TMR4243 - Marine Control Systems II

Exam

Spring 2016

Notation: Throughout this exam $|x|$ means the vector 2-norm, i.e. $|x| = \sqrt{x^\top x}$. For a scalar x, this corresponds to the absolute value.

1 Properties of nonlinear systems (30 pts)

1. Consider the nonlinear ordinary differential equation (ODE):

$$\dot{x} = h(x), \qquad x_0 = x(0),$$

where $x \in \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^n$.

(a) Suppose for each compact set $\mathcal{C} \in \mathbb{R}^n$ there exists a $k_{\mathcal{C}} > 0$ such that for any two vectors $u, v \in \mathcal{C}$ then $h(\cdot)$ satisfies that

$$|h(u) - h(v)| \leq k_{\mathcal{C}} |u - v|$$

What do you call this property for the system and what can you say about the solutions if this property is satisfied?

Answer: (2 pts) The system is then said to be Locally Lipshcitz, and this implies *existence* and *uniqueness* of solutions, i.e., for each $x_0 = x(0)$ there exists $T > 0$ and a unique solution $x(t, x_0)$ on $[0, T]$.

(b) Suppose there exists a constant $k > 0$ such that $h(\cdot)$ satisfies for all $u, v \in \mathbb{R}^n$

$$|h(u) - h(v)| \leq k |u - v|$$

What do you call this property for the system and what can you say about the solutions if this property is satisfied?

Answer: (2 pts) The system is then said to be Globally Lipshcitz, and this ensures *existence*, *uniqueness*, and *forward completeness* of solutions, i.e., for each $x_0 \in \mathbb{R}^n$ there exists a unique solution $x(t, x_0)$ for all $t \geq 0$.

(c) Suppose $h(x) = Hx$ where $H \in \mathbb{R}^{n \times n}$ is a constant matrix.

- Show which of the above conditions the system now satisfies, and what this means for the properties of the solutions.

  Answer: (2 pts) For $h(x) = Hx$ we get

$$|Hu - Hv| \leq \|H\| \, |u - v| = k \, |u - v| \, , \qquad \forall u, v \in \mathbb{R}^n,$$

  and hence $h(x)$ is Globally Lipschitz, ensuring *existence*, *uniqueness*, and *forward completeness* of solutions of linear systems.

- Suppose $n = 3$ and the eigenvalues of H is $\{-1, 0, 1\}$. What type of equilibrium points does the system have?

  Answer: (2 pts) Note that a linear system cannot have multiple isolated equilibria. Since H has a zero eigenvalue, it will be singular. Hence, any state vector in the nullspace of H will be an equilibrium point, corresponding to a continuum of equilibrium points. In particular, the eigenvector corresponding to the zero eigenvalue is a basis for the nullspace. Denoting this v_0, then $Hv_0 = 0$, and all state vectors $x = cv_0$, $c \in \mathbb{R}$, will be an eqilibrium point.

2. For the scalar ODE

$$\dot{x} = -x^3$$

we propose the solution for $t \geq 0$ and initial condition x_0:

$$x(t, x_0) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}}$$

(a) Show that the proposed solution is indeed a solution to the ODE.

Answer: (2 pts) Initial condition $x(0, x_0) = \frac{x_0}{\sqrt{1}} = x_0$, and differentiating $x(t, x_0)$ gives

$$\begin{aligned}
\dot{x}(t, x_0) &= -\frac{1}{2} x_0 \left(1 + 2x_0^2 t\right)^{-\frac{3}{2}} \cdot 2x_0^2 = -x_0^3 \left(1 + 2x_0^2 t\right)^{-\frac{3}{2}} \\
&= -\left(\frac{x_0}{\sqrt{1 + 2x_0^2 t}}\right)^3 = -x(t, x_0)^3 \qquad \text{Q.E.D.}
\end{aligned}$$

(b) Explain the Lipschitz property of this ODE.

Answer: (2 pts) The vector field $f(x) = -x^3$ is locally Lipschitz since $f(x)$ is continuously differentiable and for any compact interval $\mathcal{C} \in \mathbb{R}$ we can take $k_{\mathcal{C}} = \sup_{x \in \mathcal{C}} \left| \frac{\partial f}{\partial x} \right|$ such that for $\forall u, v \in \mathcal{C}$ we get $|f(u) - f(v)| \leq k_{\mathcal{C}} |u - v|$. However, $f(x)$ is not Globally Lipschitz since $\sup_{x \in \mathbb{R}} \left| \frac{\partial f}{\partial x} \right| = \infty$, that is, for any given $L > 0$ there always exists $c \in \mathbb{R}$ such that $\left| \frac{\partial f}{\partial x} \right|_{x=c} = 3c^2 > L$, and, thus, there does not exist a global Lipschitz constant L.

