Appendices

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A

Nonlinear Stability Theory

This appendix briefly reviews some useful results from nonlinear stability theory. The methods are classified according to:

- Lyapunov stability of nonlinear *autonomous* systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, that is systems where $\boldsymbol{f}(\boldsymbol{x})$ does not explicitly depend on the time t.
- Lyapunov stability of nonlinear *nonautonomous* systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{t})$, that is systems where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{t})$ does depend on t explicitly.

A.1 Lyapunov Stability for Autonomous Systems

Before stating the main Lyapunov theorems for *autonomous* systems, the concepts of stability and convergence are briefly reviewed (Khalil, 2002).

A.1.1 Stability and Convergence

Consider the nonlinear time-invariant system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{A.1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be *locally Lipschitz* in $\boldsymbol{x}$; that is for each point $\boldsymbol{x} \in D \subset \mathbb{R}^n$ there exists a neighborhood $D_0 \in D$ such that

$$\| \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y}) \| \leq L \, \| \boldsymbol{x} - \boldsymbol{y} \| \,, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}_0 \tag{A.2}$$

where L is called the Lipschitz constant on D_0.

Let $\boldsymbol{x}_e$ denote the equilibrium point of (A.1) given by

$$\boldsymbol{f}(\boldsymbol{x}_e) = \boldsymbol{0} \tag{A.3}$$

The solutions $\boldsymbol{x}(t)$ of (A.1) are:

- *bounded*, if there exists a nonnegative function $0 < \gamma(\boldsymbol{x}(0)) < \infty$ such that

$$\|\boldsymbol{x}(t)\| \leq \gamma(\boldsymbol{x}(0)), \quad \forall t \geq 0 \tag{A.4}$$

In addition, the equilibrium point $\boldsymbol{x}_e$ of (A.1) is:

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Table A.1 Classification of theorems for stability and convergence

Autonomous systems	$V > 0,\ \dot{V} < 0$	Lyapunov's direct method	GAS/GES
	$V > 0,\ \dot{V} \leq 0$	Krasovskii–LaSalle's theorem	GAS
Non-autonomous systems	$V > 0,\ \dot{V} < 0$	LaSalle–Yoshizawa's theorem	UGAS
	$V > 0,\ \dot{V} \leq 0$	Matrosov's theorem	UGAS
	$V \geq 0,\ \dot{V} \leq 0$	Barbalat's lemma	Convergence

- *stable,* if, for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that

$$\|\boldsymbol{x}(0)\| < \delta(\epsilon) \Rightarrow \|\boldsymbol{x}(t)\| < \epsilon, \quad \forall t \geq 0 \tag{A.5}$$

- *unstable,* if it is not stable.
- *attractive,* if, for each $r > 0, \epsilon > 0$, there exists a $T(r, \epsilon) > 0$ such that

$$\|\boldsymbol{x}(0)\| \leq r \Rightarrow \|\boldsymbol{x}(t)\| \leq \epsilon, \quad \forall t \geq T(r, \epsilon) \tag{A.6}$$

  Attractivity implies convergence, that is $\lim_{t\to\infty} \|\boldsymbol{x}(t)\| = \mathbf{0}$.
- *(locally) asymptotically stable (AS),* if the equilibrium point $\boldsymbol{x}_e$ is stable and attractive.
- *globally stable (GS),* if the equilibrium point $\boldsymbol{x}_e$ is stable and $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$.
- *global asymptotically stable (GAS),* if the equilibrium point $\boldsymbol{x}_e$ is stable for all $\boldsymbol{x}(0)$ (region of attraction $\mathbb{R}^n$).
- *(locally) exponentially stable (ES),* if there exist positive constants α, λ and r such that

$$\|\boldsymbol{x}(0)\| < r \Rightarrow \|\boldsymbol{x}(t)\| < \alpha \exp(-\lambda t) \, \|\boldsymbol{x}(0)\| \,, \ \forall t \geq 0 \tag{A.7}$$

- *globally exponentially stable (GES),* if there exist positive constants α, λ and r such that for all $\boldsymbol{x}(0)$ (region of attraction $\mathbb{R}^n$):

$$\|\boldsymbol{x}(t)\| < \alpha \exp(-\lambda t) \, \|\boldsymbol{x}(0)\| \,, \ \forall t \geq 0 \tag{A.8}$$

Different theorems for investigation of stability and convergence will now be presented. A guideline for which theorem that should be applied is given in Table A.1 whereas the different theorems are listed in the forthcoming sections.

