TMR4243 - Marine Control Systems II

Homework assignment 2

1 A time-varying Lyapunov function

Consider the Lyapunov function

$$V(t, x_1, x_2) = \phi(t)x_1^2 + \frac{1}{2}x_2^2, \tag{1}$$

where $\phi(t) > 0$ is a continuously differentiable function (we say, $\phi \in \mathcal{C}^1$, i.e. the set of all functions for which the derivative exists and is continuous), and (x_1, x_2) are the states driven by $\dot{x}_1 = f_1$ and $\dot{x}_2 = f_2$. Show that

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \tag{2}$$

and calculate this as an expression of (t, x_1, x_2, f_1, f_2).

2 A nonautonomous system

Consider the time-varying linear system

$$\dot{x}_1 = g(t)x_2 \tag{3a}$$

$$\dot{x}_2 = -cg(t)x_1 - x_2, \qquad c > 0, \quad 0 < g_0 \leq |g(t)| \leq g_1, \forall t \geq 0. \tag{3b}$$

1. Show by Lyapunov's Direct Method that $(x_1, x_2) = (0, 0)$ is UGS, using

$$V_1(x) := \frac{1}{2}cx_1^2 + \frac{1}{2}x_2^2. \tag{4}$$

2. Verify by Barbalat's Lemma that you can prove convergence of $x_2(t) \to 0$.
3. Verify by the LaSalle-Yoshizawa theorem that you can prove UGS and convergence of $x_2(t) \to 0$.
4. Verify by the Nested Matrosov theorem that you can prove that $(x_1, x_2) = (0, 0)$ is UGES.
5. Explain why Krasovskii-LaSalle's Invariance Principle is not applicable to this system.
6. Assume that $g(t) = 3$ and $c = 2$ so that the system becomes time-invariant. Show then by Krasovskii-LaSalle that $(x_1, x_2) = 0$ is indeed UGES.

3 An autonomous system

Consider the time-invariant linear system

$$\dot{x}_1 = x_2 + x_3 \tag{5a}$$

$$\dot{x}_2 = -x_1 + x_3 \tag{5b}$$

$$\dot{x}_3 = -x_1 - x_2 - x_3 \tag{5c}$$

1. Verify that $(x_1, x_2, x_3) = 0$ is the single equilibrium for this system.

2. Verify that the origin is GES by linear system methods.
3. What stability property are you able to prove by Lyapunov's Direct Method by the Lyapunov function

$$V_1(x) := \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \tag{6}$$

4. Based on V_1 and $\dot{V}_1$ above, use the Krasovskii-LaSalle's Invariance Principle to show that the origin is GES.
5. Using V_1 from above and

$$\begin{aligned} V_2(x) \quad &: \quad = x_1x_3 + \frac{1}{2}x_1^2 && (7) \\ V_3(x) \quad &: \quad = x_2x_3, && (8) \end{aligned}$$

   verify by the Nested Matrosov theorem that you can prove that the origin is UGAS.
6. Conjecture: If this system is UGAS, then it must be GES. Is this correct? Explain.

References