Marine Control Systems II

Lecture 4: ISS and Feedback Linearization

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Goals of lecture

- Understand the concept of Input-to-State-Stability (ISS)
  - ISS definition.
  - ISS-Lyapunov function.
- Understand concepts from linear systems theory and transfer some of these to nonlinear systems.
  - Poles, zeros, relative degree, and zero dynamics.
- Being able to perform feedback linearization:
  - Full state feedback linearization.
  - Partial state feedback linearization.

Literature

- Khalil, H. K. (2015). Nonlinear Control:
  - Chapters: 4.2-4.4 and 9.1-9.4
- Lecture presentation.

Nonlinear systems with inputs

Consider the system

$$\dot{x} = f(t, x, u)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m, \ \forall t \geq 0$, and the map $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is smooth.

u is a measurable, locally essentially bounded function $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$.

Space of such functions is $\mathcal{L}_\infty^m$ with norm $||u_{[t_0,\infty)}|| := \operatorname{ess\,sup} \{u(t) : \ t \geq t_0 \geq 0\}$.

Use $||u|| = ||u_{[t_0,\infty)}||$ and let $||u_{[t_0,t]}||$ be the signal norm over the truncated interval $[t_0, t]$.
For each initial condition $t_0 \in [0, \infty)$, $x_0 = x(t_0) \in \mathbb{R}^n$ and each $u \in \mathcal{L}_\infty^m$, let $x(t, t_0, x_0, u)$ denote the solution at time t.

Example 1

Consider

$$\dot{x} = f(x,u) = -x^3 + u$$

With $u = 0$ we get that $V(x) = \frac{1}{2}x^2$ is a Lyapunov function, and the origin $x = 0$ is UGAS for $\dot{x} = f(x,0)$.
Consider now $u \neq 0$. Differentiating $V(x)$ gives

$$\begin{aligned}
\dot{V} &= -x^4 + xu \\
&= -(1-\lambda)x^4 - \lambda x^4 + xu, \qquad\qquad \lambda \in (0,1) \\
&\leq -(1-\lambda)x^4 - \lambda\,|x|^4 + |x|\,\|u\| \\
&\leq -(1-\lambda)x^4, \qquad \forall\,|x| \geq \sqrt[3]{\frac{\|u\|}{\lambda}}
\end{aligned}$$

For example, $\|u\| \leq 4$ and setting $\lambda = 0.5$, then

$$\dot{V} \leq -\frac{1}{2}x^4, \qquad \forall\,|x| \geq \sqrt[3]{8} = 2$$

Input-to-State Stability (ISS)

Definition

The system $\dot{x} = f(t,x,u)$ is ISS with respect to the origin $x = 0$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for each $u \in \mathcal{L}_\infty^m$, $\forall x_0$, and $\forall t \geq t_0 \geq 0$, the solution $x(t, t_0, x_0, u)$ is defined and satisfies

$$|x(t,t_0,x_0,u)| \leq \beta\left(|x_0|\,,\; t - t_0\right) + \gamma\left(\left|\left|u_{[t_0,t]}\right|\right|\right).$$

ISS-Lyapunov function

Definition

A smooth ISS-Lyapunov function for $\dot{x} = f(t, x, u)$ with respect to the origin is a smooth function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that satisfies:

1. there exist two class-$\mathcal{K}_\infty$ functions α_1 and α_2 such that for any $x \in \mathbb{R}^n$ and $t \geq 0$,
$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|),$$
2. there exist a class-$\mathcal{K}$ function α_3 and a $\mathcal{K}_\infty$-function χ such that for all $t \geq 0$, $x \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$,
$$|x| \geq \chi(|u|) \ \Rightarrow \ V^t(t, x) + V^x(t, x) f(t, x, u) \leq -\alpha_3(|x|).$$

Example 2

Consider
$$\dot{x} = f(x, u) = -x^3 + xu$$
Then $V(x) = \frac{1}{2}x^2$ is an ISS-Lyapunov function.
Proof:
$$\begin{aligned} \dot{V} &= -x^4 + x^2 u \\ &= -(1-\lambda)x^4 - \lambda x^4 + x^2 u, \qquad\qquad \lambda \in (0, 1) \\ &\leq -(1-\lambda)x^4 - \lambda |x|^4 + |x|^2 |u| \\ &\leq -(1-\lambda)x^4, \qquad \forall\, |x| \geq \sqrt{\frac{1}{\lambda}|u|} \end{aligned}$$

