6

Maneuvering Theory

In Chapter 5 the 6 DOF seakeeping equations of motion for a ship in a seaway were presented. The seakeeping model is based on linear theory and a potential theory program is used to compute the frequency-dependent hydrodynamic forces for varying wave excitation frequencies. The time-domain representation of the seakeeping model is very useful for accurate prediction of motions and sealoads of floating structures offshore. The seakeeping theory can also be applied to displacement ships moving at constant speed. Seakeeping time-domain models are limited to linear theory since it is necessary to approximate the fluid memory effects by impulse responses or transfer functions.

An alternative to the seakeeping formalism is to use maneuvering theory to describe the motions of marine craft in 3 DOF, that is *surge*, *sway* and *yaw.* Sometimes roll is augmented to the horizontal plane model to describe more accurately the coupled lateral motions, that is *sway–roll–yaw* couplings while *surge* is left decoupled; see Section 7.4. In maneuvering theory, frequency-dependent added mass and potential damping are approximated by constant values and thus it is not necessary to compute the fluid-memory effects. The main results of this chapter are based on the assumption that the hydrodynamic forces and moments can be approximated at one frequency of oscillation such that the fluid-memory effects can be neglected. The result is a nonlinear mass–damper–spring system with constant coefficients.

In the following sections, it is shown that the maneuvering equations of motion can be represented by (Fossen, 1991, 1994)

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.1}$$

In the case of *irrotational ocean currents*, the relative velocity vector

$$\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c, \qquad \boldsymbol{\nu}_c = [u_c, v_c, w_c, 0, 0, 0]^\top$$

contributes to the hydrodynamic terms such that

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}} + \underbrace{\boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o}_{\text{hydrostatic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.2}$$

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The model (6.2) can be simplified if the *ocean currents* are assumed to be *constant* and *irrotational* in $\{n\}$ such that (see Section 8.3)

$$\dot{\boldsymbol{\nu}}_c = \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b) & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} \end{bmatrix} \boldsymbol{\nu}_c \tag{6.3}$$

According to Property 8.1, it is then possible to represent the equations of motion by relative velocities only:

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.4}$$

where

$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A$ - system inertia matrix (including added mass)
$\boldsymbol{C}(\boldsymbol{\nu}_r) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r) + \boldsymbol{C}_A(\boldsymbol{\nu}_r)$ - Coriolis–centripetal matrix (including added mass)
$\boldsymbol{D}(\boldsymbol{\nu}_r)$ - damping matrix
$\boldsymbol{g}(\boldsymbol{\eta})$ - vector of gravitational/buoyancy forces and moments
$\boldsymbol{g}_o$ - vector used for pretrimming (ballast control)
$\boldsymbol{\tau}$ - vector of control inputs
$\boldsymbol{\tau}_{\text{wind}}$ - vector of wind forces
$\boldsymbol{\tau}_{\text{wave}}$ - vector of wave-induced forces

The expressions for $\boldsymbol{M}$, $\boldsymbol{C}(\boldsymbol{\nu}_r)$, $\boldsymbol{D}(\boldsymbol{\nu}_r)$, $\boldsymbol{g}(\boldsymbol{\eta})$ and $\boldsymbol{g}_o$ are derived in the forthcoming sections while the environmental forces $\boldsymbol{\tau}_{\text{wind}}$ and $\boldsymbol{\tau}_{\text{wave}}$ are treated separately in Chapter 8. The maneuvering model presented in this chapter is mainly intended for controller–observer design, prediction and computer simulations in combination with system identification and parameter estimation. Application specific models are presented in Chapter 7.

Hydrodynamic programs compute mass, inertia, potential damping and restoring forces while a more detailed treatment of viscous dissipative forces (damping) are found in the extensive literature on hydrodynamics; see Faltinsen (1990, 2005), Newman (1977), Sarpkaya (1981) and Triantafyllou and Hover (2002). Other useful references on marine craft modeling are Lewandowski (2004) and Perez (2005).

6.1 Rigid-Body Kinetics

Recall from Chapter 3 that the rigid-body kinetics can be expressed as

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{6.5}$$

where $\boldsymbol{M}_{RB} = \boldsymbol{M}_{RB}^\top > 0$ is the rigid-body mass matrix and $\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = -\boldsymbol{C}_{RB}^\top(\boldsymbol{\nu})$ is the rigid-body Coriolis and centripetal matrix due to the rotation of $\{b\}$ about the inertial frame $\{n\}$. The horizontal motion of a maneuvering ship or semi-submersible is given by the motion components in surge, sway and yaw. Consequently, the state vectors are chosen as $\boldsymbol{\nu} = [u, v, r]^\top$ and $\boldsymbol{\eta} = [N, E, \psi]^\top$. It is also common to assume that the craft has homogeneous mass distribution and xz-plane symmetry so that

$$I_{xy} = I_{yz} = 0 \tag{6.6}$$

Let the $\{b\}$-frame coordinate origin be set in the centerline of the craft in the point CO, such that $y_g = 0$. Under the previously stated assumptions, the matrices (3.44) and (3.60) associated with the rigid-body kinetics reduce to

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & mx_g \\ 0 & mx_g & I_z \end{bmatrix}, \qquad \boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & -mr & -mx_g r \\ mr & 0 & 0 \\ mx_g r & 0 & 0 \end{bmatrix} \tag{6.7}$$

Notice that surge is decoupled from sway and yaw in $\boldsymbol{M}_{RB}$ due to symmetry considerations of the system inertia matrix (see Section 3.3).

The linear approximation to (6.5) about $u = U = \text{constant}$, $v = 0$ and $r = 0$ is

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}^{*}_{RB}\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{6.8}$$

where

$$\boldsymbol{C}^{*}_{RB} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & mU \\ 0 & 0 & mx_g U \end{bmatrix} \tag{6.9}$$

6.2 Potential Coefficients

Hydrodynamic potential theory programs can be used to compute the added mass and damping matrices by integrating the pressure of the fluid over the wetted surface of the hull; see Section 5.1. These programs assume that viscous effects can be neglected. Consequently, it is necessary to add viscous forces manually. The programs are also based on the assumptions that first- and second-order wave forces can be linearly superimposed.

The potential coefficients are usually represented as frequency-dependent matrices for 6 DOF motions. The matrices are:

- $\boldsymbol{A}(\omega)$ added mass
- $\boldsymbol{B}(\omega)$ potential damping

where ω is the wave excitation frequency of a sinusoidal (regular) wave generated by a wave maker or the ocean. Figure 6.1 illustrates the components in sway.

Surface Vessels

In seakeeping analysis, the equations of motion are formulated as perturbations

$$\boldsymbol{\xi} = \delta\boldsymbol{\eta} = [\delta x, \delta y, \delta z, \delta\phi, \delta\theta, \delta\psi]^{\top} \tag{6.10}$$

about an inertial equilibrium frame (see Section 5.3). For a floating body at zero speed this is written as

$$[\boldsymbol{M}_{RB} + \boldsymbol{A}(\omega)]\ddot{\boldsymbol{\xi}} + \boldsymbol{B}(\omega)\dot{\boldsymbol{\xi}} + \boldsymbol{C}\boldsymbol{\xi} = \boldsymbol{f}\cos(\omega t) \tag{6.11}$$

where $\boldsymbol{C}$ is the spring stiffness matrix due to *Archimedes* and the right-hand side of (6.11) is a vector of forced oscillations with amplitudes:

$$\boldsymbol{f} = [f_1, \ldots, f_6]^{\top} \tag{6.12}$$

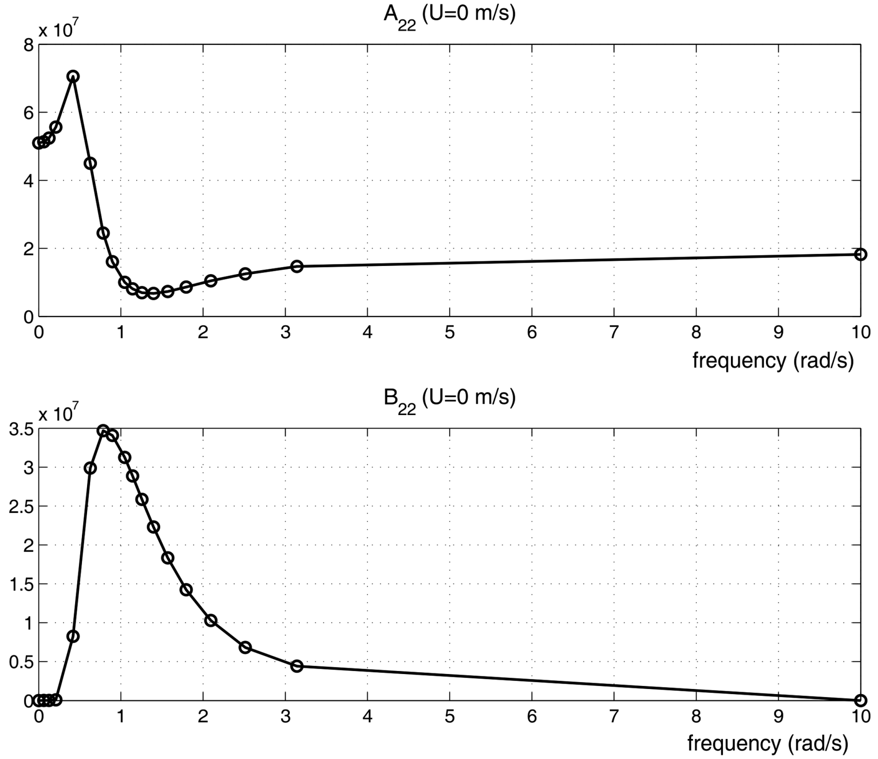

Figure 6.1 Added mass $A_{22}(\omega)$ and potential damping $B_{22}(\omega)$ in sway as a function of ω for a large tanker.

