TMR4243: Summary Lecture 11.4.2019 ①

- Example:

$$\begin{aligned}\dot{x}_1 &= x_2 + g_1 x_3\\ \dot{x}_2 &= g_2 x_3 - x_1\\ \dot{x}_3 &= -g_1 x_1 - g_2 x_2 - x_3\end{aligned}$$

$g_1, g_2 \neq 0$

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3 \Rightarrow \dot{V} = -x_3^2 \leq 0$$

Krasovskii-Lasalle: $\Omega = \{x \in \mathbb{R}^3 : x_3 = 0\}$

- $x_3 \equiv 0 \Rightarrow \dot{x}_3 = 0 \Rightarrow g_1 x_1 + g_2 x_2 \equiv 0$

$$\Rightarrow g_1 \dot{x}_1 + g_2 \dot{x}_2 = 0$$

Given $x_3 = 0 \Rightarrow g_1 x_2 - g_2 x_1 = 0$

$$x_2 = \frac{g_2}{g_1} x_1$$

$$\Rightarrow g_1 x_1 + \frac{g_2^2}{g_1} x_1 = \left(g_1 + \frac{g_2^2}{g_1}\right) x_1 = 0$$

$$\Rightarrow g_1\left(1 + \frac{g_2^2}{g_1^2}\right) x_1 = 0$$

$$\Rightarrow x_1 = 0$$

$$\Rightarrow x_2, \dot{x}_2 = 0 \qquad \text{QED.}$$

- So, $\mathcal{M} = \{x \in \mathbb{R}^3 : x_1 = x_2 = x_3 = 0\}$ is the largest invariant set.

11.4.2019 ②

Assume $g_1 = g_1(t)$ & $g_2 = g_2(t)$

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 + g_1(t)\, x_3 \\ \dot{x}_3 &= g_2(t)\, x_3 - x_1 \\ \dot{x}_3 &= -g_1(t)\, x_1 - g_2(t) x_2 - x_3 \end{aligned}}$$

$g_1(t) \neq 0 \quad \forall t$ and c

$g_2(t) \neq 0 \quad \forall t$ and c

$$V = \tfrac{1}{2} x_1^2 + \tfrac{1}{2} x_2^2 + \tfrac{1}{2} x_3^2 \implies \dot{V} = -x_3^2 \leq 0 \quad \text{UGS}$$

Matrosov's thm: 1. UGS ok.

2. Need $\mu \geq 0$, V_1, V_2, V_3 $j = 3$

Fcn. ϕ, Y_1, Y_2, Y_3 where $Y_i = Y_i(x, \phi(x,t))$

$$\forall x \in B^3(\Delta), \ \forall t \geq 0 \implies \max\{V_1, V_2, V_3, \phi\} \leq \mu$$
$$\& \quad \dot{V}_i \leq Y_i$$

Propose $V_1 = V \qquad Y_1 = -x_3^2$

$$V_2 = \frac{c}{g_2(t)} x_1 x_3 + \frac{c}{2} x_1^2 = V_2(x,t) \qquad \dot{c} = 0$$

$$V_3 = d\, x_2 x_3 = V_3(x,t) \qquad \dot{d} = 0$$

We assume:

$\exists\ g_{min}, g_{max} > 0$ s.t.

$$\left.\begin{aligned} 0 < g_{min} &\leq |g_1(t)| \leq g_{max} < \infty \\ \text{— '' —} &\leq |g_2(t)| \leq \text{— '' —} \end{aligned}\right\}$$

That is: bounded away from zero and bounded.

We get:

- $\dot{V}_1 = -x_3^2 =: Y_1(x, \Phi(x,t)) \quad \Phi_1 = 0$

  $\dot{V}_2 = \frac{d}{dt}\left(\frac{c}{g_2(t)}\right) x_1 x_3 + \frac{c}{g_2} \dot{x}_1 x_3 + \frac{c}{g_2} x_1 \left[-g_1 x_1 - g_2 x_2 - x_3\right]$
  $+ c x_1 x_2 + c g_1 x_1 x_3 =: \Phi_2(x,t)$

  $Y_2(x, \Phi) := \Phi_2$

  For $(x,t) \in \mathcal{B}(\Delta) \times \mathbb{R}$ by assumptions we have $|\Phi_2| \leq \mu$.

- We note $\dot{V}_2\big|_{Y_1=0} = -c\frac{g_1(t)}{g_2(t)} x_1^2 \leq 0$

  We let $c = \mathrm{sgn}\left(\frac{g_1(t)}{g_2(t)}\right) \Rightarrow \dot{c} = 0$ a.e.

  $\dot{V}_2\big|_{Y_1=0} = -\left|\frac{g_1}{g_2}\right| x_2^2$

  $\Rightarrow Y_1 = 0 \Rightarrow Y_2 \leq 0$

- $\dot{V}_3 = d\dot{x}_2 x_3 + d\, x_2\left[-g_1 x_1 - g_2 x_2 - x_3\right] =: \Phi_3(x,t)$

  $Y_3(x, \Phi) := \Phi_3(x,t)$

  $\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}$

  Again $|\Phi_3| \leq \mu \quad \forall (x,t) \in \mathcal{B}(\Delta) \times \mathbb{R}$.

  $\dot{V}_3\big|_{\substack{Y_1=0 \\ Y_2=0}} = -d g_2 x_2^2 \Rightarrow d = \mathrm{sgn}(g_2(t))$

  $= |g_2(t)| x_2^2 \leq 0$

  $\Rightarrow Y_1 = 0,\ Y_2 = 0 \Rightarrow Y_3 \leq 0$

- Finally $Y_1 = 0, Y_2 = 0, Y_3 = 0 \Rightarrow (x_1, x_2, x_3) = 0$

  Origin is UGAS. QED.