TMR4243 - Marine Control Systems II

Exam

Spring 2015

Notation: Throughout this exam $|x|$ means the vector 2-norm, i.e. $|x| = \sqrt{x^\top x}$.

1 Solutions to nonlinear ODEs (22 pts)

1. Consider the nonlinear ordinary differential equation (ODE):

$$\dot{x} = f(x), \qquad x_0 = x(0)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

(a) What does it mean that a solution is **forward complete**?

Answer: (2 pts) Answer: A solution $x(t, x_0)$ is forward complete if it is defined for all future time $t \geq 0$ for all $x_0 \in \mathbb{R}^n$.

(b) What does it mean that this system is **locally Lipschitz**?

Answer: (2 pts) The system is Locally Lipshcitz if for each $x \in \mathbb{R}^n$ there exists there exists a neighborhood $\mathcal{U}$ of x and a constant $L > 0$ such that

$$x_1, x_2 \in \mathcal{U} \quad \Longrightarrow \quad |f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|\,.$$

(c) What can you say about the solutions if it is *locally Lipschitz*?

Answer: (2 pts) Existence and uniqueness of solutions: For each $x_0 = x(0)$ there exists $T > 0$ and a unique solution $x(t, x_0)$ on $[0, T]$.

(d) What does it mean that this system is **globally Lipschitz**?

Answer: (2 pts) The system is Globally Lipshcitz if there exists a (global) constant $L > 0$ such that for each pair $x_1, x_2 \in \mathbb{R}^n$

$$|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|\,.$$

(e) What can you say about the solutions if it is *globally Lipschitz*?

Answer: (2 pts) Existence, uniqueness, and forward completeness of solutions: For each $x_0 \in \mathbb{R}^n$ there exists a unique solution $x(t, x_0)$ for all $t \geq 0$.

2. For the scalar system

$$\dot{x} = x^{\frac{1}{3}}, \qquad x_0 = 0$$

we propose the following solutions for $t \geq 0$:

$$x(t) = 0 \qquad \& \qquad x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$$

(a) Show that the proposed solutions are indeed solutions to the ODE.

Answer: (2 pts) Zero solution: Initial condition $x_0 = x(0) = 0$, and differentiating $x(t) = 0$ gives $\dot{x}(t) = 0 = 0^{\frac{1}{3}} = x(t)^{\frac{1}{3}}$ Q.E.D.

Answer: (2 pts) Non-zero solution: Initial condition $x_0 = x(0) = \left(\frac{2 \cdot 0}{3}\right)^{\frac{3}{2}} = 0$, and differentiating $x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$ gives

$$\dot{x}(t) = \frac{3}{2}\left(\frac{2t}{3}\right)^{\frac{1}{2}} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{\frac{3}{2}\frac{1}{3}} = x(t)^{\frac{1}{3}} \qquad \text{Q.E.D.}$$

(b) What is the Lipschitz property of this ODE?

Answer: (2 pts) Differentiating $\frac{\partial f}{\partial x} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{2/3}}$. At $x = 0$ this derivative is infinite, which implies that it cannot be locally Lipschitz at $x = 0$. For an initial condition at $x = 0$ the solutions exist but are not unique.

(c) What is the stability property of this ODE?

Answer: (1 pt) The equilibrium $x = 0$ is unstable. A slight offset from $x = 0$ will make the solution grow unbounded in either positive or negative direction.

3. For the scalar system

$$\dot{x} = kx, \qquad x_0 = -1, \quad k = \text{const.}$$

we propose the solution for $t \geq 0$:

$$x(t) = -e^{kt}$$

(a) Show that the proposed solution is indeed a solution to this ODE.

Answer: (2 pts) Initial condition $x_0 = x(0) = -e^{k \cdot 0} = -1$ OK. Differentiating $\dot{x}(t) = -ke^{kt} = kx(t)$ Q.E.D.

(b) What is the Lipschitz property of this ODE?

Answer: (2 pts) A linear ODE is Globally Lipschitz. We get

$$|f(x_1) - f(x_2)| = |kx_1 - kx_2| \leq L\,|x_1 - x_2|\,, \quad L = |k|\,,$$

which holds for all x_1, x_2.

(c) What is the stability property of this ODE for $k > 0$?

