 > 0$ are constants.

(a) Calculate the time derivative of V as a function of (x_1, x_2).

(b) Find values for c_1 and c_2 that proves UGES of $(x_1, x_2) = 0$.

2. Let a system be

$$\begin{aligned} \dot{x}_1 &= 4x_2 \\ \dot{x}_2 &= -2\,\mathrm{sat}(x_1) - \frac{x_2}{10} \end{aligned}$$

where

$$\mathrm{sat}(y) := \begin{cases} -1; & y \leq -1 \\ y & -1 < y < 1 \\ +1 & 1 \leq y \end{cases}$$

Let a Lyapunov function candidate be

$$V(x_1, x_2) = \int_0^{x_1} \mathrm{sat}(y)dy + x_2^2$$

(a) Draw $\mathrm{sat}(y)$ and show that it is globally Lipschitz.

(b) Show that $\mathrm{sat}(0) = 0$, $\mathrm{sat}(y)y > 0$ for all $y \neq 0$, and that V is radially unbounded.

(c) By differentiating V, what stability conclusion can you make for the origin by Lyapunov's direct method?

(d) Show that the origin is UGAS.

3. Consider the nonlinear time-varying system:

$$\dot{x} = G\left(x - x_d(t)\right) + H(x)\left(x - x_d(t)\right) + \dot{x}_d(t)$$

where G is a constant matrix satisfying

$$G + G^\top < 0,$$

$H(x)$ is a nonlinear matrix satisfying

$$H(x) = -H(x)^\top,$$

and $(x_d(t), \dot{x}_d(t))$ are bounded reference signals.

Let a Lyapunov function candidate be

$$V(t,x) = \left(x - x_d(t)\right)^\top P\left(x - x_d(t)\right)$$

(a) Show that G satisfies the Lyapunov equation

$$PG + G^\top P = -Q$$

with $P = I$ (identity matrix). What becomes Q?

(b) Show how to differentiate $V(t,x)$.

(c) What is the stability conclusion according to Lyapunov's direct method?

3 DP observer and control design (30 pts)

Consider the low-speed DP vessel model

$$\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= \tau - D\nu \\ z &= \eta \end{aligned}$$

where $\eta = col(x, y, \psi)$, $\nu = col(u, v, r)$, $M^\top > 0$ and $D > 0$ are the mass and damping matrices, respectively, $R(\psi)$ is the rotation matrix, and z is the measured output.

Suppose D satisfies

$$D + D^\top > 0.$$

1. Assume that $R = I$ (constant identity matrix) and investigate if the "linearized system" is *uniformly completely observable.*

2. Let an observer be

$$\begin{aligned} \dot{\hat{\eta}} &= R(\psi)\hat{\nu} + K_1 (z - \hat{\eta}) \\ M\dot{\hat{\nu}} &= \tau - D\hat{\nu} + R(\psi)^\top K_2 (z - \hat{\eta}), \end{aligned}$$

   where $K_1 = K_1^\top > 0$ and $K_2 = K_2^\top > 0$ are injection gain matrices.

   (a) Write down the equations for the corresponding observer error dynamics $\tilde{\eta} := \eta - \hat{\eta}$ and $\tilde{\nu} := \nu - \hat{\nu}$.

   (b) Let a Lyapunov function candidate be

   $$V_o = \tilde{\eta}^\top K_2 \tilde{\eta} + \tilde{\nu}^\top M \tilde{\nu},$$

   and find the time derivative of V_o as a function of the error states.

   (c) Derive and give conditions on the injection gain matrices K_1 and K_2 that ensures that the error dynamics is UGES.

3. Disregarding the observer for now, let a state feedback control law to control $(\eta, \nu) \to 0$, be

$$\tau = -R(\psi)^\top L_1 \eta - L_2 \nu$$

where $L_1 = L_1^\top > 0$ and $L_2 = L_2^\top > 0$.

(a) Write down the closed-loop system with this state feedback.

(b) Let a Lyapunov function candidate be

$$V_c = \eta^\top L_1 \eta + \nu^\top M \nu,$$

and find the time derivative of V_c as a function of (η, ν).

(c) What stability conclusion can you make from Lyapunov's direct method?

(d) Use Krasovskii-LaSalle's invariance principle to show that $(\eta, \nu) = (0, 0)$ is in fact UGAS.

4. Using output feedback, by including the observer, let the feedback control law be

$$\tau = -R(\psi)^\top L_1 \hat{\eta} - L_2 \hat{\nu}.$$

(a) Write down the overall closed-loop system in $(\tilde{\eta}, \tilde{\nu}, \eta, \nu) \in \mathbb{R}^{12}$.

(b) Show that the closed-loop system is a cascade between the UGES observer error system and the UGAS feedback control system. What is the interconnection terms?

(c) These interconnection terms will in fact satisfy a linear growth condition, which together with UGES+UGAS proves that the overall cascaded system is UGAS. Explain and discuss the *separation principle*, given this fact.

4 Adaptive control design (23 pts)

Consider a nonlinear mechanical system

$$\begin{aligned} \dot{x}_1 &= x_2 + v_0 \\ M\dot{x}_2 &= -D(x_1, x_2)x_2 + u \end{aligned}$$

where $x_1 \in \mathbb{R}^m$ contains e.g. positions and angles of the system, $x_2 \in \mathbb{R}^m$ is a relative velocity state, $v_0 \in \mathbb{R}^m$ is a constant unknown velocity reference, $u \in \mathbb{R}^m$ is the control input (e.g. typically forces and torques), $M = M^\top > 0$ is a constant mass matrix, and $D(x_1, x_2)$ is a nonlinear matrix.

Let the control objective be to control $x_1 \to x_d(t)$.

1. Assume $x_2 \equiv \alpha_1$ is a control input for the 1st equation and v_0 is an unknown constant vector to be adaptively estimated.

   Let $\hat{v}_0$ be an estimate, $\tilde{v}_0 := v_0 - \hat{v}_0$, and $z_1 := x_1 - x_d(t)$.

   (a) Using the control Lyapunov function

   $$V_1 := \frac{1}{2} z_1^\top z_1 + \frac{1}{2\gamma} \tilde{v}_0^\top \tilde{v}_0$$

   design a control law for α_1 and an adaptive update law for $\dot{\hat{v}}_0$ that solves the control objective.

   (b) What is the closed-loop system in $(z_1, \tilde{v}_0)$?

   (c) What stability and convergence property will you get for the origin $(z_1, \tilde{v}_0) = 0$?

2. Assume M and $D(\cdot,\cdot)$ are fully known, and let $z_2 := x_2 - \alpha_1$ and

$$V_2 := V_1 + \frac{1}{2} z_2^\top M z_2$$

(a) With your α_1 as you defined it above, and $z_2 = x_2 - \alpha_1$, what is now your closed-loop equation for $\dot{z}_1$ and the resulting derivative of $\dot{V}_1$?

(b) Differentiate $M z_2$ and $\dot{V}_2$, and design a control law for u by the (adaptive) *backstepping* control design method, that solves the control objective for the complete system.

(c) Write down the closed-loop system in the states $(\tilde{v}_0, z_1, z_2)$ and conclude what stability and convergence you get.