TMR4243 - MARINE CONTROL SYSTEMS II

Homework assignment 3

1 Stabilization

Consider the scalar system

$$\dot{x} = u + 2\,|x|\,x.$$

1. Let $V(x) = \frac{1}{2}x^2$ be a CLF for the system, and design a corresponding control for u that renders $x = 0$ UGES.
2. For the above quadratic CLF, derive the control law based on Sontag's formula, and discuss the achieved stability.
3. Show that the control law

$$u = -kx, \qquad k > 0$$

   achieves:

   - Local stabilization.
   - Regional stabilization.
   - Semiglobal stabilization – but not global stabilization.

2 Practical stabilization

Consider the scalar system

$$\dot{x} = u + d(t), \qquad \|d\| \leq 1.$$

1. Show that the control law

$$u = -kx, \qquad k > 0$$

   achieves global practical stabilization.

3 Diesel generator control

The (simplified) mechanical dynamics of a diesel-generator is given by

$$\begin{aligned} \dot{\delta} &= \omega_B \left(\omega - \omega_0\right) \\ 2H\dot{\omega} &= t_m - D\omega - t_e(t) \end{aligned}$$

where ω is the normalized (per-unit) electric frequency, δ is the load angle of the generator, t_m is the per-unit control torque from the cylinder combustion dynamics (our control input), t_e is the per-unit electric load torque, $H > 0$ is an inertia constant, $D > 0$ is a damping gain, $\omega_B = 120\pi$ [rad/s] is the base frequency constant, and ω_0 is the per-unit electric frequency of the connected electric power bus.

Suppose we want to control δ to δ_{ref} and ω to ω_0 and define the error states $e_\delta := \delta - \delta_{ref}$ and $e_\omega := \omega - \omega_0$. Assume that $t_e(t) = t_L + w(t)$ where t_L is a constant electric load torque and w a bounded disturbance torque.

1. Assume $w(t) \equiv 0$ and t_L is known. Write the system as a linear state-space vectorial system with $x = \mathrm{col}(e_\delta, e_\omega)$ and $u = t_m$.
2. State the control objective.
3. Let $P \in \mathbb{R}^{2\times 2}$ with $P = P^\top > 0$. For an appropriate choice of P, let $V(x) = x^\top P x$ be a CLF for the system and propose a corresponding control law for t_m that renders $x = 0$ UGES.
4. Let the bound for $\|w\| \leq w_0$. In presence of the disturbance w, show that your control law renders the closed-loop system Practically-UGES with respect to $x = 0$.

1 Solution: Stabilization

We consider the scalar system

$$\dot{x} = u + 2|x|x.$$

1. Differentiating the CLF $V(x) = \frac{1}{2}x^2$ we get

$$\dot{V} = x\left(u + 2|x|x\right).$$

   Choosing for instance

$$u = -kx - (2+l)|x|x, \qquad k, l > 0$$

   we get

$$\dot{V} = -\left(k + l|x|\right)x^2 \leq -kx^2.$$

   It follows from Lyapunov's direct method that $x = 0$ UGES.

2. Sontag's formula: We recognize $f(x) = 2|x|x$, and $g(x) = 1$. For $V^x(x) = x$, the control law according to Sontag's formula becomes

$$\begin{aligned}
u &= \begin{cases} -\frac{\left[V^x f + \sqrt{(V^x f)^2 + ((V^x G)(V^x G)^\top)^2}\right]}{(V^x G)(V^x G)^\top}(V^x G)^\top, & \text{if } V^x G \neq 0 \\ 0, & \text{if } V^x G = 0 \end{cases} \\
&= \begin{cases} -\frac{\left[2|x|x^2 + \sqrt{(2|x|x^2)^2 + (x^2)^2}\right]}{x^2}x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \\
&= \begin{cases} -2|x|x - \sqrt{4x^2+1}x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \\
&= -2|x|x - \sqrt{4x^2+1}x.
\end{aligned}$$

   Discussing stability, notice that for this closed-loop system we get

$$\dot{V} = -\sqrt{4x^2+1}x^2 \leq -|x|^2 =: \alpha_3(|x|).$$

   Hence, Sontag's formula also ensures that the origin is UGES.

3. Inserting the control law

$$u = -kx, \qquad k > 0$$

   into the dynamics results in the closed-loop system

$$\dot{x} = -kx + 2|x|x.$$

   - Local stabilization: Linearizing the closed-loop system gives

$$\dot{x} = \left[\left.\frac{\partial f}{\partial x}\right|_{x=0}\right]x = \left[-k + 4\left.|x|\right|_{x=0}\right]x = -kx,$$

     which is exponentially stable. Hence, the closed-loop nonlinear system is locally stable.

