Marine Control Systems II

Lecture 6: Observer designs

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Goals of lecture

- Understand how to characterize the concept of *Observability* and *Detectability* of LTI and LTV systems.
- Learn how to check for observability by the *Observability matrix* and the *Observability Gramian*.
- Be able to design state observers for linear systems, and understand the *separation principle*.
- Be able to design observers for nonlinear systems.
- Design a nonlinear observer for the low speed dynamics of a surface vessel.

Literature

- Khalil, H. K. (2015). Nonlinear Control:
  - Chapter: 11
- Lecture note “Observer for simplified DP model: Design and proof”.
- Lecture presentation.

LTI systems

Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$.

The objective of an observer is, given $\{A, B, C, D\}$ and present and past information from the output $y(t)$, to reconstruct the state vector x in presence of noise and uncertainties.

Questions:

- Does the system admit a convergent state estimator?
- Does past values of $y(t)$ contain enough information to reliably reconstruct the state vector x?

LTI systems

Problem

Note the nonlinear filtering problem

Predict:	$\hat{x}_{t\|t-1} = E\left(x_t \| y_0, \dots, y_{t-1}\right)$
Correct:	$\hat{x}_{t\|t} = E\left(x_t \| y_0, \dots, y_t\right)$

LTI systems

Let $u(t)$ be a known input signal, and let x_{0a} and x_{0b} be two intial conditions. Denote $y_a(t)$ and $y_b(t)$ the two output trajectories of the system corresponding to x_{0a} and x_{0b} with the same input $u(t)$.

Definition

The pair (x_{0a}, x_{0b}) are said to be **distinguishable** if the outputs $y_a(t)$ and $y_b(t)$ are different at some time $t \geq 0$.

Definition

The system is said to be **observable** if every pair (x_{0a}, x_{0b}) are distinguishable.

Observability matrix

Definition

(Brogan, 1991)A linear system is **observable** at t_0 if $x(t_0)$ can be determined from the output $y(t)$ for $t \in [t_0, t_1]$ for $0 \leq t_0 \leq t_1 < \infty$. If the system is observable for all t_0 and $x(t_0)$, then the system is **completely observable**.

Let the output of the system and its derivatives be

$$\begin{aligned}
y &= Cx + Du \\
\dot{y} &= CAx + CBu + D\dot{u} \\
\ddot{y} &= CA^2x + CABu + CB\dot{u} + D\ddot{u} \\
y^{(3)} &= CA^3x + CA^2Bu + CAB\dot{u} + CB\ddot{u} + Du^{(3)} \\
&\vdots \\
y^{(n-1)} &= CA^{n-1}x + CA^{n-2}Bu + CA^{n-3}B\dot{u} + \ldots + Du^{(n-1)}
\end{aligned}$$

Observability matrix

Move the control signals to the left-hand side

$$\begin{aligned}
z_0 &= y - Du = Cx \\
z_1 &= \dot{y} - CBu - D\dot{u} = CAx \\
z_2 &= \ddot{y} - CABu - CB\dot{u} - D\ddot{u} = CA^2x \\
z_3 &= y^{(3)} - CA^2Bu - CAB\dot{u} - CB\ddot{u} - Du^{(3)} = CA^3x \\
&\vdots \\
z_{n-1} &= y^{(n-1)} - CA^{n-2}Bu - CA^{n-3}B\dot{u} - \ldots - Du^{(n-1)} = CA^{n-1}x
\end{aligned}$$

where the z's are known since y and u are known.

Observability matrix

Without loss of generality, we consider the map from x to y. At $t = 0$ we have

$$
\begin{aligned}
z_0(0) &= Cx_0 \\
z_1(0) &= CAx_0 \\
z_2(0) &= CA^2x_0 \\
&\vdots \\
z_{n-1}(0) &= CA^{n-1}x_0
\end{aligned}
$$

We call the matrix $\mathcal{O}$ the observability matrix:

$$
\mathcal{O} := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}
$$

Observability

The following statements are equivalent:

1. The observability matrix $\mathcal{O}$ is full rank.
2. The LTI system, or the pair $[A, C]$, is completely observable.