Equation (6.11) is a pseudo-differential equation combining time and frequency (see Section 5.3.2). This equation is not intended for computer simulations as discussed in Chapter 5 where a proper time-domain representation is derived using the Cummins equation. The matrices $\boldsymbol{A}(\omega)$, $\boldsymbol{B}(\omega)$ and $\boldsymbol{C}$ can be treated as a *hydrodynamic mass–damper–spring system* which varies with the frequency ω of the forced oscillation. By exposing the craft to different oscillations it is possible to compute added mass and potential damping for all the frequencies, as shown in Section 5.3.

Underwater Vehicles

For vehicles operating at water depths below the wave-affected zone, the hydrodynamic coefficients will be independent of the wave excitation frequency. Consequently,

$$\boldsymbol{A}(\omega) = \text{ constant } \quad \forall\omega \tag{6.13}$$

$$\boldsymbol{B}(\omega) = \boldsymbol{0} \tag{6.14}$$

This means that if a seakeeping code is used to compute the potential coefficients, only one frequency is needed to obtain an estimate of the added mass matrix. In addition, there will be no potential damping. However, viscous damping $\boldsymbol{B}_V(\omega)$ will be present.

Discussion

Equation (6.11) should not be used in computer simulations. As earlier mentioned, (6.11) is not an ordinary differential equation since it combines time and frequency. As stressed in Chapter 5, the time-domain seakeeping model should be represented by the Cummins equation, which is an integro-differential equation (Cummins, 1962). For surface vessels it is common to solve the Cummins equation in the time domain under the assumption of linear theory (see Section 5.4). This introduces *fluid-memory effects*, which can be interpreted as filtered potential damping forces. These forces are retardation functions that can be approximated by transfer functions and state-space models, as shown in Section 5.4. It is standard to include the fluid-memory effects in seakeeping analysis while classical maneuvering theory neglects the fluid-memory by relying on a zero-frequency assumption.

6.2.1 3 DOF Maneuvering Model

The classical maneuvering model makes use of the following assumption:

Definition 6.1 (Zero-Frequency Models for Surge, Sway and Yaw)
The horizontal motions (surge, sway and yaw) of a marine craft moving at forward speed can be described by a zero-frequency model where:

$$\boldsymbol{M}_A = \boldsymbol{A}^{\{1,2,6\}}(0) = \begin{bmatrix} A_{11}(0) & 0 & 0 \\ 0 & A_{22}(0) & A_{26}(0) \\ 0 & A_{62}(0) & A_{66}(0) \end{bmatrix} \tag{6.15}$$

$$\boldsymbol{D}_p = \boldsymbol{B}^{\{1,2,6\}}(0) = \boldsymbol{0} \tag{6.16}$$

are constant matrices.

Discussion
When applying a feedback control system to stabilize the motions in surge, sway and yaw, the natural periods will be in the range of 100–200 s. This implies that the natural frequencies are in the range of 0.03–0.10 rad/s, which is quite close to the zero wave excitation frequency. Also note that viscous damping forces will dominate the potential damping terms at low frequency and that fluid memory effects can be neglected at higher speeds.

Definition 6.1 is frequently applied when deriving maneuvering models for ships in a seaway. It is convenient to represent the equations of motion without using frequency-dependent quantities since this reduces model complexity.

6.2.2 6 DOF Coupled Motions

One limitation of Definition 6.1 is that it cannot be applied to *heave, roll* and *pitch*. These modes are second-order mass–damper–spring systems where the dominating frequencies are the natural frequencies. Hence, the constant frequency models in heave, roll and pitch should be formulated at their respective natural frequencies and not at the zero frequency. This suggests the following definition:

Definition 6.2 (Natural Frequency Models for Heave, Roll and Pitch)
The natural frequencies for the decoupled motions in heave, roll and pitch are given by the implicit equations

$$\omega_{\text{heave}} = \sqrt{\frac{C_{33}}{m + A_{33}(\omega_{\text{heave}})}} \tag{6.17}$$

$$\omega_{\text{roll}} = \sqrt{\frac{C_{44}}{I_x + A_{44}(\omega_{\text{roll}})}} \tag{6.18}$$

$$\omega_{\text{pitch}} = \sqrt{\frac{C_{55}}{I_y + A_{55}(\omega_{\text{pitch}})}} \tag{6.19}$$

where the potential coefficients $A_{ii}(\omega)$ *and* $B_{ii}(\omega)$ $(i = 3, 4, 5)$ *are computed in the center of flotation (CF). The corresponding mass–damper–spring systems are*

$$[m + A_{33}(\omega_{\text{heave}})]\dot{z} + B_{33}(\omega_{\text{heave}})w + C_{33}z = 0 \tag{6.20}$$

$$[I_x + A_{44}(\omega_{\text{roll}})]\dot{p} + B_{44}(\omega_{\text{roll}})p + C_{44}\phi = 0 \tag{6.21}$$

$$[I_y + A_{55}(\omega_{\text{pitch}})]\dot{q} + B_{55}(\omega_{\text{pitch}})q + C_{55}\theta = 0 \tag{6.22}$$

Equations (6.20)–(6.22) are decoupled damped oscillators. However, the natural frequencies (6.17)–(6.19) can also be computed for the 6 DOF coupled model (6.11) by using a modal analysis; see Section 4.3.2.

Consider the unforced 6 DOF linear seakeeping model

$$[\boldsymbol{M}_{RB} + \boldsymbol{A}(\omega)]\ddot{\boldsymbol{\xi}} + [\boldsymbol{B}(\omega) + \boldsymbol{B}_v(\omega)]\dot{\boldsymbol{\xi}} + \boldsymbol{C}\boldsymbol{\xi} = \boldsymbol{0} \tag{6.23}$$

where viscous damping is included. Furthermore, assume that three constant matrices $\bar{\boldsymbol{A}}$, $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{B}}_V$ exist that approximate $\boldsymbol{A}(\omega)$, $\boldsymbol{B}(\omega)$ and $\boldsymbol{B}_V(\omega)$. This system can be transformed from seakeeping coordinates $\{s\}$ to body-fixed coordinates $\{b\}$ using the approach in Section 5.4. The resulting model is a linear one:

$$[\boldsymbol{M}_{RB} + \boldsymbol{M}_A]\dot{\boldsymbol{\nu}} + [\boldsymbol{C}^*_{RB} + \boldsymbol{C}^*_A]\boldsymbol{\nu} + [\boldsymbol{D}_P + \boldsymbol{D}_V]\boldsymbol{\nu} + \boldsymbol{G}\boldsymbol{\eta} = \boldsymbol{0} \tag{6.24}$$

where

$$\begin{aligned} \boldsymbol{M}_A &= \bar{\boldsymbol{A}} & \boldsymbol{D}_P &= \bar{\boldsymbol{B}} \\ \boldsymbol{C}^*_A &= U\bar{\boldsymbol{A}}\boldsymbol{L} & \boldsymbol{D}_V &= \bar{\boldsymbol{B}}_V \\ \boldsymbol{C}^*_{RB} &= U\boldsymbol{M}_{RB}\boldsymbol{L} & \boldsymbol{G} &= \boldsymbol{C} \end{aligned} \tag{6.25}$$

and $\boldsymbol{L}$ is the selection matrix (3.63).

The potential coefficients $\boldsymbol{A}(\omega)$ and $\boldsymbol{B}(\omega)$ can be computed using a hydrodynamic code. If we rely on Definitions 6.1 and 6.2 to approximate $\boldsymbol{M}_A$, $\boldsymbol{D}_P$ and $\boldsymbol{D}_V$ it is necessary to assume that there are no couplings between the surge, heave–roll–pitch and the sway–yaw subsystems. Hence, added mass and potential damping can be approximated by two constant matrices:

$$\boldsymbol{M}_A \approx \begin{bmatrix} A_{11}(0) & 0 & \cdots & 0 \\ 0 & A_{22}(0) & & A_{26}(0) \\ & \cdots & \boxed{\begin{matrix} A_{33}(\omega_{\text{heave}}) & 0 & 0 \\ 0 & A_{44}(\omega_{\text{roll}}) & 0 \\ 0 & 0 & A_{55}(\omega_{\text{pitch}}) \end{matrix}} & \cdots \\ 0 & A_{62}(0) & \cdots & A_{66}(0) \end{bmatrix} \tag{6.26}$$

$$\boldsymbol{D}_p \approx \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \cdots & & \boxed{\begin{matrix} B_{33}(\omega_{\text{heave}}) & 0 & 0 \\ 0 & B_{44}(\omega_{\text{roll}}) & 0 \\ 0 & 0 & B_{55}(\omega_{\text{pitch}}) \end{matrix}} & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{6.27}$$

The natural frequencies ω_{heave}, ω_{roll} and ω_{pitch} can be computed using the methods in Sections 4.3.1–4.3.2. The linear viscous damping terms are usually approximated by a diagonal matrix:

$$\boldsymbol{D}_V \approx \text{diag}\{B_{11v}, B_{22v}, B_{33v}, B_{44v}, B_{55v}, B_{66v}\} \tag{6.28}$$

where the elements B_{iiv} ($i = 1, ..., 6$) can be computed from the time constants and natural periods of the system (see Section 6.4).