ISS-Lyapunov function

An equivalent representation of this is:

2.' *There exist two class-$\mathcal{K}_\infty$ functions α_3 and α_4 such that for all* $t \geq 0$, $x \in \mathbb{R}^n$, *and* $u \in \mathbb{R}^m$,

$$V^t(t,x) + V^x(t,x) f(t,x,u) \leq -\alpha_3\left(|x|\right) + \alpha_4\left(|u|\right).$$

...Example 2

Consider again

$$\dot{x} = f(x,u) = -x^3 + xu$$

for which with $V(x) = \frac{1}{2}x^2$ we got

$$\dot{V} = -x^4 + x^2 u$$

Using Young's inequality $ab \leq \kappa a^2 + \frac{1}{4\kappa} b^2$, $\kappa > 0$, with $a = x^2$ and $b = u$, gives

$$\begin{aligned} \dot{V} &\leq -x^4 + \kappa x^4 + \frac{1}{4\kappa} u^2 \\ &= -(1-\kappa)\left|x\right|^4 + \frac{1}{4\lambda}\left|u\right|^2 =: -\alpha_3\left(|x|\right) + \alpha_4\left(|u|\right). \end{aligned}$$

Choosing $\kappa \in (0,1)$ ensures that ISS is proven.

ISS-Lyapunov function

Note:

$$V^t(t,x) + V^x(t,x)f(t,x,u) \le -\alpha_3(|x|) + \alpha_4(|u|)$$
$$\Downarrow$$
$$V^t(t,x) + V^x(t,x)f(t,x,u) \le -\varepsilon\alpha_3(|x|)$$
$$\forall\, |x| \ge \alpha_3^{-1}\left(\frac{1}{1-\varepsilon}\alpha_4(|u|)\right) =: \chi(|u|)$$

where $\varepsilon \in (0,1)$.

The converse is also true for stability of an equilibrium point.

ISS sufficiency theorem

Theorem

Assume that $x = 0$ is 0-invariant for $\dot{x} = f(t,x,u)$, i.e. $x = 0$ is invariant for $\dot{x} = f(t,x,0)$ which means $x_0 = 0 \implies x(t) = 0, \forall t \ge t_0$.

If the system admits a smooth ISS-Lyapunov function with respect to the origin, then it is ISS.

ISS sufficiency theorem

Corollary

Suppose the system is ISS with respect to the origin. Then the origin of $\dot{x} = f(t, x, 0)$ is UGAS.

Corollary

Suppose the system is ISS with respect to the origin. Then

$$\lim_{t \to \infty} |u(t)| = 0 \;\Rightarrow\; \lim_{t \to \infty} |x(t)| = 0.$$

GES implies ISS theorem

We consider the system

$$\dot{x} = f(x, u) \tag{1}$$

where f is locally Lipschitz in x and u.

Lemma

(Khalil, 2015, Lemma 4.5)

Suppose $f(x, u)$ is ***continuously differentiable*** *and* ***globally Lipschitz*** *in (x, u).*

If $x = 0$ is ***GES*** *for $\dot{x} = f(x, 0)$, then (1) is ISS with respect to $x = 0$.*

Linear transfer function

Consider the strictly proper transfer function

$$\frac{Y(s)}{U(s)} = G(s) = \frac{b(s)}{a(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \qquad n > m$$

The relative degree of this system is

$$r = n - m > 0.$$

We say that:

- The *zeros* are the roots of the numerator polynomial.
- The *poles* are the roots of the denominator polynomial.
- The system is *stable* if all its poles have nonpositive real part.
- A stable system is *minimum phase* if all zeros have negative real part.
- A stable system is *nonminimum phase* if some zero has positive real part.

