3

Rigid-Body Kinetics

In order to derive the marine craft equations of motion, it is necessary to study the motion of rigid bodies, hydrodynamics and hydrostatics. The overall goal of Chapter 3 is to show that the rigid-body kinetics can be expressed in a vectorial setting according to (Fossen, 1991)

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{3.1}$$

where $\boldsymbol{M}_{RB}$ is the rigid-body mass matrix, $\boldsymbol{C}_{RB}$ is the rigid-body Coriolis and centripetal matrix due to the rotation of $\{b\}$ about the inertial frame $\{n\}$, $\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ is the generalized velocity vector expressed in $\{b\}$ and $\boldsymbol{\tau}_{RB} = [X, Y, Z, K, M, N]^\top$ is a generalized vector of external forces and moments expressed in $\{b\}$.

The rigid-body equations of motion will be derived using the *Newton–Euler formulation* and *vectorial mechanics*. In this context it is convenient to define the vectors without reference to a coordinate frame (*coordinate free vector*). The velocity of the origin o_b with respect to $\{n\}$ is a vector $\vec{v}_{b/n}$ that is defined by its magnitude and the direction. The vector $\vec{v}_{b/n}$ decomposed in the inertial reference frame is denoted as $\boldsymbol{v}^i_{b/n}$, which is also referred to as a *coordinate vector* (see Section 2.1).

The equations of motion will be represented in two body-fixed reference points:

CO - origin o_b of $\{b\}$
CG - center of gravity

These points coincide if the vector $\vec{r}_g = \vec{0}$ (see Figure 3.1). The point CO is usually specified by the control engineer and it is the reference point used to design the guidance, navigation and control systems. For marine craft, it is common to locate this point on the centerline midships. It is advantageous to use a fixed reference point CO for controller–observer design since CG will depend on the load condition (see Section 4.3).

3.1 Newton–Euler Equations of Motion about CG

The *Newton–Euler formulation* is based on *Newton's second law*, which relates mass m, acceleration $\dot{\vec{v}}_{g/i}$ and force $\vec{f}_g$ according to

$$m\dot{\vec{v}}_{g/i} = \vec{f}_g \tag{3.2}$$

where $\vec{v}_{g/i}$ is the velocity of the CG with respect to the *inertial frame* $\{i\}$.

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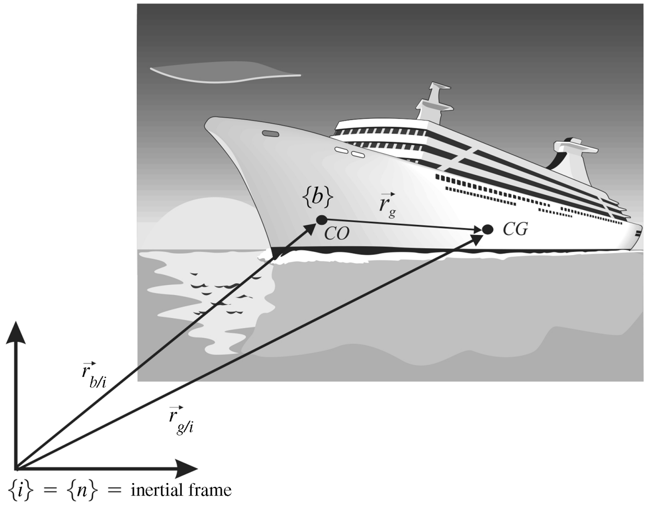

Figure 3.1 Definition of the volume element dV and the coordinate origins CO and CG.

If no force is acting ($\vec{f}_g = \vec{0}$), then the rigid body is moving with constant speed ($\vec{v}_{g/i}$ = constant) or the body is at rest ($\vec{v}_{g/i} = \vec{0}$)–a result known as *Newton's first law*. Newton's laws were published in 1687 by Isaac Newton (1643–1727) in *Philosophia Naturalis Principia Mathematica.*

Euler's First and Second Axioms

Leonhard Euler (1707–1783) showed in his *Novi Commentarii Academiae Scientarium Imperialis Petropolitane* that Newton's second law can be expressed in terms of conservation of both linear momentum $\vec{p}_g$ and angular momentum $\vec{h}_g$. These results are known as *Euler's first and second axioms*, respectively:

$$\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{p}_g = \vec{f}_g \qquad \vec{p}_g = m\vec{v}_{g/i} \tag{3.3}$$

$$\frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{h}_g = \vec{m}_g \qquad \vec{h}_g = I_g\vec{\omega}_{b/i} \tag{3.4}$$

where $\vec{f}_g$ and $\vec{m}_g$ are the forces and moments acting on the body's CG, $\vec{\omega}_{b/i}$ is the angular velocity of $\{b\}$ with respect to $\{i\}$, and I_g is the inertia dyadic about the body's CG (to be defined later). Time differentiation in the inertial frame $\{i\}$ is denoted by ${}^i\mathrm{d}/\mathrm{d}t$. The application of these equations is often referred to as *vectorial mechanics* since both conservation laws are expressed in terms of vectors.

