TMR4243 - Marine Control Systems II

Exam

Spring 2018

Notation: Throughout this exam $|x|$ means the vector 2-norm, i.e. $|x| = \sqrt{x^\top x}$. For a scalar x, this corresponds to the absolute value.

States and variables are scalars unless these are specifically defined as vectors, e.g., x_1 is a scalar while $x_2 \in \mathbb{R}^n$ is an n-dimensional vector.

1 Properties and Solution of nonlinear ODEs (21 pts)

1. Consider the nonlinear ordinary differential equation (ODE):

$$\dot{x} = f(x), \qquad x_0 = x(0)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$

(a) Explain what mathematical property this ODE satisfies if it is *Locally Lipschitz.*

(b) What can you say about the solutions if this property is satisfied?

(c) Suppose $f(x) = -c \arctan(\epsilon x)$ where $x \in \mathbb{R}^1$ and c, ϵ are positive constants.

i. Show that this is Globally Lipschitz.

ii. Explain what this means for the properties of the solutions.

2. For each of the three systems

$$\Sigma_1 \quad : \quad \boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\arctan x_1 - 2x_2 \end{aligned}}$$

$$\Sigma_2 \quad : \quad \boxed{\begin{aligned} \dot{x}_1 &= -x_1 + x_2 x_1 \\ \dot{x}_2 &= -x_2 + x_3 \\ \dot{x}_3 &= 1 - x_2 \end{aligned}}$$

$$\Sigma_3 \quad : \quad \boxed{\begin{aligned} \dot{x}_1 &= 3\cos x_2 \\ \dot{x}_2 &= -\cos x_2 \end{aligned}}$$

Find the equilibria of these systems, and explain what type of equilibria these are.

3. The scalar ODE

$$\dot{x} = \sqrt{x}, \qquad x_0 = 0$$

admits the solutions

$$x(t) = 0 \text{ and } x(t) = \frac{1}{4}t^2, \qquad t \geq 0$$

(a) Discuss the Lipschitz properties of this ODE and the corresponding existence, uniqueness, and forward completeness of its solutions.

(b) Show that these are indeed solutions to this ODE.

(c) Do the solutions experience any finite escape time?

2 Stability of Nonlinear Systems (33 pts)

1. Consider a general mass-, damper- and spring-system

$$M\ddot{y} + D\dot{y} + Ky = 0$$

where $y \in \mathbb{R}^n$ and $M,\ D,\ K$ are all symmetric positive definite matrices. Let the vector $x = [y \quad \dot{y}]^T$ be the state vector ($x \in \mathbb{R}^{2n}$)

   (a) Find all equilibrium points.

   (b) Show that the system is globally asymptotically stable at $x = 0$. (Hint: The Kinetic energy of the system is $\frac{1}{2}\dot{y}^T M\dot{y}$ and the stored potential energy of the system can be written as $\frac{1}{2}y^T Ky$.)

2. Consider a double integrator with nonlinear output feedback.

$$\begin{aligned} \ddot{y} &= u \\ u &= -k(y). \end{aligned}$$

   (a) check the stability of the origin assuming that $k(0) = 0$ and $\frac{\partial k}{\partial y}(0) > 0$.

   (Hint: Consider the function $V(\dot{y}, y) = \frac{\dot{y}^2}{2} + \int_0^y k(r)dr$.)

3. Let a system be

$$\begin{aligned}\dot{x}_1 &= 4x_2\\ \dot{x}_2 &= -2\arctan(x_1) - \frac{x_2}{10}\end{aligned}$$

Let a Lyapunov function candidate be

$$V(x_1, x_2) = \int_0^{x_1} \arctan(y)dy + x_2^2$$

(a) Draw $\arctan(y)$ and show that it is a monotonically increasing function.

(b) Show that $\arctan(y)$ is globally Lipschitz, and that V is radially unbounded.

(c) Differentiate V and conclude what stability you get for the origin by Lyapunov's direct method?

(d) Show that the origin is UGAS.

3 Lyapunov-based Nonliner Control design (15 pts)

1. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2^3 \\ \dot{x}_2 &= u. \end{aligned}$$

   Our goal is to find a globally asymptotically stabilizing control law $u = u(x)$. To this end, answer the following sub-questions:

   (a) consider a Lyapunov function candidate $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$. Show that it satisfies the properties of a candidate Lyapunov function for proving globally stability.

   (b) By compute the derivative of V along the solution of the systems, suggest a control action in form of $u = u(x)$. (Hint: In your computation, treat u as function of x and suggest a control action that leads to $V \leq 0$.)

   (c) Using the suggested controller, prove that the closed loop system is Globally Astatically Stable.

2. For the question above, try the Lyapunov function candidate $V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$, and investigate what will go wrong if we repeat the same process above.

4 Low-speed observer/control design (11 pts)

Consider the 3DOF stationkeeping model

$$\begin{aligned} \dot{\eta} &= R(\psi)\nu \\ M\dot{\nu} &= \tau - D\nu + m(t) \\ z &= \eta \end{aligned}$$

where $\eta = col(x, y, \psi)$, $\nu = col(u, v, r)$, $M^\top > 0$ and $D > 0$ are the mass and damping matrices, respectively, $R(\psi)$ is the rotation matrix, τ is the thrust control input, $m(t)$ is a measured external load, and z is the measured output.

Suppose D satisfies

$$D + D^\top > 0.$$

1. Assume that $R = I$ (constant identity matrix), and investigate if the "linearized system" is *uniformly completely observable.*

2. Assuming $\psi = \psi(t)$ is an accurate gyrocompass measurement time signal and $R(t) := R(\psi(t))$, let an observer be

$$\begin{aligned} \dot{\hat{\eta}} &= R(t)\hat{\nu} + L_1 (z - \hat{\eta}) \\ M\dot{\hat{\nu}} &= \tau - D\hat{\nu} + m(t) + R(t)^\top L_2 (z - \hat{\eta}), \end{aligned}$$

   where $L_1 = L_1^\top > 0$ and $L_2 = L_2^\top > 0$ are injection gain matrices.

   (a) Write down the equations for the corresponding observer error dynamics $\tilde{\eta} := \eta - \hat{\eta}$ and $\tilde{\nu} := \nu - \hat{\nu}$.

   (b) Let a Lyapunov function candidate be

   $$V_o = \tilde{\eta}^\top L_2 \tilde{\eta} + \tilde{\nu}^\top M \tilde{\nu},$$

   and find the time derivative of V_o as a function of the error states.

   (c) Derive and give conditions on the injection gain matrices L_1 and L_2 that ensures that the error dynamics is UGES.

5 Backstepping (20 pts)

Consider the system

$$\begin{aligned}\dot{x}_1 &= g(x_1)x_2 - |x_1|x_1 \\ \dot{x}_2 &= u - be^{-x^2}\end{aligned}$$

where $0.5 \leq g(x_1) \leq 2$ for all $x_1 \in \mathbb{R}$, $u \in \mathbb{R}$ is control input, and b is a contact coefficient.

1. Suppose $b = 5$. Let the control objective be to stabilize $(x_1, x_2) = (0, 0)$. Use backstepping to design a feedback control law for u that solves the regulation control objective.

2. Write the equation of the closed loop system (in variable Z) and verify the stability of the system. Investigate how can you simplify the selection of the first virtual reference by not cancelling the negative definite terms in the derivation of the candidate Lyapunov function in the first step of backstepping.