NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Marine Technology

Examination paper for:
TMR4243 MARINE CONTROL SYSTEMS II

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Examination date: June 1, 2015

Examination time (from-to): 09:00 – 13:00

Permitted examination support material: A – All printed and handwritten material allowed. All approved calculators allowed.

Other information:

The solutions to the 5 problems are given a maximum of 100 points. The exam counts 60% on the final grade.

Work fast; your answers should be short, clear, and concise. All statements should be explained; all mathematical answers should be derived.

Make qualified assumptions if:

- you cannot find an intermediate answer that is needed in further calculations, or
- there are missing (or obvious wrong) information in the problem text.

Language: English

Number of pages: 7

Number of pages enclosed:

Checked by:

Date Signature

TMR4243 - Marine Control Systems II

Exam

Spring 2015

Notation: Throughout this exam $|x|$ means the vector 2-norm, i.e. $|x| = \sqrt{x^\top x}$.

1 Solutions to nonlinear ODEs (22 pts)

1. Consider the nonlinear ordinary differential equation (ODE):

$$\dot{x} = f(x), \qquad x_0 = x(0)$$

   where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

   (a) What does it mean that a solution is **forward complete**?

   (b) What does it mean that this system is **locally Lipschitz**?

   (c) What can you say about the solutions if it is *locally Lipschitz*?

   (d) What does it mean that this system is **globally Lipschitz**?

   (e) What can you say about the solutions if it is *globally Lipschitz*?

2. For the scalar system

$$\dot{x} = x^{\frac{1}{3}}, \qquad x_0 = 0$$

   we propose the following solutions for $t \geq 0$:

$$x(t) = 0 \qquad \& \qquad x(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$$

   (a) Show that the proposed solutions are indeed solutions to the ODE.

   (b) What is the Lipschitz property of this ODE?

   (c) What is the stability property of this ODE?

3. For the scalar system

$$\dot{x} = kx, \qquad x_0 = -1, \quad k = \text{const.}$$

   we propose the solution for $t \geq 0$:

$$x(t) = -e^{kt}$$

   (a) Show that the proposed solution is indeed a solution to this ODE.

   (b) What is the Lipschitz property of this ODE?

   (c) What is the stability property of this ODE for $k > 0$?

2 Stability of nonlinear systems (18 pts)

1. Consider the nonlinear time-varying system:

$$\dot{x} = f(t, x), \qquad x_0 = x(t_0), \quad t_0 \geq 0$$

   where $x \in \mathbb{R}^n$ and $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$, and assume $f(t, 0) = 0, \forall t \geq 0$.

   (a) Define UGS and UGAS for the system solutions with respect to $x = 0$ based on class-$\mathcal{K}$ and class-$\mathcal{KL}$ functions.

   (b) What meaning does *Uniform* have?

   (c) What meaning does *Global* have?

   (d) What meaning does *Asymptotic* have?

2. In terms of equilibrium points you can have three types: *single point*, *multiple isolated points*, and a *continuum of points*. Explain each of these and wether they are possible for linear vs. nonlinear systems.

3. Consider the linear time-varying system:

$$\dot{x} = Q\left(x - x_d(t)\right) + \dot{x}_d(t)$$

   where $Q = Q^\top$, and $(x_d(t), \dot{x}_d(t))$ are bounded reference signals. Let

$$V(t, x) = \frac{1}{2}\left(x - x_d(t)\right)^\top \left(x - x_d(t)\right)$$

   be a Lyapunov function candidate.

   (a) Show how to differentiate $V(t, x)$.

   (b) What is your stability conclusion if $Q < 0$?

   (c) What is your stability conclusion if $Q \geq 0$?

3 Lyapunov stability (22 pts)

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + \frac{k}{\sqrt{5}} x_2^2 \end{aligned}$$

1. Assume $k = 0$. Differentiate the Lyapunov function

$$V_1(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

and discuss stability of the origin by Lyapunov's method and the Krasovskii-LaSalle theorem.

2. Assume $k = 0$. Differentiate the Lyapunov function

$$V_2(x) = \frac{3}{2}x_1^2 + x_1 x_2 + x_2^2$$

and discuss stability of the origin by Lyapunov's method.

3. Show that the system can be written in vector form with state vector $x = col(x_1, x_2)$, as

$$\dot{x} = Ax + kbx_2^2$$

by appropriate definitions of A and b.

   (a) Show that $\left|x_2^2\right| \leq |x|^2$.

   (b) Show that the Lyapunov function $V_2(x)$ above can be written

   $$V_2(x) = x^\top P x$$

   where P is symmetric positive definite.

   (c) Show that (P, A) satisfies the Lyapunov equation with $Q = I$.

