Marine Control Systems II

Lecture 2: Nonlinear systems and stability

Roger Skjetne

Department of Marine Technology
Norwegian University of Science and Technology

TMR4243

Goals of lecture

- Understand UGS, UGAS, and UGES global stability definitions.
- Understand how a Lyapunov function implies local or global stability properties.
- Derive *Region of Convergence* (RoC) from Lyapunov surfaces.
- Relate the *Lyapunov equation* to a *Lyapunov function* for a linear system.
- Demonstrate when and how to apply *Krasovskii-LaSalle's invariance principle*.
- Understand properties of *nonautonomous systems* (time-varying ODEs)
- Stability definitions redone for time-varying ODEs. Understand the concept of *Uniform*.
- Demonstrate skills to apply *Lyapunov's direct method*.
- Learn when to apply the invariance theorems:
  - *Barbalat's Lemma.*
  - *LaSalle-Yoshizawa.*
  - *Matrosov's Theorem.*

Literature

- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 3.4-3.7 and 4.1
- Lavretsky, E. and K. A. Wise (2013). Robust and Adaptive Control (With Aerospace Applications)
  - Chapters 8.4-8.8 (for alternative explanations and deeper learning)
- Lecture presentation.

Recall the Lyapunov stability result

Define $B_r := \{x \in \mathbb{R}^n : |x| \leq r\}$ be a ball set in $\mathbb{R}^n$.

$\exists V : B_r \mapsto \mathbb{R}$ continuously differentiable such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -\alpha_3(|x|), \qquad x \in B_r$$

where $\alpha_1, \alpha_2 \in \mathcal{K}$ and α_3 is a continuous function.

- If α_3 is positive semidefinite, then $x = 0$ is Locally Stable (LS).
- If α_3 is positive definite, then $x = 0$ is Locally Asymptotically Stable (LAS).

In the latter case, we get that $|x(t)| \leq \beta(|x_0|, t)$, $\beta \in \mathcal{KL}$.

Example 1

Consider the simple linear system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - x_2\end{aligned}$$

and the Lyapunov function

$$V(x) = 1.5x_1^2 + x_1x_2 + x_2^2$$

for which we can show that

$$0.5\,|x|^2 \le V(x) \le 2\,|x|^2 .$$

Differentiating, we get that

$$\begin{aligned}\dot{V}(x) &= 3x_1x_2 + x_2^2 + x_1\left(-x_1 - x_2\right) + 2x_2\left(-x_1 - x_2\right)\\ &= 3x_1x_2 + x_2^2 - x_1^2 - x_1x_2 - 2x_1x_2 - 2x_2^2\\ &= -x_1^2 - x_2^2.\end{aligned}$$

Since this holds $\forall x \in \mathbb{R}^2$, the origin $x = 0$ is GES.

Example 2

Consider the modified system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - x_2 - 8x_1^2x_2\end{aligned}$$

and the Lyapunov function

$$V(x) = 1.5x_1^2 + x_1x_2 + x_2^2$$

for which we can show that

$$0.5\,|x|^2 \le V(x) \le 2\,|x|^2 .$$

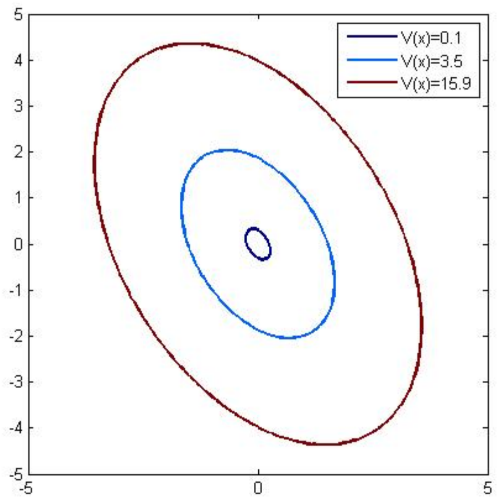
Differentiating, we get that

$$\begin{aligned}\dot{V}(x) &= 3x_1x_2 + x_2^2 + x_1\left(-x_1 - x_2 - 8x_1^2x_2\right) + 2x_2\left(-x_1 - x_2 - 8x_1^2x_2\right)\\ &= -x_1^2 - x_2^2 - 8x_1^3x_2 - 16x_1^2x_2^2\\ &\le -x_1^2 - x_2^2 + \frac{8}{4\kappa}x_1^4 + 8\kappa x_1^2x_2^2 - 16x_1^2x_2^2, \qquad \kappa = 2\\ &= -x_1^2 - x_2^2 + x_1^4\\ &\le -|x|^2 + |x|^4 < 0, \qquad \forall\,|x| < 1.\end{aligned}$$