Answer: (1 pt) For k positive the origin is unstable. The solution for $x_0 = -1$ shows directly that it will decrease unbounded to $-\infty$.

2 Stability of nonlinear systems (18 pts)

1. Consider the nonlinear time-varying system:

$$\dot{x} = f(t, x), \qquad x_0 = x(t_0), \quad t_0 \geq 0$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, and assume $f(t, 0) = 0, \forall t \geq 0$.

(a) Define UGS and UGAS for the system solutions with respect to $x = 0$ based on class-$\mathcal{K}$ and class-$\mathcal{KL}$ functions.

Answer: (4 pts) The origin is UGS if there exists a function $\varphi \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies

$$|x(t, x_0)| \leq \varphi\left(|x_0|\right), \qquad \forall t \geq 0.$$

The origin is UGAS if there exists a function $\beta \in \mathcal{KL}$ such that for all $x_0 \in \mathbb{R}^n$, the solution $x(t, x_0)$ satisfies

$$|x(t, x_0)| \leq \beta\left(|x_0|\,,\ t\right), \qquad \forall t \geq 0.$$

(b) What meaning does *Uniform* have?

Answer: (1 pt) Uniform implies that the above bounds does not change with time, that is, the bounding $\mathcal{K}/\mathcal{KL}$ functions does not depend on the initial time. For time-invariant systems this is redundant information, e.g. UGS = GS, since the system dynamics then cannot change over time.

(c) What meaning does *Global* have?

Answer: (1 pt) This means that the stability property and above bounds are valid for all initial conditions $x_0 \in \mathbb{R}^n$.

(d) What meaning does *Asymptotic* have?

Answer: (1 pt) This means that the solutions will asymptotically converge to the origin $x = 0$ as $t \to \infty$. This is also denoted as the origin being *attractive*.

2. In terms of equilibrium points you can have three types: *single point*, *multiple isolated points*, and a *continuum of points*. Explain each of these and wether they are possible for linear vs. nonlinear systems.

Answer: (3 pts) *Single point:* There is a unique single point that is an equilibrium point for the system. Both linear and nonlinear systems can have this.

Multiple isolated points: There are several points in the state space, each isolated from the others, that all are equilibria to the dynamical system. This is not possible for a linear system (due to the linearity property), but generally possible for a nonlinear system.

Continuum of points: This means that there is a continuous (closed) set of points in the state space that all are equilibria for the dynamical system, e.g. the x_1-axis. This is possible for both linear and nonlinear systems; for a linear system $\dot{x} = Ax$ such a continuum of equilibria must correspond to the nullspace of A.

3. Consider the linear time-varying system:

$$\dot{x} = Q\,(x - x_d(t)) + \dot{x}_d(t)$$

where $Q = Q^\top$, and $(x_d(t), \dot{x}_d(t))$ are bounded reference signals. Let

$$V(t,x) = \frac{1}{2}\,(x - x_d(t))^\top\,(x - x_d(t))$$

be a Lyapunov function candidate.

(a) Show how to differentiate $V(t,x)$.

Answer: (4 pts) We get

$$\begin{aligned}
\dot{V} &= V^t(t,x) + V^x(t,x)\dot{x} = -\,(x - x_d(t))^\top\,\dot{x}_d(t) + (x - x_d(t))^\top\,\dot{x} \\
&= -\,(x - x_d(t))^\top\,\dot{x}_d(t) + (x - x_d(t))^\top\,(Q\,(x - x_d(t)) + \dot{x}_d(t)) \\
&= -\,(x - x_d(t))^\top\,\dot{x}_d(t) + (x - x_d(t))^\top\,Q\,(x - x_d(t)) + (x - x_d(t))^\top\,\dot{x}_d(t) \\
&= (x - x_d(t))^\top\,Q\,(x - x_d(t))
\end{aligned}$$

(b) What is your stability conclusion if $Q < 0$?

Answer: (2 pts) The Lyapunov function $V(t,x)$ is positive definite and radially unbounded in the error $e = x - x_d(t)$, and if Q is symmetric negative definite then $\dot{V}$ is negative definite in e. Hence, we get for the error system $\dot{e} = Qe$ that the equilibrium point $e = 0$ is UGES from Lyapunov's direct method.

(c) What is your stability conclusion if $Q \geq 0$?