(c) Discuss the properties of the solutions to this ODE and the stability of $x = 0$ in terms of class $\mathcal{K}$ and $\mathcal{L}$ functions.

Answer: (3 pts) The Locally Lipschitz condition guarantee existence and uniqueness of solution. In addition, for any intial condition x_0 we find that if $x_0 < 0$ then $\dot{x} > 0$ such that $x(t)$ increases, if $x_0 > 0$ then $\dot{x} < 0$ such that $x(t)$ decreases, and $x_0 = 0$ then $\dot{x} = 0$. Hence, any solution cannot escape the set $\{x \in: \mathbb{R} \quad |x| \leq x_0\}$, and the solution must be forward complete. From the solution we also see this directly since it satisfies the class-$\mathcal{KL}$ bound

$$|x(t, x_0)| \leq \beta\left(|x_0|, t\right) := \frac{|x_0|}{\sqrt{1 + 2|x_0|^2 t}} = \frac{1}{\sqrt{\left(\frac{1}{|x_0|^2} + 2t\right)}},$$

where $\beta(r, s)$ is a class-$\mathcal{K}$ function in $r = |x_0|$ and a class-$\mathcal{L}$ function in $s = t$.

(d) Propose a Lyapunov function candidate for this system, and discuss stability of $x = 0$ in sense of Lyapunov.

Answer: (2 pts) Using $V(x) = x^2$ gives $\dot{V} = -2x^4 < 0$, $\forall x \neq 0$, and hence $x = 0$ is UGAS by Lyapunov's direct method using e.g. $\alpha_1(|x|) = \frac{1}{2}|x|^2$, $\alpha_2(|x|) = 2|x|^2$, and $\alpha_3(|x|) = 2|x|^4$.

3. Consider the scalar time-varying system

$$\begin{aligned}\dot{x}_1 &= -\phi(t)x_1 + x_2^5 \\ \dot{x}_2 &= -x_1^3 - \phi(t)x_2\end{aligned}$$

for $t \geq t_0 \geq 0$.

(a) Let $V(x_1, x_2) = x_1^4 + \frac{2}{3}x_2^6$ be a Lyapunov candidate. Differentiate this along the solutions of the system.

Answer: (2 pts) Differentiating gives

$$\begin{aligned}\dot{V} &= 4x_1^3\dot{x}_1 + \frac{2}{3}6x_2^5\dot{x}_2 = 4x_1^3\left(-\phi(t)x_1 + x_2^5\right) + 4x_2^5\left(-x_1^3 - \phi(t)x_2\right) \\ &= -4\phi(t)x_1^4 + 4x_1^3x_2^5 - 4x_2^5x_1^3 - 4\phi(t)x_2^6 \\ &= -4\phi(t)\left(x_1^4 + x_2^6\right)\end{aligned}$$

(b) Suppose $\phi(t) := e^{-t}$. Discuss Uniform Global Asymptotic Stability of the equilibrium $x = 0$.

Answer: (3 pts) For $\forall t \geq 0$ we have $\phi(t) > 0$, and hence $x = 0$ is UGS. However, we observe that $\phi(t) \to 0$ rapidly, and in the limit $\phi(t) = 0$ the system becomes an oscillator. This destroys uniform attractivity, i.e., for each $\varepsilon > 0$ and $r > 0$ there does not exist a $T \geq t_0$ that ensures that $|x(t)| \leq \varepsilon$, $\forall t \geq T$ and $|x_0| \leq r$. To better understand this, fix $|x_0| \leq r = 1$. Then for any $\varepsilon \ll 1$ you specify there always exist a future start time t_0, large enough, such that $\phi(t)$ is too small to give damping and make the solutions converge. In fact, since $\phi(t)$ is exponentially converging, the solutions will never converge - but end up in a permanent oscillation.

In the definition of a Lyapunov function you will correspondingly find that it is impossible to find a positive definite $\alpha_3(|x|)$, not depending on t or t_0, that is valid for $\forall t \geq 0$.