6.3 Nonlinear Coriolis Forces due to Added Mass in a Rotating Coordinate System

The model discussed in Section 6.2 was derived using linear theory. In order to extend this to nonlinear maneuvering theory, the Coriolis and centripetal forces will be derived in a Lagrangian framework. The Coriolis and centripetal matrix $\boldsymbol{C}_A(\boldsymbol{\nu})$ is a function of added mass $\boldsymbol{M}_A$ and depends on which reference frames are considered. Lagrangian theory considers the motion of a rotating frame $\{b\}$ with respect to $\{n\}$.

In seakeeping theory, the body frame $\{b\}$ rotates about $\{s\}$. This results in a linear Coriolis and centripetal matrix denoted by $\boldsymbol{C}_A^*$ (see Section 5.4.2). Both representations can be used depending on whether a linear or nonlinear model is needed. The rotation of $\{b\}$ about the inertial systems $\{n\}$ and alternatively $\{s\}$ are illustrated in Figure 6.2.

6.3.1 Lagrangian Mechanics

In Section 3.1, it was shown that the rigid-body kinetics of a marine craft can be derived by applying the *Newtonian* formulation. As for the rigid-body kinetics, it is advantageous to separate the added mass forces and moments in terms that belong to the *added mass matrix* $\boldsymbol{M}_A$ and a matrix of hydrodynamic Coriolis and centripetal terms denoted $\boldsymbol{C}_A(\boldsymbol{\nu})$. To derive the expressions for these two matrices, an *energy approach* based on Kirchhoff's equations will now be presented. Detailed discussions of Newtonian and Lagrangian mechanics are found in Goldstein (1980), Hughes (1986), Kane *et al.* (1983), Meirovitch (1990) and Egeland and Gravdahl (2002).

The Lagrangian L is formed by using kinetic energy T and potential energy V, according to

$$L = T - V \tag{6.29}$$

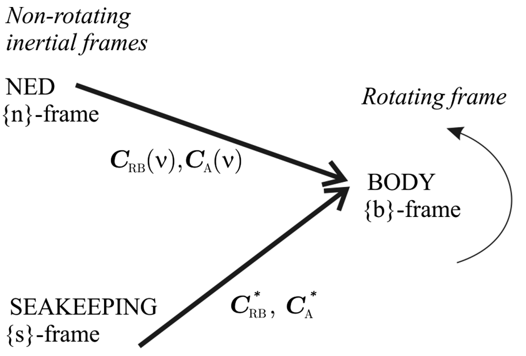

Figure 6.2 Coriolis matrices due to the rotation of the body-fixed frame $\{b\}$ about the inertial frames $\{n\}$ or $\{s\}$.

The *Euler–Lagrange equation* is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\eta}}}\right) - \frac{\partial L}{\partial \boldsymbol{\eta}} = \boldsymbol{J}_\Theta^{-\top}(\boldsymbol{\eta})\boldsymbol{\tau} \tag{6.30}$$

which in component form corresponds to a set of six second-order differential equations. From the above formula it is seen that the Lagrangian mechanics describes the system dynamics in terms of energy. Formula (6.30) is valid in any reference frame, inertial and body-fixed, as long as *generalized coordinates* are used.

For a marine craft not subject to any motion constraints, the number of independent (*generalized*) coordinates is equal to the number of DOF. For a marine craft moving in 6 DOF the generalized coordinates in $\{n\}$ can be chosen as

$$\boldsymbol{\eta} = [N, E, D, \phi, \theta, \psi]^\top \tag{6.31}$$

It should be noted that the alternative representation

$$\boldsymbol{\eta} = [N, E, D, \eta, \varepsilon_1, \varepsilon_2, \varepsilon_3]^\top \tag{6.32}$$

using unit quaternions cannot be used in a Lagrangian approach since this representation is defined by seven parameters. Hence, these parameters are not *generalized coordinates*. It is not straightforward to formulate the equations of motion in $\{b\}$ since

$$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top \tag{6.33}$$

cannot be integrated to yield a set of generalized coordinates in terms of position and orientation. In fact $\int_0^t \boldsymbol{\nu}\mathrm{d}\tau$ has no immediate physical interpretation. Consequently, the Lagrange equation cannot be directly used to formulate the equations of motion in $\{b\}$. However, this problem is circumvented by applying Kirchhoff's equations of motion, or the so-called *quasi-Lagrangian* approach; see Meirovitch and Kwak (1989) for details.

6.3.2 Kirchhoff's Equations in Vector Form

Consider a marine craft with linear velocity $\boldsymbol{\nu}_1 := [u, v, w]^\top$ and angular velocity $\boldsymbol{\nu}_2 := [p, q, r]^\top$ expressed in $\{b\}$. Hence, the force $\boldsymbol{\tau}_1 := [X, Y, Z]^\top$ and moment $\boldsymbol{\tau}_2 := [K, M, N]^\top$ are related to the kinetic

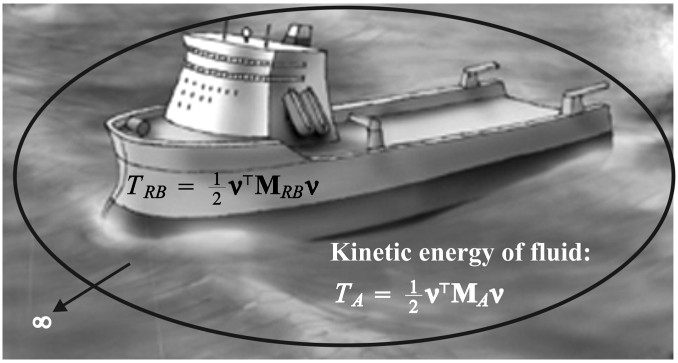

Figure 6.3 Rigid-body and fluid kinetic energy (ocean surrounding the ship). Illustration by Bjarne Stenberg.

energy (Kirchhoff, 1869)

$$T = \frac{1}{2}\boldsymbol{\nu}^{\top}\boldsymbol{M}\boldsymbol{\nu} \tag{6.34}$$

by the vector equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_1}\right) + \boldsymbol{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{\tau}_1 \tag{6.35}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_2}\right) + \boldsymbol{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_2} + \boldsymbol{S}(\boldsymbol{\nu}_1)\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{\tau}_2 \tag{6.36}$$

where $\boldsymbol{S}$ is the skew-symmetric cross-product operator in Definition 2.2. *Kirchhoff's equations* will prove to be very useful in the derivation of the expression for added inertia. Notice that Kirchhoff's equations do not include the gravitational forces.

6.3.3 Added Mass and Coriolis–Centripetal Forces due to the Rotation of BODY Relative to NED

The matrix $\boldsymbol{C}_A^*$ in (6.25) represents linearized forces due to a rotation of $\{b\}$ about the seakeeping frame $\{s\}$. Instead of using $\{s\}$ as the inertial frame, we will assume that $\{n\}$ is the inertial frame and that $\{b\}$ rotates about $\{n\}$. The nonlinear Coriolis and centripetal matrix $\boldsymbol{C}_A(\boldsymbol{\nu})$ due to a rotation of $\{b\}$ about the inertial frame $\{n\}$ can be derived using an energy formulation based on the constant matrix $\boldsymbol{M}_A$. Since any motion of the marine craft will induce a motion in the otherwise stationary fluid, the fluid must move aside and then close behind the craft in order to let the craft pass through the fluid. As a consequence, the fluid motion possesses kinetic energy that it would lack otherwise (see Figure 6.3). The expression for the fluid kinetic energy T_A is written as a quadratic form (Lamb, 1932)

$$T_A = \frac{1}{2}\boldsymbol{\nu}^{\top}\boldsymbol{M}_A\boldsymbol{\nu}, \qquad \dot{\boldsymbol{M}}_A = \boldsymbol{0} \tag{6.37}$$

where $\boldsymbol{M}_A = \boldsymbol{M}_A^\top \geq 0$ is the 6×6 *system inertia matrix* of added mass terms:

$$\boldsymbol{M}_A = - \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix} \tag{6.38}$$

The notation of SNAME (1950) for the hydrodynamic derivatives is used in this expression; for instance the hydrodynamic added mass force Y along the y axis due to an acceleration $\dot{u}$ in the x direction is written as

$$Y = -Y_{\dot{u}}\dot{u}, \qquad Y_{\dot{u}} := \frac{\partial Y}{\partial \dot{u}} \tag{6.39}$$

This implies that $\{M_A\}_{21} = -Y_{\dot{u}}$ in the example above.