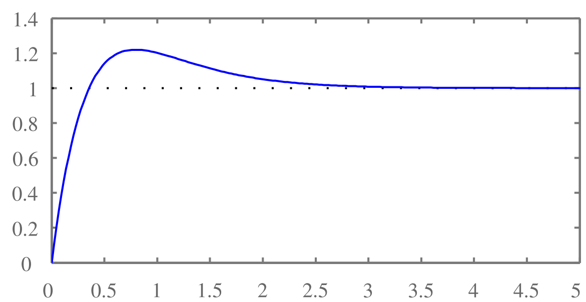
Example 3

Consider

$$\frac{Y(s)}{U(s)} = G(s) = \frac{24}{5}\frac{(s+1)(s+5)}{(s+2)(s+3)(s+4)} = \frac{24}{5}\left(\frac{s^2+6s+5}{s^3+9s^2+26s+24}\right)$$

- Relative degree: $r = 3 - 2 = 1$.
- Zeros: $z_1 = -1$, $z_2 = -5$
- Poles: $p_1 = -2$, $p_2 = -3$, $p_3 = -4$
- System is stable and minimum phase.

Step response:



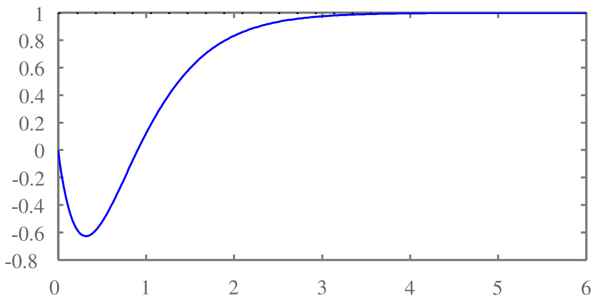
Example 4

Consider

$$\frac{Y(s)}{U(s)} = G(s) = \frac{-24}{5}\frac{(s-1)(s+5)}{(s+2)(s+3)(s+4)} = \frac{-24}{5}\left(\frac{s^2+4s-5}{s^3+9s^2+26s+24}\right)$$

- Zeros: $z_1 = +1$, $z_2 = -5$
- Poles: $p_1 = -2$, $p_2 = -3$, $p_3 = -4$
- System is stable but **nonminimum** phase.

Step response:



Zero dynamics

Consider the strictly proper transfer function

$$\frac{Y(s)}{U(s)} = G(s) = k\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}, \qquad n > m$$

Suppose $z_2 = p_2$. Then we have a pole-zero cancellation. This gives rise to *zero dynamics* (or *internal dynamics*):

Definition

Zero dynamics are (internal) states that are not observable from the output of the system.

...Zero dynamics

For the zero dynamics we have:

- The zero dynamics can be found by setting the output $y(t)$ and its derivatives identically equal to zero. The remaining dynamics is then the zero dynamics.
- The zero dynamics can be stable or unstable. Even if the zero dynamics does not affect the output, unstable zero dynamics (that may grow unbounded) will typically have a detrimental effect on the system.
- If not stable, then one should aim for controllability of the zero dynamics to render it stable through control action.

Example 5: Stable zero dynamics

Consider
$$\frac{Y(s)}{U(s)} = G(s) = \frac{2\,(s+1)}{(s+1)\,(s+2)} = \frac{2}{(s+2)}$$
which has a pole-zero cancellation of a stable pole. A state space realization of the complete system is:

$$\begin{aligned}\dot{x}_1 &= -3x_1 - \sqrt{2}x_2 + u, \qquad \dot{x}_2 = \sqrt{2}x_1 \\ y &= 2x_1 + \sqrt{2}x_2\end{aligned}$$

Differentiating y gives

$$\begin{aligned}\dot{y} &= 2\dot{x}_1 + \sqrt{2}\dot{x}_2 = 2\left(-3x_1 - \sqrt{2}x_2 + u\right) + \sqrt{2}\sqrt{2}x_1 \\ &= -6x_1 - 2\sqrt{2}x_2 + 2x_1 + 2u = -4x_1 - 2\sqrt{2}x_2 + 2u \\ &= -2\,(y-u)\end{aligned}$$