Notice that for nonautonomous systems GAS is replaced by *uniform global asymptotic stability* (UGAS) since uniformity is a necessary requirement in the case of time-varying nonlinear systems.

A.1.2 Lyapunov's Direct Method

Theorem A.1 (Lyapunov's Direct Method)
Let $\boldsymbol{x}_e$ be the equilibrium point of (A.1) and assume that $\boldsymbol{f}(\boldsymbol{x})$ is locally Lipschitz in $\boldsymbol{x}$. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable function $V(\boldsymbol{x})$ satisfying:

$$(i)\ V(x) > 0 \ \textit{(positive definite) and} \ \ V(0) = 0 \tag{A.9}$$

$$(ii)\ \dot{V}(x) = \frac{\partial V(x)}{\partial x} f(x) \leq -W(x) \leq 0 \tag{A.10}$$

$$(iii)\ V(x) \to \infty \ \ as \ \ \|x\| \to \infty \quad \textit{(radially unbounded)} \tag{A.11}$$

Then the equilibrium point x_e is GS if $W(x) \geq 0$ (positive semi-definite) and GAS if $W(x) > 0$ (positive definite) for all $x \neq 0$.

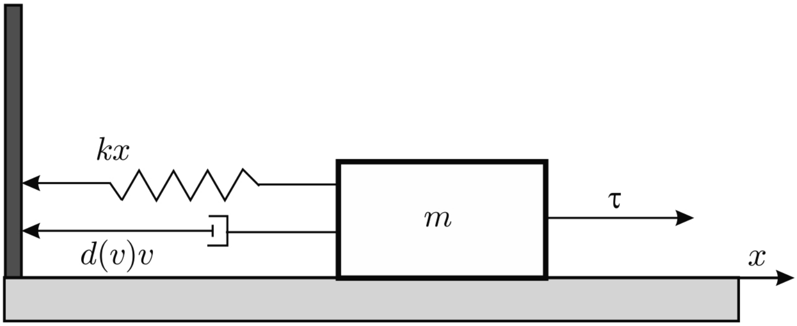

Figure A.1 Mass–damper–spring system.

Proof. See Khalil (2002) or Lyapunov (1907).

The requirement that $W(\boldsymbol{x}) > 0$ such that $\dot{V}(\boldsymbol{x}) < 0$ is in many cases difficult to satisfy. This is illustrated in the following example.

Example A.1 (Stability of a Mass–Damper–Spring System)
Consider the nonlinear mass–damper–spring system

$$\dot{x} = v \tag{A.12}$$

$$m\dot{v} + d(v)v + kx^2 = 0 \tag{A.13}$$

where $m > 0$, $d(v) > 0$, $\forall v$ and $k > 0$, see Figure A.1. Let us choose $V(\boldsymbol{x})$ as the sum of kinetic energy $\frac{1}{2}mv^2$ and potential energy $\frac{1}{2}kx^2$ such that

$$V(\boldsymbol{x}) = \frac{1}{2}\left(mv^2 + kx^2\right) = \frac{1}{2}\boldsymbol{x}^\top \begin{bmatrix} m & 0 \\ 0 & k \end{bmatrix} \boldsymbol{x} \tag{A.14}$$

where $\boldsymbol{x} = [v, x]^\top$ results in

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= mv\dot{v} + kx\dot{x} \\ &= v(m\dot{v} + kx) \\ &= -d(v)v^2 \\ &= -\boldsymbol{x}^\top \begin{bmatrix} d(v) & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} \end{aligned} \tag{A.15}$$

Hence, only stability can be concluded from Theorem A.1, since $\dot{V}(\boldsymbol{x}) = 0$ for all $v = 0$. However, GAS can in many cases also be proven for systems with a negative semi-definite $\dot{V}(\boldsymbol{x})$ thanks to the invariant set theorem of Krasovskii–LaSalle; *see LaSalle and Lefschetz (1961) and LaSalle (1966).*

A.1.3 Krasovskii–LaSalle's Theorem

The theorem of Krasovskii–LaSalle can be used to check a nonlinear *autonomous* system for GAS in the case of a negative semi-definite $\dot{V}(\boldsymbol{x})$.