Property 6.1 (Hydrodynamic System Inertia Matrix $\boldsymbol{M}_A$)
For a rigid body at rest or moving at forward speed $U \geq 0$ in ideal fluid, the hydrodynamic system inertia matrix $\boldsymbol{M}_A$ is positive semi-definite:

$$\boldsymbol{M}_A = \boldsymbol{M}_A^\top \geq 0$$

Proof. Newman (1977) has shown this for zero speed. The results extend to forward speed by using the approach presented in Chapter 5.

Remark 6.1
In a real fluid the 36 elements of $\boldsymbol{M}_A$ may all be distinct but still $\boldsymbol{M}_A \geq 0$. Experience has shown that the numerical values of the added mass derivatives in a real fluid are usually in good agreement with those obtained from ideal theory (see Wendel, 1956).

Remark 6.2
If experimental data are used, the inertia matrix can be symmetrized by using:

$$\boldsymbol{M}_A = \frac{1}{2}\left(\boldsymbol{M}_{A,\mathrm{exp}} + \boldsymbol{M}_{A,\mathrm{exp}}^\top\right) \tag{6.40}$$

where $\boldsymbol{M}_{A,\mathrm{exp}}$ contains the experimentally data.

Added Mass Forces and Moments

Based on the kinetic energy T_A of the fluid, it is straightforward to derive the added mass forces and moments. Substituting (6.37) into (6.35)–(6.36) gives the following expressions for the added mass terms

(Imlay, 1961):

$$
\begin{aligned}
X_A &= X_{\dot{u}}\dot{u} + X_{\dot{w}}(\dot{w} + uq) + X_{\dot{q}}\dot{q} + Z_{\dot{w}}wq + Z_{\dot{q}}q^2 \\
&\quad + X_{\dot{v}}\dot{v} + X_{\dot{p}}\dot{p} + X_{\dot{r}}\dot{r} - Y_{\dot{v}}vr - Y_{\dot{p}}rp - Y_{\dot{r}}r^2 \\
&\quad - X_{\dot{v}}ur - Y_{\dot{w}}wr \\
&\quad + Y_{\dot{w}}vq + Z_{\dot{p}}pq - (Y_{\dot{q}} - Z_{\dot{r}})qr \\
Y_A &= X_{\dot{v}}\dot{u} + Y_{\dot{w}}\dot{w} + Y_{\dot{q}}\dot{q} \\
&\quad + Y_{\dot{v}}\dot{v} + Y_{\dot{p}}\dot{p} + Y_{\dot{r}}\dot{r} + X_{\dot{v}}vr - Y_{\dot{w}}vp + X_{\dot{r}}r^2 + (X_{\dot{p}} - Z_{\dot{r}})rp - Z_{\dot{p}}p^2 \\
&\quad - X_{\dot{w}}(up - wr) + X_{\dot{u}}ur - Z_{\dot{w}}wp \\
&\quad - Z_{\dot{q}}pq + X_{\dot{q}}qr \\
Z_A &= X_{\dot{w}}(\dot{u} - wq) + Z_{\dot{w}}\dot{w} + Z_{\dot{q}}\dot{q} - X_{\dot{u}}uq - X_{\dot{q}}q^2 \\
&\quad + Y_{\dot{w}}\dot{v} + Z_{\dot{p}}\dot{p} + Z_{\dot{r}}\dot{r} + Y_{\dot{v}}vp + Y_{\dot{r}}rp + Y_{\dot{p}}p^2 \\
&\quad + X_{\dot{v}}up + Y_{\dot{w}}wp \\
&\quad - X_{\dot{v}}vq - (X_{\dot{p}} - Y_{\dot{q}})pq - X_{\dot{r}}qr \\
K_A &= X_{\dot{p}}\dot{u} + Z_{\dot{p}}\dot{w} + K_{\dot{q}}\dot{q} - X_{\dot{v}}wu + X_{\dot{r}}uq - Y_{\dot{w}}w^2 - (Y_{\dot{q}} - Z_{\dot{r}})wq + M_{\dot{r}}q^2 \\
&\quad + Y_{\dot{p}}\dot{v} + K_{\dot{p}}\dot{p} + K_{\dot{r}}\dot{r} + Y_{\dot{w}}v^2 - (Y_{\dot{q}} - Z_{\dot{r}})vr + Z_{\dot{p}}vp - M_{\dot{r}}r^2 - K_{\dot{q}}rp \\
&\quad + X_{\dot{w}}uv - (Y_{\dot{v}} - Z_{\dot{w}})vw - (Y_{\dot{r}} + Z_{\dot{q}})wr - Y_{\dot{p}}wp - X_{\dot{q}}ur \\
&\quad + (Y_{\dot{r}} + Z_{\dot{q}})vq + K_{\dot{r}}pq - (M_{\dot{q}} - N_{\dot{r}})qr \\
M_A &= X_{\dot{q}}(\dot{u} + wq) + Z_{\dot{q}}(\dot{w} - uq) + M_{\dot{q}}\dot{q} - X_{\dot{w}}(u^2 - w^2) - (Z_{\dot{w}} - X_{\dot{u}})wu \\
&\quad + Y_{\dot{q}}\dot{v} + K_{\dot{q}}\dot{p} + M_{\dot{r}}\dot{r} + Y_{\dot{p}}vr - Y_{\dot{r}}vp - K_{\dot{r}}(p^2 - r^2) + (K_{\dot{p}} - N_{\dot{r}})rp \\
&\quad - Y_{\dot{w}}uv + X_{\dot{v}}vw - (X_{\dot{r}} + Z_{\dot{p}})(up - wr) + (X_{\dot{p}} - Z_{\dot{r}})(wp + ur) \\
&\quad - M_{\dot{r}}pq + K_{\dot{q}}qr \\
N_A &= X_{\dot{r}}\dot{u} + Z_{\dot{r}}\dot{w} + M_{\dot{r}}\dot{q} + X_{\dot{v}}u^2 + Y_{\dot{w}}wu - (X_{\dot{p}} - Y_{\dot{q}})uq - Z_{\dot{p}}wq - K_{\dot{q}}q^2 \\
&\quad + Y_{\dot{r}}\dot{v} + K_{\dot{r}}\dot{p} + N_{\dot{r}}\dot{r} - X_{\dot{v}}v^2 - X_{\dot{r}}vr - (X_{\dot{p}} - Y_{\dot{q}})vp + M_{\dot{r}}rp + K_{\dot{q}}p^2 \\
&\quad - (X_{\dot{u}} - Y_{\dot{v}})uv - X_{\dot{w}}vw + (X_{\dot{q}} + Y_{\dot{p}})up + Y_{\dot{r}}ur + Z_{\dot{q}}wp \\
&\quad - (X_{\dot{q}} + Y_{\dot{p}})vq - (K_{\dot{p}} - M_{\dot{q}})pq - K_{\dot{r}}qr
\end{aligned}
\tag{6.41}
$$

Imlay (1961) arranged the equations in four lines with longitudinal components on the first line and lateral components on the second. The third line consists of mixed terms involving u or w as one factor. If one or both of these velocities are large enough to be treated as constants, the third line may be treated as an additional term to the lateral equations of motion. The fourth line contains mixed terms that usually can be neglected as second-order terms.

It should be noted that the off-diagonal elements of $\boldsymbol{M}_A$ will be small compared to the diagonal elements for most practical applications. A more detailed discussion on the different added mass derivatives can be found in Humphreys and Watkinson (1978).

Property 6.2 (Hydrodynamic Coriolis–Centripetal Matrix $\boldsymbol{C}_A(\boldsymbol{\nu})$)
For a rigid body moving through an ideal fluid the hydrodynamic Coriolis and centripetal matrix $\boldsymbol{C}_A(\boldsymbol{\nu})$ can always be parameterized such that it is skew-symmetric:

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = -\boldsymbol{C}_A^\top(\boldsymbol{\nu}), \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6 \tag{6.42}$$

One parametrization satisfying (6.42) is

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -\boldsymbol{S}(\boldsymbol{A}_{11}\boldsymbol{\nu}_1 + \boldsymbol{A}_{12}\boldsymbol{\nu}_2) \\ -\boldsymbol{S}(\boldsymbol{A}_{11}\boldsymbol{\nu}_1 + \boldsymbol{A}_{12}\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{A}_{21}\boldsymbol{\nu}_1 + \boldsymbol{A}_{22}\boldsymbol{\nu}_2) \end{bmatrix} \tag{6.43}$$

where $\boldsymbol{A}_{ij} \in \mathbb{R}^{3\times3}$ is given by

$$\boldsymbol{M}_A = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \tag{6.44}$$

Proof. Substituting $\boldsymbol{M}_A$ for $\boldsymbol{M}$ in (3.46) in Theorem 3.2 directly proves (6.43).