   - Regional stabilization: We use $V(x)$ to estimate a region of convergence. We have

$$\begin{aligned}
V(x) &= \frac{1}{2}x^2 \\
\dot{V} &= -kx^2 + 2|x|x^2
\end{aligned}$$

     This is negative semidefinite on the set

$$\Omega = \left\{x \in \mathbb{R} : \dot{V} \leq 0\right\} = \left\{x \in \mathbb{R} : |x| \leq \frac{k}{2}\right\}.$$

The largest level set contained in Ω is for some value $c > 0$ given by

$$\mathcal{G} = \{x \in \mathbb{R} : V(x) \leq c\} = \left\{x \in \mathbb{R} : |x|^2 \leq 2c\right\} = \left\{x \in \mathbb{R} : |x| \leq \sqrt{2c}\right\}$$

Choosing $\sqrt{2c} = \frac{k}{2} - \varepsilon$ for $\varepsilon \ll \frac{k}{2}$, shows that the level set $\mathcal{G}$ can be chosen as Region of Convergence (RoC). Hence, the closed-loop system is regionally stable with RoC $\mathcal{G}$ (meaning that $\forall x(0) \in \mathcal{G}$ then $\lim_{t\to\infty} x(t) = 0$).

- Semiglobal stabilization – but not global stabilization: For $x > \frac{k}{2}$ then $\dot{x} > 0$ and $x < -\frac{k}{2}$ then $\dot{x} < 0$. Hence, the system is unstable $\forall x(0) \notin \Omega$. However, we note that we can write the region of convergence as

$$\mathcal{G} = \mathcal{G}(k) = \left\{x \in \mathbb{R} : |x| \leq \frac{k}{2} - \varepsilon\right\}.$$

Hence, for any compact set $\mathcal{H} \subset \mathbb{R}$ of desired initial conditions, we can choose a control gain k such that $\mathcal{H} \subseteq \mathcal{G}(k)$. It follows that the closed-loop system is semiglobally stable.

2 Solution: Practical stabilization

We consider the scalar system

$$\dot{x} = u + d(t), \qquad \|d\| \leq 1.$$

1. Global practical stabilization: For the control law $u = -kx$ we get

$$\dot{x} = -kx + d(t),$$

and using $V(x) = \frac{1}{2}x^2$ we get

$$\begin{aligned}
\dot{V} &\leq -k|x|^2 + |x| \|d\| \\
&= -(1-\lambda)k|x|^2 - k\lambda |x|^2 + |x| \\
&\leq -(1-\lambda)k|x|^2, \quad \forall |x| \geq \frac{1}{k\lambda}, \quad \lambda \in (0,1).
\end{aligned}$$

Hence, we get that the solution is uniformly ultimately bounded by $|x(t)| < \frac{1}{k}$. For any chosen $\varepsilon > 0$ we can then choose $k = \frac{1}{\varepsilon}$ to make the solutions bounded by ε, thus achieving global practical stabilization.

3 Solution: Diesel generator control

We are given the diesel-generator dynamics

$$\dot{\delta} = \omega_B (\omega - \omega_0)$$
$$2H\dot{\omega} = t_m - D\omega - t_e(t)$$

where ω is the electric frequency, δ is the load angle, t_m is the control input, t_e is the electric load, (H, D, ω_B) are constants, and ω_0 is the electric frequency of the connected electric bus. Defining the error states $e_\delta := \delta - \delta_{ref}$ and $e_\omega := \omega - \omega_0$, and assuming $t_e(t) = t_L + w(t)$ where t_L is a constant load and w a bounded disturbance, then we get:

1. For $w(t) \equiv 0$ and t_L known, we have

$$\begin{aligned}
\dot{e}_\delta &= \omega_B e_\omega \\
\dot{e}_\omega &= \frac{1}{2H} t_m - \frac{D}{2H}(e_\omega + \omega_0) - \frac{1}{2H} t_L - \frac{1}{2H} w(t)
\end{aligned}$$

Let $\sigma = -D\omega_0 - t_L$ and $w(t) = 0$. Then we get

$$\begin{aligned} \dot{x} &= Ax + B\left(u + \sigma\right) \\ A &= \begin{bmatrix} 0 & \omega_B \\ 0 & \frac{-D}{2H} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{2H} \end{bmatrix} \end{aligned} \tag{1}$$

where σ is treated as a known perturbation.

2. For δ_{ref} and ω_0 constant references, the control problem becomes a regulation problem of $\delta \to \delta_{ref}$ and $\omega \to \omega_0$. This is transformed into a stabilization problem in the error states of $(e_\delta, e_\omega) = (0, 0)$. The control objective is for (1) to design a state feedback control law for the control input $u = t_m$ that renders $(e_\delta, e_\omega) = (0, 0)$ UGES.