Note also the weaker condition of **detectability**:

Definition

(Brogan, 1991, Def. 11.4)A linear system is detectable if all of its unstable modes, if any, are observable.

Observability

We define the $n \times n$ observability Gramian matrix (see "Jørgen Pedersen Gram") for an LTI system:

$$Q_t := \int_0^t e^{\tau A^\top} C^\top C e^{A\tau} d\tau$$

which is symmetric positive semidefinite for every $t \geq 0$. We then have:

- A state x_0 is unobservable iff $x_0 \in \mathcal{N}(Q_t)$ for every $t \geq 0$.
- The following statements are equivalent:
  1. The LTI system, or the pair $[A, C]$, is completely observable.
  2. There exists $t \geq 0$ such that $\operatorname{rank}(Q_t) = n$.
  3. There exist positive constants α, β, and $t \geq$ such that $\alpha I \leq Q_t \leq \beta I$.

Observability

Note that if A is Hurwitz, then we can check the observability Gramian for $t = \infty$, that is,

$$Q = Q_\infty = \int_0^\infty e^{\tau A^\top} C^\top C e^{A\tau} d\tau.$$

In this case we get that Q satisfies the Lyapunov equation

$$A^\top Q + QA = -C^\top C$$

Definition UCO

Definition

(Anderson et al., 1986) For an LTV system

$$\begin{aligned} \dot{x} &= A(t)x \\ y &= C(t)x, \qquad t \geq 0 \end{aligned}$$

the pair $[A(t), C(t)]$ is said to be Uniformly Completely Observable (UCO) if there exist positive constants α, β, and $t_1 \geq t_0$ such that

$$\alpha I \leq \int_{t_0}^{t_1} \Phi(\tau, t)^\top C(\tau)^\top C(\tau)\Phi(\tau, t) d\tau \leq \beta I$$

holds for all $t_0 \geq 0$, where $\Phi(t, t_0)$ is the transition matrix for the above system, that is,

$$x(t) = \Phi(t, t_0)x(t_0)$$

UCO is invariant under output feedback

Lemma

The pair $[A(t), C(t)]$ is UCO if and only if the pair

$$[A(t) - K(t)C(t), C(t)],$$

with $K(\cdot)$ bounded and locally integrable, is UCO.

See (Anderson et al., 1986) for proof.

Luenberger observer for LTI system

Consider the deterministic system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$.
A Luenberger observer is constructed as

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - \hat{y}\right)$$
$$\hat{y} = C\hat{x} + Du$$

where the matrix $L \in \mathbb{R}^{n \times p}$ is known as the *observer gain*, *injection gain*, or *innovation gain*.

Luenberger observer for LTI system

Defining then $\tilde{x} := x - \hat{x}$ and $\tilde{y} := y - \hat{y}$ we get $y - \hat{y} = \tilde{y} = C\tilde{x}$ and the resulting error dynamics

$$\dot{\tilde{x}} = \left(A - LC\right)\tilde{x}$$

The following is equivalent:

1. The pair $[A, C]$ is completely observable.
2. For any set of n complex numbers $\mathcal{P}$, there exists $L \in \mathbb{R}^{n \times p}$ such that $\mathrm{eig}(A - LC) = \mathcal{P}$.

Obviously, L is designed so that $A_o := A - LC$ is Hurwitz. This ensures that $\tilde{x} = 0$ is UGES.
In Matlab, we can e.g. use the `place.m` command to find L.

