TMR4243 - Marine Control Systems II

Homework assignment 5

1 Backstepping

1.1 Task: Integrator backstepping

For the following systems, determine the relative degree and perform an integrator backstepping design:

1. Objective: Control $x_1 \to x_d(t)$. Plant:

$$\begin{aligned} \dot{x}_1 &= x_2 && (1) \\ \dot{x}_2 &= \left(1 + x_1^2\right) u - 2x_1 - 3\left|x_2\right| x_2 + e^{1-t} x_1 x_2 && (2) \end{aligned}$$

2. Objective: Control $x_1 \to 0$. Plant:

$$\begin{aligned} \dot{x}_1 &= x_2 && (3) \\ \dot{x}_2 &= x_3 && (4) \\ \dot{x}_3 &= u - x_1 - 3x_2 - 2x_3 && (5) \end{aligned}$$

Draw the block-diagrams for the closed-loop systems in the z-dynamics.

1.2 Task: Vectorial backstepping

For the Euler-Lagrange system

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D(q, \dot{q})\dot{q} + g(q) = \tau \tag{6}$$

where $q \in \mathbb{R}^n$ and $\dot{q}$ are generalized coordinates and velocities in an inertial frame, $M(q) \in \mathbb{R}^{n\times n}$ is an invertible inertia matrix, $C(q, \dot{q}) \in \mathbb{R}^{n\times n}$ is a Coriolis matriix, $D(q, \dot{q}) \in \mathbb{R}^{n\times n}$ model damping/friction coefficients, $g(q)$ model restoring/spring forces, and $\tau \in \mathbb{R}^n$ is an actuated control input force vector. The objective is to control the coordinate vector $q(t)$ to track a desired vector $q_d(t) \in \mathbb{R}^n$.

Choose state variables, and set the system up on state space form.

Perform a vectorial backstepping design that solves the tracking objective and ensures that the closed-loop error system is UGES.

1.3 Task: LgV backstepping

With the objective to control $x_1 \to 0$, design a feedback control law based on LgV-backstepping (Arcak and Kokotović, 2001) for the plant

$$\begin{aligned} \dot{x}_1 &= x_2 && (7) \\ \dot{x}_2 &= x_3 && (8) \\ \dot{x}_3 &= u - x_1 - 3x_2 - 2x_3 && (9) \end{aligned}$$

Draw the block-diagram for the closed-loop system in the z-dynamics, compare and contrast this to your answer above.

1.4 Task: Integral action

For the simplified DP plant

$$
\begin{aligned}
\dot{\xi} &= \eta && (10)\\
\dot{\eta} &= R(\psi)\nu && (11)\\
M\dot{\nu} &= \tau && (12)
\end{aligned}
$$

where $\eta \in \mathbb{R}^3$ is the position/heading of the vessel in an inertial frame, $\nu \in \mathbb{R}^3$ is the body-fixed velocity vector, $R(\psi)$ is the rotation matrix, $M = M^\top > 0$ is the mass matrix, and ξ is an integral action state. The control objective is to control $\eta \to 0$ and include integral action in the control law.

Instead of a typical 3-step design, you shall perform a 2-step design that deviates from the traditional backstepping procedure. You will then design a PI control law in the 1st step that renders the CLF negative semidefinite and satisfies the LaSalle-Yoshizawa theorem. Then you shall backstep this in a 2nd step leading to the control law for $\tau \in \mathbb{R}^3$.

Correspondingly, let $z_1 := \eta$, $z_2 := \nu - \alpha_1$, and

$$
\begin{aligned}
V_1 \quad &: \; = \frac{1}{2} z_1^\top z_1 + \frac{1}{2}\xi^\top K_i \xi, \qquad K_i = K_i^\top > 0 && (13)\\
V_2 \quad &: \; = V_1 + \frac{1}{2} z_2^\top M z_2. && (14)
\end{aligned}
$$

Perform a 2-step backstepping design, where you use V_1 in the 1st step and V_2 in the 2nd step, to design a control law with integral action, and conclude on stability and convergence properties.

