TMR4243 - Marine Control Systems II

Exam

Spring 2017

Notation: Throughout this exam $|x|$ means the vector 2-norm, i.e. $|x| = \sqrt{x^\top x}$. For a scalar x, this corresponds to the absolute value.

States and variables are scalars unless these are specifically defined as vectors, e.g., x_1 is a scalar while $x_2 \in \mathbb{R}^n$ is an n-dimensional vector.

1 Properties of nonlinear systems (20 pts)

1. Consider the three ordinary differential equations (ODEs):

$$\begin{aligned} \dot{z} &= g_1(z) = -cz \qquad (1) \\ \dot{z} &= g_2(z) = -cz^3 \qquad (2) \\ \dot{z} &= g_3(z) = -c\sin(z) \qquad (3) \end{aligned}$$

where $z \in \mathbb{R}$, $z_0 = z(0)$, and c is a positive constant.

(a) For each of the three ODEs, explain if these are *Locally Lipschitz* and/or *Globally Lipschitz* or not Lipschitz at all.

Answer: (6 pts.) We get

$$|g_1(z) - g_1(y)| = |-cz - (-cy)| = c\,|z - y|$$

is Globally Lipschitz with the global Lipschitz constant $L = c$ (since this is a linear scalar system).

We get that $g_2(z)$ is not Globally Lipschitz since

$$\frac{\partial g_2}{\partial z} = -3z^2 \rightarrow -\infty \quad \text{as} \quad z \rightarrow \infty.$$

However, on any compact interval $[z, y]$ this derivative is bounded, say by $\left|\frac{\partial g_2}{\partial z}\right| \leq L$, and thus by the the mean value theorem

$$|g_2(z) - g_2(y)| \leq L\,|z - y|\,.$$

Hence it is Locally Lipschitz.

We get for $g_3(z)$, again by the mean value theorem

$$\begin{aligned} |g_3(z) - g_3(y)| &\leq \sup_x \left(\left|\frac{\partial g_3(x)}{\partial z}\right|\right)|z - y| \\ &= c\sup\left(|\cos x|\right)|z - y| = c\,|z - y|\,, \end{aligned}$$

which shows that $g_3(z)$ is Globally Lipschitz.

(b) What can you say about *existence*, *uniqueness*, and *forward completeness* of the solutions for the three ODEs?

Answer: (3 pts.) Global Lipschitz properties of 1st and 3rd ODEs ensures *existence*, *uniqueness*, and *forward completeness* of the solutions. This is also the case for the 2nd ODE, since for any initial condition $z_0 = a$ its solution will be bounded within the set $[-a, a]$ for all time (it is UGAS).

2. For each of the three systems

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - 2x_2 \end{aligned}} \tag{4}$$

$$\boxed{\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \cos x_1 \\ \dot{x}_2 &= x_1 \cos x_1 - x_2 (\cos x_1)^2 + x_3 \\ \dot{x}_3 &= -x_2^3 \end{aligned}} \tag{5}$$

$$\boxed{\begin{aligned} \dot{x}_1 &= 3\sin x_2 \\ \dot{x}_2 &= -\sin x_2 \end{aligned}} \tag{6}$$

Find the equilibria of these systems, and explain what type of equilibria these are.

Answer: (6 pts.) System 1:

$$\mathcal{E}_1 = \{(x_1, x_2) : x_2 = 0, \quad x_1 = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots\}$$

These are multiple isolated equilibria at $x_1 = k\pi$ along the x_1-axis.
System 2:

$$\mathcal{E}_2 = \{(x_1, x_2, x_3) : x_2 = 0, \quad x_1 = 0, \quad x_3 = 0\}$$

The origin is a single equilibrium.
System 3:

$$\mathcal{E}_3 = \{(x_1, x_2) : x_2 = k\pi, \quad k = 0, \pm 1, \pm 2, \ldots\}$$

These are multiple isolated continuums of equilibrium points for every $x_2 = k\pi$ where x_1 is arbitrary.

3. The scalar ODE

$$\dot{x} = -x^2, \qquad x_0 = -1$$

admits the solution

$$x(t) = \frac{1}{t-1}, \qquad t \geq 0$$

(a) Discuss the Lipschitz properties of this ODE and the corresponding existence, uniqueness, and forward completeness of its solutions.