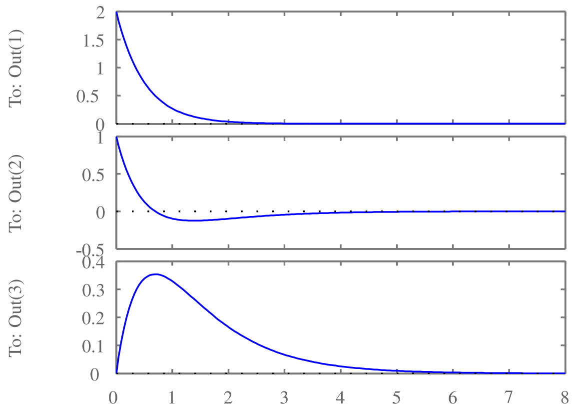
Setting $y = \dot{y} = 0$ gives

$$2x_1 + \sqrt{2}x_2 = 0 \qquad \Rightarrow \qquad x_1 = -\frac{1}{2}\sqrt{2}x_2, \quad x_2 = -\sqrt{2}x_1$$

$$\dot{x}_1 = -\frac{1}{2}\sqrt{2}\dot{x}_2 = -\frac{1}{2}\sqrt{2}\sqrt{2}x_1 = -x_1$$

$$\dot{x}_2 = -\sqrt{2}\dot{x}_1 = -\sqrt{2}\left[-3\left(-\frac{1}{2}\sqrt{2}x_2\right) - \sqrt{2}x_2 + u\right] = -x_2 - \sqrt{2}u$$

Impulse responses (y, x_1, x_2):



Example 6: Unstable zero dynamics

Consider

$$\frac{Y(s)}{U(s)} = G(s) = \frac{2(s-1)}{(s-1)(s+2)} = \frac{2}{(s+2)}$$

which has a pole-zero cancellation of an unstable pole. A state space realization is:

$$\dot{x}_1 = -x_1 + \sqrt{2}x_2 + u, \qquad \dot{x}_2 = \sqrt{2}x_1$$

$$y = 2x_1 - \sqrt{2}x_2$$

Differentiating y gives

$$\begin{aligned}\dot{y} &= 2\dot{x}_1 - \sqrt{2}\dot{x}_2 = 2\left(-x_1 + \sqrt{2}x_2 + u\right) - \sqrt{2}\sqrt{2}x_1 \\ &= -2x_1 + 2\sqrt{2}x_2 - 2x_1 + 2u = -4x_1 + 2\sqrt{2}x_2 + 2u \\ &= -2(y-u)\end{aligned}$$

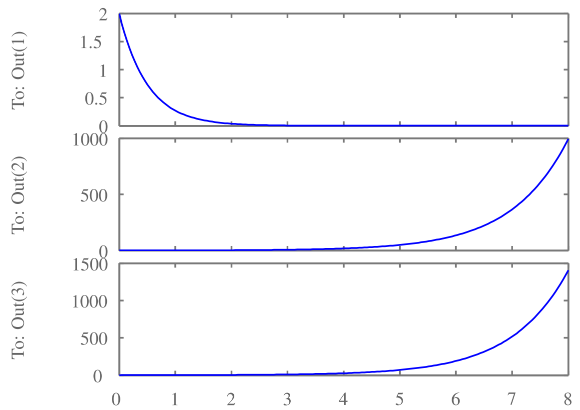
Setting $y = \dot{y} = 0$ gives

$$2x_1 - \sqrt{2}x_2 = 0 \qquad \Rightarrow \qquad x_1 = \frac{1}{2}\sqrt{2}x_2, \quad x_2 = \sqrt{2}x_1$$

$$\dot{x}_1 = \frac{1}{2}\sqrt{2}\dot{x}_2 = \frac{1}{2}\sqrt{2}\sqrt{2}x_1 = x_1$$

$$\dot{x}_2 = \sqrt{2}\dot{x}_1 = \sqrt{2}\left[-\left(\frac{1}{2}\sqrt{2}x_2\right) + \sqrt{2}x_2 + u\right] = x_2 + \sqrt{2}u$$

Impulse responses (y, x_1, x_2):