Expect that the derivative of the Lyapunov function will only be negative semidefinite, but by application of the LaSalle-Yoshizawa theorem you should get the result you need.

1 Solution: Backstepping

1.1 Task: Integrator backstepping

We perform integrator backstepping for the systems:

1. Objective: Control $x_1 \to x_d(t)$. Plant:

$$\begin{aligned} \dot{x}_1 &= x_2 & (15) \\ \dot{x}_2 &= \left(1+x_1^2\right)u - 2x_1 - 3\left|x_2\right|x_2 + e^{1-t}x_1x_2 & (16) \end{aligned}$$

Design: This is a relative degree two system and needs to steps of backstepping.
Step 1: Let $z_1 = x_1 - x_d(t)$ and $z_2 := x_2 - \alpha_1(x_1, t)$. Differentiating gives

$$\dot{z}_1 = \dot{x}_1 - \dot{x}_d(t) = z_2 + \alpha_1 - \dot{x}_d(t) \tag{17}$$

Let the step 1 CLF be $V_1 = \frac{1}{2}z_1^2$. This gives

$$\dot{V}_1 = z_1\left(\alpha_1 - \dot{x}_d(t)\right) + z_1z_2 \tag{18}$$

Choosing

$$\begin{aligned} \alpha_1 &= -c_1z_1 + \dot{x}_d(t) & (19) \\ &= -c_1x_1 + c_1x_d(t) + \dot{x}_d(t) = \alpha_1(x_1, t) \end{aligned}$$

results in

$$\begin{aligned} \dot{V}_1 &= -c_1z_1^2 + z_1z_2 & (20) \\ \dot{z}_1 &= -c_1z_1 + z_2 & (21) \end{aligned}$$

Assuming $z_2 = 0$ (we postpone handling z_2 to Step 2), we have that $z_1 = 0$ is UGES. In aid of Step 2, we differentiate α_1 to get

$$\dot{\alpha}_1 = -c_1x_2 + c_1\dot{x}_d(t) + \ddot{x}_d(t). \tag{22}$$

Step 2: Differentiating z_2 and $V_2 = V_1 + \frac{1}{2}z_2^2$ gives

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = \left(1+x_1^2\right)u - 2x_1 - 3\left|x_2\right|x_2 + e^{1-t}x_1x_2 - \dot{\alpha}_1 & (23) \\ \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 & (24) \\ &= -c_1z_1^2 + z_2\left[z_1 + \left(1+x_1^2\right)u - 2x_1 - 3\left|x_2\right|x_2 + e^{1-t}x_1x_2 - \dot{\alpha}_1\right] \end{aligned}$$

Letting

$$u = \frac{1}{\left(1+x_1^2\right)}\left[-z_1 - c_2z_2 + 2x_1 + 3\left|x_2\right|x_2 - e^{1-t}x_1x_2 + \dot{\alpha}_1\right] \tag{25}$$

results in

$$\begin{aligned} \dot{V}_2 &= -c_1z_1^2 - c_2z_2^2 & (26) \\ \dot{z}_2 &= -z_1 - c_2z_2 & (27) \end{aligned}$$

It follows from Lyapunov's Direct Method with

$$\begin{aligned} V_2 &= \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 & (28) \\ \dot{V}_2 &= -c_1z_1^2 - c_2z_2^2 & (29) \end{aligned}$$

that $(z_1, z_2) = 0$ is UGES.