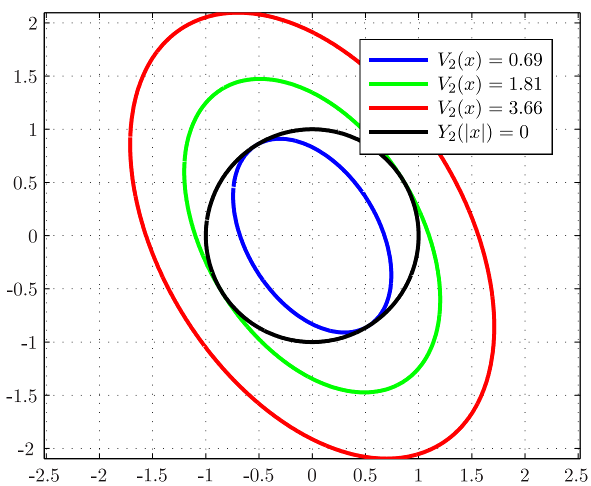
   (d) Assume $k = 1$. Show that $|Pb| = \frac{1}{2}$ and that

   $$\dot{V}_2 \leq -|x|^2 + |x|^3 =: Y_2(|x|)$$

   For what state values $x \in \mathbb{R}^2$ is $Y_2(|x|)$ negative definite?

   (e) For $k = 1$, what stability of $x = 0$ can you conclude for the nonlinear system based on $V_2(x)$?

4. The figure below shows three ellipsoidal level curves of $V_2(x)$, i.e. the curves $\left\{x \in \mathbb{R}^2 : V(x) = c\right\}$ for $c = \{0.69, 1.81, 3.66\}$, and the level curve $\{x \in \mathbb{R}^2 \backslash \{0\} : Y_2(|x|) = 0\}$.



Which of the 4 ellipses can be used as an estimate of the Region of Convergence (ROC)? Justify your answer.

4 Nonlinear feedback control (22 pts)

Consider a marine system

$$\begin{aligned} \dot{\eta} &= R(\eta)\nu \\ M\dot{\nu} &= \tau + \rho(\eta,\nu) + d(t) \end{aligned}$$

where $\eta \in \mathbb{R}^n$ is a position/orientation vector, $\nu \in \mathbb{R}^n$ is a velocity vector, $\tau \in \mathbb{R}^n$ is the control input, $\rho(\eta,\nu) \in \mathbb{R}^n$ is a locally Lipschitz vector function, $d(t)$ is a bounded bias, $M = M^\top > 0$, and $R(\eta)$ is a rotation matrix with the properties $R(\eta)^\top R(\eta) = R(\eta)R(\eta)^\top = I$ and $\dot{R} = R(\eta)S(\nu)$ where $S(\nu) = -S(\nu)^\top$.

1. Choosing $x_1 = \eta$, $x_2 = R(\eta)\nu$, and $x = col(x_1, x_2)$, show that you can transform the system into the controller form:

   $$\dot{x} = Ax + B\Gamma(\eta,\nu)\left[\tau + \varphi(\eta,\nu) + d(t)\right]$$

   where Γ is nonsingular. Show that (A, B) is a controllable pair.

2. Assume all states and the model is fully known, and $d(t) = 0$. Design a static state feedback control law $\tau = \alpha(\eta,\nu)$, using *feedback linearization*, that renders the closed-loop system linear and $x = 0$ UGES.

3. For $d(t)$ a bounded unknown bias, show that your static feedback control law renders the system Input-to-State-Stable (ISS) from $d(t)$ as input.

4. Suppose $d(t) = d$ =constant and unknown, and that the static control law $\tau = \alpha(\eta,\nu) + \tau_0$ renders the closed-loop system into

   $$\dot{x} = A_0 x + B\Gamma(\eta,\nu)\left[\tau_0 + d\right]$$

   where A_0 is Hurwitz. Augmenting the control law with integral action τ_0, let

   $$\begin{aligned} \dot{\xi} &= \gamma \\ \tau_0 &= -K_i\xi, \qquad K_i = K_i^\top > 0 \end{aligned}$$

   where the function γ shall be designed. Define $\tilde{\xi} = \xi - K_i^{-1}d$, let $P = P^\top > 0$ satisfy $PA_0 + A_0^\top P = -I$, and define the CLF

   $$V(x,\tilde{\xi}) = x^\top P x + \frac{1}{2}\tilde{\xi}^\top K_i \tilde{\xi}$$

   (a) Write down the state equations for $(\tilde{\xi}, x)$.

   (b) Differentiate $V(x,\tilde{\xi})$ and design γ so that $\dot{V}$ becomes

   $$\dot{V} = -x^\top x$$

   (c) What stability properties can you conclude for $(\tilde{\xi}, x) = (0,0)$ (Hint: Use the LaSalle-Yoshizawa theorem).

5 Backstepping (16 pts)

Consider the system

$$\begin{aligned} \dot{x}_1 &= g(x_1)x_2 - \sin(x_1) \\ \dot{x}_2 &= u - |x_2|\, x_2 + b \end{aligned}$$

where $1 \leq g(x_1) \leq 10$ for all $x_1 \in \mathbb{R}$, $u \in \mathbb{R}$ is the control input, and b is a constant bias.

1. Suppose $b = 0$, and let the control objective be to stabilize $(x_1, x_2) = (0, 0)$. Use *backstepping* to design a feedback control law for u that solves the regulation control objective.

2. Suppose $b \neq 0$, let $\hat{b}$ be an estimate of b, and define the estimation error $\tilde{b} = b - \hat{b}$. Augment the Step 2 CLF V_2 with a term $\frac{1}{2\gamma}\tilde{b}^2$, and design an adaptive update law for $\dot{\hat{b}}$ that renders $(z_1, z_2, \tilde{b}) = 0$ UGS and ensures the convergence $(x_1(t), x_2(t)) \to 0$.