Formula (6.43) can be written in component form according to

$$\boldsymbol{C}_A(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & 0 & 0 & -a_3 & a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_1 \\ 0 & 0 & 0 & -a_2 & a_1 & 0 \\ 0 & -a_3 & a_2 & 0 & -b_3 & b_2 \\ a_3 & 0 & -a_1 & b_3 & 0 & -b_1 \\ -a_2 & a_1 & 0 & -b_2 & b_1 & 0 \end{bmatrix} \tag{6.45}$$

where

$$\begin{aligned} a_1 &= X_{\dot u}u + X_{\dot v}v + X_{\dot w}w + X_{\dot p}p + X_{\dot q}q + X_{\dot r}r \\ a_2 &= Y_{\dot u}u + Y_{\dot v}v + Y_{\dot w}w + Y_{\dot p}p + Y_{\dot q}q + Y_{\dot r}r \\ a_3 &= Z_{\dot u}u + Z_{\dot v}v + Z_{\dot w}w + Z_{\dot p}p + Z_{\dot q}q + Z_{\dot r}r \\ b_1 &= K_{\dot u}u + K_{\dot v}v + K_{\dot w}w + K_{\dot p}p + K_{\dot q}q + K_{\dot r}r \\ b_2 &= M_{\dot u}u + M_{\dot v}v + M_{\dot w}w + M_{\dot p}p + M_{\dot q}q + M_{\dot r}r \\ b_3 &= N_{\dot u}u + N_{\dot v}v + N_{\dot w}w + N_{\dot p}p + N_{\dot q}q + N_{\dot r}r \end{aligned} \tag{6.46}$$

Properties 6.2 and 8.1 imply that the marine craft dynamics can be represented in terms of nonlinear Coriolis and centripetal forces using relative velocity:

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{g}(\boldsymbol{\eta}) = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.47}$$

where

$$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A \tag{6.48}$$

$$\boldsymbol{C}(\boldsymbol{\nu}_r) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r) + \boldsymbol{C}_A(\boldsymbol{\nu}_r) \tag{6.49}$$

while classical seakeeping theory uses linear matrices $\boldsymbol{C}_{RB}^*$ and $\boldsymbol{C}_A^*$ as explained in Section 6.2.

Example 6.1 (Added Mass for Surface Vessels)
For surface ships such as tankers, cargo ships and cruise-liners it is common to decouple the surge mode from the steering dynamics due to xz-plane symmetry. Similarly, the heave, pitch, and roll modes are neglected under the assumption that these motion variables are small. Hence, $\boldsymbol{\nu}_r = [u_r, v_r, r]^\top$ *implies that the added mass derivatives for a surface ship are*

$$\boldsymbol{M}_A = \boldsymbol{M}_A^\top = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 \\ 0 & Y_{\dot{v}} & Y_{\dot{r}} \\ 0 & Y_{\dot{r}} & N_{\dot{r}} \end{bmatrix} \qquad (N_{\dot{v}} = Y_{\dot{r}}) \tag{6.50}$$

$$\boldsymbol{C}_A(\boldsymbol{\nu}_r) = -\boldsymbol{C}_A^\top(\boldsymbol{\nu}_r) = \begin{bmatrix} 0 & 0 & Y_{\dot{v}} v_r + Y_{\dot{r}} r \\ 0 & 0 & -X_{\dot{u}} u_r \\ -Y_{\dot{v}} v_r - Y_{\dot{r}} r & X_{\dot{u}} u_r & 0 \end{bmatrix} \tag{6.51}$$

The Coriolis and centripetal forces are recognized as

$$\boldsymbol{C}_A(\boldsymbol{\nu})\boldsymbol{\nu} = \begin{bmatrix} Y_{\dot{v}} v_r r + Y_{\dot{r}} r^2 \\ -X_{\dot{u}} u_r r \\ \underbrace{(X_{\dot{u}} - Y_{\dot{v}}) u_r v_r}_{\textit{Munk moment}} - Y_{\dot{r}} u_r r \end{bmatrix} \tag{6.52}$$

where the first term in the yaw moment is the nonlinear Munk moment, which is known to have destabilizing effects.

Example 6.2 (Added Mass for Underwater Vehicles)
In general, the motion of an underwater vehicle moving in 6 DOF at high speed will be highly nonlinear and coupled. However, in many AUV and ROV applications the vehicle will only be allowed to move at low speed. If the vehicle also has three planes of symmetry, this suggests that the contribution from the off-diagonal elements in the matrix $\boldsymbol{M}_A$ *can be neglected. Hence, the following simple expressions for the matrices* $\boldsymbol{M}_A$ *and* $\boldsymbol{C}_A$ *are obtained:*

$$\boldsymbol{M}_A = \boldsymbol{M}_A^\top = -\mathrm{diag}\{X_{\dot{u}}, Y_{\dot{v}}, Z_{\dot{w}}, K_{\dot{p}}, M_{\dot{q}}, N_{\dot{r}}\} \tag{6.53}$$

$$\boldsymbol{C}_A(\boldsymbol{\nu}_r) = -\boldsymbol{C}_A^\top(\boldsymbol{\nu}_r) = \begin{bmatrix} 0 & 0 & 0 & 0 & -Z_{\dot{w}} w_r & Y_{\dot{v}} v_r \\ 0 & 0 & 0 & Z_{\dot{w}} w_r & 0 & -X_{\dot{u}} u_r \\ 0 & 0 & 0 & -Y_{\dot{v}} v_r & X_{\dot{u}} u_r & 0 \\ 0 & -Z_{\dot{w}} w_r & Y_{\dot{v}} v_r & 0 & -N_{\dot{r}} r & M_{\dot{q}} q \\ Z_{\dot{w}} w_r & 0 & -X_{\dot{u}} u_r & N_{\dot{r}} r & 0 & -K_{\dot{p}} p \\ -Y_{\dot{v}} v_r & X_{\dot{u}} u_r & 0 & -M_{\dot{q}} q & K_{\dot{p}} p & 0 \end{bmatrix} \tag{6.54}$$

The diagonal structure is often used since it is time consuming to determine the off-diagonal elements from experiments as well as theory. In practice, the diagonal approximation is found to be quite good for many applications. This is due to the fact that the off-diagonal elements of a positive inertia matrix will be much smaller than their diagonal counterparts.

6.4 Viscous Damping and Ocean Current Forces

Hydrodynamic damping for marine craft is mainly caused by:

Potential Damping: We recall from the beginning of Section 6.2 that *added mass, damping* and *restoring* forces and moments are encountered when a body is forced to oscillate with the wave excitation frequency in the absence of incident waves. The radiation-induced damping term is usually referred to as *linear frequency-dependent potential damping* $\boldsymbol{B}(\omega)$.

Skin Friction: Linear frequency-dependent skin friction $\boldsymbol{B}_v(\omega)$ due to laminar boundary layer theory is important when considering the low-frequency motion of marine craft (Faltinsen and Sortland, 1987). In addition to linear skin friction, there will be a high-frequency contribution due to a turbulent boundary layer. This is usually referred to as a quadratic or nonlinear skin friction.

Wave Drift Damping: Wave drift damping can be interpreted as added resistance for surface vessels advancing in waves. This type of damping is derived from second-order wave theory. Wave drift damping is the most important damping contribution to surge for higher sea states. This is due to the fact that the wave drift forces are proportional to the square of the significant wave height H_s. Wave drift damping in sway and yaw is small relative to eddy-making damping (vortex shedding). A rule of thumb is that second-order wave drift forces are less than 1 % of the first-order wave forces when the significant wave height is equal to 1 m and 10 % when the significant wave height is equal to 10 m.

Damping Due to Vortex Shedding: *D'Alambert's paradox* states that no hydrodynamic forces act on a solid moving completely submerged with constant velocity in a nonviscous fluid. In a viscous fluid, frictional forces are present such that the system is not conservative with respect to energy. This is commonly referred to as *interference drag*. It arises due to the shedding of vortex sheets at sharp edges. The viscous damping force due to vortex shedding can be modeled as

$$f(u) = -\frac{1}{2}\rho\, C_D(\mathrm{R}_n)\, A|u|u \tag{6.55}$$

where u is the velocity of the craft, A is the projected cross-sectional area under water, $C_D(\mathrm{R}_n)$ is the drag coefficient based on the representative area and ρ is the water density. This expression is recognized as one of the terms in *Morison's equation* (see Faltinsen, 1990). The drag coefficient $C_D(\mathrm{R}_n)$ is a function of the *Reynolds number:*

$$\mathrm{R}_n = \frac{uD}{\nu} \tag{6.56}$$

where D is the characteristic length of the body and ν is the kinematic viscosity coefficient ($\nu = 1.56 \times 10^{-6}$ for salt water at 5 °C with salinity 3.5 %).

Lifting Forces: Hydrodynamic lift forces arise from two physical mechanisms. The first is due to the linear circulation of water around the hull. The second is a nonlinear effect, commonly called cross-flow drag, which acts from a momentum transfer from the body to the fluid. This secondary effect is closely linked to vortex shedding.