(c) Suppose $\phi(t) := 10^{-3} + e^{-t}$. Discuss Uniform Global Asymptotic Stability of the equilibrium $x = 0$.

Answer: (2 pts) In this case we get

$$\begin{aligned}\dot{V} &= -4\left(10^{-3} + e^{-t}\right)\left(x_1^4 + x_2^6\right) \\ &\leq -4 \cdot 10^{-3}\left(x_1^4 + x_2^6\right) =: -\alpha_3(|x|) < 0, \quad \forall x \neq 0\end{aligned}$$

and hence, the $x = 0$ is UGAS.

4. Consider the linear time-varying system:

$$\dot{x} = A\left(x - x_d(t)\right) + \dot{x}_d(t)$$

where A is Hurwitz, and $(x_d(t), \dot{x}_d(t))$ are bounded reference signals. Suppose the triple (P, A, Q) satisfies the Lyapunov equation, and let

$$V(t, x) = \frac{1}{2}\left(x - x_d(t)\right)^\top P\left(x - x_d(t)\right)$$

be a Lyapunov function candidate.

(a) What does it mean that (P, A, Q) satisfies the Lyapunov equation?

Answer: (2 pts) If A is Hurwitz, then for each $Q = Q^\top > 0$ there exists $P = P^\top > 0$ such that

$$PA + A^\top P = -Q$$

(b) Show how to differentiate $V(t, x)$ and make a stability conclusion from this.

Answer: (4 pts) We get

$$\begin{aligned}
\dot{V} &= \frac{1}{2}\left(\dot{x} - \dot{x}_d(t)\right)^\top P\left(x - x_d(t)\right) + \frac{1}{2}\left(x - x_d(t)\right)^\top P\left(\dot{x} - \dot{x}_d(t)\right) \\
&= \frac{1}{2}\left(A\left(x - x_d(t)\right)\right)^\top P\left(x - x_d(t)\right) + \frac{1}{2}\left(x - x_d(t)\right)^\top P\left(A\left(x - x_d(t)\right)\right) \\
&= \frac{1}{2}\left(x - x_d(t)\right)^\top \left[A^\top P + PA\right]\left(x - x_d(t)\right) \\
&= -\frac{1}{2}\left(x - x_d(t)\right)^\top Q\left(x - x_d(t)\right) \\
&\leq -\frac{1}{2}\lambda_{\min}(Q)\left|x - x_d(t)\right|^2
\end{aligned}$$

The Lyapunov function $V(t, x)$ is positive definite and radially unbounded in the error $e = x - x_d(t)$, and $\dot{V}$ is negative definite in e. Hence, we get for the error system $\dot{e} = Ae$ that the equilibrium point $e = 0$ is UGES from Lyapunov's direct method.

2 Observer design (30 pts)

Consider the pendulum equations with unity mass, length, and friction coefficients $m = l = k = 1$:

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -10 \sin\theta - \omega + u\cos\theta \\ y &= \theta \end{aligned}$$

where θ is the angle from vertical hanging condition, ω is the angular rate, and u is a control torque.

1. Let $x = col(\theta, \omega)$, and assume small angular deviations $\theta \approx 0$. Write down the corresponding linearized model for the pendulum

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Answer: (2 pts) For small $\theta \approx 0$, we get $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ such that

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) Show that A is a Hurwitz matrix and that the pair (A, C) is completely observable.

Answer: (2 pts) The characteristic polynomial for A is given by

$$\det(sI - A) = \det \begin{bmatrix} s & -1 \\ 10 & s+1 \end{bmatrix} = s^2 + s + 10 = 0.$$

The real part of the roots of this polynominal is -0.5, strictly negative, and hence A is Hurwitz.
To show complete observability we check the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is full rank. Hence, (A, C) is completely observable.

(b) What is the rank of the infinite time observability Gramian Q_∞ for this system? Is there an easy way to calculate the matrix Q_∞?

Answer: (2 pts) (A, C) being completely observable is equivalent to $rank\,(Q_\infty) = n = 2$ when A is Hurwitz. We can in this case calculate Q_∞ from the Lyapunov equation

$$A^\top Q_\infty + Q_\infty A = -C^\top C.$$

(c) Letting $\hat{x}$ be the estimate of x, design a *Luenberger observer* for the linearized system.

