Marine Control Systems II

Lecture 1: Properties and stability of time-invariant ODEs

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TMR4243

Lectures

See:

- TMR4243 course description.
- Note on "Mathematical notations and preliminaries."

Goals of lecture

- Understand properties of solutions to ordinary differential equations (ODEs).
- Understand Lipschitz properties of ODEs and relate this to existense, uniqueness, and completeness of solutions.
- Assess equilibrium points of ODEs; types and properties.
- Contrast nonlinear, linear, and linearized ODEs.
- Describe stability characterizations of equilibrium points of ODEs.
  - Comparison functions.
  - Stability definitions.
  - Stability in the sense of Lyapunov.

Literature

- Note on “Mathematical notations and preliminaries”.
- Note on "Some inequalities".
- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 1, 2, 3.1-3.3
- Lavretsky, E. and K. A. Wise (2013). Robust and Adaptive Control (With Aerospace Applications)
  - Chapters 8.1-8.3 (for alternative explanations and deeper learning)
- Lecture presentation.

Solutions of ODEs

Consider the *ordinary differential equation*

$$\dot{x}(t) = f(x(t)), \qquad f : \mathbb{R}^n \mapsto \mathbb{R}^n$$

where $f(\,\cdot\,)$ is locally Lipschitz on $\mathbb{R}^n$, and for each $t \geq 0$ the vector $x(t) \in \mathbb{R}^n$ is the state.

Let $x(t, x_0)$ denote the solution at time t with initial state $x_0 = x(0, x_0)$.

If there is no ambiguity from the context, the solution is simply written as $x(t)$.

It is defined on some maximal interval of existence $(T_{\min}(x_0), T_{\max}(x_0))$ where $T_{\min}(x_0) < 0 < T_{\max}(x_0)$.

- The system is said to be *forward complete* if $T_{\max}(x_0) = +\infty$ for all x_0,
- *backward complete* if $T_{\min}(x_0) = -\infty$ for all x_0, and
- *complete* if it is both forward and backward complete.

Example 1

The linear system

$$\dot{x} = Ax + Bu(t)$$

admits the solution

$$x(t, x_0) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

For $u(t)$, say, bounded, then this system is complete.

That is to say, there does not exist a finite time $t' \geq 0$ such that, for any $x_0 \in \mathbb{R}^n$, then $x(t', x_0) = \infty$.

Solutions of ODEs

A solution $t \mapsto x(t, x_0)$ is an absolute continuous function, satisfying:

- $x(0, x_0) = x_0$,
- $x(\,\cdot\,, x_0)$ is differentiable a.e. on $(T_{\min}(x_0), T_{\max}(x_0))$,
- $\frac{d}{dt}x(t, x_0) = f(x(t, x_0))$ is Lebesgue integrable on $(T_{\min}(x_0), T_{\max}(x_0))$, and
- $x(t, x_0) - x_0 = \int_0^t \frac{d}{dt}x(\tau, x_0)d\tau = \int_0^t f(x(\tau, x_0))d\tau$.

For simplicity in this course, this boils down to checking:

Initial condition constraint:	$x(0, x_0) = x_0$
Differential constraint:	$\frac{d}{dt}x(t, x_0) = f(x(t, x_0))$

Locally Lipschitz

A function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is locally Lipschitz at x if there exists a neighborhood $\mathcal{U}$ of x and a constant $L > 0$ such that

$$x_1, x_2 \in \mathcal{U} \quad \Longrightarrow \quad |f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|$$

The function is said to be *locally Lipschitz* if it is locally Lipschitz at x for every $x \in \mathbb{R}^n$.

The function is said to be *globally Lipschitz* if we can take $\mathcal{U} = \mathbb{R}^n$. [Teel, 2002]

Locally Lipschitz

Lemma

A function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ is locally Lipschitz if and only if for each compact set $\mathcal{X}$ there exists $L > 0$ such that

$$x_1, x_2 \in \mathcal{X} \quad \Longrightarrow \quad |f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|\,.$$

Existence and uniqueness of solutions

Theorem

If $f(\cdot)$ is locally Lipschitz, then for each $x_0 = x(0)$ there exists $T > 0$ and a unique $x(t, x_0)$ that is a solution on $[0, T]$.

Example 2

The solution of

$$\dot{x} = -x^2, \qquad x_0 = x(0)$$

exists locally in time for all x_0.