Answer: (3 pts.) The ODE is Locally Lipschitz, and thus for every intial condition there exist $T > 0$ and a unique solution on $[0, T]$. However, for the above unique solution starting from $x_0 = -1$, we see that it has a finite escape time at $t = 1$, and hence the solution is not forward complete.

(b) Is it stable?

Answer: (2 pts.) The ODE is not stable - if it were, forward completeness would be ensured by definition.

2 Lyapunov stability (27 pts)

1. For the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -\left(x_1 + x_1^3\right) - 2x_1x_2^2 \\ \dot{x}_2 &= x_1^2x_2 - \left(x_2 + x_2^5\right)\end{aligned}$$

let a Lyapunov function candidate be

$$V(x_1, x_2) = c_1x_1^2 + c_2x_2^2$$

where $c_1 > 0$ and $c_2 > 0$ are constants.

(a) Calculate the time derivative of V as a function of (x_1, x_2).

Answer: (3 pts.) We get

$$\begin{aligned}\dot{V} &= 2c_1x_1\dot{x}_1 + 2c_2x_2\dot{x}_2 \\ &= 2c_1x_1\left(-\left(x_1 + x_1^3\right) - 2x_1x_2^2\right) + 2c_2x_2\left(x_1^2x_2 - \left(x_2 + x_2^5\right)\right) \\ &= -2c_1\left(x_1^2 + x_1^4\right) - 4c_1x_1^2x_2^2 + 2c_2x_1^2x_2^2 - 2c_2\left(x_2^2 + x_2^6\right) \\ &= -2c_1\left(x_1^2 + x_1^4\right) - 2\left(2c_1 - c_2\right)x_1^2x_2^2 - 2c_2\left(x_2^2 + x_2^6\right)\end{aligned}$$

(b) Find values for c_1 and c_2 that proves UGES of $(x_1, x_2) = 0$.

Answer: (3 pts.) Choosing $c_1, c_2 > 0$ with $c_1 \geq \frac{1}{2}c_2$ will ensure UGES, e.g. using $c_1 = 1$ and $c_2 = 2$ gives

$$\begin{aligned}|x|^2 &\leq V(x_1, x_2) \leq 2|x|^2 \\ \dot{V} &= -2\left(x_1^2 + x_1^4\right) - 4\left(x_2^2 + x_2^6\right) \leq -2x_1^2 - 4x_2^2 \leq -2|x|^2.\end{aligned}$$

Also, e.g. choosing $c_1 = c_2 = 1$ works, giving

$$\begin{aligned}|x|^2 &\leq V(x_1, x_2) \leq |x|^2 \\ \dot{V} &= -2\left(x_1^2 + x_1^4\right) - 2x_1^2x_2^2 - 2\left(x_2^2 + x_2^6\right) \leq -2x_1^2 - 2x_2^2 = -2|x|^2.\end{aligned}$$

2. Let a system be

$$\begin{aligned} \dot{x}_1 &= 4x_2 \\ \dot{x}_2 &= -2\,\mathrm{sat}(x_1) - \frac{x_2}{10} \end{aligned}$$

where

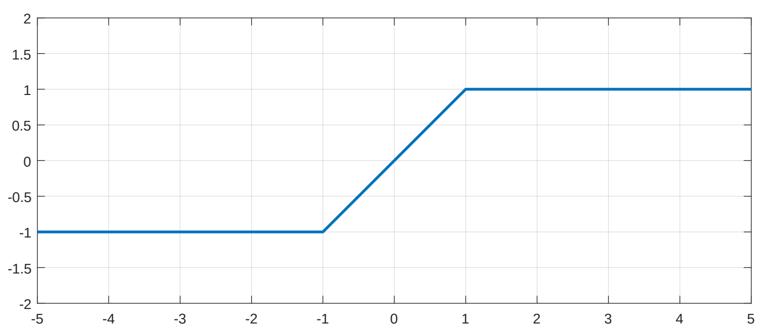
$$\mathrm{sat}(y) := \begin{cases} -1; & y \leq -1 \\ y & -1 < y < 1 \\ +1 & 1 \leq y \end{cases}$$

Let a Lyapunov function candidate be

$$V(x_1, x_2) = \int_0^{x_1} \mathrm{sat}(y)dy + x_2^2$$

(a) Draw $\mathrm{sat}(y)$ and show that it is Globally Lipschitz.