Thus we get LES. But what is the ROC?

Lyapunov stability

A continuously differentiable function $V(x)$ satisfying the above conditions for stability is called a *Lyapunov function*. The surface given by $V(x) = c$, for $c > 0$, is called a *Lyapunov surface* or *level surface*.



Example 2 ... contd.

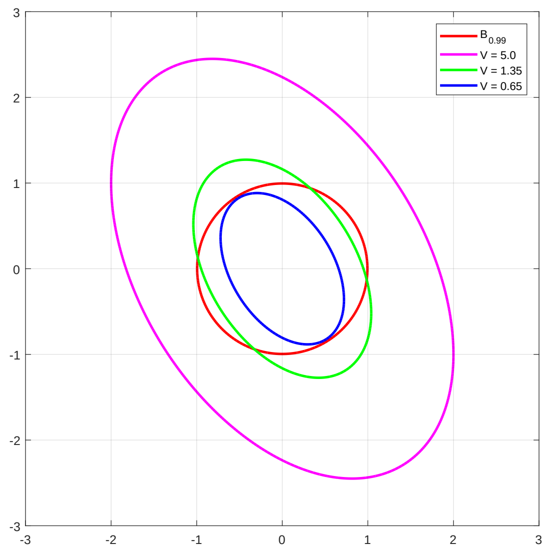
In Example 2 above, we used the Lyapunov function

$$V(x) = 1.5x_1^2 + x_1x_2 + x_2^2$$

and got

$$\dot{V}(x) \leq -|x|^2 + |x|^4 < 0, \qquad \forall\, |x| < 1.$$

If we plot the level set of $V(x)$ and the set $B_{0.99} = \{x : \ |x| \leq 0.99\}$, for which $B_{0.99} \subset \left\{x : \ \dot{V}(x) < 0\right\}$, we get



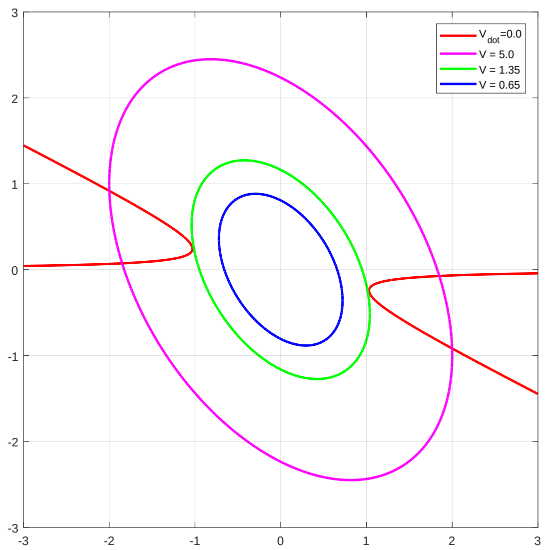
Which set is an estimate of the ROC?

Example 2 ... contd.

If we in the Example 2 above plot the level surface using the exact value of $\dot{V}$, given by

$$\dot{V}(x) = -x_1^2 - x_2^2 - 8x_1^3 x_2 - 16x_1^2 x_2^2$$

we get the set $\left\{x : \dot{V}(x) < 0\right\}$ more accurately as



Which set is now an estimate of the ROC?