For instance, $x_0 = -1$ gives

$$x(t) = \frac{1}{t-1}$$

having $T_{\max}(x_0) = 1.0$ for which $t = T_{\max}(x_0)$ gives finite escape. [De Persis, 2005]

Such a phenomenon (finite escape time) cannot happen for linear systems.

Example 3

Solving

$$\dot{x} = x^{\frac{1}{3}}, \qquad x_0 = 0$$

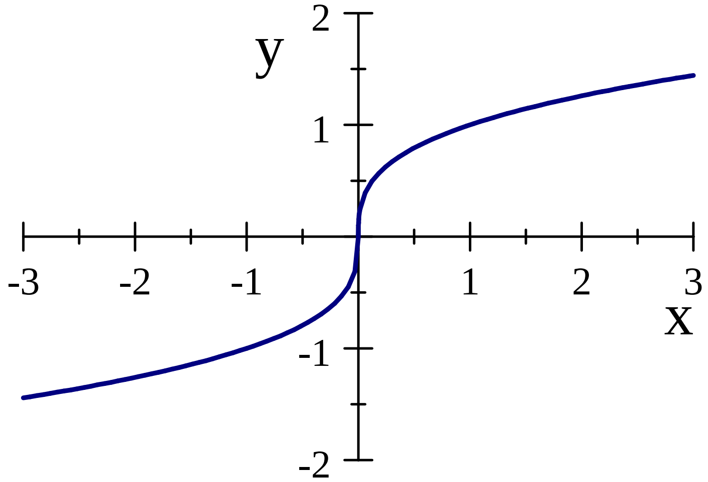
gives two solutions

$$x(t) = 0 \qquad \text{and} \qquad x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}.$$

Hence, solutions starting at the origin is not unique.

...Example 3

Looking at $f(x) = x^{\frac{1}{3}}$, we find it is not locally Lipschitz on any neighborhood of the origin:



Guaranteeing forward completeness

Lemma

If $f(\cdot)$ is globally Lipschitz, then for each $x_0 \in \mathbb{R}^n$, there exists a unique solution $x(t, x_0)$ of

$$\dot{x} = f(x), \qquad x_0 = x(0)$$

for all $t \geq 0$.

This theorem is somewhat restrictive, as the following example will show.

Example 4

The system

$$\dot{x} = f(x) := -x^3$$

has $f(x)$ locally Lipschitz on $\mathbb{R}$, but not globally Lipschitz since $\frac{\partial f}{\partial x} = -3x^2$ is not globally bounded [Khalil, 2002a].

However, if at any time instant

- $x(t) \geq 0$ then $f(x(t)) \leq 0$, and similarly
- $x(t) < 0$ then $f(x(t)) > 0$.

Hence, starting from any initial condition $x(0) = a$, the solution cannot leave the set $\{x \in \mathbb{R} : \ |x| \leq a\}$, and it follows that there is a unique solution $\forall t \geq 0$.

Guaranteeing forward completeness

Lemma

Suppose the function $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies the conditions for existence and uniqueness of solutions.
Suppose also that $f(\cdot)$ satisfies a global sector bound, that is, there exist $L \geq 0$ and $c \geq 0$ such that for all x,

$$|f(x)| \leq L\,|x| + c.$$

Then all solutions are defined for all $t \geq 0$. [Teel, 2002]

Showing backward completeness

Suppose you have good tools for showing *forward completeness*.
Let $s := -t$ such that

$$ds = -dt \qquad \text{or} \qquad dt = -ds$$

$$x' := \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = -\dot{x} = -f(x) =: g(x)$$

It follows that:

- as $t : 0 \to -\infty$, then $s : 0 \to +\infty$, and
- showing *backward completeness* for $\dot{x} = f(x)$ is equivalent to showing *forward completness* for $x' = g(x)$.

Equilibrium points

A point $x = x^* \in \mathbb{R}^n$ is called an equilibrium point of $\dot{x} = f(x)$ if

$$x(0) = x^* \quad \Longrightarrow \quad x(t) \equiv x^*, \ \forall t \geq 0.$$

For such autonomous systems, the equilibrium points are the real solutions of

$$f(x) = 0.$$

Equilibrium points:

- Single point.
- Multiple isolated points; no other equilibrium points in the neighborhood of each point.
- A continuum of equilibrium points.