Answer: (3 pts.) The saturation function plotted next shows



that

$$\left| \frac{\partial\,\mathrm{sat}(y)}{\partial y} \right| \leq 1, \quad \forall y,$$

and, hence, for any two points (y, z) we have by the mean value theorem

$$|\mathrm{sat}(y) - \mathrm{sat}(z)| \leq |y - z|\,, \quad \forall (y, z).$$

Hence, the saturation function is Glopbally Lipschitz.

(b) Show that $\mathrm{sat}(0) = 0$, $\mathrm{sat}(y)y > 0$ for all $y \neq 0$, and that V is radially unbounded.

Answer: (4 pts.) Obviously, $\mathrm{sat}(0) = 0$ from the definition. Moreover,

$$\mathrm{sat}(y)y = \begin{cases} -y; & y \leq -1 \\ y^2 & -1 < y < 1 \\ +y & 1 \leq y \end{cases} > 0, \quad \forall y \neq 0.$$

To check that V is radially unbounded, we check if $V \to \infty$ if $|x_1| \to \infty$ or $|x_2| \to \infty$. Obviously,

$$|x_2| \to \infty \implies V(x_1, x_2) \geq |x_2|^2 \to \infty$$

We also get for $|x_1| \to \infty$ that

$$\begin{aligned} x_1 = a \to +\infty \implies V(x_1, x_2) &\geq \int_0^1 y dy + \int_1^a dy \\ &= \frac{1}{2} + a - 1 \to \infty \\ x_1 = -a \to -\infty \implies V(x_1, x_2) &\geq \int_0^{-1} y dy + \int_{-1}^{-a} (-1) dy \\ &= \frac{1}{2} + a - 1 \to \infty \end{aligned}$$

Hence, V is radially unbounded.

(c) By differentiating V, what stability conclusion can you make for the origin by Lyapunov's direct method?

Answer: (3 pts.) We get

$$\begin{aligned} \dot{V} &= \mathrm{sat}(x_1)\dot{x}_1 + 2x_2\dot{x}_2 = \mathrm{sat}(x_1)(4x_2) + 2x_2\left(-2\,\mathrm{sat}(x_1) - \frac{x_2}{10}\right) \\ &= 4x_2\,\mathrm{sat}(x_1) - 4x_2\,\mathrm{sat}(x_1) - \frac{2}{10}x_2^2 \\ &= -\frac{1}{5}x_2^2 \leq 0. \end{aligned}$$

This shows that $\dot{V}$ is negative semidefinite, and hence, by Lyapunov's direct method, we can only prove that the origin is (Uniformly) Globally Stable (UGS).

(d) Show that the origin is UGAS.

Answer: (3 pts.) We invoke the Krasovskii-LaSalle's invariance principle and define the set where $\dot{V} = 0$, that is,

$$\Omega = \{(x_1, x_2) : x_2 = 0\}$$

and look for the largest INVARIANT set within Ω. This is given by

$$\begin{array}{ll} 1^{st} \text{ eq.:} & x_2 = 0 \implies \dot{x}_1 = 0 \\ 2^{nd} \text{ eq.:} & x_2 \equiv 0 \implies \dot{x}_2 = 0 \implies -2\,\mathrm{sat}(x_1) = 0 \implies x_1 = 0 \end{array}$$

Hence, the largest invariant set $\mathcal{M} \subset \Omega$ is the origin,

$$\mathcal{M} = \{(x_1, x_2) : (x_1, x_2) = 0\},$$

which then must be UGAS.

3. Consider the nonlinear time-varying system:

$$\dot{x} = G\,(x - x_d(t)) + H(x)\,(x - x_d(t)) + \dot{x}_d(t)$$

where G is a constant matrix satisfying

$$G + G^\top < 0,$$

$H(x)$ is a nonlinear matrix satisfying

$$H(x) = -H(x)^\top,$$

and $(x_d(t), \dot{x}_d(t))$ are bounded reference signals.

Let a Lyapunov function candidate be

$$V(t, x) = (x - x_d(t))^\top\, P\,(x - x_d(t))$$

(a) Show that G satisfies the Lyapunov equation

$$PG + G^\top P = -Q$$

with $P = I$ (identity matrix). What becomes Q?