Luenberger observer is a lowpass filter

Setting $u = 0$, then we find that the Luenberger observer becomes

$$\dot{\hat{x}} = (A - LC)\,\hat{x} + Ly$$
$$\hat{y} = C\hat{x}$$

- For A_o Hurwitz, this is a stable lowpass filter. Hence, it has some filtering capacities.
- In feedback control we say that the observer should be significantly faster that the closed-loop feedback control system, typically 5-10 times faster. However, if too fast it will amplify the noise in the measurement of y. The designer must balance these two criteria.

Separation principle

Suppose for our system we have a linear state feedback control law $u = -Kx$ such that the closed-loop system becomes

$$\dot{x} = Ax - BKx = (A - BK)\,x =: A_c x$$

where the feedback gain matrix $K \in \mathbb{R}^{m \times n}$ is designed so that A_c is Hurwitz.
However, since we don't measure x we use a Luenberger observer and the feedback law $u = -K\hat{x}$.

Using $\tilde{x} := x - \hat{x}$, the closed-loop system now becomes

$$\begin{aligned}
\dot{\tilde{x}} &= A_o \tilde{x} \\
\dot{x} &= Ax - BK\hat{x} + BKx - BKx \\
&= Ax - BKx - BK\hat{x} + BKx \\
&= A_c x + BK\tilde{x}
\end{aligned}$$

Separation principle

Having

$$\dot{\tilde{x}} = A_o \tilde{x}$$
$$\dot{x} = A_c x + BK\tilde{x}$$

we clearly find that this is a cascade with a linear interaction gain. As one big system, with $\chi := \mathrm{col}(\tilde{x}, x)$, we get

$$\dot{\chi} = \begin{bmatrix} A_o & 0 \\ BK & A_c \end{bmatrix} \chi$$

which is block lower-triangular. The eigenvalues of this matrix corresponds to the eigenvalues of the matrices A_o and A_c along the diagonal.

As a result we have that the poles of the observer and the poles of the feedback controller can be tuned independently.
This is known as the separation principle.

Stochastic LTI model

We now consider the stochastic system

$$\dot{x} = Ax + Bu + \Gamma w$$
$$y = Cx + v$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, w represent process noise and v measurement noise – both assumed Gaussian zero-mean white noise.
Let $E\left[ww^\top\right] = Q$ and $E\left[vv^\top\right] = R$ denote the covariances of w and v, respectively. These are real, symmetric, and positive definite matrices. Since the stochastic processes v and w are assumed independent and uncorrelated, we also have $E\left[wv^\top\right] = 0$.

Kalman-Bucy observer

The Kalman-Bucy observer (continous time Kalman filter) is now given by

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - C\hat{x}\right) = \left(A - LC\right)\hat{x} + Bu + Ly$$

The estimation error covariance is

$$E\left[\left(x - \hat{x}\right)\left(x - \hat{x}\right)^\top\right]$$

This is minimized when the injection gain L is chosen as

$$L = P_0 C^\top R^{-1}$$

where P_0 is the solution to the **algebraic Riccati equation**

$$P_0 A^\top + A P_0 - P_0 C^\top R^{-1} C P_0 + \Gamma Q \Gamma^{-1} = 0$$

Kalman-Bucy observer for LTV system

Consider the linear system being time-varying,

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u(t) + w(t) \\ y &= C(t)x + v(t). \end{aligned}$$

In this case the Kalman-Bucy filter will consist of the two differential equations

$$\begin{aligned} \dot{\hat{x}} &= \left(A(t) - L(t)C(t)\right)\hat{x} + B(t)u(t) + L(t)y, \qquad && \hat{x}(t_0) = \hat{x}_0 \\ \dot{P} &= PA(t)^\top + A(t)P - L(t)R(t)L(t)^\top + Q(t), \qquad && P(t_0) = P_0 \end{aligned}$$

where the 2nd equation is the **differential Riccati equation**, and

$$L(t) = PC(t)^\top R(t)^{-1}$$

Note that the differential equations for $\hat{x}$ and P must be solved simultaneously due to the dependence on $\{A(t), B(t), C(t)\}$.