Global stability

Definition

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is:

1. *Uniformly Globally Stable* (UGS) if $\exists \varphi \in \mathcal{K}_\infty$ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies
$$|x(t, x_0)| \le \varphi(|x_0|), \qquad \forall t \ge 0.$$
2. *Uniformly Globally Asymptotically Stable* (UGAS) if $\exists \beta \in \mathcal{KL}$ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies
$$|x(t, x_0)| \le \beta(|x_0|, t), \qquad \forall t \ge 0,$$
3. *Uniformly Globally Exponentially Stable* (UGES) if $\exists k, \lambda > 0$ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies
$$|x(t, x_0)| \le k\,|x_0|\, e^{-\lambda t}, \qquad \forall t \ge 0.$$

Global Lyapunov stability

$\exists V : \mathbb{R}^n \mapsto \mathbb{R}$ continuously differentiable such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$$
$$\frac{\partial V}{\partial x} f(x) \leq -\alpha_3(|x|), \qquad x \in \mathbb{R}^n$$

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and α_3 is a continuous *positive semidefinite* function. Then $x = 0$ is GS (or UGS).

- If α_3 is strengthened to a *positive definite* function. Then $x = 0$ is GAS (or UGAS).
- If $\alpha_i(|x|) = c_i |x|^r$, $i = 1, 2, 3$ with $c_i > 0$ and $r \geq 1$. Then $x = 0$ is GES (or UGES).

Lyapunov stability

Radial unboundedness is essential.

Definition

A function $V : \mathbb{R}^n \mapsto \mathbb{R}$ satisfying $V(x) \to \infty$ as $|x| \to \infty$ is said to be *radially unbounded*.

Example 3

$$\dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2 \qquad \dot{x}_2 = \frac{-2\,(x_1+x_2)}{(1+x_1^2)^2} \quad \text{[Hahn, 1967]}$$

is not GAS.

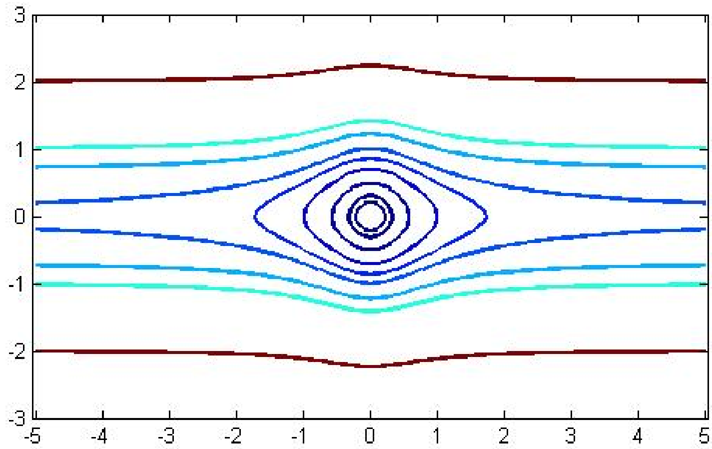
Trajectories outside the branch $x_2 = \frac{2}{x_1 - \sqrt{2}}$ in the first quadrant cannot cross that branch towards the axes.

But

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

satisfies $V(x) > 0$ and $\dot{V} < 0, \qquad \forall x \in \mathbb{R}^2 \backslash \{0\}.$

...Example 3



The contour plot giving the level sets of V shows that it is not radially unbounded. Impossible to find $\alpha_1 \in \mathcal{K}_\infty$.

Example 4

Consider the linear system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n$$

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Hurwitz* if all of its eigenvalues have negative real part.

Definition

Lyapunov's equation: For a Hurwitz matrix A and for any $Q = Q^\top > 0$ there exists a symmetric positive definite matrix $P = P^\top > 0$ such that

$$PA + A^\top P = -Q$$

Let $V(x) = x^\top P x$. This gives

$$\lambda_{\min}(P)\,|x|^2 \leq V(x) \leq \lambda_{\max}(P)\,|x|^2$$

and

$$\begin{aligned}\dot{V} &= x^\top P\dot{x} + \dot{x}^\top P x = x^\top PAx + x^\top A^\top P x \\ &= x^\top \left(PA + A^\top P\right) x = -x^\top Q x \\ &\leq -\lambda_{\min}(Q)\,|x|^2\end{aligned}$$

Hence, $x = 0$ is GES.