Answer: (2 pts.) With $P = I$ we get

$$\begin{aligned} PG + G^\top P &= G + G^\top = -Q \\ Q &= Q^\top = -\left(G + G^\top\right) > 0 \end{aligned}$$

(b) Show how to differentiate $V(t, x)$.

Answer: (3 pts.) Differentiating gives

$$\begin{aligned} \dot{V} &= 2\left(x - x_d\right)^\top P\left(\dot{x} - \dot{x}_d\right) \\ &= 2\left(x - x_d\right)^\top P\left(G\left(x - x_d\right) + H(x)\left(x - x_d\right)\right) \\ &= \left(x - x_d\right)^\top \left(G + G^\top\right)\left(x - x_d\right) + \left(x - x_d\right)^\top \left(H(x) + H(x)^\top\right)\left(x - x_d\right) \\ &= -\left(x - x_d\right)^\top Q\left(x - x_d\right) \\ &\leq -\lambda_{\min}\left(Q\right)\left|x - x_d\right|^2 \end{aligned}$$

since $2z^\top Hz = z^\top\left(H + H^\top\right)z = 0$ and
$2z^\top Gz = z^\top\left(G + G^\top\right)z = -z^\top Qz$.

(c) What is the stability conclusion according to Lyapunov's direct method?

Answer: (3 pts.) Using $e = x - x_d(t)$ we get

$$\dot{e} = Ge + H(e + x_d(t))e,$$

and $V = e^\top e$ then gives

$$\begin{aligned} &\alpha_1(|e|) = \alpha_2(|e|) := |e|^2 \\ &\alpha_3(|e|) := \lambda_{\min}\left(Q\right)|e|^2 \end{aligned}$$

$$\begin{aligned} \alpha_1(|e|) &\leq V(e) \leq \alpha_2(|e|) \\ \dot{V} &\leq -\alpha_3(|e|). \end{aligned}$$

Hence, the origin $e = 0$ is UGES according to Lyapunov's direct method, using $c_1 = c_2 = 1$, $c_3 = \lambda_{\min}\left(Q\right)$, and $r = 2$.

3 DP observer and control design (30 pts)

Consider the low-speed DP vessel model

$$\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= \tau - D\nu \\ z &= \eta \end{aligned}$$

where $\eta = col(x, y, \psi)$, $\nu = col(u, v, r)$, $M^\top > 0$ and $D > 0$ are the mass and damping matrices, respectively, $R(\psi)$ is the rotation matrix, and z is the measured output.

Suppose D satisfies

$$D + D^\top > 0.$$

1. Assume that $R = I$ (constant identity matrix) and investigate if the "linearized system" is *uniformly completely observable.*

   Answer: (3 pts.) We get the linear system, with $x = col(\eta, \nu)$,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & I \\ 0 & -M^{-1}D \end{bmatrix} x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tau \\ z &= \begin{bmatrix} I & 0 \end{bmatrix} x, \end{aligned}$$

   which gives the (vectorial) observability matrix

$$\mathcal{O} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

   which clearly is full rank. Hence, the "linearized system" is UCO.

2. Let an observer be

$$\begin{aligned}\dot{\hat{\eta}} &= R(\psi)\hat{\nu} + K_1 (z - \hat{\eta}) \\ M\dot{\hat{\nu}} &= \tau - D\hat{\nu} + R(\psi)^\top K_2 (z - \hat{\eta}),\end{aligned}$$

where $K_1 = K_1^\top > 0$ and $K_2 = K_2^\top > 0$ are injection gain matrices.

(a) Write down the equations for the corresponding observer error dynamics $\tilde{\eta} := \eta - \hat{\eta}$ and $\tilde{\nu} := \nu - \hat{\nu}$.

Answer: (2 pts.) Error dynamics:

$$\begin{aligned}\dot{\tilde{\eta}} &= \dot{\eta} - \dot{\hat{\eta}} = R(\psi)\nu - R(\psi)\hat{\eta} - K_1 (\eta - \hat{\eta}) = R(\psi)\tilde{\nu} - K_1\tilde{\eta} \\ M\dot{\tilde{\nu}} &= M\dot{\nu} - M\dot{\hat{\nu}} = \tau - D\nu - \tau + D\hat{\nu} - R(\psi)^\top K_2 (\eta - \hat{\eta}) \\ &= -D\tilde{\nu} - R(\psi)^\top K_2\tilde{\eta}\end{aligned}$$

(b) Let a Lyapunov function candidate be

$$V_o = \tilde{\eta}^\top K_2 \tilde{\eta} + \tilde{\nu}^\top M \tilde{\nu},$$

and find the time derivative of V_o as a function of the error states.