Theorem A.2 (Krazovskii–LaSalle's Theorem)
Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable positive definite function such that

$$V(\boldsymbol{x}) \to \infty \text{ as } \|\boldsymbol{x}\| \to \infty \tag{A.16}$$

$$\dot{V}(\boldsymbol{x}) \leq 0, \ \forall \boldsymbol{x} \tag{A.17}$$

Let Ω be the set of all points where $\dot{V}(\boldsymbol{x}) = 0$, that is

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^n | \dot{V}(\boldsymbol{x}) = 0 \right\} \tag{A.18}$$

and M be the largest invariant set in Ω. Then all solutions $\boldsymbol{x}(t)$ converge to M. If $M = \{\boldsymbol{x}_e\}$ then the equilibrium point $\boldsymbol{x}_e$ of (A.1) is GAS.

Proof. See LaSalle (1966).

Example A.2 (Continued Example A.1: Stability of a Mass–Damper–Spring System)
Again consider the mass–damper–spring system of Example A.1. The set Ω is found by requiring that

$$\dot{V}(\boldsymbol{x}) = -d(v)v^2 \equiv 0 \tag{A.19}$$

which is true for $v = 0$. Therefore,

$$\Omega = \{(x \in \mathbb{R}, v = 0)\} \tag{A.20}$$

Now, $v = 0$ implies that $m\dot{v} = -kx$, which is nonzero when $x \neq 0$. Hence, the system cannot get "stuck" at a point other than $x = 0$. Since the equilibrium point of the mass–damper–spring system is $(x, v) = (0, 0)$, the largest invariant set M in Ω contains only one point, namely $(x, v) = (0, 0)$. Hence, the equilibrium point of (A.1) is GAS according to Theorem A.2.

A.1.4 Global Exponential Stability

The following theorem is useful to guarantee exponential convergence.

Theorem A.3 (Global Exponential Stability)
Let $\boldsymbol{x}_e$ be the equilibrium point of (A.1) and assume that $\boldsymbol{f}(\boldsymbol{x})$ is locally Lipschitz in $\boldsymbol{x}$. Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be a continuously differentiable and radially unbounded function satisfying

$$V(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{P} \boldsymbol{x} > 0, \quad \forall \boldsymbol{x} \neq \boldsymbol{0} \tag{A.21}$$

$$\dot{V}(\boldsymbol{x}) \leq -\boldsymbol{x}^\top \boldsymbol{Q} \boldsymbol{x} < 0, \quad \forall \boldsymbol{x} \neq \boldsymbol{0} \tag{A.22}$$

with constant matrices $\boldsymbol{P} = \boldsymbol{P}^\top > 0$ and $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$. Then the equilibrium point $\boldsymbol{x}_e$ is GES and the state vector satisfies

$$\|\boldsymbol{x}(t)\|_2 \leq \sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}} \exp(-\alpha t) \, \|\boldsymbol{x}(0)\|_2 \tag{A.23}$$

where

$$\alpha = \frac{\lambda_{\min}(\boldsymbol{Q})}{2\lambda_{\max}(\boldsymbol{P})} > 0 \tag{A.24}$$

is a bound on the convergence rate.

Proof. Since $V(\boldsymbol{x})$ is bounded by

$$0 < \lambda_{\min}(\boldsymbol{P})\,\|\boldsymbol{x}(t)\|_2^2 \le V(\boldsymbol{x}) \le \lambda_{\max}(\boldsymbol{P})\,\|\boldsymbol{x}(t)\|_2^2\,, \quad \forall \boldsymbol{x} \neq \boldsymbol{0} \tag{A.25}$$

it is seen that

$$-\,\|\boldsymbol{x}(t)\|_2^2 \le -\frac{1}{\lambda_{\max}(\boldsymbol{P})}V(\boldsymbol{x}) \tag{A.26}$$