Answer: (2 pts) If $Q \geq 0$ then no stability conclusion can be made from using $V(t,x)$ as a Lyapunov candidate. However, considering the error system $\dot{e} = Qe$ we know that $e = 0$ will generally be unstable and marginally stable, at best, for $Q = 0$.

3 Lyapunov stability (22 pts)

Consider the system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2 + \frac{k}{\sqrt{5}} x_2^2
\end{aligned}
$$

1. Assume $k = 0$. Differentiate the Lyapunov function

$$V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

and discuss stability of the origin by Lyapunov's method and the Krasovskii-LaSalle theorem.

Answer: (4 pts) Differentiating we get

$$\dot{V}_1 = x_1 x_2 - x_2 x_1 - x_2^2 = -x_2^2 \leq 0$$

Since the derivative is negative semidefinite we have that the origin $x = col(x_1, x_2) = 0$ is UGS. Since the system is time-invariant, we can apply the Krasovskii-LaSalle theorem. We let

$$\Omega = \left\{ x \in \mathbb{R}^2 : \dot{V}_1 = 0 \right\} = \{x_2 = 0\}$$

and look for the largest invariant set in Ω by considering the state equations. For Ω to be invariant we need $x_2 = \dot{x}_2 = 0$. Then we get $\dot{x}_1 = x_2 = 0 \Rightarrow \dot{x}_1 = 0$ and $\dot{x}_2 = -x_1 - x_2 = 0 \Rightarrow x_1 = 0$. Hence, the largest invariant set in Ω is the origin, which then is GAS. Since for $k = 0$ the system is linear, then GAS=GES.

2. Assume $k = 0$. Differentiate the Lyapunov function

$$V_2(x) = \frac{3}{2}x_1^2 + x_1 x_2 + x_2^2$$

and discuss stability of the origin by Lyapunov's method.

Answer: (4 pts) Differentiating we get

$$
\begin{aligned}
\dot{V}_2 &= 3x_1\dot{x}_1 + \dot{x}_1 x_2 + x_1 \dot{x}_2 + 2x_2\dot{x}_2 \\
&= 3x_1 x_2 + x_2^2 + x_1(-x_1 - x_2) + 2x_2(-x_1 - x_2) \\
&= 3x_1 x_2 + x_2^2 - x_1^2 - x_1 x_2 - 2x_2 x_1 - 2x_2^2 \\
&= -x_1^2 - x_2^2 = -|x|^2
\end{aligned}
$$

For V_2 we also get

$$
\begin{aligned}
V_2(x) &= \left(\frac{1}{\sqrt{2}} x_1 + \frac{1}{\sqrt{2}} x_2 \right)^2 + x_1^2 + \frac{1}{2}x_2^2 \geq \frac{1}{2}|x|^2 \\
V_2(x) &= \frac{3}{2}x_1^2 + x_1 x_2 + x_2^2 \leq \frac{3}{2}x_1^2 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_2^2 \leq 2|x|^2 \\
\frac{1}{2}|x|^2 &\leq V_2(x) \leq 2|x|^2
\end{aligned}
$$

It follows from Lyapunov's Direct Method that the origin $x = 0$ is GES.

3. Show that the system can be written in vector form with state vector $x = col(x_1, x_2)$, as
$$\dot{x} = Ax + kbx_2^2$$
by appropriate definitions of A and b.

Answer: (2 pts) We get
$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad b = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Show that $\left|x_2^2\right| \leq |x|^2$.

Answer: (1 pt) $\left|x_2^2\right| = x_2^2 \leq x_1^2 + x_2^2 = |x|^2$.

(b) Show that the Lyapunov function $V_2(x)$ above can be written
$$V_2(x) = x^\top P x$$
where P is symmetric positive definite.

Answer: (1 pt) We get
$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

(c) Show that (P, A) satisfies the Lyapunov equation with $Q = I$.

Answer: (2 pts) Lyapunov equation:
$$\begin{aligned} PA + A^\top P &= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -Q. \end{aligned}$$

(d) Assume $k = 1$. Show that $|Pb| = \frac{1}{2}$ and that
$$\dot{V}_2 \leq -|x|^2 + |x|^3 =: Y_2(|x|)$$
For what state values $x \in \mathbb{R}^2$ is $Y_2(|x|)$ negative definite?