Equilibrium points

Linear system $\dot{x} = Ax$ can have:

- an isolated equilibrium at $x = 0$ if A is nonsingular, or
- a continuum of equilibrium points, if A is singular, in the null space of A.

It cannot have multiple isolated eqilibrium points, since if x_1 and x_2 are two such points, then by linearity any point on the line $\alpha x_1 + (1 - \alpha)x_2$ connecting these points will also be an equilibrium point.

Linearization

A common engineering practice in analyzing or simplifying a nonlinear system is to linearize it around a working point and then use the resulting linear system.
Limitations of this approach:

- Since this results in an approximation in the neighborhood of the working point, it will only predict the *local* behavior of the nonlinear system. It cannot predict the *nonlocal* or *global* behavior.
- There are *nonlinear phenomena* that can occur in the presence of a nonlinearity, that is not captured by the linear model.

Nonlinear phenomena

The following essential nonlinear phenomena cannot occur for a linear system: See [Khalil, 2002b, Khalil, 2015] and [Teel, 2002].

- Finite escape times.
- Multiple isolated equilibrium points.
- Lack of existence of solutions.
- Lack of uniqueness of solutions.
- Limit cycles or isolated periodic solutions.
- Subharmonic, harmonic, or almost periodic oscillations.
- Bounded regions of attraction.
- Asymptotic but not exponential stability.
- Convergence without stability.
- Bifurcations.
- Chaos.
- Multiple modes of behavior.

Comparison functions

A function $\gamma : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ is a *positive definite* function if

- $\gamma(0) = 0$
- $\gamma(r) > 0$ for all $r \neq 0$

A function $\gamma : \mathbb{R} \mapsto \mathbb{R}_{\geq 0}$ is a *positive semidefinite* function if

- $\gamma(0) = 0$
- $\gamma(r) \geq 0$ for all $r \neq 0$

Negative definite and *negative semidefinite* follow similarly.

Examples ($k, \omega > 0$):

- $\gamma(r) = k\,|r|$ is positive definite.
- $\gamma(r) = k \sin(\omega r)^2$ is positive semidefinite.
- $\gamma(r) = \int_0^r \sigma(s)ds$ where $\sigma : \mathbb{R} \mapsto \mathbb{R}$ is C^1, $\sigma(0) = 0$, and strictly increasing. This is positive definite.

Comparison functions

A continuous function $\alpha : [0, a) \mapsto [0, \infty)$ is called a class-$\mathcal{K}$ function if

- $\alpha(0) = 0$
- it is strictly increasing

Examples:

- $\alpha(r) = \ln \frac{1}{1-r}$, defined on $r \in [0, 1)$.
- $\alpha(r) = 1 - e^{-r}$, which is bounded by 1.

Comparison functions

A class-$\mathcal{K}$ function $\alpha : [0, a) \mapsto [0, \infty)$ is called a class-$\mathcal{K}_\infty$ function if

- $a = \infty$
- $\lim_{r \to \infty} \alpha(r) = \infty$

Examples:

- $\alpha(r) = \sqrt{r}$
- $\alpha(r) = r^2$

Comparison functions

A continuous function $\varphi : [0, \infty) \mapsto [0, \infty)$ is called a class-$\mathcal{L}$ function if

- it is decreasing
- $\lim_{s \to \infty} \varphi(s) = 0$

Example:

- $\varphi(s) = ke^{-\lambda s}$

Comparison functions

A function $\beta : [0, \infty) \times [0, \infty) \mapsto [0, \infty)$ is called a class-$\mathcal{KL}$ function if

- $\beta\,(\,\cdot\,, s) \in \mathcal{K}$ for all $s \geq 0$.
- $\beta\,(r, \,\cdot\,) \in \mathcal{L}$ for all $r \geq 0$.

Example:

- $\beta(r, s) = kre^{-\lambda s}$

Comparison functions

Properties:

- $\alpha_1, \alpha_2 \in \mathcal{K}\ (\mathcal{K}_\infty) \implies \alpha_1 \circ \alpha_2(r) := \alpha_1\,(\alpha_2(r)) \in \mathcal{K}\ (\mathcal{K}_\infty)$.
- $\alpha \in \mathcal{K}$ on $[0, a) \implies \exists! \alpha^{-1} \in \mathcal{K}$ on $[0, \lim_{r \to a} \alpha(r))$.