Hence, it follows from (A.22) that

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &\le -\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x}\\ &\le -\lambda_{\min}(\boldsymbol{Q})\,\|\boldsymbol{x}(t)\|_2^2\\ &\le -\underbrace{\frac{\lambda_{\min}(\boldsymbol{Q})}{\lambda_{\max}(\boldsymbol{P})}}_{2\alpha}V(\boldsymbol{x})\end{aligned} \tag{A.27}$$

Integration of $\dot{V}(\boldsymbol{x}(t))$ yields

$$V(\boldsymbol{x}(t)) \le \exp(-2\alpha t)\,V(\boldsymbol{x}(0)) \tag{A.28}$$

Finally, (A.25) implies

$$\lambda_{\min}(\boldsymbol{P})\,\|\boldsymbol{x}(t)\|_2^2 \le \exp(-2\alpha t)\,\lambda_{\max}(\boldsymbol{P})\,\|\boldsymbol{x}(0)\|_2^2 \tag{A.29}$$

$$\|\boldsymbol{x}(t)\|_2 \le \sqrt{\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}}\mathrm{e}^{-\alpha t}\,\|\boldsymbol{x}(0)\|_2 \tag{A.30}$$

This shows that $\|\boldsymbol{x}(t)\|_2$ will converge exponentially to zero with convergence rate α.

A.2 Lyapunov Stability of Nonautonomous Systems

In this section several useful theorems for convergence and stability of time-varying nonlinear systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{A.31}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is assumed to be *locally Lipschitz* in $\boldsymbol{x}$ and uniformly in t, are briefly reviewed.

A.2.1 Barbălat's Lemma

Lemma A.1 (Barbălat's Lemma)
Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a uniformly continuous function and suppose that $\lim_{t\to\infty}\int_0^t \phi(\tau)\mathrm{d}\tau$ exists and is finite; then

$$\lim_{t\to\infty}\phi(t) = 0 \tag{A.32}$$

Proof. See Barbălat (1959).

Notice that *Barbălat's lemma* only guarantees *global convergence*. This result is particularly useful if there exists a uniformly continuous function $V\colon \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying:

$$
\begin{aligned}
&\text{(i)} \quad V(\boldsymbol{x}, t) \geq 0 \\
&\text{(ii)} \quad \dot{V}(\boldsymbol{x}, t) \leq 0 \\
&\text{(iii)} \quad \dot{V}(\boldsymbol{x}, t) \text{ is uniformly continuous}
\end{aligned}
$$

Hence, according to Barbălat's lemma, $\lim_{t\to\infty} \dot{V}(\boldsymbol{x}, t) = 0$. The requirement that $\dot{V}$ should be uniformly continuous can easily be checked by using

$$\ddot{V}(\boldsymbol{x}, t) \text{ is bounded } \Longrightarrow \dot{V}(\boldsymbol{x}, t) \text{ is uniformly continuous}$$

A.2.2 LaSalle–Yoshizawa's Theorem

For nonautonomous systems the following theorem of LaSalle (1966) and Yoshizawa (1968) is quite useful

Theorem A.4 (LaSalle–Yoshizawa's Theorem)
Let $\boldsymbol{x}_e = \boldsymbol{0}$ be the equilibrium point of (A.31) and assume that $\boldsymbol{f}(\boldsymbol{x}, t)$ is locally Lipschitz in $\boldsymbol{x}$. Let $V : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuously differentiable function $V(\boldsymbol{x}, t)$ satisfying

$$(i) \;\; V(\boldsymbol{x}, t) > 0 \text{ (positive definite) and } \; V(0) = 0 \tag{A.33}$$

$$(ii) \;\; \dot{V}(\boldsymbol{x}, t) = \frac{\partial V(\boldsymbol{x}, t)}{\partial t} + \frac{\partial V(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \boldsymbol{f}(x, t) \leq -W(\boldsymbol{x}) \leq 0 \tag{A.34}$$

$$(iii) \;\; V(\boldsymbol{x}, t) \rightarrow \infty \;\; as \;\; \|\boldsymbol{x}\| \rightarrow \infty \quad \text{(radially unbounded)} \tag{A.35}$$

where $W(\boldsymbol{x})$ is a continuous function. Then all solutions $\boldsymbol{x}(t)$ of (A.31) are uniformly globally bounded and

$$\lim_{t\to\infty} W(\boldsymbol{x}(t)) = 0 \tag{A.36}$$

In addition, if $W(\boldsymbol{x}) > 0$ (positive definite), then the equilibrium point $\boldsymbol{x}_e = \boldsymbol{0}$ of (A.31) is UGAS.