The different damping terms contribute to both linear and quadratic damping. However, it is in general difficult to separate these effects. In many cases, it is convenient to write total hydrodynamic damping as

$$\boldsymbol{D}(\boldsymbol{\nu}_r) = \boldsymbol{D} + \boldsymbol{D}_n(\boldsymbol{\nu}_r) \tag{6.57}$$

where $\boldsymbol{D}$ is the *linear damping matrix* due to potential damping and possible skin friction and $\boldsymbol{D}_n(\boldsymbol{\nu}_r)$ is the *nonlinear damping matrix* due to quadratic damping and higher-order terms. Hydrodynamic damping satisfies the following dissipative property:

Property 6.3 (Hydrodynamic Damping Matrix $\boldsymbol{D}(\boldsymbol{\nu}_r)$)
For a rigid body moving through an ideal fluid the hydrodynamic damping matrix,

$$\boldsymbol{D}(\boldsymbol{\nu}_r) = \frac{1}{2}\left[\boldsymbol{D}(\boldsymbol{\nu}_r) + \boldsymbol{D}(\boldsymbol{\nu}_r)^\top\right] + \frac{1}{2}\left[\boldsymbol{D}(\boldsymbol{\nu}_r) - \boldsymbol{D}(\boldsymbol{\nu}_r)^\top\right] \tag{6.58}$$

will be real, nonsymmetric and strictly positive:

$$\boldsymbol{D}(\boldsymbol{\nu}_r) > 0, \quad \forall\, \boldsymbol{\nu} \in \mathbb{R}^6 \tag{6.59}$$

or

$$\boldsymbol{x}^\top \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{x} = \frac{1}{2}\boldsymbol{x}^\top \left[\boldsymbol{D}(\boldsymbol{\nu}_r) + \boldsymbol{D}(\boldsymbol{\nu}_r)^\top\right] \boldsymbol{x} > 0 \ \ \forall\, \boldsymbol{x} \neq \boldsymbol{0} \tag{6.60}$$

Some of the damping terms can be determined by using well-established methods from the literature and experimental techniques.

6.4.1 Linear Viscous Damping

As shown in Section 6.2, the linear damping matrix in CO with decoupled surge dynamics can be written

$$\boldsymbol{D} = \boldsymbol{D}_P + \boldsymbol{D}_V \tag{6.61}$$

$$= -\begin{bmatrix} X_u & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ 0 & 0 & Z_w & 0 & Z_q & 0 \\ 0 & K_v & 0 & K_p & 0 & K_r \\ 0 & 0 & M_w & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{bmatrix} \tag{6.62}$$

where the diagonal terms relate to seakeeping theory according to

$$-X_u = B_{11v} \tag{6.63}$$

$$-Y_v = B_{22v} \tag{6.64}$$

$$-Z_w = B_{33v} + B_{33}(\omega_{\text{heave}}) \tag{6.65}$$

$$-K_p = B_{44v} + B_{44}(\omega_{\text{roll}}) \tag{6.66}$$

$$-M_q = B_{55v} + B_{55}(\omega_{\text{pitch}}) \tag{6.67}$$

$$-N_r = B_{66v} \tag{6.68}$$

Consider a second-order test system stabilized by a PD controller:

$$m\ddot{x} + (d + K_d)\dot{x} + K_p x = 0 \tag{6.69}$$

This system satisfies

$$2\zeta\omega_n = \frac{d + K_d}{m}, \qquad \omega_n^2 = \frac{K_p}{m} \tag{6.70}$$

In the uncontrolled case $K_p = K_d = 0$. Hence, the time constant becomes

$$T = \frac{m}{d} \tag{6.71}$$

In closed loop, K_p and K_d are positive constants satisfying

$$\begin{aligned} \frac{1}{T} + \frac{K_d}{m} &= 2\zeta\left(\frac{2\pi}{T_n}\right) \\ &= \frac{4\pi\zeta}{T_n} \end{aligned} \tag{6.72}$$

If K_d is specified as $K_d/m = 1/T$, the bandwidth of the closed-loop system is approximately doubled and it follows that

$$\frac{2}{T} = \frac{4\pi\zeta}{T_n} \tag{6.73}$$

The relationship between the time constant T, relative damping ratio ζ and the natural period T_n under feedback control is

$$T = \frac{T_n}{8\pi\zeta} \tag{6.74}$$

The corresponding feedback gains are $K_p = m\omega_n^2$ and $K_d = m/T$. This implies that a PD-controlled ship in surge, sway and yaw with relative damping ratio $\zeta = 0.1$ and natural periods in the range $100 \text{ s} \leq T_n \leq 150$ s has time constants in the range

$$39.8 \text{ s} \leq T \leq 59.7 \text{ s} \tag{6.75}$$

From (6.71), it is seen that the linear viscous damping terms can be specified as three time constants in surge, sway and yaw ($T_{\text{surge}}, T_{\text{sway}}, T_{\text{yaw}}$) while additional damping can be added in heave, roll and pitch as $(\Delta\zeta_{\text{heave}}, \Delta\zeta_{\text{roll}}, \Delta\zeta_{\text{pitch}})$. Consequently, we can use the following formulae to estimate linear viscous damping:

$$B_{11v} = \frac{m + A_{11}(0)}{T_{\text{surge}}} = \frac{8\pi\zeta_{\text{surge}}[m + A_{11}(0)]}{T_{n,\text{surge}}} \tag{6.76}$$

$$B_{22v} = \frac{m + A_{22}(0)}{T_{\text{sway}}} = \frac{8\pi\zeta_{sway}[m + A_{22}(0)]}{T_{n,\text{sway}}} \tag{6.77}$$

$$B_{33v} = 2\Delta\zeta_{\text{heave}}\omega_{\text{heave}}[m + A_{33}(\omega_{\text{heave}})] \tag{6.78}$$

$$B_{44v} = 2\Delta\zeta_{\text{roll}}\omega_{\text{roll}}[I_x + A_{44}(\omega_{\text{roll}})] \tag{6.79}$$

$$B_{55v} = 2\Delta\zeta_{\text{pitch}}\omega_{\text{pitch}}[I_y + A_{55}(\omega_{\text{pitch}})] \tag{6.80}$$

$$B_{66v} = \frac{I_z + A_{66}(0)}{T_{\text{yaw}}} = \frac{8\pi\zeta_{\text{yaw}}[I_z + A_{66}(0)]}{T_{n,\text{yaw}}} \tag{6.81}$$

where typical values for $T_{\text{surge}}, T_{\text{sway}}$ and T_{yaw} are 100–250 s, $\Delta\zeta_{\text{heave}} = \Delta\zeta_{\text{pitch}} = 0$ while additional roll damping $\Delta\zeta_{\text{roll}}$ could be added to obtain a total roll damping of 0.05–0.10, which is quite common for offshore supply vessels, tankers, semi-submersibles and container ships. For ships with anti-roll tanks a relative damping factor of 0.4–0.6 at the resonance frequency ω_{roll} is common.

6.4.2 Nonlinear Surge Damping

In surge, the viscous damping for ships may be modeled as (Lewis, 1989)

$$X = -\frac{1}{2}\rho S(1 + k)C_f(u_r)|u_r|u_r \tag{6.82}$$

$$C_f(u_r) = \underbrace{\frac{0.075}{(\log_{10} \mathrm{R}_n - 2)^2}}_{C_F} + C_R \tag{6.83}$$

where ρ is the density of water, S is the wetted surface of the hull,

$$\begin{aligned} u_r &= u - u_c \\ &= u - V_c \cos(\beta_c - \psi) \end{aligned} \tag{6.84}$$

is the relative surge velocity (see Section 8.3), k is the form factor giving a viscous correction, C_F is the flat plate friction from the ITTC 1957 line and C_R represents *residual friction* due to hull roughness, pressure resistance, wave-making resistance and wave-breaking resistance. For ships in transit k is typically 0.1 whereas this value is much higher in DP, typically $k = 0.25$ (Hoerner, 1965). The friction coefficient C_F depends on the *Reynolds number:*

$$\mathrm{R}_n = \frac{u_r L_{pp}}{\nu} \tag{6.85}$$

where $\nu = 1 \times 10^{-6}$ m/s^2 is the *kinematic viscosity* at 20 °C. For small values of $(\log_{10} \mathrm{R}_n - 2)$ in the expression for C_F, a minimum value of R_n should be used in order to avoid the condition where C_F blows up at low speed.