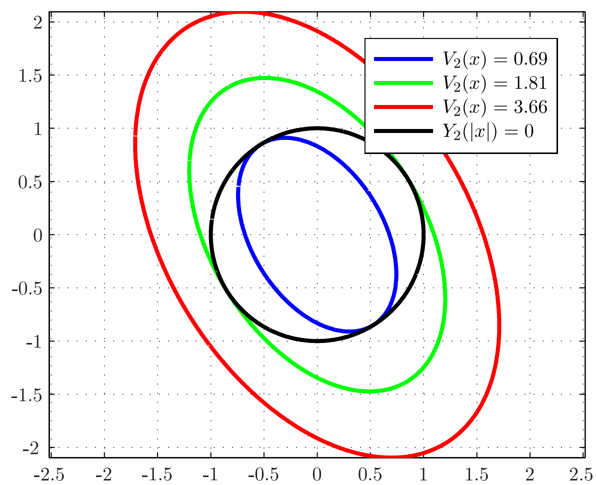
Answer: (3 pts) We get
$$\begin{aligned} Pb &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \\ |Pb| &= \sqrt{\left(\frac{1}{2\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{4 \cdot 5} + \frac{1}{5}} = \sqrt{\frac{1}{4 \cdot 5} + \frac{4}{4 \cdot 5}} = \sqrt{\frac{1}{4}} = \frac{1}{2} \\ \dot{V}_2 &= 2x^\top PAx + 2x^\top Pkbx_2^2 = x^\top \left(PA + A^\top P\right) x + 2x^\top Pbx_2^2 \\ &\leq -|x|^2 + 2\,|Pb|\,|x|\,\left|x_2^2\right| \leq -|x|^2 + |x|^3 \end{aligned}$$

where we used $k = 1$, $|Pb| = \frac{1}{2}$, and $\left|x_2^2\right| \leq |x|^2$.
We get that $Y_2(|x|)$ is negative definite on $\mathcal{D} = \left\{x \in \mathbb{R}^2 : \ |x| < 1\right\}$, i.e. all points strictly inside the unit circle.

(e) For $k = 1$, what stability of $x = 0$ can you conclude for the nonlinear system based on $V_2(x)$?

Answer: (2 pts) We have the $V_2(x)$ is positive definite and radially unbounded, and $\dot{V}_2 < 0$ on $\mathcal{D}\backslash\{0\}$. Hence, the origin is LES.

4. The figure below shows three ellipsoidal level curves of $V_2(x)$, i.e. the curves $\left\{x \in \mathbb{R}^2 : V(x) = c\right\}$ for $c = \{0.69, 1.81, 3.66\}$, and the level curve $\{x \in \mathbb{R}^2\backslash\{0\} : Y_2(|x|) = 0\}$.



Which of the 4 ellipses can be used as an estimate of the Region of Convergence (ROC)? Justify your answer.

Answer: (3 pts) Any level set of $V_2(x)$ contained in $\mathcal{D}$ can be used as an estimmate of the ROC, i.e. in this case the set

$$\mathcal{G} = \left\{x \in \mathbb{R}^2 : V(x) \leq 0.69\right\}.$$

This guarantees that for any $x_0 \in \mathcal{G}$ then the solutions can never cross the boundary for which $\dot{V}_2$ becomes positive. Hence, since $\dot{V}_2 < 0$ in $\mathcal{G}\backslash\{0\}$ the solutions $|x(t)|$ must monotonically decrease to the origin.

4 Nonlinear feedback control (22 pts)

Consider a marine system

$$\begin{aligned}\dot{\eta} &= R(\eta)\nu \\ M\dot{\nu} &= \tau + \rho(\eta,\nu) + d(t)\end{aligned}$$

where $\eta \in \mathbb{R}^n$ is a position/orientation vector, $\nu \in \mathbb{R}^n$ is a velocity vector, $\tau \in \mathbb{R}^n$ is the control input, $\rho(\eta,\nu) \in \mathbb{R}^n$ is a locally Lipschitz vector function, $d(t)$ is a bounded bias, $M = M^\top > 0$, and $R(\eta)$ is a rotation matrix with the properties $R(\eta)^\top R(\eta) = R(\eta)R(\eta)^\top = I$ and $\dot{R} = R(\eta)S(\nu)$ where $S(\nu) = -S(\nu)^\top$.