Example 7

Consider the nonlinear pendulum system

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -\frac{g}{l}\sin\theta - \frac{k}{ml^2}\omega + \frac{\cos\theta}{ml}u$$

on the domain $D = \left\{(\theta, \omega) \in \mathbb{R}^2 : \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\right\}$. The control

$$u = \frac{ml}{\cos\theta}\left[\frac{g}{l}\sin\theta + \frac{k}{ml^2}\omega + v\right], \qquad (\theta, \omega) \in D$$

renders the system into the linear double integrator

$$\dot{\theta} = \omega$$
$$\dot{\omega} = v.$$

Setting $v = -k_1\theta - k_2\omega$ will then make the system exponentially stable.

Full state feedback linearization

We follow the lecture by Khalil (2002, Lecture 26) and consider a nonlinear system on affine form

$$\dot{x} = f(x) + G(x)u, \qquad f(0) = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

Suppose there is a coordinate transformation $z = T(x)$, defined $\forall x \in \mathcal{D} \subset \mathbb{R}^n$, that transforms the state into *controller form*

$$\dot{z} = Az + B\gamma(x)\left[u - \alpha(x)\right]$$

where $[A, B]$ is a controllable pair, and $\gamma(x)$ is nonsingular $\forall x \in \mathcal{D}$. Then we can apply the feedback linearizing control

$$u = \alpha(x) + \gamma(x)^{-1}v$$

where v is a new control input, resulting in the new system

$$\dot{z} = Az + Bv$$

Full state feedback linearization

Now we can apply a linear state feedback control

$$v = -Kz$$

where K is designed to make $A - BK$ Hurwitz.

Result: the origin $z = 0$ of the closed-loop system

$$\dot{z} = (A - BK)\, z$$

is UGES.

In the x-dynamics we have designed the control and closed-loop

$$\begin{aligned} u &= \alpha(x) - \gamma(x)^{-1}KT(x) \\ \dot{x} &= f(x) + G(x)\left[\alpha(x) - \gamma(x)^{-1}KT(x)\right] \end{aligned}$$

But what about stability of $x = 0$?

Full state feedback linearization

Since $x \mapsto T(x)$ is a *local diffeomorphism* on $\mathcal{D}$, $x = 0$ is LAS. However, note that it is not in general exponentially stable. Exponential convergence in the z-state is transformed to asymptotic convergence in the x-state.

In general $x = 0$ is not globally stable. However, if $T(x)$ is a *global diffeomorphism*, then $x = 0$ is GAS.

Robustness

Uncertainties in the models for α, γ, and T is typically the criticism for feedback linearization; it relies on exact cancellation of nonlinear dynamics.

Let $\alpha_0(x)$, $\gamma_0(x)$, and $T_0(x)$ be your nominal models for α, γ, and T, so that the realized control and closed-loop become

$$\begin{aligned} u &= \alpha_0(x) - \gamma_0(x)^{-1} K T_0(x) \\ \dot{z} &= (A - BK)\, z + B\delta(z) \\ \delta &= \gamma \left[\alpha_0 - \alpha + \gamma^{-1} K z - \gamma_0^{-1} K T_0\right] \end{aligned} \tag{2}$$

Let

$$V(z) = z^\top P z, \qquad P(A - BK) + (A - BK)^\top P = -qI$$
$$\dot{V} = -q z^\top z + 2 z^\top P B \delta(z) \le -q\, |z|^2 + 2\, \|PB\|\, |z|\, |\delta(z)|$$

Robustness

Lemma

If $\exists k$ satisfying $0 \le k < \frac{\lambda q}{2\|PB\|}$, for $\lambda \in (0,1)$, such that $|\delta(z)| \le k\,|z|$, $\forall z \in \mathbb{R}^n$, then $z = 0$ of (2) is GES.

Proof: We get

$$
\begin{aligned}
\dot{V} &= -q z^\top z + 2 z^\top P B \delta(z) \\
&\le -q\,|z|^2 + 2\,\|PB\|\,|z|\,|\delta(z)| \\
&\le -(1-\lambda) q\,|z|^2 - \lambda q\,|z|^2 + 2k\,\|PB\|\,|z|^2 \\
&= -(1-\lambda) q\,|z|^2 - (\lambda q - 2k\,\|PB\|)\,|z|^2 \\
&\le -(1-\lambda) q\,|z|^2, \qquad \forall k < \frac{\lambda q}{2\,\|PB\|},
\end{aligned}
$$

Q.E.D.