Answer: (3 pts) A Luenberger observer is given by

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L\left(y - \hat{y}\right) = \left(A - LC\right)\hat{x} + Bu + Ly \\
\hat{y} &= C\hat{x}
\end{aligned}
$$

where $L \in \mathbb{R}^2$ is an injection gain matrix designed such that $A - LC$ is Hurwitz (e.g. by the `place.m` command in Matlab).

(d) What stability do you get for $\tilde{x} = x - \hat{x} = 0$ for the linearized error dynamics?

Answer: (2 pts) We get

$$
\begin{aligned}
\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - \left(A\hat{x} + Bu + L\left(y - \hat{y}\right)\right) \\
&= Ax - A\hat{x} - L\left(Cx - C\hat{x}\right) \\
&= \left(A - LC\right)\tilde{x}
\end{aligned}
$$

Since $A - LC$ is Hurwitz, $\tilde{x} = 0$ is UGES.

(e) What stability can you claim for $\tilde{x} = 0$ for the real nonlinear error dynamics?

Answer: (2 pts) Since the linearization is an approximation of the real system locally around $x = 0$, the stability is locally exponentially stable (LES) in a small neighborhood of the set $\left\{(x, \tilde{x}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \quad x = 0,\ \tilde{x} = 0\right\}$.

2. For the linearized pendulum model above, propose a state-feedback control that renders $x = 0$ UGES. How would you select your feedback gains?

Answer: (3 pts) We check that (A, B) is a controllable pair, by the controllability matrix

$$
C = [B, AB] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},
$$

which is full rank. A typical state feedback control is then given by

$$
u = -Kx
$$

where $K \in \mathbb{R}^{1\times 2}$ is a feedback gain matrix designed such that $A - BK$ is Hurwitz. I would select K e.g. by pole placement using the `place.m` command in Matlab, or by the LQR technique.

3. Show that the *separation principle* holds for the linear closed-loop system with feedback taken from the estimated states by the Luenberger observer.

Answer: (3 pts) We then use $u = -K\hat{x}$ such that the closed-loop system becomes

$$\begin{aligned}\dot{\tilde{x}} &= (A - LC)\,\tilde{x} \\ \dot{x} &= Ax - BK\hat{x} + BKx - BKx = (A - BK)\,x + BK\tilde{x}\end{aligned}$$

Letting $\chi = col(\tilde{x}, x) \in \mathbb{R}^4$ we get the total system

$$\dot{\chi} = \begin{bmatrix} A - LC & 0 \\ BK & A - BK \end{bmatrix} \chi.$$

The eigenvalues of this system is given individually by the block-diagonal elements $A-LC$ and $A-BK$, which shows that the observer poles and the state-feedback poles can be tuned individually without affecting the stability of the overall system. This is the separation principle.

4. Aiming for a global observer, show now that the nonlinear system can be written on the form

$$\begin{aligned}\dot{x} &= Hx + \phi(x) + \psi(u, y) \\ y &= Cx\end{aligned}$$

Answer: (2 pts) We get

$$H = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad \phi(x) = \begin{bmatrix} 0 \\ -10\sin x_1 \end{bmatrix}, \quad \psi(u, y) = \begin{bmatrix} 0 \\ u\cos y \end{bmatrix}$$

and note that (H, C) is observable.

(a) Show that the nonlinearity $\phi(x)$ is Globally Lipschitz.

Answer: (2 pts) We get

$$\begin{aligned}|\phi(x) - \phi(z)|_2 &= 10\,|(-\sin x_1) - (-\sin z_1)| = 10\,|\sin x_1 - \sin z_1| \\ &\leq 10\,|(x_1 - z_1)| \leq 10\,|x - z|_2\end{aligned}$$

(b) Propose a nonlinear observer, with linear injection term, for the nonlinear system, that takes advantage of the global Lipschitz property of $\phi(x)$.

Answer: (3 pts) Designing $L \in \mathbb{R}^2$ such that $H - LC$ is Hurwitz, we propose

$$\begin{aligned}\dot{\hat{x}} &= H\hat{x} + \phi(\hat{x}) + \psi(u, y) + L\,(y - \hat{y}) \\ &= (H - LC)\,\hat{x} + \phi(\hat{x}) + \psi(u, y) + Ly\end{aligned}$$

(c) Use a Lyapunov argument to find a parameter bound related to the global Lipschitz constant for $\phi(x)$ to ensure UGES of $\tilde{x} = 0$.