Answer: (3 pts.) Differentiating V_o gives

$$\begin{aligned}\dot{V}_o &= 2\tilde{\eta}^\top K_2 \dot{\tilde{\eta}} + 2\tilde{\nu}^\top M \dot{\tilde{\nu}} \\ &= 2\tilde{\eta}^\top K_2 (R(\psi)\tilde{\nu} - K_1\tilde{\eta}) + 2\tilde{\nu}^\top \left(-D\tilde{\nu} - R(\psi)^\top K_2\tilde{\eta}\right) \\ &= 2\tilde{\eta}^\top K_2 R(\psi)\tilde{\nu} - 2\tilde{\eta}^\top K_2 K_1\tilde{\eta} - 2\tilde{\nu}^\top D\tilde{\nu} - 2\tilde{\nu}^\top R(\psi)^\top K_2\tilde{\eta} \\ &= -2\tilde{\eta}^\top K_2 K_1\tilde{\eta} - \tilde{\nu}^\top \left(D + D^\top\right)\tilde{\nu} \\ &= -2\tilde{\eta}^\top K_1 K_2\tilde{\eta} - \tilde{\nu}^\top \left(D + D^\top\right)\tilde{\nu} \\ &= -\tilde{\eta}^\top (K_2 K_1 + K_1 K_2)\tilde{\eta} - \tilde{\nu}^\top \left(D + D^\top\right)\tilde{\nu}\end{aligned}$$

(c) Derive and give conditions on the injection gain matrices K_1 and K_2 that ensures that the error dynamics is UGES.

Answer: (3 pts.) Letting $\tilde{x} := col(\tilde{\eta}, \tilde{\nu})$ and requiring $K_2 > 0$, then we get

$$\lambda_{\min}(K_2, M) |\tilde{x}|^2 \leq V_o \leq \lambda_{\max}(K_2, M) |\tilde{x}|^2 .$$

The error dynamics is UGES if either of the conditions are met:

$$\begin{array}{ll} 1) & K_2K_1 > 0 \quad \text{or} \quad K_1K_2 > 0 \\ 2) & K_2K_1 + K_1K_2 > 0 \end{array}$$

In both these cases we get

$$\dot{V}_o \leq -c_3 |\tilde{x}|^2$$

for some c_3 that depends on K_1, K_2, and D.

3. Disregarding the observer for now, let a state feedback control law to control $(\eta, \nu) \to 0$, be

$$\tau = -R(\psi)^\top L_1 \eta - L_2 \nu$$

where $L_1 = L_1^\top > 0$ and $L_2 = L_2^\top > 0$.

(a) Write down the closed-loop system with this state feedback.

Answer: (2 pts.) The state-feedback closed loop system becomes

$$\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= -R(\psi)^\top L_1 \eta - L_2 \nu - D\nu \end{aligned}$$

(b) Let a Lyapunov function candidate be

$$V_c = \eta^\top L_1 \eta + \nu^\top M \nu,$$

and find the time derivative of V_c as a function of (η, ν).

Answer: (3 pts.) Differentiating V_c gives

$$\begin{aligned}
\dot{V}_c &= 2\eta^\top L_1 \dot{\eta} + 2\nu^\top M \dot{\nu} \\
&= 2\eta^\top L_1 R(\psi)\nu + 2\nu^\top \left(-R(\psi)^\top L_1 \eta - L_2 \nu - D\nu\right) \\
&= -2\nu^\top L_2 \nu - \nu^\top \left(D + D^\top\right) \nu \\
&\leq -2\nu^\top L_2 \nu \\
&\leq -2\lambda_{\min}\left(L_2\right) |\nu|^2
\end{aligned}$$

(c) What stability conclusion can you make from Lyapunov's direct method?

Answer: (2 pts.) Since $\dot{V}_c$ is only negative semidefinite, the origin $(\eta, \nu) = 0$ is UGS by Lyapunov's direct method.

(d) Use Krasovskii-LaSalle's invariance principle to show that $(\eta, \nu) = (0, 0)$ is in fact UGAS.