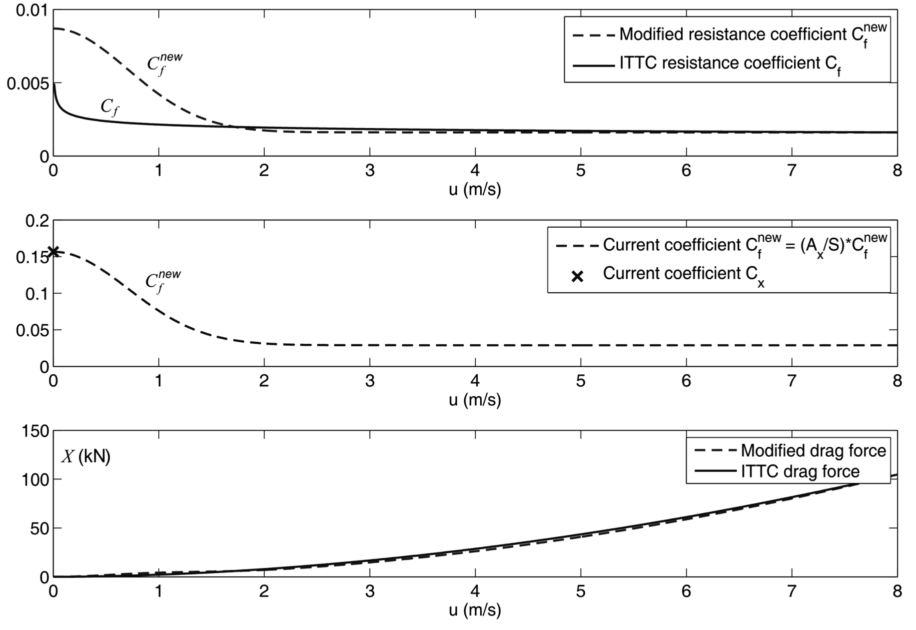

Figure 6.4 Modified resistance curve $C_f^{\text{new}}(u_r)$ and $C_f(u_r)$ as a function of u_r when $C_R = 0$. The zero speed value $C_f^{\text{new}}(0) = (A_x/S)C_X$ where $C_X = 0.16$ is the current coefficient.

For ships, a typically value is $\mathrm{R}_{n,\min} = 10^6$, which limits the friction coefficient to $C_{F,\max} < 0.05$ at lower speeds (see Figure 6.4). The damping model in surge can also be written as

$$X = X_{|u|u}|u_r|u_r \tag{6.86}$$

$$X_{|u|u} = -\frac{1}{2}\rho S(1+k)C_f < 0 \tag{6.87}$$

For low-speed maneuvering, this formula gives too little damping compared to the quadratic drag formula

$$X_{|u|u} = \frac{1}{2}\rho A_x C_x \tag{6.88}$$

where $C_X > 0$ is the current coefficient and A_x is the frontal project area (see Section 7.3.1). The current coefficients are usually found from experiments using a model ship in up to 1.0 m/s currents. The resistance and current coefficients (6.87) and (6.88) are related by

$$C_X = \frac{S}{A_x}C_f \tag{6.89}$$

One way to obtain sufficient damping at low speed is to modify the resistance curve according to

$$C_f^{\text{new}}(u_r) = C_f(u_r^{\max}) + \left(\frac{A_x}{S} C_X - C(u_r^{\max})\right) \exp(-\alpha u_r^2) \tag{6.90}$$

where $\alpha > 0$ (typically 1.0) and the maximum friction coefficient $C_f(u_r^{\max})$ is computed for maximum relative velocity $u_r^{\max}$. The modified resistance curve $C_f^{\text{new}}(u_r)$ is plotted together with $C_f(u_r)$ in Figure 6.4. Notice that the resistance curve is increased at lower velocities due to the contribution of the current coefficient C_X. The second plot shows the current coefficient C_X at zero speed together with $C_X^{\text{new}} = 1/2\rho A_x C_f^{\text{new}}$. The effect of the current coefficient vanishes at higher speeds thanks to the exponentially decaying weight $\exp(-\alpha u_r^2)$.

6.4.3 *Cross-Flow Drag Principle*

For relative current angles $|\beta_c - \psi| \gg 0$, where β_c is the current direction, the cross-flow drag principle may be applied to calculate the nonlinear damping force in sway and the yaw moment (Faltinsen, 1990):

$$Y = -\frac{1}{2}\rho \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} T(x) C_d^{2D}(x) |v_r + xr|(v_r + xr) dx \tag{6.91}$$

$$N = -\frac{1}{2}\rho \int_{-\frac{L_{pp}}{2}}^{\frac{L_{pp}}{2}} T(x) C_d^{2D}(x) x |v_r + xr|(v_r + xr) dx \tag{6.92}$$

where $C_d^{2D}(x)$ is the 2-D drag coefficient, $T(x)$ is the draft and

$$\begin{aligned} v_r &= v - v_c \\ &= v - V_c \sin(\beta_c - \psi) \end{aligned} \tag{6.93}$$

is the relative sway velocity (see Section 8.3). This is a strip theory approach where each hull section contributes to the integral. Drag coefficients for different hull forms are found in Hooft (1994). A constant 2-D current coefficient can also be estimated using Hoerner's curve (see Figure 6.5).

Matlab

The 2-D drag coefficients C_d^{2D} can be computed as a function of beam B and length T using Hoerner's curve. This is implemented in the Matlab MSS toolbox as

```
Cd=Hoerner(B,T)
```

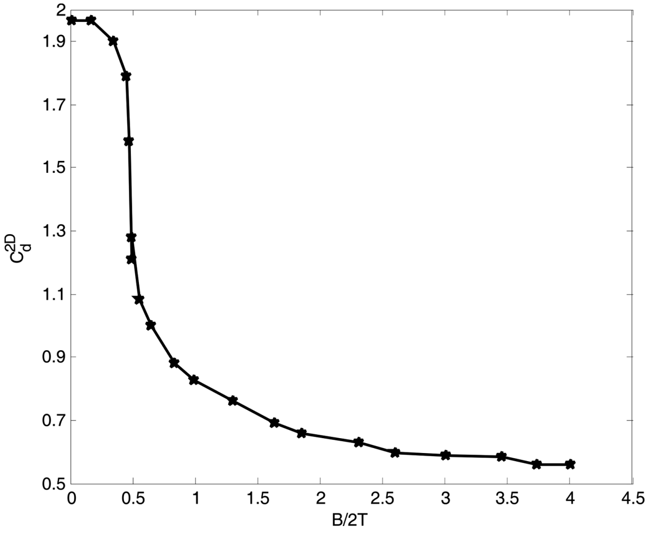

Figure 6.5 2-D cross-flow coefficient C_d^{2D} as a function of $B/2T$ (Hoerner, 1965).

A 3-D representation of (6.91)–(6.92) eliminating the integrals can be found by curve fitting formula (6.91) and (6.92) to *second-order modulus terms* to obtain a maneuvering model similar to that of Fedyaevsky and Sobolev (1963):

$$Y = Y_{|v|v}|v_r|v_r + Y_{|v|r}|v_r|r + Y_{v|r|}v_r|r| + Y_{|r|r}|r|r \tag{6.94}$$

$$N = N_{|v|v}|v_r|v_r + N_{|v|r}|v_r|r + N_{v|r|}v_r|r| + N_{|r|r}|r|r \tag{6.95}$$

where $Y_{|v|v}$, $Y_{|v|r}$, $Y_{v|r|}$, $Y_{|r|r}|r|r|$, $N_{|v|v}$, $N_{|v|r}$, $N_{v|r|}$, and $N_{|r|r}$ are maneuvering coefficients defined using the SNAME notation. In the next section, this approach will be used to derive maneuvering models in 3 DOF.

6.5 Maneuvering Equations

This section summarizes the linear and nonlinear maneuvering equations using the results in Sections 6.1–6.4. Application specific models for ships and underwater vehicles are presented in Chapter 7.

6.5.1 Hydrodynamic Mass–Damper–Spring System

In hydrodynamics it is common to assume that the hydrodynamic forces and moments on a rigid body can be linearly superimposed (see Faltinsen, 1990, 2005). This results in a *hydrodynamic mass–damper–spring* system that can be explained as:

> *Forces on the body when the body is forced to oscillate with the wave excitation frequency and there are no incident waves*

The contribution to the hydrodynamic mass–damper–spring forces is as follows:

Hydrodynamic Mass–Damper

- *Added mass* $\boldsymbol{M}_A$ due to the inertia of the surrounding fluid (see Section 6.2). The corresponding Coriolis and centripetal matrix due to added mass is due to the rotation of $\{b\}$ with respect to $\{n\}$ and is denoted $\boldsymbol{C}_A(\boldsymbol{\nu}_r)$ (see Section 6.3).
- *Radiation-induced potential damping* $\boldsymbol{D}_P$ due to the energy carried away by generated surface waves.
- *Viscous damping* caused by skin friction, wave drift damping, vortex shedding and lift/drag (see Section 6.4). The resulting hydrodynamic force is written as

$$\boldsymbol{\tau}_{\text{hyd}} = -\underbrace{\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r - \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{added mass}} - \underbrace{\boldsymbol{D}_P\boldsymbol{\nu}_r}_{\text{potential damping}} + \boldsymbol{\tau}_{\text{visc}} \tag{6.96}$$

where $\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$ with $\boldsymbol{\nu}_c = [u, v_c, w_c, 0, 0, 0]^\top$ is the relative velocity due to an irrotational constant ocean current (see Section 8.3) and

$$\boldsymbol{\tau}_{\text{visc}} = -\underbrace{\boldsymbol{D}_V\boldsymbol{\nu}_r}_{\substack{\text{linear}\\ \text{viscous friction}}} - \underbrace{\boldsymbol{D}_n(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\substack{\text{nonlinear}\\ \text{viscous damping}}} \tag{6.97}$$