3. Using $V(x) = x^\top P x$ as a CLF for the system, we differentiate to get

$$\dot{V} = x^\top P(Ax + B\left(u + \sigma\right)) + \left(Ax + B\left(u + \sigma\right)\right)^\top Px$$

Letting

$$\begin{aligned} u &= -K^\top x - \sigma \\ \dot{x} &= \left(A - BK^\top\right)x \end{aligned}$$

where the state-feedback gain $K \in \mathbb{R}^2$ is designed so that $A - BK^\top$ is Hurwitz. This is always possible since the pair (A, B) is controllable. The matrix $P = P^\top > 0$ is chosen to satisfy the Lyapunov equation

$$P(A - BK^\top) + (A - BK^\top)^\top P = -qI, \qquad q > 0.$$

This yields

$$\begin{aligned} \dot{V} &= x^\top P(A - BK^\top)x + x^\top \left(A - BK^\top\right)^\top Px \\ &= x^\top \left[P(A - BK^\top) + \left(A - BK^\top\right)^\top P\right] x \\ &= -qx^\top x, \end{aligned}$$

which by Lyapunov's Direct Method proves UGES of $x = 0$.

4. In presence of the disturbance $w(t)$, with $\|w\| \leq w_0$, we get

$$\begin{aligned} \dot{x} &= \left(A - BK^\top\right)x - Bw(t) \\ \lambda_{\min}(P)\,|x|^2 &\leq V(x) \leq \lambda_{\max}(P)\,|x|^2 \end{aligned}$$

Differentiating $V(x)$ gives

$$\begin{aligned} \dot{V} &= 2x^\top P\left[\left(A - BK^\top\right)x - Bw(t)\right] \\ &= 2x^\top P\left(A - BK^\top\right)x - 2x^\top PBw(t) \\ &\leq -q\,|x|^2 + 2\,|x|\,\|PB\|\,\|w\| \\ &\leq -q\,|x|^2 + 2\,\|PB\|\,|x|\,w_0 \\ &\leq 0, \qquad \forall\,|x| \geq \gamma w_0 \end{aligned}$$

where $\gamma := 2\frac{\|PB\|}{q}$. These bounds imply that the closed-loop system is Practically-UGES, since increasing the feedback gain K will decrease the ratio $\frac{\|PB\|}{q}$.
However, to make this explicit we can proceed the analysis. Since the level sets of $V(x)$ can be elliptic, while the set for which $\dot{V}$ is sign-indefinite is circular, given by $\Omega = \left\{x \in \mathbb{R}^2 : |x| \leq \gamma w_0\right\}$, we get that an estimate of the ultimate bound of $|x(t)|$ is given by any level set of $V(x)$ that contains Ω. The reason for this is that if $x(t)$ is on the boundary of Ω, for some $t \geq 0$, then $\dot{V}$ may be zero and the solutions will traverse along the level curve $V(x(t)) = c_0$. But due to the elliptic shape of this curve, the solutions may then leave Ω, which means that Ω cannot be the ultimate bounding set. This is illustrated in Figure 1, showing the circular set Ω (bounded by $|x| \leq \gamma w_0$), the smallest

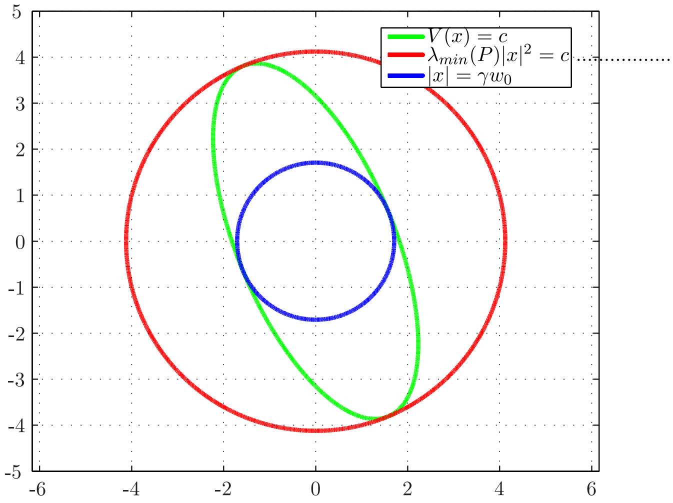

Figure 1: Level curves

elliptic level set of $V(x)$ containing Ω, and the smallest circular set that again contains this level set.
The largest circular set is calculated as follows: On the boundary of Ω we have $|x|^2 = \gamma^2 w_0^2$. Since the bounds $\lambda_{\min}(P)|x|^2 \leq V(x) \leq \lambda_{\max}(P)|x|^2$ are tight, we use

$$V(x) = c = \lambda_{\max}(P)|x|^2 = \lambda_{\max}(P)\gamma^2 w_0^2$$

to calculate the level set value c. Then we can use

$$\lambda_{\min}(P)|x|^2 = c$$

to calculate the radius of the largest set. It follows that an estimate of the ultimate bound becomes

$$|x(t)| \leq \sqrt{\frac{c}{\lambda_{\min}(P)}} = \sqrt{\frac{\lambda_{\max}(P)\gamma^2 w_0^2}{\lambda_{\min}(P)}} = 2\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\|PB\|}{q}w_0.$$

It can be shown that the ultimate bound can be made smaller by increasing the feedback gain K. Hence, the closed-loop system is Practically-UGES with respect to $x = 0$.

References