Krasovskii-LaSalle

For a **time-invariant system** $\dot{x} = f(x)$, let $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a continuously differentiable positive definite function such that

$$V(x) \to \infty \text{ as } |x| \to \infty$$
$$\dot{V} \leq 0, \qquad \forall x \in \mathbb{R}^n.$$

Let $\Omega := \left\{ x \in \mathbb{R}^n : \ \dot{V}(x) = 0 \right\}$ and $\mathcal{M}$ the largest invariant set in Ω. Then all solutions $x(t, x_0)$ converge to $\mathcal{M}$.

If $\mathcal{M}$ is the equilibrium point $x = 0$, then this is GAS.

Example 5

Recall the linear system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2$$

which we know is UGES by using the Lyapunov equation. Suppose we were not that clever and just choose

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

$$\dot{V} = x_1 x_2 - x_1 x_2 - x_2^2 = -x_2^2 \leq 0,$$

which gives UGS. We now look at the largest invariant set in $\Omega = \{x : \ x_2 = 0\}$. We get $x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow x_1(t) = 0$, and thus the only solution that can stay identically in Ω is $(x_1, x_2) = (0, 0)$ which then is GAS. Linear system: UGAS $\Rightarrow$ UGES.

...Example 5

However, if we define the transformation

$$z = Tx \qquad \begin{cases} z_1 = x_1 \\ z_2 = x_2 + \frac{1}{2}x_1 \end{cases} \qquad T = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = x_2 = z_2 - \frac{1}{2}z_1 \\ \dot{z}_2 &= \dot{x}_2 + \frac{1}{2}\dot{x}_1 = -z_1 - \frac{1}{2}\left(z_2 - \frac{1}{2}z_1\right) = -\frac{3}{4}z_1 - \frac{1}{2}z_2 \end{aligned}$$

$$V(z) = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$$

$$\begin{aligned} \dot{V} &= -\frac{1}{2}z_1^2 + \frac{1}{4}z_1 z_2 - \frac{1}{2}z_2^2 = -\frac{1}{4}z_1^2 - \frac{7}{16}z_2^2 - \left(\frac{1}{2}z_1 - \frac{1}{4}z_2\right)^2 \\ &\leq -\frac{1}{4}z_1^2 - \frac{7}{16}z_2^2 \end{aligned}$$

...Example 5

This shows that $(z_1, z_2) = 0$ is UGES, which implies that there exist $k_z > 0$ and $\lambda > 0$ s.t.

$$|z(t)| \leq k_z |z_0| e^{-\lambda t}$$

Since T is a linear nonsingular transformation, we get

$$\begin{aligned} |x(t)| = \left|T^{-1}z(t)\right| &\leq \left\|T^{-1}\right\| |z(t)| \leq k_z \left\|T^{-1}\right\| |z_0| e^{-\lambda t} \\ &\leq k_z \left\|T^{-1}\right\| \|T\| |x_0| e^{-\lambda t} \\ &=: k_x |x_0| e^{-\lambda t} \end{aligned}$$

which shows that $(x_1, x_2) = 0$ is also UGES.

Nonlinear nonautonomous system

Time-varying ODE

$$\dot{x} = f(t, x), \qquad x(t) \in \mathbb{R}^n,\ t \geq 0.$$

The solution at time t is written $x(t, t_0, x_0)$ with initial time and state $x(t_0) = x_0$ where $0 \leq t_0 < \infty$.
No ambiguity: the solution is simply $x(t)$ with IC (t_0, x_0).

The solution is defined on some maximal interval of existence $(T_{\min}(x_0), T_{\max}(x_0))$ where $T_{\min}(x_0) < t_0 < T_{\max}(x_0)$.
The system is said to be

- *forward complete* if $T_{\max}(x_0) = +\infty$ for all x_0,
- *backward complete* if $T_{\min}(x_0) = -\infty$ for all x_0, and
- *complete* if it is both forward and backward complete

Nonlinear nonautonomous system

Time-varying ODE

$$\dot{x} = f(t, x), \qquad x(t) \in \mathbb{R}^n,\ t \geq 0.$$

For each starting point $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and each compact set $\mathcal{T} \times \mathcal{X}$ containing (t_0, x_0) then:

- for all $(t, x) \in \mathcal{T} \times \mathcal{X}$, the function $f(t, x)$ is continuous in x and piecewise continuous in t,
- there exists $L > 0$ such that

$$|f(t, x) - f(t, y)| \leq L\,|x - y|\,, \qquad \forall (t, x, y) \in \mathcal{T} \times \mathcal{X} \times \mathcal{X}$$

- f is bounded on $\mathcal{X} \times \mathcal{T}$.