Local observer by linearization (Khalil, 2015, Ch. 11.1)

We consider the nonlinear system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

and assume this is shifted so that the origin is an equilibrium, i.e. $f(0, 0) = 0$ and $h(0) = 0$.
Assume for each $\varepsilon > 0$ there exists $\delta_1 > 0$ and $\delta_2 > 0$ s.t. $|x_0| \leq \delta_1$ and $\|u\| \leq \delta_2$ ensure that the solution $x(t)$ is defined $\forall t \geq 0$ and $|x(t)| \leq \varepsilon$, $\forall t \geq 0$.
Let

$$A = \left.\frac{\partial f(x, u)}{\partial x}\right|_{x=0, u=0}, \qquad C = \left.\frac{\partial h(x)}{\partial x}\right|_{x=0}$$

and assume the pair $[A, C]$ is detectable, and design L such that $A - LC$ is Hurwitz. Then, for sufficiently small $|\tilde{x}_0| = |x_0 - \hat{x}_0|$, $|x_0|$, and $\|u\|$, the observer

$$\dot{\hat{x}} = f(\hat{x}, u) + L\left(y - h(\hat{x})\right)$$

ensures that $|\tilde{x}(t)| \to 0$ as $t \to \infty$. (See Khalil, 2015, Lemma 11.1).

Local observer by linearization

For the proof you consider

$$\begin{aligned}
\dot{\tilde{x}} &= f(x, u) - f(\hat{x}, u) - L\left(h(x) - h(\hat{x})\right) \\
&= A\tilde{x} - A\tilde{x} + f(x, u) - f(\hat{x}, u) - L\left(C\tilde{x} - C\tilde{x} + h(x) - h(\hat{x})\right) \\
&= (A - LC)\,\tilde{x} + \Delta(\tilde{x}, x, u)
\end{aligned}$$

with

$$\Delta(\tilde{x}, x, u) := f(x, u) - f(\hat{x}, u) - A\tilde{x} - L\left(h(x) - h(\hat{x}) - C\tilde{x}\right)$$

If then δ_1, δ_2, ε, and $|\tilde{x}_0|$ are small enough, it can be shown that the linear term $(A - LC)\,\tilde{x}$ will dominate the nonlinear residual $\Delta(\tilde{x}, x, u)$.

Extended Kalman Filter

With the previous local observer in mind, let the observer be

$$\dot{\hat{x}} = f(\hat{x}, u) + L(t)\left[y - h(\hat{x})\right]$$

where $L(t)$ will be a time-varying gain. We now let

$$A(t) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x=\hat{x}(t)}, \qquad C(t) = \left.\frac{\partial h(x)}{\partial x}\right|_{x=\hat{x}(t)}$$

be the Jacobian matrices evaluated along $\tilde{x} = 0$ or $x = \hat{x}$. In the error dynamics $\tilde{x} = x - \hat{x}$ we get

$$\dot{\tilde{x}} = \left(A(t) - L(t)C(t)\right)\tilde{x} + \Delta(\tilde{x}, x, u, t)$$

with

$$\Delta(\tilde{x}, x, u, t) := f(x,u) - f(\hat{x}, u) - A(t)\tilde{x} - L(t)\left(h(x) - h(\hat{x}) - C(t)\tilde{x}\right)$$

EKF

As for our earlier Kalman-Bucy filter design for LTV systems, we take

$$L(t) = P(t)C(t)^{\top}R^{-1}$$

where Q and R are the symmetric positive definite covariance matrices for the process and measurement noise, respectively, and P satisfy the DRE

$$\dot{P} = PA(t)^{\top} + A(t)P - PC(t)^{\top}R^{-1}C(t)P + Q, \qquad P(t_0) = P_0$$