2. Objective: Control $x_1 \to 0$. Plant:

$$\begin{aligned} \dot{x}_1 &= x_2 && (30) \\ \dot{x}_2 &= x_3 && (31) \\ \dot{x}_3 &= u - x_1 - 3x_2 - 2x_3 && (32) \end{aligned}$$

Design: This is arelative degree 3 system and needs three steps of backstepping.
Step 1: Let $z_1 = x_1$, $z_2 = x_2 - \alpha_1(x_1)$, and $V_1 = \frac{1}{2}z_1^2$. Differentiating gives

$$\begin{aligned} \dot{z}_1 &= z_2 + \alpha_1 && (33) \\ \dot{V}_1 &= z_1\alpha_1 + z_1 z_2 && (34) \end{aligned}$$

Choosing

$$\alpha_1(x_1) = -c_1 z_1 = -c_1 x_1 \tag{35}$$

gives

$$\begin{aligned} \dot{z}_1 &= -c_1 z_1 + z_2 && (36) \\ \dot{V}_1 &= -c_1 z_1^2 + z_1 z_2 && (37) \\ \dot{\alpha}_1 &= -c_1 \dot{z}_1 = -c_1 x_2. && (38) \end{aligned}$$

Step 2: Define $z_3 = x_3 - \alpha_2(x_1, x_2)$. Differentiating z_2 and $V_2 = V_1 + \frac{1}{2}z_2^2$ gives

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = z_3 + \alpha_2 - \dot{\alpha}_1 && (39) \\ \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 = -c_1 z_1^2 + z_2\left[z_1 + \alpha_2 - \dot{\alpha}_1\right] + z_2 z_3 && (40) \end{aligned}$$

Choosing

$$\begin{aligned} \alpha_2(x_1, x_2) &= -z_1 - c_2 z_2 + \dot{\alpha}_1 && (41) \\ &= -x_1 - c_2\left(x_2 + c_1 x_1\right) - c_1 x_2 \end{aligned}$$

gives

$$\begin{aligned} \dot{z}_2 &= -z_1 - c_2 z_2 + z_3 && (42) \\ \dot{V}_2 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 && (43) \\ \dot{\alpha}_2 &= -\dot{x}_1 - c_2\left(\dot{x}_2 + c_1\dot{x}_1\right) - c_1\dot{x}_2 \\ &= -\left(1 - c_2 c_1\right)x_2 - \left(c_1 + c_2\right)x_3. && (44) \end{aligned}$$

Step 3: Differentiating z_3 and $V_3 = V_2 + \frac{1}{2}z_3^2$ gives

$$\begin{aligned} \dot{z}_3 &= u - x_1 - 3x_2 - 2x_3 - \dot{\alpha}_2 && (45) \\ \dot{V}_3 &= \dot{V}_2 + z_3\dot{z}_3 && (46) \\ &= -c_1 z_1^2 - c_2 z_2^2 + z_3\left[z_2 + u - x_1 - 3x_2 - 2x_3 - \dot{\alpha}_2\right] \end{aligned}$$

Choosing now the control law

$$u = -z_2 - c_3 z_3 + x_1 + 3x_2 + 2x_3 + \dot{\alpha}_2 \tag{47}$$

results in

$$\begin{aligned} \dot{z}_3 &= -z_2 - c_3 z_3 && (48) \\ \dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 && (49) \end{aligned}$$

Note that the closed-loop overall system can be written

$$\begin{aligned} \dot{z} &= -Cz + Sz && (50) \\ V &= \frac{1}{2}z^\top z, \qquad \dot{V} = -z^\top C z && (51) \\ C &= \operatorname{diag}\left(c_1, c_2, c_3\right), \qquad S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = -S^\top && (52) \end{aligned}$$

The block-diagram for the closed-loop system in the z-dynamics to be drawn!

1.2 Task: Vectorial backstepping

We have the Euler-Lagrange system

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D(q,\dot{q})\dot{q} + g(q) = \tau \tag{53}$$

with the objective to control $q(t)$ to track a desired vector $q_d(t) \in \mathbb{R}^n$.