Answer: (4 pts) The error dynamics become

$$\dot{\tilde{x}} = (H - LC)\,\tilde{x} + \phi(x) - \phi(\hat{x})$$

Let $P = P^\top > 0$ satisfy the Lyapunov equation

$$P\,(H - LC) + (H - LC)^\top P = -I,$$

where we note that L is the degree of freedom to find a suitable P. Then using the Lyapunov function $V(\tilde{x}) = \tilde{x}^\top P \tilde{x}$ gives

$$\begin{aligned}
\dot{V} &= 2\tilde{x}^\top P\,(H - LC)\,\tilde{x} + 2\tilde{x}^\top P\,[\phi(x) - \phi(\hat{x})] \\
&= \tilde{x}^\top \left[P\,(H - LC) + (H - LC)^\top P\right] \tilde{x} + 2\tilde{x}^\top P\,[\phi(x) - \phi(\hat{x})] \\
&\leq -|\tilde{x}|^2 + 2\lambda_{\max}(P)\,|\tilde{x}|\,|\phi(x) - \phi(\hat{x})| \\
&\leq -|\tilde{x}|^2 + 20\lambda_{\max}(P)\,|\tilde{x}|^2 \leq -(1 - 20\lambda_{\max}(P))\,|\tilde{x}|^2
\end{aligned}$$

Hence, for $\tilde{x} = 0$ to be proven UGES by this Lyapunov function, we must find P such that

$$\lambda_{\max}(P) < \frac{1}{20}.$$

3 Control design (40 pts)

Consider the Nomoto steering model of a ship

$$\begin{aligned}\dot{\psi} &= r \\ \dot{r} &= -\frac{1}{\tau}r + \frac{\kappa}{\tau}(\delta + b) \\ y &= \psi + \psi_w\end{aligned}$$

where ψ is the ship yaw, r is the yaw rate, δ is the rudder angle, and (τ, κ) are model parameters. In the tasks below, the control objective is stabilization of $(\psi, r) = (\psi_{ref}, 0)$, where ψ_{ref} is constant.

1. The disturbances of the model are b, a slowly-varying rudder bias, and ψ_w, an oscillatory motion due to waves. What category of disturbances do we name these?

   Answer: (2 pts) The bias is an *input disturbance*, and the wave-oscillation is an *output disturbance*.

2. Assume $\psi_w = b = 0$:

   (a) Use *LgV backstepping* to design a state feedback control law for δ that solves the regulation control objective.

   Answer: (6 pts) We note that the system is in strict feedback form. An LgV backstepping design goes as follows:
   Step 1:
   We let $z_1 = \psi - \psi_{ref}$ and $z_2 = r - \alpha_1(\psi)$, and differentiate z_1 and $V_1 = \frac{1}{2}z_1^2$ to get

$$\begin{aligned}\dot{z}_1 &= z_2 + \alpha_1 \\ \dot{V}_1 &= z_1\dot{z}_1 = z_1 z_2 + z_1\alpha_1\end{aligned}$$

   We choose

$$\alpha_1 = -c_1 z_1 + \alpha_{10}$$

   where α_{10} is yet to be designed. This gives

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 + z_1\alpha_{10} \leq -c_1 z_1^2 + \kappa_1 z_1^2 + \frac{1}{4\kappa_1}z_2^2 + z_1\alpha_{10}$$

   Selecting $\alpha_{10} = -\kappa_1 z_1$ gives

$$\dot{V}_1 \leq -c_1 z_1^2 + \frac{1}{4\kappa_1}z_2^2$$

   which renders the z_1-subsystem ISS from z_2 to z_1.
   Step 2:
   Differentiating z_2 and $V_2 = V_1 + \frac{1}{2}z_2^2$ gives

$$\begin{aligned}\dot{z}_2 &= \dot{r} - \dot{\alpha}_1 = -\frac{1}{\tau}r + \frac{\kappa}{\tau}\delta - \dot{\alpha}_1 \\ \dot{V}_2 &\leq -c_1 z_1^2 + \frac{1}{4\kappa_1}z_2^2 + z_2\dot{z}_2 = -c_1 z_1^2 + \frac{1}{4\kappa_1}z_2^2 + z_2\left[-\frac{1}{\tau}r + \frac{\kappa}{\tau}\delta - \dot{\alpha}_1\right]\end{aligned}$$