Lemma

Suppose $P(t)$ exists and satisfies $\alpha I \leq P(t) \leq \beta I$ for all $t \geq t_0$. The origin $\tilde{x} = 0$ is then exponentially stable and satisfy

$$|\tilde{x}_0| \leq c \quad \Rightarrow \quad |\tilde{x}(t)| \leq ke^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0$$

for some positve constants c, k, and λ. (Khalil, 2015, Lemma 11.2)

Observer with linear error dynamics

Consider a nonlinear system in the *observer form*

$$\dot{x} = Ax + \psi(u, y), \qquad y = Cx$$

assumed forward complete, where $[A, C]$ is observable and ψ is locally Lipschitz.
A Luenberger-like observer is then

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + L\left(y - C\hat{x}\right), \qquad \hat{y} = C\hat{x}$$

such that the estimation error becomes

$$\dot{\tilde{x}} = (A - LC)\,\tilde{x}$$

This is known as a *nonlinear observer with linear error dynamics*.
Obviously, if $[A, C]$ is observable, then we can design L such that $A_o := A - LC$ is Hurwitz.

Observer with linear error dynamics

Note that the nonlinear system may not be in the observer form from the beginning. In this case a technique called *output injection* must be used. One must then find a change of coordinates around a fixed point x_{ss},

$$z = z(x), \qquad z(x_{ss}) = 0$$

which transforms the system into

$$\dot{z} = Az + \psi(u, y), \qquad y = Cz.$$

Not all nonlinear systems can be transformed in this way with output injection. Krener & Isidori (1983) gives conditions for when this is possible.

Observer with global Lipshitz condition

Suppose the nonlinear system is a bit more generally written

$$\dot{x} = Ax + \phi(x,u) + \psi(u,y), \qquad y = Cx$$

assumed forward complete, where $[A, C]$ is observable, ψ and ϕ are locally Lipschitz. Suppose also ϕ is globally Lipschitz in x, uniformly in u, that is

$$|\phi(x,u) - \phi(z,u)| \leq K\,|x - z|\,, \qquad \forall (x,z,u).$$

Let the observer be

$$\dot{\hat{x}} = A\hat{x} + \phi(\hat{x},u) + \psi(u,y) + L\,(y - C\hat{x})\,, \qquad \hat{y} = C\hat{x}$$

where L is designed so that $A_o := A - LC$ is Hurwitz.

Observer with global Lipshitz condition

The error dynamics now become

$$\dot{\tilde{x}} = A_o\tilde{x} + \phi(x,u) - \phi(\hat{x},u)$$

Let $P_o = P_o^\top > 0$ satisfy $P_oA_o + A_o^\top P_o = -qI$, and let $V(\tilde{x}) = \tilde{x}^\top P_o \tilde{x}$ be a Lyapunov function candidate. We get

$$\begin{aligned}\dot{V} &= -q\tilde{x}^\top\tilde{x} + 2\tilde{x}^\top P_o\,[\phi(x,u) - \phi(\hat{x},u)] \\ &\leq -q\,|\tilde{x}|^2 + 2\lambda_{\max}(P_o)\,|\tilde{x}|\,|\phi(x,u) - \phi(\hat{x},u)| \\ &\leq -q\,|\tilde{x}|^2 + 2K\lambda_{\max}(P_o)\,|\tilde{x}|^2\end{aligned}$$

If then we have that

$$K < \frac{q}{2\lambda_{\max}(P_o)}$$

then $\tilde{x} = 0$ is globally exponentially stable. Given $[A, C]$ and since P_o depends on the observer gain L, this means that we should design L to make $\frac{q}{2\lambda_{\max}(P_o)}$ sufficiently large.

Example: Observer with global Lipshitz condition

Consider

$$\dot{x}_1 = x_2 - \frac{1}{10}\sin x_1$$
$$\dot{x}_2 = -\frac{1}{10}\cos x_2 + u$$
$$y = x_1$$

which can be written

$$\dot{x} = Ax + Bu + \phi(x)$$
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \phi(x) = \frac{1}{10}\begin{bmatrix} -\sin x_1 \\ -\cos x_2 \end{bmatrix}$$

and we check observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is full rank.