This ensures $\exists T > t_0 \geq 0$ such that $\exists!$ solution on $[t_0, T]$.

Often we simply assume that $f(\,\cdot\,,\,\cdot\,)$ is smooth which implies all the above conditions.

Uniform stability

Definition

The origin $x = 0$ of $\dot{x} = f(t, x)$ is:

- *Uniformly Stable* (US) if there exists $\delta(\,\cdot\,) \in \mathcal{K}_\infty$ such that for any $\varepsilon > 0$,
$$|x_0| \leq \delta(\varepsilon),\ t \geq t_0 \geq 0 \ \Rightarrow\ |x(t, t_0, x_0)| \leq \varepsilon.$$
- *Uniformly Globally Asymptotically Stable* (UGAS) if it is US and *Uniformly Attractive* (UA), that is, for each $\varepsilon > 0$ and $r > 0$ there exists $T > t_0 \geq 0$ such that
$$|x_0| \leq r,\ t \geq T \ \Rightarrow\ |x(t, t_0, x_0)| \leq \varepsilon.$$

Uniform global stability

Definition

The origin $x = 0$ of $\dot{x} = f(t, x)$, with $x_0 = x(t_0)$, is:

- Uniformly Globally Stable (UGS) if there exists a class-$\mathcal{K}_\infty$ function φ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, t_0, x_0)$ satisfies
$$|x(t, t_0, x_0)| \leq \varphi\left(|x_0|\right), \qquad \forall t \geq t_0 \geq 0.$$
- Uniformly Globally Asymptotically Stable (UGAS) if there exists a class-$\mathcal{KL}$ function β such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, t_0, x_0)$ satisfies
$$|x(t, t_0, x_0)| \leq \beta\left(|x_0|\,,\ t - t_0\right), \qquad \forall t \geq t_0 \geq 0.$$

Uniform global stability

Definition

The origin $x = 0$ of $\dot{x} = f(t, x)$, with $x_0 = x(t_0)$, is:

- Uniformly Globally Exponentially Stable (UGES) if there exist strictly positive real numbers $k > 0$ and $\lambda > 0$ such that, $\forall x_0 \in \mathbb{R}^n$, the solution $x(t, t_0, x_0)$ satisfies

$$|x(t, x_0)| \leq k\, |x_0|\, e^{-\lambda(t-t_0)}, \qquad \forall t \geq t_0 \geq 0.$$

Lyapunov function

Definition

A smooth Lyapunov function for $\dot{x} = f(t, x)$ with respect to the origin is a smooth function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satisfies:

1. there exist two $\mathcal{K}_\infty$-functions α_1 and α_2 such that for any $x \in \mathbb{R}^n$ and $t \geq 0$,
$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|),$$
2. there exists a continuous and, at least, positive semidefinite function α_3 such that for any $x \in \mathbb{R}^n$ and $t \geq 0$,
$$V^t(t, x) + V^x(t, x) f(t, x) \leq -\alpha_3(|x|).$$

Lyapunov's direct method

Theorem

Assume the system $\dot{x} = f(t, x)$ is forward complete.