To perform vectorial backstepping we first write the system in strict parametric feedback form, by simply choosing $x_1 = q$ and $x_2 = \dot{q}$. This gives state-space system

$$\begin{aligned} \dot{x}_1 &= x_2 && (54) \\ \dot{x}_2 &= M(x_1)^{-1}\left[\tau - C(x)x_2 - D(x)x_2 - g(x_1)\right]. && (55) \end{aligned}$$

Step 1: Let $z_1 = x_1 - q_d(t)$ and $z_2 = x_2 - \alpha_1(x_1, t)$, and define the first CLF $V_1 = \frac{1}{2}z_1^\top z_1$ Then

$$\begin{aligned} \dot{z}_1 &= z_2 + \alpha_1 - \dot{q}_d(t) && (56) \\ \dot{V}_1 &= z_1^\top\left[\alpha_1 - \dot{q}_d(t)\right] + z_1^\top z_2. && (57) \end{aligned}$$

Choosing

$$\alpha_1 = -C_1 z_1 + \dot{q}_d(t) = -C_1\left(x_1 - q_d(t)\right) + \dot{q}_d(t) \tag{58}$$

where $C_1 = C_1^\top > 0$, gives

$$\begin{aligned} \dot{z}_1 &= -C_1 z_1 + z_2 && (59) \\ \dot{V}_1 &= -z_1^\top C_1 z_1 + z_1^\top z_2 && (60) \\ \dot{\alpha}_1 &= -C_1 x_2 + C_1 \dot{q}_d(t) + \ddot{q}_d(t). && (61) \end{aligned}$$

Step 2: Differentiating z_2 and $V_2 = V_1 + \frac{1}{2}z_2^\top z_2$ gives

$$\begin{aligned} \dot{z}_2 &= M(x_1)^{-1}\left[\tau - C(x)x_2 - D(x)x_2 - g(x_1)\right] - \dot{\alpha}_1 && (62) \\ \dot{V}_2 &= \dot{V}_1 + z_2^\top \dot{z}_2 && (63) \\ &= -z_1^\top C_1 z_1 + z_2^\top\left\{z_1 + M(q)^{-1}\left[\tau - C(x)x_2 - D(x)x_2 - g(x_1)\right] - \dot{\alpha}_1\right\} \end{aligned}$$

Choosing then the control law

$$\tau = C(x)x_2 + D(x)x_2 + g(x_1) + M(x_1)\left[-z_1 - C_2 z_2 + \dot{\alpha}_1\right] \tag{64}$$

where $C_2 = C_2^\top > 0$, gives

$$\begin{aligned} \dot{z}_2 &= -z_1 - C_2 z_2 && (65) \\ \dot{V}_2 &= -z_1^\top C_1 z_1 - z_2^\top C_2 z_2 && (66) \end{aligned}$$

It follows that $(z_1, z_2) = 0$ is UGES.

Note that we could also use the Lyapunov equation in each step, choosing $V_1 = z_1^\top P_1 z_1$, $V_2 = V_1 + z_2^\top P_2 z_2$, $C_1 = -A_1$ and $C_2 = -A_2$ where (A_1, A_2) just need to be Hurwitz.

1.3 Task: LgV backstepping

We now use LgV-backstepping to control $x_1 \to 0$ for

$$\begin{aligned} \dot{x}_1 &= x_2 && (67) \\ \dot{x}_2 &= x_3 && (68) \\ \dot{x}_3 &= u - x_1 - 3x_2 - 2x_3 && (69) \end{aligned}$$

Before we start, remember Youngs inequality:

$$x^\top y \le \kappa x^\top x + \frac{1}{4\kappa} y^\top y, \qquad \kappa > 0 \tag{70}$$

Step 1: Let $z_1 = x_1$, $z_2 = x_2 - \alpha_1(x_1)$, and $V_1 = \frac{1}{2}z_1^2$. Differentiating gives

$$\begin{aligned} \dot{z}_1 &= z_2 + \alpha_1 & (71)\\ \dot{V}_1 &= z_1\alpha_1 + z_1 z_2 & (72) \end{aligned}$$

Choosing

$$\alpha_1(x_1) = -c_1 z_1 + \alpha_{10} \tag{73}$$

gives

$$\begin{aligned} \dot{V}_1 &= -c_1 z_1^2 + z_1\alpha_{10} + z_1 z_2 & (74)\\ &\le -c_1 z_1^2 + z_1\alpha_{10} + \kappa_1 z_1^2 + \frac{1}{4\kappa_1} z_2^2 & (75) \end{aligned}$$