Example:

$$\alpha(r) = 1 - e^{-r} \implies \alpha^{-1}(r) = \ln \frac{1}{1-r}$$

and it is verified that

$$\alpha \circ \alpha^{-1}(r) = 1 - e^{-\ln \frac{1}{1-r}} = r.$$

Basic stability result

Scalar nonlinear system

$$\dot{y} = -\alpha(y)$$

where $\alpha : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz and $\alpha \in \mathcal{K}$ on $[0, a)$.

For all $y_0 \in [0, a)$, $\exists!$ solution $y(t, y_0)$ given by

$$y(t, y_0) = \beta\left(|y_0|, t\right), \qquad t \geq 0$$

where $\beta \in \mathcal{KL}$.

Example 5

- Linear scalar system

  $$\dot{y} = -ay, \qquad a > 0$$

  admits the unique solution $y(t, y_0) = e^{-at} y_0 \in \mathcal{KL}$.
- Since $\alpha(y) := \sin(ky) \in \mathcal{K}$ on $[0, \frac{\pi}{2k})$, the nonlinear system

  $$\dot{y} = -\sin(ky)$$

  admits a unique solution $y(t, y_0) = \beta\left(|y_0|, t\right)$, $t \geq 0$ for all $y_0 \in [0, \frac{\pi}{2k})$.

Example 6

- The nonlinear system
$$\dot{y} = -y^3$$
admits the solution
$y(t, y_0) = \mathrm{sgn}(y_0)\sqrt{\frac{y_0^2}{1+2y_0^2 t}} = \mathrm{sgn}(y_0)\sqrt{\frac{1}{2t+\frac{1}{y_0^2}}}.$

Hence, for all $y_0 \in \mathbb{R}_{\geq 0}$,
$$y(t, y_0) = \beta\left(|y_0|, t\right)$$
where $\beta(r, s) := \sqrt{\frac{r^2}{1+2r^2 s}} \in \mathcal{KL}$.

Basic stability result

Lemma

For each continuous positive definite function α there exists a $\mathcal{KL}$ function $\beta_\alpha(r, s)$ with the following properties:

- *if $y(\,\cdot\,)$ is any (locally) absolutely continuous function defined for each $t \geq 0$ and with $y(t) \geq 0$, $\forall t \geq 0$, and*
- *$y(\,\cdot\,)$ satisfies the differential inequality*
$$\dot{y}(t) \leq -\alpha(y(t)), \qquad \forall t \geq 0$$
with $y(0) = y_0 \geq 0$,
- *then*
$$y(t) \leq \beta_\alpha\left(y_0, t\right), \qquad \forall t \geq 0.$$

See [Lin et al., 1996].

Stability

Definition

The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is:

- *stable* if, for each $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that

$$|x(0)| < \delta(\varepsilon) \quad \Longrightarrow \quad |x(t)| < \varepsilon, \qquad \forall t \geq 0,$$

- *unstable* if not stable, and
- *asymptotically stable* if it is stable and δ can be chosen such that

$$|x(0)| < \delta \quad \Longrightarrow \quad \lim_{t \to \infty} x(t) = 0.$$

See [Khalil, 2002b, Khalil, 2015, Def. 3.1].

Lyapunov stability

[Khalil, 2015, Thm. 3.3] Define $B_r := \{x \in \mathbb{R}^n : |x| \leq r\}$ be a ball set in $\mathbb{R}^n$.

$\exists V : B_r \mapsto \mathbb{R}$ continuously differentiable such that

$$\begin{aligned} \alpha_1 (|x|) \leq V(x) \leq \alpha_2 (|x|) \\ \frac{\partial V}{\partial x} f(x) \leq -\alpha_3 (|x|), \qquad x \in B_r \end{aligned}$$

where $\alpha_1, \alpha_2 \in \mathcal{K}$ and α_3 is a continuous function.

- If α_3 is positive semidefinite, then $x = 0$ is Locally Stable (LS).
- If α_3 is positive definite, then $x = 0$ is Locally Asymptotically Stable (LAS).