- *If there exists a smooth Lyapunov function for the system $\dot{x} = f(t, x)$ with respect to the origin, then the origin $x = 0$ is UGS.*
- *If α_3 is strengthened to a positive definite function, then the origin $x = 0$ is UGAS.*
- *If $\alpha_i(|x|_{\mathcal{A}}) = c_i \, |x|_{\mathcal{A}}^r$ for $i = 1, 2, 3,$ where c_1, c_2, c_3, r are strictly positive reals with $r \geq 1$, then the origin $x = 0$ is UGES.*

Example 6

Consider the nonlinear system

$$\dot{x} = -x^3 + u$$

Control objective: Tracking, i.e. $\lim_{t\to\infty} |x(t) - x_d(t)| = 0$; and $x(t) - x_d(t) = 0$ is UGES. Using feedback linearization we choose

$$u = x^3 - k\,(x - x_d(t)) + \dot{x}_d(t)$$

gives the time-varying closed loop:

$$\dot{x} = f(t, x) = -k\,(x - x_d(t)) + \dot{x}_d(t)$$

Defining the "error" state $e := x - x_d(t)$, however, gives

$$\dot{e} = -ke$$

and a simple exercise with $V(e) = \frac{1}{2}e^2$ shows that $e = 0$ is UGES.

...Example 6

However, since feedback linearization relies on exact cancellation of nonlinear state-dependent terms, it is sensitive to measurement noise and uncertainties. A better choice is perhaps

$$u = x_d(t)^3 - k\,(x - x_d(t)) + \dot{x}_d(t)$$

which does not cancel state-dependent terms by feedforward. This gives the time-varying closed loop:

$$\begin{aligned}\dot{x} &= f(t,x) = -k\,(x - x_d(t)) - \left(x^3 - x_d(t)^3\right) + \dot{x}_d(t)\\ \dot{e} &= g(t,x) = -ke - \left((e + x_d(t))^3 - x_d(t)^3\right)\end{aligned}$$

which is time-varying in the original state x and error state e.

...Example 6

To analyze stability, note that $\rho(r) = r^3$ is a monotonically strictly increasing function with $\rho(0) = 0$:

$$\begin{aligned} r\rho(r) > 0, &\quad \forall r \neq 0\\ \left[\rho(r) - \rho(s)\right](r - s) > 0, &\quad \forall r \neq s. \end{aligned}$$

Define $\delta(t) = kx_d(t) + x_d(t)^3 + \dot{x}_d(t)$ such that $f(t,x) = -kx - x^3 + \delta(t)$. We first analyze forward completeness, using $W(x) = \frac{1}{2}x^2$,

$$\dot{W} = -kx^2 - x^4 + x\delta(t) \leq -x^4, \quad \forall\, |x| \geq \frac{1}{k}\,\|\delta\|$$

Thus; $x(t)$ cannot grow unbounded. Thus; the system must be forward complete.

...Example 6

Defining $V(t,x) = \frac{1}{2}(x - x_d(t))^2$ gives

$$\begin{aligned}\dot{V} &= -k(x - x_d(t))^2 - \left(x^3 - x_d(t)^3\right)(x - x_d(t)) \\ &\le -k(x - x_d(t))^2\end{aligned}$$

which proves UGES.

Invariance theorems

A UGS system may also have internal signals that converge to some value, often to zero. For such convergence analysis the most commonly used result is Barbalat's Lemma:

Lemma (Barbălat)

Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau$ exists and is finite. Then

$$\phi(t) \to 0 \text{ as } t \to \infty.$$

Example 7

Consider the time-varying linear system

$$\begin{aligned}\dot{x}_1 &= g(t)x_2 \\ \dot{x}_2 &= -cg(t)x_1 - x_2, \qquad |g(t)| \leq g_0, \quad c > 0\end{aligned}$$

Using $V(x) = \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2$ we get

$$\dot{V} = cx_1 g(t)x_2 - cg(t)x_1x_2 - x_2^2 = -x_2^2 \leq 0,$$

which shows that $(x_1, x_2) = (0, 0)$ is UGS.
Hypothesis: $x_2(t)$ converges to zero. **Can we show this?**

...Example 7

Using Barbalat's lemma we consider $\phi(t) = x_2(t)^2$.
Obviously, this function is uniformly continuous, since $x_2(t)$ is a solution to the above system.
Integrating, we get

$$\int_{t_0}^{t} \phi(\tau)d\tau = -\int_{t_0}^{t} \dot{V}(\tau)d\tau = V(x(t_0)) - V(x(t)).$$

Since, $V(x(t))$ is monotonically nonincreasing and bounded from below by zero, it must converge. It then follows that $\lim_{t\to\infty} \int_{t_0}^{t} \phi(\tau)d\tau$ exists and is finite, and $\phi(t) = x_2(t)^2 \to 0$. Q.E.D.