Then we choose

$$\alpha_{10} = -\kappa_1 z_1 \tag{76}$$

resulting in

$$\begin{aligned} \dot{z}_1 &= -\left(c_1 + \kappa_1\right) z_1 + z_2 & (77)\\ \dot{V}_1 &\le -c_1 z_1^2 + \frac{1}{4\kappa_1} z_2^2 & (78)\\ \dot{\alpha}_1 &= -\left(c_1 + \kappa_1\right)\dot{z}_1 = -\left(c_1 + \kappa_1\right) x_2. & (79) \end{aligned}$$

Step 2: Define $z_3 = x_3 - \alpha_2(x_1, x_2)$. Differentiating z_2 and $V_2 = V_1 + \frac{1}{2}z_2^2$ gives

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = z_3 + \alpha_2 - \dot{\alpha}_1 & (80)\\ \dot{V}_2 &= \dot{V}_1 + z_2\dot{z}_2 \le -c_1 z_1^2 + z_2\left[\frac{1}{4\kappa_1} z_2 + \alpha_2 - \dot{\alpha}_1\right] + z_2 z_3 & (81) \end{aligned}$$

Choosing

$$\alpha_2(x_1, x_2) = -\left(c_2 + \frac{1}{4\kappa_1}\right) z_2 + \dot{\alpha}_1 + \alpha_{20} \tag{82}$$

gives

$$\begin{aligned} \dot{V}_2 &\le -c_1 z_1^2 - c_2 z_2^2 + z_2\alpha_{20} + z_2 z_3 & (83)\\ &\le -c_1 z_1^2 - c_2 z_2^2 + z_2\alpha_{20} + \kappa_2 z_2^2 + \frac{1}{4\kappa_2} z_3^2 & (84) \end{aligned}$$

Then we choose

$$\alpha_{20} = -\kappa_2 z_2 \tag{85}$$

resulting in

$$\begin{aligned} \dot{z}_2 &= -\left(c_2 + \frac{1}{4\kappa_1} + \kappa_2\right) z_2 + z_3 & (86)\\ \dot{V}_2 &\le -c_1 z_1^2 - c_2 z_2^2 + \frac{1}{4\kappa_2} z_3^2 & (87)\\ \dot{\alpha}_2 &= -\left(c_2 + \frac{1}{4\kappa_1} + \kappa_2\right)\left(x_3 + \left(c_1 + \kappa_1\right) x_2\right) - \left(c_1 + \kappa_1\right) x_3. & (88) \end{aligned}$$

Step 3: Differentiating z_3 and $V_3 = V_2 + \frac{1}{2}z_3^2$ gives

$$\begin{aligned} \dot{z}_3 &= u - x_1 - 3x_2 - 2x_3 - \dot{\alpha}_2 & (89)\\ \dot{V}_3 &= \dot{V}_2 + z_3\dot{z}_3 & (90)\\ &\le -c_1 z_1^2 - c_2 z_2^2 + z_3\left[\frac{1}{4\kappa_2} z_3 + u - x_1 - 3x_2 - 2x_3 - \dot{\alpha}_2\right] \end{aligned}$$

Choosing now the control law

$$u = -\left(c_3 + \frac{1}{4\kappa_2}\right) z_3 + x_1 + 3x_2 + 2x_3 + \dot{\alpha}_2 \tag{91}$$

results in

$$\dot{z}_3 = -\left(c_3 + \frac{1}{4\kappa_2}\right) z_3 \tag{92}$$

$$\dot{V}_3 \leq -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 \tag{93}$$

Note that the closed-loop overall system becomes

$$\dot{z} = \begin{bmatrix} -\left(c_1 + \kappa_1\right) & 1 & 0 \\ 0 & -\left(c_2 + \frac{1}{4\kappa_1} + \kappa_2\right) & 1 \\ 0 & 0 & -\left(c_3 + \frac{1}{4\kappa_2}\right) \end{bmatrix} z \tag{94}$$

The block-diagram for the closed-loop system in the z-dynamics should be drawn. You will note from this that the $z_3 - z_2 - z_1$ systems make out a cascade. You see this also from their closed-loop equations, where z_3 has become an independent exponentially converging subsystem, which is input to the exponentially stable z_2-subsystem – implying that z_2 also converges exponentially to zero. This is again input to the exponentially stable z_1-subsystem, which in the end also converges exponentially to zero.