   where $\dot{\alpha}_1 = -(c_1 + \kappa_1)\, r$. We choose the control law

$$\delta = \frac{\tau}{\kappa}\left[-c_2 z_2 + \frac{1}{\tau}r + \dot{\alpha}_1\right]$$

   which gives

$$\dot{V}_2 \leq -c_1 z_1^2 - \left(c_2 - \frac{1}{4\kappa_1}\right)z_2^2.$$

   Correspondingly, the origin $(z_1, z_2) = (0, 0)$ of the closed-loop system

$$\begin{aligned}\dot{z}_1 &= -(c_1 + \kappa_1)\, z_1 + z_2 \\ \dot{z}_2 &= -c_2 z_2\end{aligned}$$

   is UGES.

(b) Suppose b is an unknown constant bias, let $\hat{b}$ be an estimate of b, and define the adaptation error $\tilde{b} = b - \hat{b}$. Redo Step 2 of the LgV backstepping design, by designing an adaptive update law for $\dot{\hat{b}}$ that renders $(z_1, z_2, \tilde{b}) = 0$ UGS and ensures the convergence $(\psi(t), r(t)) \to (\psi_{ref}, 0)$.

Answer: (5 pts) Redoing Step 2 for b unknown, we get

$$\dot{z}_2 = \dot{r} - \dot{\alpha}_1 = -\frac{1}{\tau} r + \frac{\kappa}{\tau}\delta + \frac{\kappa}{\tau} b - \dot{\alpha}_1$$

and augmenting with the adaptive term, we define $V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma}\tilde{b}^2$ and differentiate

$$\begin{aligned} \dot{V}_2 &\le -c_1 z_1^2 + \frac{1}{4\kappa_1} z_2^2 + z_2 \left[-\frac{1}{\tau} r + \frac{\kappa}{\tau}\delta + \frac{\kappa}{\tau} b - \dot{\alpha}_1 \right] + \frac{1}{\gamma} \tilde{b}\dot{\tilde{b}} \\ &= -c_1 z_1^2 + \frac{1}{4\kappa_1} z_2^2 + z_2 \left[-\frac{1}{\tau} r + \frac{\kappa}{\tau}\delta + \frac{\kappa}{\tau} \hat{b} - \dot{\alpha}_1 \right] + \tilde{b} \left(\frac{\kappa}{\tau} z_2 - \frac{1}{\gamma} \dot{\hat{b}} \right) \end{aligned}$$

Choosing then the adaptive control law

$$\begin{aligned} \dot{\hat{b}} &= \gamma \frac{\kappa}{\tau} z_2 \\ \delta &= \frac{\tau}{\kappa} \left[-c_2 z_2 + \frac{1}{\tau} r + \dot{\alpha}_1 \right] - \hat{b} \end{aligned}$$

gives

$$\begin{aligned} \dot{V}_2 &\le -c_1 z_1^2 - \left(c_2 - \frac{1}{4\kappa_1} \right) z_2^2 \le 0 \\ \dot{z}_1 &= -\left(c_1 + \kappa_1 \right) z_1 + z_2 \\ \dot{z}_2 &= -c_2 z_2 + \frac{\kappa}{\tau} \tilde{b} \\ \dot{\tilde{b}} &= -\gamma \frac{\kappa}{\tau} z_2 \end{aligned}$$

According to the LaSalle-Yoshizawa theorem, the origin $(z_1, z_2, \tilde{b}) = 0$ is UGS and $\lim_{t\to\infty} (z_1(t), z_2(t)) = 0$. Since

$$\begin{aligned} \psi(t) &= z_1(t) + \psi_{ref} \\ r(t) &= z_2(t) - \left(c_1 + \kappa_1 \right) z_1(t) \end{aligned}$$

we also get that $\lim_{t\to\infty} (\psi(t), r(t)) = (\psi_{ref}, 0)$.

Can you show that $(z_1, z_2, \tilde{b}) = 0$ is in fact UGAS?

3. Provide a vectorial state-space model (A, B, C, D) for the Nomoto plant with $x = col(\psi, r)$:

Answer: (2 pts) The state-space model is given by

$$\begin{aligned} \dot{x} &= Ax + B(\delta + b), \qquad y = Cx + \psi_w \\ A &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{\kappa}{\tau} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0 \end{aligned}$$

(a) Show that the pair (A, B) is controllable.

