PYTHON FOR THE FINANCIAL ECONOMIST, ORDINARY EXAM 2023
DYNAMIC PORTFOLIO STRATEGIES

Copenhagen Business School
19. December 2023

3 weeks, home assignment

The home assignment is to be answered in groups of two students (maximum of 25 A4-pages) or individually (maximum of 15 A4-pages). The students must individualize the assignment.

The take-home assignment should take the form of an academic report written in either Word or Latex converted to pdf format. It is expected that students present relevant formulas, present results using visualizations and tables, include relevant references, etc. The overall impression of how the results are presented will count in the assessment.

The analysis should be performed using Python. Please attach code / Jupyter notebooks.

If you think that you do not have all the necessary information to answer a problem, make the necessary assumptions in order to proceed and state these assumptions in the solution.

Good luck!

Buy-and-Hold versus Constant-Mix portfolios

Assume that the market can be represented by two assets taking the value $V_{1,t}$ and $V_{2,t}$ at time t respectively with dynamics given by

$$dV_{1,t} = V_{1,t}\mu_1 dt + \sigma_1 V_{1,t} dZ_{1,t}$$
$$dV_{2,t} = V_{2,t}\mu_2 dt + \sigma_2 V_{2,t}(\rho dZ_{1,t} + \sqrt{1-\rho^2} dZ_{2,t})$$

where $Z_{1,t}$ and $Z_{2,t}$ are standard Brownian motions with $\mathrm{E}[dZ_{1,t} dZ_{2,t}] = 0$. What is the distribution of the value vector $\mathbf{V}_t = (V_{1,t}, V_{2,t})^\top$ when assuming $\mathbf{V}_0 = (1,1)^\top$?

In order to calculate buy-and-hold mean-variance optimal portfolios, we have to find $\mathrm{E}[\mathbf{V}_t]$ and $\mathrm{Cov}[\mathbf{V}_t]$. Using your knowledge of the distribution of $\mathbf{V}_t$, present the relevant formulas.

Assume the parameters

$$\begin{aligned} &\mu_1 = 0.03, \quad &&\mu_2 = 0.06 \\ &\sigma_1 = 0.075, \quad &&\sigma_2 = 0.15 \\ &\rho = 0.2 \end{aligned}$$

Use simulations to validate the analytical expression for the expectation and variance-covariance matrix.

The investor can at time $t = 0$ choose the portfolio holdings $\boldsymbol{h}$ and hold that portfolio for the entire investment horizon. The value of the buy-and-hold portfolio at a given point in time is given by

$$V_t^{BH} = \mathbf{h}^\top \mathbf{V}_t$$

with corresponding portfolio weights[1]

$$\mathbf{w}_t = \frac{1}{V_t^{BH}} \mathbf{h} \odot \mathbf{V}_t$$

What is the expectation and variance of the buy-and-hold portfolio for a given horizon?

Alternatively, the investor can choose the weights $\mathbf{w}$ to which he / she continuously rebalances. This is known as a constant-mix portfolio. The weights are constant, but the portfolio holdings will change such that

$$\mathbf{h}_t = \mathbf{w} \odot \frac{V_t^{CM}}{\mathbf{V}_t}$$

[1] $\odot$ is the Hadamard (element-by-element) product

The dynamics of the constant-mix portfolio can be written as

$$dV_t^{CM} = V_t^{CM}\mathbf{w}^\top\boldsymbol{\mu}dt + V_t^{CM}\sqrt{\mathbf{w}^\top\boldsymbol{\sigma}^2\mathbf{w}}dZ_t$$

where

$$\boldsymbol{\mu} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix},\ \boldsymbol{\sigma}^2 = \begin{pmatrix}\sigma_1^2 & \sigma_1\sigma_2\rho\\ \sigma_1\sigma_2\rho & \sigma_2^2\end{pmatrix},$$

What is the distribution of V_t^{CM}? What is the expectation and variance? Check using simulations that the formulas for the expectation and variance are correct. Also check that the simulated percentiles align with the theoretical ones.

Assume that $V_0^{CM} = V_0^{BH} = 1$. Define the minimum-variance optimization problem for respectively the buy-and-hold and constant-mix case using a suitable budget constraint and a non-negative constraint. Solve the minimum-variance problem for different investment horizons and present results.

Define the mean-variance optimization problem where the investor targets a portfolio value $\mathrm{E}[V_T^{BH}] = \mathrm{E}[V_T^{CM}] = e^{0.05T}$ where T defines the investment horizon. Solve it and present results.