Example: Observer with global Lipshitz condition

Checking for global Lipschitz:

$$|\phi(x) - \phi(z)|_2 \le \frac{1}{10}|\sin x_1 - \sin z_1| + \frac{1}{10}|\cos x_2 - \cos z_2|$$
$$= \frac{1}{10}\left|2\sin\frac{x_1 - z_1}{2}\cos\frac{x_1 + z_1}{2}\right| + \frac{1}{10}\left|-2\sin\frac{x_2 + z_2}{2}\sin\frac{x_2 - z_2}{2}\right|$$
$$\le \frac{1}{10}|x_1 - z_1| + 0.1|x_2 - z_2| \le \frac{\sqrt{2}}{10}|x - z|$$

So global Lipschitz condition satisfied with gain $K = \frac{\sqrt{2}}{10} = 0.141$.

When designing the gain L, we find the poles and determine L, i.e.

$$\det(sI - A + LC) = \det\begin{bmatrix} s + l_1 & -1 \\ l_2 & s \end{bmatrix} = s^2 + l_1 s + l_2 = s^2 + 2\xi\omega s + \omega^2$$

Letting e.g. $\omega = 5$ and $\xi = 0.75$ gives $l_2 = 25$ and $l_1 = 7.5$. This gives

$$A_o = A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 7.5 \\ 25 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -7.5 & 1 \\ -25 & 0 \end{bmatrix}$$

Example: Observer with global Lipshitz condition

Letting $q = 1$ we then get

$$P_o = \begin{bmatrix} 1.733 & -0.5 \\ -0.5 & 0.219 \end{bmatrix}, \qquad \lambda_{\max}(P_o) = 1.884$$

This gives

$$\frac{q}{2\lambda_{\max}(P_o)} = \frac{1}{2 \cdot 1.884} = 0.266 > K = 0.141 \qquad \text{Ok!}$$

In this case the observer

$$\dot{\hat{x}} = A\hat{x} + Bu + \phi(\hat{x}) + L(y - C\hat{x}), \qquad \hat{y} = C\hat{x}$$

will ensure that $\tilde{x} = 0$ is UGES.

Observer based on circle criterion

(Arcak, 2002) Consider the nonlinear system

$$\dot{x} = Ax + G\gamma(Hx) + \psi(u, y), \qquad y = Cx$$

assumed forward complete, where $[A, C]$ is observable, ψ and γ are locally Lipschitz. Suppose now that γ is a nondecreasing function, that is

$$(v - w)[\gamma(v) - \gamma(w)] \geq 0, \qquad \forall(v, w).$$

Let the observer be

$$\dot{\hat{x}} = A\hat{x} + G\gamma(H\hat{x} + K(y - C\hat{x})) + \psi(u, y) + L(y - C\hat{x}), \qquad \hat{y} = C\hat{x}$$

We now want to determine the observer gains L and K such that $\tilde{x} = 0$ is GES.

Observer based on circle criterion

The error dynamics now become

$$\begin{aligned}\dot{\tilde{x}} &= A\tilde{x} + G\gamma(Hx) - G\gamma(H\hat{x} + K\left(y - C\hat{x}\right)) - LC\tilde{x} \\ &= \left(A - LC\right)\tilde{x} + G\left[\gamma(v) - \gamma(w)\right]\end{aligned}$$

where $v = Hx$ and $w = H\hat{x} + K\left(y - C\hat{x}\right)$. Let

$$z := v - w = Hx - H\hat{x} - K\left(y - C\hat{x}\right) = \left(H - KC\right)\tilde{x}$$