Answer: (2 pts) We check the rank of the controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{\kappa}{\tau} \\ \frac{\kappa}{\tau} & -\frac{\kappa}{\tau^2} \end{bmatrix},$$

which is full rank for $(\tau, \kappa) \neq 0$.

(b) Assume $\psi_w = b = 0$. Let a linear control be

$$\delta = -Kx + Ly_{ref}$$

and give conditions on the feedback gain K and feedforward gain L such that in steady-state we achieve stabilization of $(\psi - \psi_{ref}, r) = 0$.

Answer: (4 pts) Using $K = row(k_1, k_2)$ and $L \in \mathbb{R}$, we get the closed-loop and steady-state solution

$$\begin{aligned} \dot{x} &= (A - BK)\,x + BLy_{ref} \\ y_{ss} &= Cx_{ss} = -C\,(A - BK)^{-1}\,BLy_{ref} \\ &= -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{\kappa}{\tau}k_1 & -\frac{1}{\tau} - \frac{\kappa}{\tau}k_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{\kappa}{\tau} \end{bmatrix} Ly_{ref} \\ &= -\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -\frac{(\kappa k_2 + 1)}{\kappa k_1} & -\frac{\tau}{\kappa k_1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{\kappa}{\tau} \end{bmatrix} Ly_{ref} \\ &= \frac{1}{k_1} Ly_{ref} \end{aligned}$$

This was the general way of calculating L. An easier way to calculate it for this particular plant is seen by

$$\begin{aligned} \dot{x}_{ss} &= 0 \implies r_{ss} = 0 \implies Ax_{ss} = 0 \implies -BKx_{ss} + BLy_{ref} = 0 \\ \implies & \frac{\kappa}{\tau}k_1\psi_{ss} = \frac{\kappa}{\tau}Ly_{ref} \implies k_1\psi_{ss} = Ly_{ref} \implies L = k_1 \end{aligned}$$

Yet another way can be done by noticing that

$$\begin{aligned} x_{ss} &= C^\top y_{ss} = \begin{bmatrix} \psi_{ss} \\ 0 \end{bmatrix} \\ &\implies (A - BK)\,x_{ss} + BLy_{ref} = -BKx_{ss} + BLy_{ref} \\ &\implies BKC^\top y_{ss} + BLy_{ref} \implies L = KC^\top = k_1 \end{aligned}$$

In conclusion, K is chosen such that $A - BK$ is Hurwitz, and L is chosen such that the output regulation gain $y_{ref} \to y_{ss}$: $-C\,(A - BK)^{-1}\,BL = 1$, that is, $L = k_1$.

(c) Assume $\psi_w = b = 0$. Let a PID control law (with reference feedforward) be

$$\begin{aligned} \dot{\xi} &= y - y_{ref} \\ \delta &= -Kx - K_i\xi + Ly_{ref} \end{aligned}$$

Write down the closed-loop system, and state design conditions on K and K_i such that the closed-loop is stable.

Answer: (3 pts) We get

$$\begin{bmatrix} \dot{\xi} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & C \\ -BK_i & A - BK \end{bmatrix} \begin{bmatrix} \xi \\ x \end{bmatrix} + \begin{bmatrix} -1 \\ BL \end{bmatrix} y_{ref}$$

and, hence, $K \in \mathbb{R}^{1\times 2}$ and $K_i \in \mathbb{R}$ must be designed to render the matrix $\begin{bmatrix} 0 & C \\ -BK_i & A - BK \end{bmatrix}$ Hurwitz.

(d) Suppose $\psi_w = 0$, but b is constant nonzero. What is the steady-state solution for (ξ, ψ, r)? Is the regulation control objective met?