Robustness

Lemma

If $\exists k$ satisfying $0 \le k < \frac{\lambda q}{2\|PB\|}$ and $\varepsilon > 0$ such that $|\delta(z)| \le k\,|z| + \varepsilon$, $\forall z \in \mathbb{R}^n$, then $z = 0$ of (2) is globally ultimately bounded.

Proof: We get

$$
\begin{aligned}
\dot{V} &\le -(1-\lambda) q\,|z|^2 - \lambda q\,|z|^2 + 2\,\|PB\|\,|z|\,(k\,|z| + \varepsilon) \\
&\le -(1-\lambda) q\,|z|^2 - (\lambda q + 2k\,\|PB\|)\,|z|^2 + 2\varepsilon\,\|PB\|\,|z| \\
&\le -(1-\lambda) q\,|z|^2 + 2\varepsilon\,\|PB\|\,|z|, \qquad \forall k < \frac{\lambda q}{2\,\|PB\|} \\
&\le -\frac{1-\lambda}{2} q\,|z|^2 - \frac{1-\lambda}{2} q\,|z|^2 + 2\varepsilon\,\|PB\|\,|z| \\
&\le -\frac{1-\lambda}{2} q\,|z|^2, \qquad \forall\,|z| \ge \frac{4\varepsilon\,\|PB\|}{q\,(1-\lambda)},
\end{aligned}
$$

Q.E.D.

Example 8: Is feedback linearization a good idea?

See Khalil (2002, Lecture 26). Let

$$\begin{aligned}
\dot{x} &= ax - bx^3 + u, \qquad a, b > 0 \\
u &= -(k+a)x + bx^3, \qquad k > 0 \\
\dot{x} &= -kx
\end{aligned}$$

But we notice that $-bx^3$ is a stabilizing nonlinear damping term. Why cancel this?
Instead

$$\begin{aligned}
u &= -(k+a)x, \qquad k > 0 \\
\dot{x} &= -kx - bx^3
\end{aligned}$$

Which design is better?

Partial feedback linearization

We follow the lecture by Khalil (2002, Lecture 27) and consider a nonlinear system on affine form

$$\dot{x} = f(x) + G(x)u, \qquad f(0) = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

Suppose there is a coordinate transformation

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

defined $\forall x \in \mathcal{D} \subset \mathbb{R}^n$, that transforms the system into

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A\xi + B\gamma(x)\left[u - \alpha(x)\right]
\end{aligned}$$

where $[A, B]$ is a controllable pair, and $\gamma(x)$ is nonsingular $\forall x \in \mathcal{D}$.

Partial feedback linearization

Applying the partial feedback linearizing control

$$u = \alpha(x) + \gamma(x)^{-1} v$$
$$v = -K\xi$$

where $A - BK$ is Hurwitz. Then

Lemma

The origin $(\eta, \xi) = 0$ *of*

$$\dot{\eta} = f_0(\eta, \xi)$$
$$\dot{\xi} = (A - BK)\,\xi$$

is LAS if $\eta = 0$ *of the zero-dynamics* $\dot{\eta} = f_0(\eta, 0)$ *is LAS.*

Proof: We have $V_1(\eta)$, with bounds $V_1^{\eta} f_0(\eta, 0) \leq -\alpha_1(\eta)$ and $|V_1^{\eta}| \leq \rho(|\eta|)$. We use

$$V(z) = V_1(\eta) + c\sqrt{\xi^{\top} P \xi}$$

Differentiating. . .