Letting $\varphi(v, z) = \gamma(v) - \gamma(v - z)$ we can write the system

$$\begin{aligned}\dot{\tilde{x}} &= \left(A - LC\right)\tilde{x} + G\varphi(v, z) \\ z &= \left(H - KC\right)\tilde{x}\end{aligned}$$

where $\varphi(v, z)$ now satsifies the sector nonlinearity property

$$z\varphi(v, z) \geq 0.$$

Observer based in circle criterion

Arcak (2002) gives the following result:

Theorem

Given the nonlinear system and observer, with its assumptions. If $\exists P = P^\top > 0$ and a constant $q > 0$ such that

$$\begin{bmatrix} \left(A - LC\right)^\top P + P\left(A - LC\right) + qI & PG + \left(H - KC\right)^\top \\ G^\top P + \left(H - KC\right) & 0 \end{bmatrix} \leq 0$$

then $\tilde{x}(t)$ satisfies

$$|\tilde{x}(t)| \leq \sqrt{\frac{\lambda_{\max}\left(P\right)}{\lambda_{\min}\left(P\right)}}\,|\tilde{x}(0)|\,e^{-\frac{q}{2\lambda_{\max}(P)}t}, \qquad \forall t \geq 0.$$

Observer with monotone damping property

Aamo et al. (2001) was inspired by the previous work, and considered the nonlinear mechanical system

$$\begin{aligned} \dot{q} &= J(q)\nu \\ M\dot{\nu} + D\nu + d(\nu) + v(q) &= \tau \\ y &= q \end{aligned}$$

where $q, \nu \in \mathbb{R}^n$, M and D are constant matrices satisfying $M = M^\top > 0$ and $D + D^\top \geq 0$, and d satisfies the nondecreasing property:

$$(v - w)\left[d(v) - d(w)\right] \geq 0, \qquad \forall (v, w).$$

Also, J has the following property:

1. $J(q)$ is invertible and satisfy $0 < \alpha \leq \|J(q)\| \leq \beta$ for all $q \in \mathbb{R}^n$ and some α, β.
2. $\frac{d}{dt} J(q) = \dot{J}(q, \dot{q})$ is globally Lipschitz in $\dot{q}$, uniformly in q.

An observer is propsed next...

Observer with monotone damping property

The proposed observer is

$$\begin{aligned} \dot{\hat{q}} &= J(q)\hat{\nu} + L_1\left(y - \hat{q}\right) \\ M\dot{\hat{\nu}} + D\hat{\nu} + d(\hat{\nu}) + v(q) &= \tau + ML_2(q)\left(y - \hat{q}\right) \end{aligned}$$

The error dynamics become

$$\begin{aligned} \dot{\tilde{q}} &= J(q)\tilde{\nu} - L_1\tilde{q} \\ \dot{\tilde{\nu}} &= -M^{-1}\left[D\tilde{\nu} + d(\nu) - d(\hat{\nu})\right] - L_2(q)\tilde{q} \end{aligned}$$

Then Aamo et al. (2001) gives the following result...

Observer with monotone damping property

Theorem

Let $P = P^\top > 0$, and suppose the observer gains can be chosen to satisfy

$$PL_1 + L_1^\top P > 0$$
$$L_2(q) = M^{-1} J(q)^\top P$$

Assume also that the state ν is uniformly globally bounded. Then $(\tilde{q}, \tilde{\nu}) = 0$ is UGAS and ULES. If also $D = D^\top > 0$, then it is UGES.

See (Aamo et al., 2001, Proposition 1) for proof.

Observer with monotone damping property

Theorem

Suppose a locally Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}^n$, is known and for some $\varepsilon > 0$ satisfy

$$\frac{\partial \phi(q)}{\partial q} J(q) + \left(\frac{\partial \phi(q)}{\partial q} J(q) \right)^\top > \varepsilon I, \qquad \forall q \in \mathbb{R}^n.$$

Then, with the same definition of observer gain matrices as above, the origin $(\tilde{q}, \tilde{\nu}) = 0$ is UGES.

See (Aamo et al., 2001, Proposition 2) for proof.