Answer: (4 pts) For $b \neq 0$ we get

$$\begin{aligned} \dot{\xi} &= y - y_{ref} \\ \dot{x} &= (A - BK)\,x - B\,(K_i\xi - b) + BLy_{ref} \end{aligned}$$

Using $-C\,(A - BK)^{-1}\,BL = 1$, the steady-state solution is

$$\begin{aligned} y_{ss} &= y_{ref} \\ x_{ss} &= (A - BK)^{-1}\,B\,(K_i\xi_{ss} - b) - (A - BK)^{-1}\,BLy_{ref} \\ y_{ss} &= Cx_{ss} = C\,(A - BK)^{-1}\,B\,(K_i\xi_{ss} - b) + y_{ref} = y_{ref} \\ &\Downarrow \\ 0 &= C\,(A - BK)^{-1}\,B\,(K_i\xi_{ss} - b) = -\frac{K_i\xi_{ss} - b}{k_1} \\ &\Downarrow \\ \xi_{ss} &= \frac{1}{K_i}b \end{aligned}$$

We see that the integral action forces y to regulate to y_{ref} by the compensation $\xi(t) \to \xi_{ss} = \frac{1}{K_i}b$.

4. To compensate the wave motion ψ_w we can model this as a damped harmonic oscillator, and the slowly-varying bias b can be modeld as a Markov process:

   (a) Using the *internal model principle*, write up a model for the total system on state-space form.

   Answer: (3 pts) The overall model becomes

$$
\begin{aligned}
\dot{\xi}_w &= \psi_w \\
\dot{\psi}_w &= -\omega_0^2 \xi_w - 2\lambda\omega_0\psi_w \\
\dot{b} &= -\frac{1}{\tau_b} b \\
\dot{x} &= Ax + B(\delta + b) \\
y &= \psi + \psi_w \qquad \text{Measured output} \\
z &= \psi, \qquad\quad\;\; \text{Controlled output}
\end{aligned}
$$

   where the bias is a Markov process with time constant τ_b, and ψ_w is generated by a 2nd order damped harmonic oscillator with ω_0 the fundamental wave frequency and λ a damping coefficient.

   (b) Propose an observer for estimating the system state and the disturbances.

   Answer: (3 pts) Once again we find that the system is LTI. Hence, we can for instance apply the Kalman-Bucy filter. Letting $\chi = col(\xi_w, \psi_w, b, x) \in \mathbb{R}^5$ such that the system can be written

$$\dot{\chi} = \mathcal{A}\chi + \mathcal{B}\delta, \qquad y = \mathcal{C}\chi,$$

   then we can use the filter

$$\dot{\hat{\chi}} = (\mathcal{A} - L\mathcal{C})\,\hat{\chi} + \mathcal{B}\delta + Ly$$

   where L is the injection gain calculated by

$$L = P\mathcal{C}^\top R^{-1}.$$

   Here, P is the solution to the algebraic Riccati equation

$$P\mathcal{A}^\top + \mathcal{A}P - P\mathcal{C}^\top R^{-1}\mathcal{C}P + Q = 0,$$

   where R is the covariance matrix of the measurement noise, and Q the covariance matrix for process noise (not modeled here).

(c) Design a linear control law, based on the estimated states, that solves the regulation objective while rejecting the influence of b and ψ_w.

Answer: (3 pts) With the optimal state estimate $\hat{\chi}$ available, we can use the feedback control

$$\delta = -K\hat{x} - \hat{b} + Lz_{ref}$$

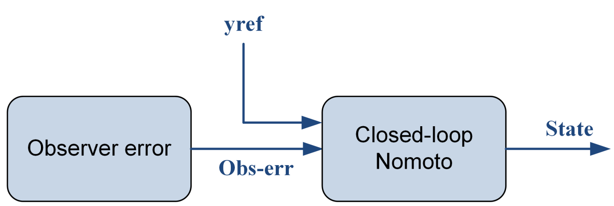
where $z_{ref} = \psi_{ref}$. Note that $(\hat{\xi}_w, \hat{\psi}_w)$ is not included in the control law. Hence, this directly compensates the estimated bias $\hat{b}$, while the influence of ψ_w is filtered from the low-frequency estimate $\hat{x}$.

(d) Write down the overall closed-loop state-space system on vectorial form, and draw the block diagram of the system as a cascade of two subsystems. Discuss stability of the subsystems and if the separation principle is satisfied.

Answer: (3 pts) The overall closed-loop system becomes

$$\begin{aligned} \dot{\tilde{\chi}} &= (\mathcal{A}-L\mathcal{C})\,\tilde{\chi} \\ \dot{x} &= (A - BK)\,x + BK\tilde{x} + B\tilde{b} + BLz_{ref} \end{aligned}$$

This is a cascade between two exponentially stable subsystems, as shown in the following figure. Clearly, the separation principle



holds, since we can tune these two subsystems individually while maintaining overall stability.