Compare the optimal mean-variance portfolios at the 5-year horizon using different relevant metrics.

Portfolio insurance strategies

Time series strategies such as option based portfolio insurance or constant proportion portfolio insurance are typically defensive strategies that dynamically protect profits from risky offensive strategies. The goal is to implement an allocation rule which rebalances between a low-risk and risky investment based on the information available at that given point in time. Given a budget constraint, the allocation rule is fully determined by the holding of the risky investment h_t^{risky}.

Assume that the risky investment follows a Geometric Brownian motion ($V_{1,0} = 1$)

$$dV_{1,t} = V_{1,t}\mu_1 dt + \sigma_1 V_{1,t} dZ_{1,t}$$

with

$$\mu_1 = 0.06, \quad \sigma_1 = 0.15$$

Furthermore assume that the risk-free rate, r_t, follows a Vasicek process

$$dr_t = \kappa[\theta - r_t]dt + \beta dZ_{2,t}$$

with $\mathrm{E}[dZ_{1,t} dZ_{2,t}] = 0$ and the parameters.

$$r_0 = 0.03, \quad \kappa = 1.0, \quad \theta = 0.03, \quad \beta = 0.02$$

What is the value of the bank account that earns the risk free rate if it has an initial value of 1? Visualize the distribution of the short rate and bank account for the next five years. Comment on your findings.

The investor seeks to allocate, potentially dynamically, between the risk-free and risky investment to optimize the utility at a 5-year horizon. The investor has an initial budget of $W_0 = 1000000$ USD and a constant relative risk aversion (CRRA) utility function

$$U^{CRRA}(W_5) = \frac{W_5^{1-\gamma}}{1-\gamma}, \ \gamma = 2$$

where W_5 is the wealth at the end of the investment horizon. First, the investor considers buy-and-hold and constant-mix strategies. Describe how to find the optimal portfolios when the investor is maximizing the expected utility. Find the optimal buy-and-hold and constant-mix allocations and compare them using relevant metrics and visualizations.

A constant proportion portfolio insurance strategy can be defined by an allocation to the risky investment using the rule

$$h_t^{risky} = \min\{mC_t, bW_t\}\frac{1}{V_{1,t}}$$

where C_t is the cushion, i.e. the different between current wealth and the lower limit of wealth that the investor wants to protect, and m is a multiplier defining the wealth allocated to the risky asset. The allocation to the risky investment is limited by a leverage constraint such that the wealth allocated to the risky asset cannot exceed bW_t at a given point in time. The cushion is defined as

$$C_t = W_t - c \cdot V_{t,2}$$

where c is a constant and $V_{t,2}$ is the value of the bank account. The remaining wealth is invested in the risk free asset. Note, that if the cushion is zero, the investor will be fully invested in the risk free investment. Consider the strategy with $m = 2.5$, $b = 1$, $c = 700000$ and weekly rebalancing. Explain the intuition behind the listed strategy. How does the distribution of wealth and utility differ at the 5-year horizon compared to the constant-mix and buy-and-hold strategy?

Optimize the buy-and-hold and constant-mix strategy given the constraint that the final wealth is larger than $c \cdot V_{2,5}$ at the end of the investment horizon. How does the distribution of wealth and utility differ at the 5-year horizon compared to the constant-mix and buy-and-hold strategy?

In the Vasicek model, the zero-coupon yield can be written as

$$y_t(\tau) = \frac{1}{\tau}(a(\tau) + b(\tau)r_t)$$

where

$$a(\tau) = y_\infty[\tau - b(\tau)] + \frac{\beta^2}{4\kappa}b(\tau)^2$$
$$b(\tau) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right)$$

with

$$y_\infty = \hat{\theta} - \frac{\beta^2}{2\kappa^2} = \theta - \frac{\lambda\beta}{\kappa} - \frac{\beta^2}{2\kappa^2}$$

Assume $\lambda = -0.2$. Plot the initial yield curve.

Visualize the evolution of the zero-coupon bond price that has an initial time to maturity of 5 years.

Adapt the constant proportion portfolio insurance strategy by using the zero-coupon bond as the risk free asset. The goal is to have terminal wealth larger than 800000 USD. How does this change the results?

Implement a monthly rebalancing frequency instead of the weekly rebalancing frequency and increase the multiplier to $m = 5$. What do you observe?

Finally, discuss some of the limitations with above analysis if the investor had to implement the strategy in realistic markets where market frictions are present.

References