STABILITY: Recall definition, for each $\varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ such that

$$|x(0)| < \delta(\varepsilon) \quad \Longrightarrow \quad |x(t)| < \varepsilon.$$

Let $x_0 = x(0)$, fix (wlog) $\varepsilon < r$, and choose

$$\begin{aligned}
&\delta(\varepsilon) \leq \alpha_2^{-1} \circ \alpha_1(\varepsilon) \\
&\Downarrow \\
&\alpha_1(|x_0|) \leq \alpha_2\left(|x_0|\right) \leq \alpha_2\left(\delta\right) \leq \alpha_2 \circ \alpha_2^{-1} \circ \alpha_1(\varepsilon) = \alpha_1\left(\varepsilon\right) \\
&\Downarrow \\
&|x_0| \leq \varepsilon < r \\
&\Downarrow \\
&x_0 \in B_r \quad \text{and} \quad \dot{V} \leq 0
\end{aligned}$$

Hence

$$V(x(t)) \leq V(x_0)$$

Hence

$$\alpha_1\left(|x(t)|\right) \leq V(x(t)) \leq V(x_0) \leq \alpha_2\left(|x_0|\right)$$

Hence

$$|x(t)| \leq \alpha_1^{-1} \circ \alpha_2\left(|x_0|\right) \leq \alpha_1^{-1} \circ \alpha_2\left(\delta(\varepsilon)\right) \leq \alpha_1^{-1} \circ \alpha_1\left(\varepsilon\right) \leq \varepsilon$$

$$\Downarrow$$

$$|x(t)| \in B_r, \quad \forall t \geq 0.$$

QED.

CONVERGENCE: If α_3 is positive definite, let

$$V(x_0) < \alpha_1(r)$$
$$\alpha_1(|x_0|) \leq V(x_0) < \alpha_1(r) \implies x_0 \in B_r.$$

Then

$$\begin{aligned} V(x(t)) \leq \alpha_2(|x(t)|) &\implies |x(t)| \geq \alpha_2^{-1}(V(x(t))) \\ &\implies \alpha_3(|x(t)|) \geq \alpha_3\left(\alpha_2^{-1}(V(x(t)))\right) \end{aligned}$$
$$\dot{V}(t) \leq -\alpha_3(|x(t)|) \leq -\alpha_3 \circ \alpha_2^{-1}(V(x(t)))$$

Then from the earlier (comparison) lemma, letting $\beta := \beta_{\alpha_3 \circ \alpha_2^{-1}}$:

$$\begin{aligned} V(x(t)) &\leq \beta(V(x_0), t) \\ &\Downarrow \\ |x(t)| &\leq \alpha_1^{-1}(V(x(t))) \leq \alpha_1^{-1} \circ \beta(V(x_0), t) \\ &\leq \alpha_1^{-1} \circ \beta(\alpha_2(|x_0|), t) =: \tilde{\beta}(|x_0|, t) \end{aligned}$$

QED.

Example 7

Consider the unforced nonlinear surge dynamics of a ship

$$m\dot{u} + d_1|u|u + d_2 u = 0, \qquad u_0 = 5\,\mathrm{m}/s$$

We want to analyze the stability of $u = 0$ and if the speed decays to zero.
As a Lyapunov function we consider the kinetic energy

$$V(u) = \frac{1}{2}mu^2$$

This should intuitively drop to zero as the speed drops to zero.

Example 7

We select

$$\alpha_1(|u|) := \frac{1}{4}m\,|u|^2 \leq V(u) \leq m\,|u|^2 =: \alpha_2(|u|)$$

Differentiating $V(u)$ along the solutions of the surge dynamics, we get

$$\begin{aligned}\dot{V}(u) &= m\dot{u}u \\ &= -d_1\,|u|\,u^2 - d_2u^2 = -d_1\,|u|^3 - d_2\,|u|^2 =: -\alpha_3(|u|)\end{aligned}$$

Since $\alpha_3(|u|)$ is positive definite, the equilbrium $u = 0$ is asymptotically stable.

Preparations for next lecture

Global stability of nonlinear systems:

- Read note on “Mathematical notations and preliminaries”.
- Check out note on "Some inequalities".
- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 1, 2, and 3.1-3.3 (from today)
  - Chapters: 3.4-3.7 and 4.1 (for next lecture)
- Lavretsky, E. and K. A. Wise (2013). Robust and Adaptive Control (With Aerospace Applications)
  - Chapters 8.1-8.3 (for alternative explanations and deeper learning of todays subject)
  - Chapters 8.4-8.8 (for alternative explanations next lecture)
- Lecture presentation.

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