Invariance theorems

Recall the signal norm:

$$\|\phi\|_{p,t_0} := \left(\int_{t_0}^{\infty} |\phi(t)|^p \, dt \right)^{\frac{1}{p}}$$

$$\|\phi\|_{\infty,t_0} := \sup_{t \geq t_0} |\phi(t)| \,, \qquad \|\phi\|_{2,t_0} := \sqrt{\int_{t_0}^{\infty} |\phi(t)|^2 \, dt}$$

Corollary

If a function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ satisfies $\phi, \dot{\phi} \in \mathcal{L}_\infty$ and $\phi \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then $\phi(t) \to 0$ as $t \to \infty$.

Example 8

Consider again the time-varying linear system

$$\begin{aligned} \dot{x}_1 &= g(t) x_2 \\ \dot{x}_2 &= -c g(t) x_1 - x_2, \qquad |g(t)| \leq g_0, \quad c > 0 \end{aligned}$$

for which $(x_1, x_2) = (0, 0)$ is UGS.
Letting $\phi(t) := x_2(t)$, then UGS proves directly that $(\phi, \dot{\phi}) = (x_2, \dot{x}_2) \in \mathcal{L}_\infty$.
Let us then check if $x_2 \in \mathcal{L}_2$. We get

$$\|x_2\|_{2,t_0} = \sqrt{\int_{t_0}^{\infty} x_2(t)^2 dt} = \sqrt{V(x(t_0)) - \lim_{t \to \infty} V(x(t))}$$

which limit, as argued in Example 1, exists and is finite. Hence, $x_2 \in \mathcal{L}_2$ and, thus, must converge, $x_2(t) \to 0$. Q.E.D.

LaSalle-Yoshizawa

Blending Lyapunov's direct method and Barbalat's Lemma:
If there exists a smooth function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$$
$$\dot{V} = V^t(t,x) + V^x(t,x)f(t,x) \leq -\alpha_3(|x|) \leq 0,$$

$\forall x \in \mathbb{R}^n$ and $\forall t \geq 0$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and α_3 is a continuous positive semidefinite function, then $x = 0$ is UGS and

$$\lim_{t\to\infty} \alpha_3(|x(t,t_0,x_0)|) = 0.$$

If α_3 is strengthened to continuous positive definite, then $x = 0$ is UGAS.

Example 9

Consider again the time-varying linear system

$$\dot{x}_1 = g(t)x_2$$
$$\dot{x}_2 = -cg(t)x_1 - x_2, \qquad |g(t)| \leq g_0, \quad c > 0$$

for which using $V(x) = \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2$ we get

$$\dot{V} = x_1 g(t) x_2 - g(t) x_1 x_2 - x_2^2 = -x_2^2 \leq 0,$$

which shows that $(x_1, x_2) = (0,0)$ is UGS. By LaSalle-Yoshizawa, defining $\alpha_3(|x|) := |x_2|^2$, we get directly that

$$\lim_{t\to\infty} x_2(t)^2 = 0.$$

Nested Matrosov Theorem [Loría et al., 2005]

The origin of $\dot{x} = f(t,x)$ is UGAS if:

1. The origin of the system is UGS.
2. There exist integers $j, m > 0$ and for each $\Delta > 0$ there exist
   - a number $\mu > 0$,
   - locally Lipschitz continuous functions $V_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, \dots, j\}$,
   - a function $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, $i \in \{1, \dots, j\}$,
   - continuous functions $Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \{1, \dots, j\}$,

   such that, for a.a. $(x,t) \in \mathcal{B}^n(\Delta) \times \mathbb{R}$,

$$\begin{aligned} \max\{|V_i(x,t)|, |\phi(x,t)|\} &\le \mu \\ V_i^x(x,t)f(x,t) + V_i^t(x,t) &\le Y_i(x, \phi(x,t)) \end{aligned}$$

   where $\mathcal{B}^n(r) := \{x \in \mathbb{R}^n : \ |x| \le r\}$.