Answer: (3 pts.) We look for the largest invariant set within

$$\Omega = \{(\eta, \nu) : \nu = 0\}$$

This is given by

$$\begin{array}{ll}
1^{st} \text{ eq.:} & \nu = 0 \implies \dot{\eta} = 0 \\
2^{nd} \text{ eq.:} & \nu \equiv 0 \implies \dot{\nu} = 0 \implies R(\psi)^\top L_1 \eta = 0 \implies \eta = 0
\end{array}$$

Hence, the largest invariant set $\mathcal{M} \subset \Omega$ is the origin,

$$\mathcal{M} = \{(\eta, \nu) : (\eta, \nu) = 0\},$$

which then must be UGAS.

4. Using output feedback, by including the observer, let the feedback control law be

$$\tau = -R(\psi)^\top L_1 \hat{\eta} - L_2 \hat{\nu}.$$

(a) Write down the overall closed-loop system in $(\tilde{\eta}, \tilde{\nu}, \eta, \nu) \in \mathbb{R}^{12}$.

Answer: (3 pts.) We get

$$\begin{aligned} M\dot{\nu} &= -R(\psi)^\top L_1 (\hat{\eta} + \eta - \eta) - L_2 (\hat{\nu} + \nu - \nu) - D\nu \\ &= -R(\psi)^\top L_1 \eta - L_2\nu - D\nu + R(\psi)^\top L_1 \tilde{\eta} + L_2 \tilde{\nu} \end{aligned}$$

and thus

$$\begin{aligned} \dot{\tilde{\eta}} &= R(\psi)\tilde{\nu} - K_1\tilde{\eta} \\ M\dot{\tilde{\nu}} &= -D\tilde{\nu} - R(\psi)^\top K_2 \tilde{\eta} \\ \dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= -R(\psi)^\top L_1 \eta - L_2\nu - D\nu + R(\psi)^\top L_1 \tilde{\eta} + L_2 \tilde{\nu} \end{aligned}$$

(b) Show that the closed-loop system is a cascade between the UGES observer error system and the UGAS feedback control system. What is the interconnection terms?

Answer: (3 pts.) Letting $\tilde{x} := col(\tilde{\eta}, \tilde{\nu})$ and $x := col(\eta, \nu)$, and also let $z_3(t) = \psi(t)$ be an accurate gyrocompass measurement, we get

$$\begin{aligned} A_1(t) &:= \begin{bmatrix} -K_1 & R(z_3(t)) \\ -M^{-1}R(z_3(t))^\top K_2 & -M^{-1}D \end{bmatrix} \\ A_2(t) &:= \begin{bmatrix} 0 & R(z_3(t)) \\ -M^{-1}R(z_3(t))^\top L_1 & -M^{-1}(L_2 + D) \end{bmatrix} \\ B_2(t) &:= \begin{bmatrix} 0 & 0 \\ M^{-1}R(z_3(t))^\top L_1 & M^{-1}L_2 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \Sigma_1 : \quad & \dot{\tilde{x}} = A_1(t)\tilde{x} \\ \Sigma_2 : \quad & \dot{x} = A_2(t)x + B_2(t)\tilde{x} \end{aligned}$$

The interconnection terms are $M^{-1}R(z_3(t))^\top L_1 \tilde{\eta} + M^{-1}L_2\tilde{\nu}$, which for c sufficiently large clearly satisfies the linear growth condition

$$\left| M^{-1}R(z_3(t))^\top L_1 \tilde{\eta} + M^{-1}L_2\tilde{\nu} \right| \le c\,|\tilde{x}|$$

(c) These interconnection terms will in fact satisfy a linear growth condition, which together with UGES+UGAS proves that the overall cascaded system is UGAS. Explain and discuss the *separation principle*, given this fact.

Answer: (3 pts.) The separation principle implies that if UGES and UGAS of the two subsystems are kept intact, whatever gains chosen, the overall stability is maintained. Certainly, choosing K_1 and K_2 such that the $\tilde{x}$-dynamics is UGES and $L_1 = L_1^\top > 0$ and $L_2 = L_2^\top > 0$ so that the 0-input x-dynamics is UGAS, then the linearly bounded interconnection terms will ensure that the overall interconnected system is UGAS.