Partial feedback linearization

. . .

$$\begin{aligned}
\dot{V} &= V_1^{\eta}\dot{\eta} + c\frac{1}{2\sqrt{\xi^{\top}P\xi}} 2\xi^{\top}P\dot{\xi} \\
&= V_1^{\eta} f_0(\eta,\xi) \pm V_1^{\eta} f_0(\eta,0) + \frac{c}{2\sqrt{\xi^{\top}P\xi}}\xi^{\top}\left[P\,(A-BK) + (A-BK)^{\top}P\right]\xi \\
&\leq -\alpha_1(\eta) + |V_1^{\eta}|\,|f_0(\eta,\xi) - f_0(\eta,0)| - \frac{c}{2\sqrt{\xi^{\top}P\xi}}\xi^{\top}\xi \\
&\leq -\alpha_1(\eta) + L\rho(|\eta|)\,|\xi| - \frac{c\,|\xi|^2}{2\sqrt{\lambda_{\max}(P)}\sqrt{|\xi|^2}}, \qquad L \text{ a Lipschitz const.} \\
&\leq -\alpha_1(\eta) - \left[\frac{c}{2\sqrt{\lambda_{\max}(P)}} - L\rho(|\eta|)\right]|\xi| \\
&< 0, \qquad c > 2\sqrt{\lambda_{\max}(P)}Lr
\end{aligned}$$

where r is a bound on $\rho(|\eta|)$ on a neighborhood of the origin.

Partial feedback linearization

If $\eta = 0$ of the zero-dynamics $\dot{\eta} = f_0(\eta, 0)$ is GAS,

will then the origin $(\eta, \xi) = 0$ of

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= (A - BK)\,\xi \end{aligned}$$

be GAS?

Partial feedback linearization

In general, NO!

Example 9: Consider

$$\dot{\eta} = f_0(\eta, \xi) = -\eta + \eta^2 \xi, \qquad \dot{\xi} = v.$$

The origin of the zero dynamics is GES,

$$\dot{\eta} = f_0(\eta, 0) = -\eta,$$

but the origin of

$$\dot{\eta} = -\eta + \eta^2 \xi, \qquad \dot{\xi} = -k\xi, \quad k > 0$$

is not GAS. It has an ROC

$$\{(\eta, \xi) : \eta\xi < 1 + k\}.$$

Partial feedback linearization

Sufficiency can be assured by ...

Lemma

The origin $(\eta, \xi) = 0$ *of*

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= (A - BK)\,\xi\end{aligned}$$

is GAS if the system $\dot{\eta} = f_0(\eta, \xi)$ *is ISS.*

Proof: Use Lemma 4.6 in Khalil (2015):
If $\dot{x}_1 = f_1(x_1, x_2)$ is ISS, and the origin of $\dot{x}_2 = f_2(x_2)$ is GAS, then the origin of the cascaded system is GAS.

See also (Loría and Panteley, 2005) for more relevant results on stability of cascades and interconnections.

Effect of uncertainties

See Khalil (2002, Lecture 27) Let $\alpha_0(x)$, $\gamma_0(x)$, and $T_{20}(x)$ be your nominal models for α, γ, and T_2, so that the realized control and closed-loop become

$$\begin{aligned}u &= \alpha_0(x) - \gamma_0(x)^{-1} K T_{20}(x) \\ \dot{\eta} &= f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\,\xi + B\delta(z) \\ \delta &= \gamma\left[\alpha_0 - \alpha + \gamma^{-1} K \xi - \gamma_0^{-1} K T_{20}\right]\end{aligned} \tag{3}$$

Lemma

- *If* $|\delta(z)| \leq \varepsilon$, $\forall z$, *and* $\dot{\eta} = f_0(\eta, \xi)$ *is ISS, then the state* $z = (\eta, \xi)$ *is globally ultimately bounded by a class-*$\mathcal{K}$ *function of* ε.
- *If* $|\delta(z)| \leq k\,|z|$ *in some neighborhood of* $z = 0$, *with* k *sufficiently small, and* $\dot{\eta} = f_0(\eta, 0)$ *is LES, then* $z = 0$ *is LES for (3).*

Example 10: Linearized pendulum-cart

The linearized inverted pendulum-cart system can be written (see Lecture 3)

$$\begin{aligned}\dot{p} &= v\\ \dot{v} &= \frac{gm}{M}\theta + \frac{1}{M}u\\ \dot{\theta} &= \omega\\ \dot{\omega} &= \frac{g}{L}\left(\frac{m}{M}+1\right)\theta + \frac{1}{ML}u\end{aligned}$$