Proof. See LaSalle (1966) and Yoshizawa (1968).

A.2.3 Matrosov's Theorem

Nonautonomous systems where $\dot{V}(\boldsymbol{x}, t) \leq 0$ are UGAS if Matrosov's theorem is satisfied (Matrosov, 1962).

Definition A.1 (Class $\mathcal{K}$ Function)
A continuous function $\alpha : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Given two constants $0 \leq \delta \leq \Delta < \infty$ and $\mathcal{H}(\delta, \Delta) := \{\boldsymbol{x} \in \mathbb{R}^n : \delta \leq |\boldsymbol{x}| \leq \Delta\}$, then Matrosov's theorem can be stated according to:

Theorem A.5 (Matrosov's Theorem)
Consider the system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \;\; \boldsymbol{x} \in \mathbb{R}^n \tag{A.37}$$

If for this system there exist:
- *a locally Lipschitz function* $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$
- *a continuous positive semi-definite function* $U : \mathbb{R}^n \to \mathbb{R}_+$
- *functions* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$

such that:

1. $\alpha_1 (\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \alpha_2 (\|\boldsymbol{x}\|) \quad \forall\, (t, x) \in \mathbb{R} \times \mathbb{R}^n$
2. $\dot{V}(t, \boldsymbol{x}) \leq -U(\boldsymbol{x})$ *for almost all* $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n$

and for each $0 < \delta \leq \Delta$ *and* $\mathcal{H}(0, \Delta) \subseteq \mathbb{R}^n$ *there exist:*
- *a locally Lipschitz function* $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$
- *a continuous function* $Y : \mathbb{R}^n \to \mathbb{R}$
- *strictly positive numbers* $\varepsilon_1, \varepsilon_2, \psi > 0$

such that:

3. $\max \{|W(t, \boldsymbol{x})|, |Y(\boldsymbol{x})|\} \leq \psi \;\; \forall\, (t, x) \in \mathbb{R} \times \mathcal{H}(0, \Delta)$
4. $\dot{W}(t, \boldsymbol{x}) \leq Y(\boldsymbol{x})$ *for all* $(t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n$.
5. $\boldsymbol{x} \in \mathcal{H}(\delta, \Delta) \cap \{\boldsymbol{x} : U(\boldsymbol{x}) \leq \varepsilon_1\} \Longrightarrow Y(\boldsymbol{x}) \leq -\varepsilon_2$.

then the origin of (A.37) is UGAS.

Remark: If the system (A.37) is time-invariant, that is $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, then Condition 5 can be replaced by:

5. $\boldsymbol{x} \in \mathcal{H}(\delta, \Delta) \cap \{\boldsymbol{x} : U(\boldsymbol{x}) = 0\} \Longrightarrow Y(\boldsymbol{x}) < 0$

A.2.4 UGAS when Backstepping with Integral Action

When designing industrial control systems it is important to include integral action in the control law in order to compensate for slowly varying and constant disturbances. This is necessary to avoid steady-state errors both in regulation and tracking. The integral part of the controller can be provided by using *adaptive backstepping* (Krstic *et al.*, 1995) under the assumption of constant disturbances (see Section 13.3.4). Unfortunately, the resulting error dynamics in this case often becomes nonautonomous, which again implies that *Krasovskii–LaSalle's theorem* cannot be used. An alternative theorem for this case will be stated by considering the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\theta}, t) \tag{A.38}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{\theta} \in \mathbb{R}^p$ $(p \leq n)$ is a constant *unknown* parameter vector. Furthermore, assume that there exists an adaptive control law