1. Choosing $x_1 = \eta$, $x_2 = R(\eta)\nu$, and $x = col(x_1, x_2)$, show that you can transform the system into the controller form:

   $$\dot{x} = Ax + B\Gamma(\eta,\nu)\left[\tau + \varphi(\eta,\nu) + d(t)\right]$$

   where Γ is nonsingular. Show that (A,B) is a controllable pair.

   Answer: (8 pts) We get

   $$\begin{aligned}\dot{x}_1 &= \dot{\eta} = R(\eta)\nu = x_2 \\ \dot{x}_2 &= \dot{R}\nu + R(\eta)\dot{\nu} = R(\eta)S(\nu)\nu + R(\eta)M^{-1}\left(\tau + \rho(\eta,\nu) + d(t)\right) \\ &= R(\eta)M^{-1}MS(\nu)\nu + R(\eta)M^{-1}\left(\tau + \rho(\eta,\nu) + d(t)\right) \\ &= R(\eta)M^{-1}\left[\tau + MS(\nu)\nu + \rho(\eta,\nu) + d(t)\right]\end{aligned}$$

   Hence, we can assign

   $$\begin{aligned}A &= \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{2n\times n} \\ \Gamma(\eta,\nu) &= R(\eta)M^{-1}, \quad \varphi(\eta,\nu) = MS(\nu)\nu + \rho(\eta,\nu)\end{aligned}$$

   where $\Gamma(\eta,\nu)$ is invertible.
   To show that (A,B) is controllable we form the controllability matrix,

   $$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{2n-1}B \end{bmatrix}$$

   and require that $rank(\mathcal{C}) = 2n$. This means that we need $\mathcal{C}$ to have full row rank. Since $A^2 = A^3 = \ldots A^{2n-1} = 0$ we get

   $$\mathcal{C} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \end{bmatrix}$$

   which clearly is full row-rank. Hence, (A,B) is controllable.

2. Assume all states and the model is fully known, and $d(t) = 0$. Design a static state feedback control law $\tau = \alpha(\eta,\nu)$, using *feedback linearization*, that renders the closed-loop system linear and $x = 0$ UGES.

Answer: (4 pts) A state feedback control law is

$$\begin{aligned}\tau &= \Gamma(\eta,\nu)^{-1}\alpha - \varphi(\eta,\nu) \\ \alpha &= -Kx\end{aligned}$$

gives

$$\dot{x} = Ax + B\alpha = (A - BK)\,x$$

where the linear feedback gain $K \in \mathbb{R}^{n\times 2n}$ is designed such that the closed-loop matrix $A - BK$ is Hurwitz.

3. For $d(t)$ a bounded unknown bias, show that your static feedback control law renders the system Input-to-State-Stable (ISS) from $d(t)$ as input.

Answer: (4 pts) We let $A_0 = A - BK$ such that

$$\dot{x} = A_0 x + B\Gamma(\eta,\nu)d(t)$$

Let $P = P^\top > 0$ satisfy the Lyapunov equation $PA_0 + A_0^\top P = -I$. We will then show that $V_0(x) = x^\top P x$ is an ISS-Lyapunov function. Differentiating gives

$$\begin{aligned}\dot{V}_0 &= 2x^\top PA_0x + 2x^\top PB\Gamma(\eta,\nu)d(t) \\ &= -x^\top x + 2x^\top PB\Gamma(\eta,\nu)d(t) \\ &\leq -\left|x\right|^2 + 2\left|x\right|\left\|PB\Gamma(\eta,\nu)\right\|\left|d(t)\right| \\ &\leq -\frac{1}{2}\left|x\right|^2 - \frac{1}{2}\left|x\right|^2 + L\left|x\right|\left|d(t)\right| \\ &\leq -\frac{1}{2}\left|x\right|^2, \qquad \forall \left|x\right| \geq 2L\left|d(t)\right|\end{aligned}$$

where $L \geq \|PB\Gamma(\eta,\nu)\|$ (e.g. $L = \frac{\lambda_{\max}(P)}{\lambda_{\min}(M)}$). Hence, $V_0(x)$ is an ISS-Lyapunov function.