Let the control objective be to regulate $(\theta, \omega) \to 0$, and apply the control

$$\begin{aligned}u &= ML\left[-\frac{g}{L}\left(\frac{m}{M}+1\right)\theta + u_0\right]\\ u_0 &= -k_1\theta - k_2\omega.\end{aligned}$$

Letting $\eta = (p, v)$ and $\xi = (\theta, \omega)$ renders the closed-loop system

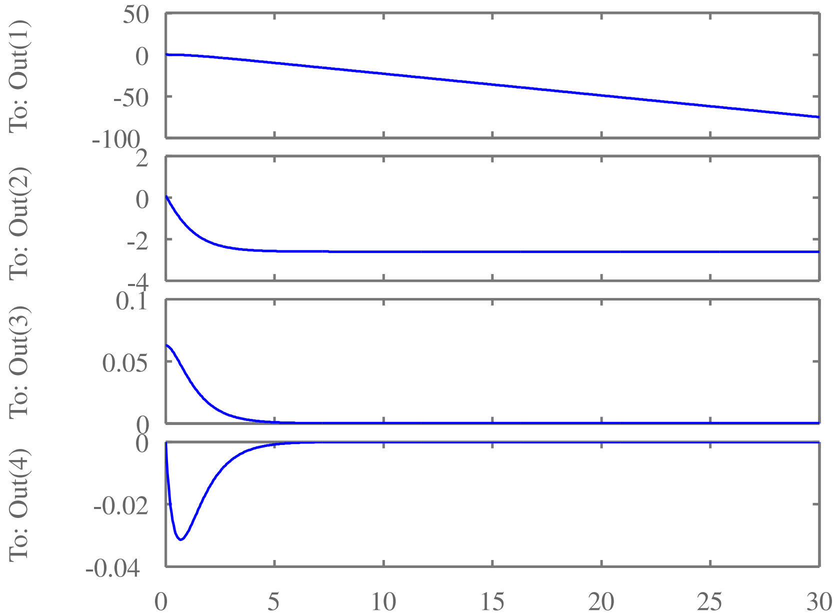
$$\begin{aligned}\dot{\eta} &= \begin{bmatrix}\dot{p}\\ \dot{v}\end{bmatrix} = \begin{bmatrix} v \\ -g\,(1+Lk_1)\,\theta - Lk_2\omega\end{bmatrix} = f_0(\eta, \xi)\\ \dot{\xi} &= \begin{bmatrix}\dot{\theta}\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix} 0 & 1\\ -k_1 & -k_2\end{bmatrix}\begin{bmatrix}\theta\\ \omega\end{bmatrix} = A\xi\end{aligned}$$

In this case, the zero dynamics, given by $\xi \equiv 0$, is

$$\dot{\eta} = f_0(\eta, 0) \Rightarrow \begin{bmatrix}\dot{p}\\ \dot{v}\end{bmatrix} = \begin{bmatrix} v\\ 0\end{bmatrix},$$

which is not stable (position will drift).

Simulating the response of the system to the initial conditions $p_0 = 0$, $v_0 = 0.1$, $\theta_0 = \frac{\pi}{50}$, and $\omega_0 = 0$ gives the responses for (p, v, θ, ω):



Preparations for next lecture

Backstepping:

- Note on Mathematical notations, section on inequalities: Especially Young's inequality.
- Khalil, H. K. (2015). Nonlinear Control.
  - Chapters: 9.7 and 9.5.
- Skjetne, R. (2005). The Maneuvering Problem.
  - Ch. 4.1 (for presentation of systematic ISS backstepping design).
- Lecture presentation.

Bibliography

Khalil, H. K. (2002). Lecture notes on nonlinear systems. Michigan State Univ.

Khalil, H. K. (2015). *Nonlinear Control*. Pearson Education Ltd., global edition.

Loría, A. and Panteley, E. (2005). Cascaded systems: Stability and stabilization. Lecture notes, NTNU, Dept. Eng. Cybernetics. April 21, 2005.