4 Adaptive control design (23 pts)

Consider a nonlinear mechanical system

$$\begin{aligned}\dot{x}_1 &= x_2 + v_0 \\ M\dot{x}_2 &= -D(x_1, x_2)x_2 + u\end{aligned}$$

where $x_1 \in \mathbb{R}^m$ contains e.g. positions and angles of the system, $x_2 \in \mathbb{R}^m$ is a relative velocity state, $v_0 \in \mathbb{R}^m$ is a constant unknown velocity reference, $u \in \mathbb{R}^m$ is the control input (e.g. typically forces and torques), $M = M^\top > 0$ is a constant mass matrix, and $D(x_1, x_2)$ is a nonlinear matrix.

Let the control objective be to control $x_1 \to x_d(t)$.

1. Assume $x_2 \equiv \alpha_1$ is a control input for the 1st equation and v_0 is an unknown constant vector to be adaptively estimated.

   Let $\hat{v}_0$ be an estimate, $\tilde{v}_0 := v_0 - \hat{v}_0$, and $z_1 := x_1 - x_d(t)$.

   (a) Using the control Lyapunov function

   $$V_1 := \frac{1}{2} z_1^\top z_1 + \frac{1}{2\gamma} \tilde{v}_0^\top \tilde{v}_0$$

   design a control law for α_1 and an adaptive update law for $\dot{\hat{v}}_0$ that solves the control objective.

   Answer: (6 pts.) We now assume that x_2 is a direct control input to the kinematic equation, i.e. $\dot{x}_1 = \alpha_1 + v_0$. Differentiating V_1 we get

   $$\begin{aligned}\dot{V}_1 &= z_1^\top (\dot{x}_1 - \dot{x}_d) + \frac{1}{\gamma} \tilde{v}_0^\top \dot{\tilde{\nu}}_0 \\ &= z_1^\top (\alpha_1 + v_0 - \dot{x}_d) - \frac{1}{\gamma} \tilde{v}_0^\top \dot{\hat{v}}_0 \\ &= z_1^\top (\alpha_1 + \hat{v}_0 - \dot{x}_d) + z_1^\top \tilde{v}_0 - \frac{1}{\gamma} \tilde{v}_0^\top \dot{\hat{v}}_0,\end{aligned}$$

and assigning

$$\alpha_1 = -K_1 z_1 - \hat{v}_0 + \dot{x}_d,$$

results in

$$\begin{aligned} \dot{V}_1 &= -z_1^\top K_1 z_1 + z_1^\top \tilde{v}_0 - \frac{1}{\gamma}\tilde{v}_0^\top \dot{\hat{v}}_0 \\ &= -z_1^\top K_1 z_1 + \tilde{v}_0^\top \left(z_1 - \frac{1}{\gamma}\dot{\hat{v}}_0 \right). \end{aligned}$$

Letting the adaptive update law be defined as

$$\dot{\hat{v}}_0 = \gamma z_1$$

gives

$$\dot{V}_1 = -z_1^\top K_1 z_1 \leq 0.$$

By the LaSalle-Yoshizawa theorem this proves that $(z_1, \tilde{v}_0) = 0$ is UGS and $z_1(t) \to 0$ as $t \to \infty$, which solves the control objective.

(b) What is the closed-loop system in $(z_1, \tilde{v}_0)$?

Answer: (2 pts.) We get

$$\begin{aligned} \dot{z}_1 &= -K_1 z_1 + \tilde{v}_0 \\ \dot{\tilde{v}}_0 &= -\gamma z_1 \end{aligned}$$

(c) What stability and convergence property will you get for the origin $(z_1, \tilde{v}_0) = 0$?

Answer: (4 pts.) We already have UGS and convergence of $z_1(t)$ by the LaSalle-Yoshizawa theorem. Since this is a time-invariant system, we can again invoke Krasovskii-LaSalles invariance principle. We then look for the largest invariant set within

$$\Omega = \{(z_1, \tilde{v}_0) : z_1 = 0\}$$

This is given by

$$
\begin{array}{ll}
2^{nd} \text{ eq.:} & z_1 = 0 \implies \dot{\tilde{v}}_0 = 0 \\
1^{st} \text{ eq.:} & z_1 \equiv 0 \implies \dot{z}_1 = 0 \implies \tilde{v}_0 = 0
\end{array}
$$

Hence, the largest invariant set $\mathcal{M} \subset \Omega$ is the origin,

$$
\mathcal{M} = \{(z_1, \tilde{v}_0) : (z_1, \tilde{v}_0) = 0\},
$$

which then must be UGAS. Since the error dynamics is linear, UGAS implies UGES.