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x}_d, \hat{\boldsymbol{\theta}}) \tag{A.39}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\phi}(\boldsymbol{x}, \boldsymbol{x}_d) \tag{A.40}$$

where $\boldsymbol{x}_d \in C^r$ and $\hat{\boldsymbol{\theta}} \in \mathbb{R}^p$, such that the error dynamics can be written

$$\dot{z} = \boldsymbol{h}(z, t) + \boldsymbol{B}(t)\,\tilde{\boldsymbol{\theta}} \tag{A.41}$$

$$\dot{\tilde{\boldsymbol{\theta}}} = -\boldsymbol{P}\boldsymbol{B}(t)^\top \left(\frac{\partial W(z, t)}{\partial z}\right)^\top, \qquad \boldsymbol{P} = \boldsymbol{P}^\top > 0 \tag{A.42}$$

where $W(z, t)$ is a suitable C^1 function and $\tilde{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ is the parameter estimation error. The parameter estimate $\hat{\boldsymbol{\theta}}$ can be used to compensate for a constant disturbance, that is integral action. Hence, the conditions in the following theorem can be used to establish UGAS when backstepping with integral action. The conditions are based on Loria *et al.* (1999) or alternatively Fossen *et al.* (2001). This can also be proven by applying Matrosov's theorem.

Theorem A.6 (UGAS/LES when Backstepping with Integral Action)
The origin of the system (A.41)–(A.42) is UGAS if $\boldsymbol{B}^\top(t)\boldsymbol{B}(t)$ is invertible for all t, $\boldsymbol{P} = \boldsymbol{P}^\top > 0$, there exists a continuous, nondecreasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max\left\{ \|\boldsymbol{h}(z, t)\|, \left\| \frac{\partial W(z, t)}{\partial z} \right\| \right\} \leq \rho(\|z\|)\,\|z\| \tag{A.43}$$

and there exist class-$\mathcal{K}_\infty$ functions α_1 and α_2 and a strictly positive real number $c > 0$ such that $W(z, t)$ satisfy

$$\alpha_1(\|z\|) \leq W(z, t) \leq \alpha_2(\|z\|) \tag{A.44}$$

$$\frac{\partial W(z, t)}{\partial t} + \frac{\partial W(z, t)}{\partial z}\boldsymbol{h}(z, t) \leq -c\,\|z\|^2. \tag{A.45}$$

If, in addition, $\alpha_2(s) \propto s^2$ for sufficiently small s then the origin is LES.

Proof. See Fossen *et al.* (2001).

Theorem A.6 implies that both $z \to \mathbf{0}$ and $\tilde{\boldsymbol{\theta}} \to \mathbf{0}$ when $t \to \infty$. The following example illustrates how a UGAS integral controller can be derived:

Example A.3 (UGAS Integral Controller)
Consider the nonautonomous system

$$\dot{x} = -a(t)x + \theta + u \tag{A.46}$$

$$u = -K_p x - \hat{\theta} \tag{A.47}$$

$$\dot{\hat{\theta}} = px \tag{A.48}$$

where $0 < a(t) \leq a_{\max}$, $\theta =$ constant, $K_p > 0$ and $p > 0$. This is a PI controller since

$$u = -K_p x - p\int_0^t x(\tau)\mathrm{d}\tau \tag{A.49}$$

Choosing $z = x$, the error dynamics can be written

$$\dot{z} = -(a(t) + K_p)z - \tilde{\theta} \tag{A.50}$$

$$\dot{\tilde{\theta}} = pz \tag{A.51}$$

which is in the form (A.41)–(A.42) with $W(z) = \frac{1}{2}z^2$ and $\boldsymbol{B} = 1$. Since $\boldsymbol{B}^\top\boldsymbol{B} = 1 > 0$ and

$$\max\left\{ |a(t)z + K_p z|, |z| \right\} \leq \rho|z| \tag{A.52}$$

with $\rho = a_{\max} + K_p$, *the equilibrium point* $z = 0$ *is UGAS according to Theorem A.6. Notice that the LaSalle–Yoshizawa theorem fails for this case since*

$$V(z,t) = W(z) + \frac{1}{2p}\tilde{\theta}^2 \tag{A.53}$$

$$\begin{aligned}\dot{V}(z,t) &= z\dot{z} + \frac{1}{p}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -[a(t) + K_p]z^2 \\ &\leq 0\end{aligned} \tag{A.54}$$

which by LaSalle–Yoshizawa only shows UGS and $z(t) \to 0$, *but not* $\tilde{\theta} \to 0$.