4. Suppose $d(t) = d =$constant and unknown, and that the static control law $\tau = \alpha(\eta,\nu) + \tau_0$ renders the closed-loop system into

$$\dot{x} = A_0x + B\Gamma(\eta,\nu)\left[\tau_0 + d\right]$$

where A_0 is Hurwitz. Augmenting the control law with integral action τ_0, let

$$\begin{aligned}\dot{\xi} &= \gamma \\ \tau_0 &= -K_i\xi, \qquad K_i = K_i^\top > 0\end{aligned}$$

where the function γ shall be designed. Define $\tilde{\xi} = \xi - K_i^{-1}d$, let $P = P^\top > 0$ satisfy $PA_0 + A_0^\top P = -I$, and define the CLF

$$V(x,\tilde{\xi}) = x^\top Px + \frac{1}{2}\tilde{\xi}^\top K_i\tilde{\xi}$$

(a) Write down the state equations for $(\tilde{\xi}, x)$.

Answer: (2 pts) State equations becomes

$$\begin{aligned}
\dot{\tilde{\xi}} &= \dot{\xi} = \gamma \\
\dot{x} &= A_0 x + B\Gamma(\eta,\nu)\left[-K_i\xi + d\right] \\
&= A_0 x - B\Gamma(\eta,\nu)K_i\left[\xi - K_i^{-1}d\right] \\
&= A_0 x - B\Gamma(\eta,\nu)K_i\tilde{\xi}
\end{aligned}$$

(b) Differentiate $V(x, \tilde{\xi})$ and design γ so that $\dot{V}$ becomes

$$\dot{V} = -x^\top x$$

Answer: (2 pts) Differentiating $V(x, \tilde{\xi})$ gives

$$\begin{aligned}
\dot{V} &= 2x^\top P\dot{x} + \tilde{\xi}^\top K_i\dot{\tilde{\xi}} \\
&= 2x^\top PA_0 x - 2x^\top PB\Gamma(\eta,\nu)K_i\tilde{\xi} + \tilde{\xi}^\top K_i\gamma \\
&= x^\top\left(PA_0 + A_0^\top P\right)x - 2\tilde{\xi}^\top K_i\Gamma(\eta,\nu)^\top B^\top Px + \tilde{\xi}^\top K_i\gamma \\
&= -x^\top x + \tilde{\xi}^\top K_i\left(\gamma - 2\Gamma(\eta,\nu)^\top B^\top Px\right)
\end{aligned}$$

Assigning then the integral action

$$\gamma = 2\Gamma(\eta,\nu)^\top B^\top Px$$

gives the desired result.

(c) What stability properties can you conclude for $(\tilde{\xi}, x) = (0, 0)$ (Hint: Use the LaSalle-Yoshizawa theorem).

Answer: (2 pts) We have $V(x, \tilde{\xi})$ positive definite and radially unbounded, and $\dot{V}$ is negative semidefinite. Hence, the origin $(\tilde{\xi}, x) = (0, 0)$ is UGS. In addition, from the LaSalle-Yoshizawa theorem we get that $\dot{V} = -x^\top x$ converges to zero, implying that $\lim_{t\to\infty} x(t) = 0$.
In fact, since $\eta = x_1$ and $\nu = R(x_1)^\top x_2$, we find that the closed-loop system is time-invariant. Hence, we can use the Krasovskii-LaSalle invariance principle to conclude that $B\Gamma(\eta,\nu)K_i\tilde{\xi}$ must converge to zero. Since ΓK_i is nonsingular, we can conclude that $\lim_{t\to\infty} \tilde{\xi}(t) = 0$.

5 Solution: Backstepping (16 pts)

Consider the system

$$\begin{aligned}\dot{x}_1 &= g(x_1)x_2 - \sin(x_1)\\ \dot{x}_2 &= u - |x_2|\,x_2 + b\end{aligned}$$

where $1 \leq g(x_1) \leq 10$ for all $x_1 \in \mathbb{R}$, $u \in \mathbb{R}$ is the control input, and b is a constant bias.

1. Suppose $b = 0$, and let the control objective be to stabilize $(x_1, x_2) = (0, 0)$. Use *backstepping* to design a feedback control law for u that solves the regulation control objective.