1.4 Task: Integral action

We have the simplified DP plant

$$\dot{\xi} = \eta \tag{95}$$

$$\dot{\eta} = R(\psi)\nu \tag{96}$$

$$M\dot{\nu} = \tau \tag{97}$$

where ξ is an augmented integral action state. Objective: control $\eta \to 0$ with integral action. We will perform a 2-step design.

Step 1: Let $z_1 := \eta$, $z_2 := \nu - \alpha_1$. This gives

$$\dot{z}_1 = R(\psi)\left(z_2 + \alpha_1\right) \tag{98}$$

Differentiating $V_1 = \frac{1}{2} z_1^\top z_1 + \frac{1}{2}\xi^\top K_i \xi$ gives

$$\dot{V}_1 = z_1^\top \dot{z}_1 + \xi^\top K_i \dot{\xi} = z_1^\top R(\psi)\left(z_2 + \alpha_1\right) + \xi^\top K_i z_1 \tag{99}$$

$$= z_1^\top \left(R(\psi)\alpha_1 + K_i \xi\right) + z_1^\top R(\psi) z_2 \tag{100}$$

Choosing

$$\alpha_1 = R(\psi)^\top \left[-K_i \xi - K_p z_1\right] \tag{101}$$

results in

$$\dot{\xi} = z_1 \tag{102}$$

$$\dot{z}_1 = -K_i \xi - K_p z_1 + R(\psi) z_2 \tag{103}$$

$$\dot{V}_1 = -z_1^\top K_p z_1 + z_1^\top R(\psi) z_2 \tag{104}$$

where we find, for $z_2 = 0$, that $\dot{V}_1 \leq 0$. Hence, the preliminary conclusion is by LaSalle-Yoshizawa, that $(\xi, z_1) = 0$ is UGS and $z_1(t) \to 0$.

Step 2: Differentiating z_2 and $V_2 = V_1 + \frac{1}{2} z_2^\top M z_2$ gives

$$
\begin{aligned}
M\dot{z}_2 &= M\dot{\nu} - M\dot{\alpha}_1 = \tau - M\dot{\alpha}_1 && (105)\\
\dot{V}_2 &= \dot{V}_1 + z_2^\top M\dot{z}_2 = -z_1^\top K_p z_1 + z_2^\top \left[R(\psi)^\top z_1 + \tau - M\dot{\alpha}_1\right] && (106)
\end{aligned}
$$

Choosing

$$\tau = -R(\psi)^\top z_1 - K_d z_2 + M\dot{\alpha}_1 \tag{107}$$

gives

$$
\begin{aligned}
M\dot{z}_2 &= -R(\psi)^\top z_1 - K_d z_2 && (108)\\
\dot{V}_2 &= -z_1^\top K_p z_1 - z_2^\top K_d z_2 \leq 0 && (109)
\end{aligned}
$$

We then get from LaSalle–Yoshizawa that $(\xi, z_1, z_2) = 0$ is UGES for the closed-loop system, and $\lim_{t\to\infty}(z_1(t), z_2(t)) = 0$.

Open question: Can we by application of Matrosov's theorem in fact show that $(\xi, z_1, z_2) = 0$ is UGAS?

References

Arcak, M. and Kokotović, P. (2001). Redesign of backstepping for robustness against unmodelled dynamics. *Int. J. Robust Nonlinear Contr.*, 11(7):633–643. Robustness in identification and control.