   ...

Nested Matrosov theorem

...

3. For each integer $k \in \{1, \dots, j\}$ we have that

$$\begin{gathered} \{(z,\psi) \in \mathcal{B}^n(\Delta) \times \mathcal{B}^m(\mu), \ Y_i(z,\psi) = 0, \ \forall i \in \{1, \dots, k-1\}\} \\ \Downarrow \\ \{Y_k(z,\psi) \le 0\}. \end{gathered}$$

4. We have that

$$\begin{gathered} \{(z,\psi) \in \mathcal{B}^n(\Delta) \times \mathcal{B}^m(\mu), \ Y_i(z,\psi) = 0, \ \forall i \in \{1, \dots, j\}\} \\ \Downarrow \\ \{z = 0\}. \end{gathered}$$

   Remark: For $k = 1$, then 3. should read $Y_1(z,\psi) \le 0$ for all $(z,\psi) \in \mathcal{B}^n(\Delta) \times \mathcal{B}^m(\mu)$.

   For proof, see [Loría et al., 2005]

Example 10

Consider again the time-varying linear system

$$\begin{aligned} \dot{x}_1 &= g(t)x_2 \\ \dot{x}_2 &= -cg(t)x_1 - x_2, \qquad |g(t)| \le g_0, \quad c > 0 \end{aligned}$$

and assume $t \mapsto g(t)$ is nonzero a.e. Show that $(x_1, x_2) = 0$ is UGES.
We have shown that $(x_1, x_2) = (0, 0)$ is UGS.
Let

$$\begin{aligned} V_1(x) &:= \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2 \\ V_2(x,t) &:= \mathrm{sgn}(g(t))x_1x_2 \\ \phi(x,t) &:= g(t) \end{aligned}$$

Differentiating, we get a.a. $t \ge 0$

$$\begin{aligned} \dot{V}_1 &= V_1^x A(t)x \le -x_2^2 := Y_1(x) \\ \dot{V}_2 &= \mathrm{sgn}(g(t))\dot{x}_1x_2 + \mathrm{sgn}(g(t))x_1\dot{x}_2 \\ &= |g(t)|\, x_2^2 - c\,|g(t)|\, x_1^2 - \mathrm{sgn}(g(t))x_1x_2 =: Y_2\left(x, \phi(x,t)\right) \end{aligned}$$

...Example 10

For a.a. $(x,t) \in \mathcal{B}^n(\Delta) \times \mathbb{R}$ there exists $\mu > 0$ s.t.
$\max\left\{|V_1(x)|, |V_2(x,t)|, |\phi(x,t)|\right\} \le \mu.$

We have that $Y_1(x) \le 0$ and get

$$Y_1(x) = 0 \implies x_2 = 0 \implies Y_2\left(x, g(t)\right) = -c\,|g(t)|\, x_1^2 \le 0.$$

Finally, $Y_1(x) = Y_2\left(x, \phi(x,t)\right) = 0$ implies that $(x_1, x_2) = 0$ and the system is thus UGAS.
Linear system: UGAS $\Rightarrow$ UGES. Q.E.D.

Summary Invariance theorems

Method	Time-var.	Time-inv.	Signals	UGS	Conv.	UGAS
Lyap. direct meth.	Yes	Yes	-	Yes	Yes	Yes
Barbalat's Lemma	-	-	Yes	-	Yes	-
Krasovskii-LaSalle	No	Yes	-	-	-	Yes
LaSalle-Yoshizawa	Yes	Yes	-	Yes	Yes	No
Matrosov's Thm.	Yes	Yes	-	-	-	Yes

Preparations for next lecture

Nonlinear control:

- Read note on “Mathematical notations and preliminaries”.
- Check out "The Matrix Cookbook".
- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 8, 9.1-9.2, and intro of 10.
- Lecture presentation.

Bibliography

Hahn, W. (1967).
Stability of Motion.
Springer-Verlag, Berlin.

Loría, A., Panteley, E., Popović, D., and Teel, A. R. (2005).
A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems.
IEEE Trans. Autom. Ctrl., 50(2):183–198.