Hydrostatic Spring Stiffness

- *Restoring forces* due to Archimedes (weight and buoyancy); see Section 4.1:

$$\boldsymbol{\tau}_{hs} = -\boldsymbol{g}(\boldsymbol{\eta}) - \boldsymbol{g}_o \tag{6.98}$$

The potential coefficient matrices $\boldsymbol{A}(\omega)$ and $\boldsymbol{B}(\omega)$ can be computed using a hydrodynamic code while approximate expressions for $\boldsymbol{M}_A$ and $\boldsymbol{D}_P$ as well as $\boldsymbol{C}_A(\boldsymbol{\nu}_r)$ can be computed using (6.26) and (6.27), which are based on Definitions 6.1 and 6.2. Fully coupled matrices $\boldsymbol{M}_A$ and $\boldsymbol{D}_P$ in 6 DOF can, however, be computed using model experiments or curve fitting to experimental data. This results in constant (frequency-independent) matrices in the following form:

$$\boldsymbol{M}_A = -\begin{bmatrix} X_{\dot{u}} & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{\dot{v}} & 0 & Y_{\dot{p}} & 0 & Y_{\dot{r}} \\ 0 & 0 & Z_{\dot{w}} & 0 & Z_{\dot{q}} & 0 \\ 0 & K_{\dot{v}} & 0 & K_{\dot{p}} & 0 & K_{\dot{r}} \\ 0 & 0 & M_{\dot{w}} & 0 & M_{\dot{q}} & 0 \\ 0 & N_{\dot{v}} & 0 & N_{\dot{p}} & 0 & N_{\dot{r}} \end{bmatrix} \tag{6.99}$$

$$\boldsymbol{D} = -\begin{bmatrix} X_u & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_v & 0 & Y_p & 0 & Y_r \\ 0 & 0 & Z_w & 0 & Z_q & 0 \\ 0 & K_v & 0 & K_p & 0 & K_r \\ 0 & 0 & M_w & 0 & M_q & 0 \\ 0 & N_v & 0 & N_p & 0 & N_r \end{bmatrix} \tag{6.100}$$

where the coefficients are called *hydrodynamic derivatives.* Again it is convenient to assume that the surge motion is decoupled and that the marine craft is symmetric about the xz plane. This reduces the number of parameters in the model.

The total hydrodynamic damping matrix $\boldsymbol{D}(\boldsymbol{\nu}_r)$ is the sum of the linear part $\boldsymbol{D}$ and the nonlinear part $\boldsymbol{D}_n(\boldsymbol{\nu}_r)$ such that

$$\boldsymbol{D}(\boldsymbol{\nu}_r) := \boldsymbol{D} + \boldsymbol{D}_n(\boldsymbol{\nu}_r) \tag{6.101}$$

If *nonlinear damping* is modeled using the ITTC resistance law in Section 6.4.2 and cross-flow drag formulae in Section 6.4.3, the following expression is obtained:

$$\boldsymbol{D}_n(\boldsymbol{\nu}_r) = -\begin{bmatrix} X_{|u|u}\,|u_r| & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{|v|v}\,|v_r| + Y_{|r|v}\,|r| & 0 & 0 & 0 & Y_{|v|r}\,|v_r| + Y_{|r|r}\,|r| \\ 0 & 0 & Z_{|w|w}|w_r| & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{|p|p}|p| & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{|q|q}|q| & 0 \\ 0 & N_{|v|v}\,|v_r| + N_{|r|v}\,|r| & 0 & 0 & 0 & N_{|v|r}\,|v_r| + N_{|r|r}\,|r| \end{bmatrix}$$

where we have included additional nonlinear damping terms on the diagonal in heave, roll and pitch. In general, there will be coupling terms in all DOF. However, many of these are small so engineering judgement must be used when deriving the model. Standard models for marine craft are discussed in Chapter 7.

The resulting nonlinear hydrodynamic mass–damper–spring forces can be expressed by

$$\boldsymbol{\tau}_{\text{hyd}} = -\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r - \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r - \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r \tag{6.102}$$

$$\boldsymbol{\tau}_{\text{hs}} = -\boldsymbol{g}(\boldsymbol{\eta}) - \boldsymbol{g}_o \tag{6.103}$$

6.5.2 Nonlinear Maneuvering Equations

The hydrodynamic forces (6.102) and (6.103) must be included in the equations of motion in order to integrate acceleration $\dot{\boldsymbol{\nu}}$ to velocity and position. Consider the rigid-body kinetics (6.5):

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{6.104}$$

where

$$\boldsymbol{\tau}_{RB} = \boldsymbol{\tau}_{\text{hyd}} + \boldsymbol{\tau}_{\text{hs}} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} + \boldsymbol{\tau} \tag{6.105}$$

The vector $\boldsymbol{\tau}$ represents the *propulsion* forces and moments. Substituting (6.102) and (6.103) into (6.105) gives the nonlinear maneuvering equations:

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}} + \underbrace{\boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o}_{\text{hydrostatic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.106}$$

A special case of (6.106) is obtained for *ocean currents* that are assumed to be *constant* and *irrotational* in $\{n\}$ such that (see Section 8.3)

$$\dot{\boldsymbol{\nu}}_c = \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{\omega}^b_{b/n}) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \end{bmatrix} \boldsymbol{\nu}_c \tag{6.107}$$

According to Property 8.1, it is then possible to represent the equations of motion by relative velocities according to

$$\boldsymbol{M}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{g}(\boldsymbol{\eta}) + \boldsymbol{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.108}$$

where

$$\boldsymbol{M} = \boldsymbol{M}_{RB} + \boldsymbol{M}_A \tag{6.109}$$

$$\boldsymbol{C}(\boldsymbol{\nu}_r) = \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r) + \boldsymbol{C}_A(\boldsymbol{\nu}_r) \tag{6.110}$$

The assumption that the potential coefficients are constant (frequency independent) implies that

$$\boldsymbol{M} = \boldsymbol{M}^\top > 0, \qquad \dot{\boldsymbol{M}} = \boldsymbol{0} \tag{6.111}$$

which are very useful properties when designing energy-based control laws where the sum of kinetic and potential energy is a natural Lyapunov function candidate.

Models for simulation and control of marine craft are treated in more detail in Chapter 7 where focus is on tailor-made models for dynamic positioning, roll damping, ship maneuvering, path following and autopilot design.

6.5.3 Linearized Maneuvering Equations

The linearized maneuvering equations in *surge*, *sway* and *yaw* is a special case of the nonlinear model (6.106):

$$\underbrace{\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}^*_{RB}\boldsymbol{\nu}}_{\text{rigid-body forces}} + \underbrace{\boldsymbol{M}_A\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}^*_A\boldsymbol{\nu}_r + \boldsymbol{D}\boldsymbol{\nu}_r}_{\text{hydrodynamic forces}} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.112}$$

where restoring forces are neglected, nonlinear Coriolis and centripetal forces are linearized about the cruise speed U and nonlinear damping is approximated by a linear damping matrix. If ocean currents are neglected, Equation (6.112) reduces to

$$\underbrace{(\boldsymbol{M}_{RB} + \boldsymbol{M}_A)}_{\boldsymbol{M}}\dot{\boldsymbol{\nu}} + \underbrace{(\boldsymbol{C}^*_{RB} + \boldsymbol{C}^*_A + \boldsymbol{D})}_{\boldsymbol{N}}\boldsymbol{\nu} = \boldsymbol{\tau} + \boldsymbol{\tau}_{\text{wind}} + \boldsymbol{\tau}_{\text{wave}} \tag{6.113}$$

The expressions for $\boldsymbol{C}^*_{RB}$ and $\boldsymbol{C}^*_A$ are computed using the selection matrix $\boldsymbol{L}$ given by (3.63). This gives

$$\begin{aligned}\boldsymbol{C}^*_{RB} &= U\boldsymbol{M}_{RB}\boldsymbol{L} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & mU \\ 0 & 0 & mx_g U \end{bmatrix}\end{aligned} \tag{6.114}$$

$$\begin{aligned}\boldsymbol{C}^*_{A} &= U\boldsymbol{M}_{A}\boldsymbol{L} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -Y_{\dot{v}}U \\ 0 & 0 & -Y_{\dot{r}}U \end{bmatrix}\end{aligned} \tag{6.115}$$

such that

$$\begin{bmatrix} m - X_{\dot{u}} & 0 & 0 \\ 0 & m - Y_{\dot{v}} & mx_g - Y_{\dot{r}} \\ 0 & mx_g - N_{\dot{v}} & I_z - N_{\dot{r}} \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{r} \end{bmatrix} + \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & (m - Y_{\dot{v}})U - Y_r \\ 0 & -N_v & (mx_g - Y_{\dot{r}})U - N_r \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_6 \end{bmatrix}$$

Notice that surge is assumed to be decoupled from the sway–yaw subsystem.