   Answer: (8 pts) Backstepping design for regulation:

   Step 1:
   We let $z_1 = x_1$ and $z_2 = x_2 - \alpha_1(x_1)$ and differentiate z_1 and $V_1 = \frac{1}{2}z_1^2$ to get

   $$\begin{aligned}\dot{z}_1 &= g(x_1)z_2 + g(x_1)\alpha_1 - \sin(x_1)\\ \dot{V}_1 &= z_1\dot{z}_1 = z_1 g(x_1)z_2 + z_1\left[g(x_1)\alpha_1 - \sin(x_1)\right]\end{aligned}$$

   We choose
   $$\alpha_1 = \frac{1}{g(x_1)}\left[-c_1 z_1 + \sin(x_1)\right]$$
   which gives
   $$\dot{V}_1 = g(x_1)z_1 z_2 - c_1 z_1^2$$
   This shows for $z_2 = 0$ that $z_1 = 0$ is UGES.

   Step 2:
   Differentiating z_2 and $V_2 = V_1 + \frac{1}{2}z_2^2$ gives

   $$\begin{aligned}\dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 = u - |x_2|\,x_2 - \dot{\alpha}_1\\ \dot{V}_2 &= -c_1 z_1^2 + g(x_1)z_1 z_2 + z_2\dot{z}_2 = -c_1 z_1^2 + z_2\left[g(x_1)z_1 + u - |x_2|\,x_2 - \dot{\alpha}_1\right]\end{aligned}$$

   where we assume an expression for $\dot{\alpha}_1$ is available by some differentiation-work. We choose the control law

   $$u = -g(x_1)z_1 - c_2 z_2 + |x_2|\,x_2 + \dot{\alpha}_1$$

   which gives
   $$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 < 0.$$
   Correspondingly, the origin $(z_1, z_2) = (0, 0)$ of the closed-loop system

   $$\begin{aligned}\dot{z}_1 &= -c_1 z_1 + g(x_1)z_2\\ \dot{z}_2 &= -g(x_1)z_1 - c_2 z_2\end{aligned}$$

   is UGES.

2. Suppose $b \neq 0$, let $\hat{b}$ be an estimate of b, and define the estimation error $\tilde{b} = b - \hat{b}$. Augment the Step 2 CLF V_2 with a term $\frac{1}{2\gamma}\tilde{b}^2$, and design an adaptive update law for $\dot{\hat{b}}$ that renders $(z_1, z_2, \tilde{b}) = 0$ UGS and ensures the convergence $(x_1(t), x_2(t)) \to 0$.

   Answer: (8 pts) Redoing Step 2 for $b \neq 0$, we will get

$$\dot{z}_2 = u - |x_2|\, x_2 - \dot{\alpha}_1 + b$$

   and augmenting with the adaptive term, we define $V_2 = V_1 + \frac{1}{2}z_2^2 + \frac{1}{2\gamma}\tilde{b}^2$ and differentiate

$$\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 + z_2\left[g(x_1)z_1 + u - |x_2|\, x_2 - \dot{\alpha}_1 + b\right] + \frac{1}{\gamma}\tilde{b}\dot{\tilde{b}} \\
&= -c_1 z_1^2 + z_2\left[g(x_1)z_1 + u - |x_2|\, x_2 - \dot{\alpha}_1 + \hat{b}\right] + \tilde{b}\left(z_2 - \frac{1}{\gamma}\dot{\hat{b}}\right)
\end{aligned}$$

   Choosing then the adaptive control law

$$\begin{aligned}
\dot{\hat{b}} &= \gamma z_2 \\
u &= -\hat{b} - g(x_1)z_1 - c_2 z_2 + |x_2|\, x_2 + \dot{\alpha}_1
\end{aligned}$$

   gives

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 \leq 0$$

   According to the LaSalle-Yoshizawa theorem, the origin $(z_1, z_2, \tilde{b}) = 0$ is UGS and $\lim_{t\to\infty}(z_1(t), z_2(t)) = 0$. Since

$$\begin{aligned}
x_1(t) &= z_1(t) \\
x_2(t) &= z_2(t) + \frac{1}{g(z_1(t))}\left[-c_1 z_1(t) + \sin(z_1(t))\right]
\end{aligned}$$

   we also get that $\lim_{t\to\infty}(x_1(t), x_2(t)) = 0$.
   Can you show that $(z_1, z_2, \tilde{b}) = 0$ is in fact UGAS?