2. Assume M and $D(\cdot, \cdot)$ are fully known, and let $z_2 := x_2 - \alpha_1$ and

$$
V_2 := V_1 + \frac{1}{2} z_2^\top M z_2
$$

(a) With your α_1 as you defined it above, and $z_2 = x_2 - \alpha_1$, what is now your closed-loop equation for $\dot{z}_1$ and the resulting derivative of $\dot{V}_1$?

Answer: (3 pts.) Substituting $x_2 = z_2 + \alpha_1$ into the x_1-dynamics, gives

$$
\begin{aligned}
\dot{x}_1 &= z_2 + \alpha_1 + v_0 \\
\dot{z}_1 &= \dot{x}_1 - \dot{x}_d = z_2 + \alpha_1 + v_0 - \dot{x}_d \\
&= -K_1 z_1 + z_2 + \tilde{v}_0 \\
\dot{V}_1 &= -z_1^\top K_1 z_1 + z_1^\top z_2
\end{aligned}
$$

(b) Differentiate $M z_2$ and V_2, and design a control law for u by the (adaptive) *backstepping* control design method, that solves the control objective for the complete system.

Answer: (4 pts.) We get

$$\begin{aligned} M\dot{z}_2 &= M\dot{x}_2 - M\dot{\alpha}_1 = -D(x_1,x_2)x_2 + u - M\dot{\alpha}_1 \\ \dot{V}_2 &= \dot{V}_1 + z_2^\top M\dot{z}_2 \\ &= -z_1^\top K_1 z_1 + z_1^\top z_2 + z_2^\top\left[-D(x_1,x_2)x_2 + u - M\dot{\alpha}_1\right] \\ &= -z_1^\top K_1 z_1 + z_2^\top\left[z_1 - D(x_1,x_2)x_2 + u - M\dot{\alpha}_1\right] \end{aligned}$$

Choosing

$$u = -z_1 - K_2 z_2 + D(x_1,x_2)x_2 + M\dot{\alpha}_1$$

gives

$$\dot{V}_2 = -z_1^\top K_1 z_1 - z_2^\top K_2 z_2 \le 0.$$

By the LaSalle-Yoshizawa theorem this proves that $(\tilde{v}_0, z_1, z_2) = 0$ is UGS and $(z_1(t), z_2(t)) \to 0$ as $t \to \infty$, which solves the control objective.

(c) Write down the closed-loop system in the states $(\tilde{v}_0, z_1, z_2)$ and conclude what stability and convergence you get.

Answer: (4 pts.) We get

$$\begin{aligned} \dot{\tilde{v}}_0 &= -\gamma z_1 \\ \dot{z}_1 &= -K_1 z_1 + z_2 + \tilde{v}_0 \\ M\dot{z}_2 &= -z_1 - K_2 z_2 \end{aligned}$$

By the LaSalle-Yoshizawa theorem we have that $(\tilde{v}_0, z_1, z_2) = 0$ is UGS and $(z_1(t), z_2(t)) \to 0$ as $t \to \infty$. However, again this is a time-invariant system so that Krasovskii-LaSalle's invariance principle is applicable. We then look for the largest invariant set within

$$\Omega = \left\{(\tilde{v}_0, z_1, z_2) : \; \dot{V}_2 = 0\right\} = \{(\tilde{v}_0, z_1, z_2) : \; (z_1, z_2) = 0\}$$

This is given by

$$\begin{aligned} &1^{st} \text{ eq.:} && z_1 = 0 \implies \dot{\tilde{v}}_0 = 0 \\ &3^{rd} \text{ eq.:} && z_1 = z_2 = 0 \implies \dot{z}_2 = 0 \\ &2^{nd} \text{ eq.:} && z_1 \equiv 0 \implies \dot{z}_1 = 0 \implies \tilde{v}_0 = 0 \end{aligned}$$

Hence, the largest invariant set $\mathcal{M} \subset \Omega$ is the origin,

$$\mathcal{M} = \{(\tilde{v}_0, z_1, z_2) : (\tilde{v}_0, z_1, z_2) = 0\},$$

which then must be UGAS. Since the error dynamics is linear, UGAS implies UGES.