Low-speed vessel model

We consider the typical low speed dynamics of a marine vessel of the form

$$\begin{aligned}\dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= -D\nu + R(\psi)^\top b + \tau \\ \dot{b} &= 0,\end{aligned}$$

where $\eta = \mathrm{col}\,(\eta_N, \eta_E, \psi) \in \mathbb{R}^3$ contains the North/East position and heading angle of the vessel, and $\nu = \mathrm{col}\,(u, v, r) \in \mathbb{R}^3$ contains the surge/sway velocity and yaw rate in the body frame of the vessel, and η is measured.

We assume ψ is measured with high precision so that it can be used directly as a signal in the observer.

DP observer algorithm

The classical observer is designed by copying the plant dynamics and adding injection terms, as follows

$$\begin{aligned}\dot{\hat{\eta}} &= R(\psi)\hat{\nu} + L_1\tilde{\eta} \\ M\dot{\hat{\nu}} &= -D\hat{\nu} + R(\psi)^\top \hat{b} + \tau + R(\psi)^\top L_2\tilde{\eta} \\ \dot{\hat{b}} &= L_3\tilde{\eta},\end{aligned}$$

where $(\hat{\eta}, \hat{\nu}, \hat{b})$ are the state estimates, and $\tilde{\eta} := \eta - \hat{\eta}$. The objective of the observer design problem is to find conditions on the injection gains L_1, L_2, L_3, together with necessary assumptions on the plant, in order to achieve global stability and attractivity of the equilibrium $(\tilde{\eta}, \tilde{\nu}, \tilde{b}) = 0$.

First we consider the case where $b = 0$, and do the design on the blackboard...

DP observer without bias

In conclusion, letting $\tilde{\eta} := \eta - \hat{\eta}$ and $\tilde{\nu} := \nu - \hat{\nu}$ be the estimation error states, the closed-loop error system becomes

$$\begin{aligned} \dot{\tilde{\eta}} &= R(\psi)\tilde{\nu} - L_1\tilde{\eta} \\ M\dot{\tilde{\nu}} &= -D\tilde{\nu} - R(\psi)^\top L_2\tilde{\eta}, \end{aligned}$$

for which the following result apply.

Theorem

The equilibrium $(\tilde{\eta}, \tilde{\nu}) = 0$ is UGAS under the following conditions:

- *The damping matrix satisfy $D + D^\top \geq 0$.*
- *The injection gain matrices L_1 and L_2 are symmetric positive definite and satisfy $L_1L_2 + L_2L_1 > 0$.*

If the damping matrix satisfies $D + D^\top > 0$, then the equilibrium $(\tilde{\eta}, \tilde{\nu}) = 0$ is UGES.

DP observer with bias

In the case with bias, the error dynamics becomes

$$\begin{aligned} \dot{\tilde{\eta}} &= R(\psi)\tilde{\nu} - L_1\tilde{\eta} \\ M\dot{\tilde{\nu}} &= -D\tilde{\nu} + R(\psi)^\top\tilde{b} - R(\psi)^\top L_2\tilde{\eta} \\ \dot{\tilde{b}} &= -L_3\tilde{\eta}, \end{aligned}$$

for which the following apply (Værnø & Skjetne, 2017):

Theorem

The equilibrium $(\tilde{\eta}, \tilde{\nu}, \tilde{b}) = 0$ is UGAS under the following conditions:

- *The damping matrix satisfy $D + D^\top \geq 0$.*
- *The injection gain matrices L_1, L_2, L_3 are symmetric positive definite and satisfy:*
  - *L_3 and L_1 are commutative, and*
  - *the symmetric matrices $L_1L_2 + L_2L_1 - 2L_3$ and $L_3^{-1}L_1 - L_2^{-1}$ are positive definite.*

Preparations for next lecture

Maneuvering control design:

- Skjetne (2005).
  - Chapters: 1.1, 1.3-1.4, 2, 3.1-3.2
- Lecture presentation.

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