When deriving the equations of motion it will be assumed: (1) that the craft is rigid and (2) that the NED frame $\{n\}$ is inertial; see Section 2.1. The first assumption eliminates the consideration of forces acting between individual elements of mass while the second eliminates forces due to the Earth's motion relative to a star-fixed inertial reference system. Consequently,

$$\vec{v}_{g/i} \approx \vec{v}_{g/n} \tag{3.5}$$

$$\vec{\omega}_{b/i} \approx \vec{\omega}_{b/n} \tag{3.6}$$

Time differentiation of a vector $\vec{a}$ in a moving reference frame $\{b\}$ satisfies

$$\frac{{}^{i}\mathrm{d}}{\mathrm{d}t}\vec{a} = \frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{a} + \vec{\omega}_{b/i} \times \vec{a} \tag{3.7}$$

where time differentiation in $\{b\}$ is denoted as

$$\dot{\vec{a}} := \frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{a} \tag{3.8}$$

For guidance and navigation applications in space it is usual to use a star-fixed reference frame or a reference frame rotating with the Earth. Marine craft are, on the other hand, usually related to $\{n\}$. This is a good assumption since the forces on marine craft due to the Earth's rotation:

$$\omega_{e/i} = 7.2921 \times 10^{-5} \text{ rad/s} \tag{3.9}$$

are quite small compared to the hydrodynamic forces. The Earth's rotation should, however, not be neglected in global navigation or if the equations of motion of a drifting ship are analyzed.

3.1.1 Translational Motion about CG

From Figure 3.1 it follows that

$$\vec{r}_{g/i} = \vec{r}_{b/i} + \vec{r}_g \tag{3.10}$$

where $\vec{r}_g$ is the distance vector from CO (origin o_b) to CG. Consequently, the assumption that $\{n\}$ is inertial implies that (3.10) can be rewritten as

$$\vec{r}_{g/n} = \vec{r}_{b/n} + \vec{r}_g \tag{3.11}$$

Time differentiation of $\vec{r}_{g/n}$ in a moving reference frame $\{b\}$ using (3.7) gives

$$\vec{v}_{g/n} = \vec{v}_{b/n} + \left(\frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{r}_g + \vec{\omega}_{b/n} \times \vec{r}_g \right) \tag{3.12}$$

For a rigid body, CG satisfies

$$\frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{r}_g = \vec{0} \tag{3.13}$$

such that

$$\vec{v}_{g/n} = \vec{v}_{b/n} + \vec{\omega}_{b/n} \times \vec{r}_g \tag{3.14}$$

From Euler's first axiom (3.3) it follows that

$$
\begin{aligned}
\vec{f}_g &= \frac{^i\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{g/i}) \\
&= \frac{^i\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{g/n}) \\
&= \frac{^b\mathrm{d}}{\mathrm{d}t}(m\vec{v}_{g/n}) + m\vec{\omega}_{b/n} \times \vec{v}_{g/n} \\
&= m(\dot{\vec{v}}_{g/n} + \vec{\omega}_{b/n} \times \vec{v}_{g/n})
\end{aligned}
\tag{3.15}
$$

Finally, the vectors can be expressed in $\{b\}$ such that the translational motion in CG becomes

$$
m[\dot{\boldsymbol{v}}^b_{g/n} + \boldsymbol{S}(\boldsymbol{\omega}^b_{b/n})\boldsymbol{v}^b_{g/n}] = \boldsymbol{f}^b_g \tag{3.16}
$$

where the cross-product is written in matrix form using the skew-symmetric matrix (2.10), that is $\boldsymbol{S}(\boldsymbol{\omega}^b_{b/n})\boldsymbol{v}^b_{g/n} = \boldsymbol{\omega}^b_{b/n} \times \boldsymbol{v}^b_{g/n}$.

3.1.2 Rotational Motion about CG

The rotational dynamics (attitude dynamics) follows a similar approach. From Euler's second axiom (3.4), it is seen that

$$
\begin{aligned}
\vec{m}_g &= \frac{^i\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{b/i}) \\
&= \frac{^i\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{b/n}) \\
&= \frac{^b\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{b/n}) + \vec{\omega}_{b/n} \times (I_g\vec{\omega}_{b/n}) \\
&= I_g\dot{\vec{\omega}}_{b/n} - (I_g\vec{\omega}_{b/n}) \times \vec{\omega}_{b/n}
\end{aligned}
\tag{3.17}
$$

From this it follows that

$$
\boldsymbol{I}_g\dot{\boldsymbol{\omega}}^b_{b/n} - \boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega}^b_{b/n})\boldsymbol{\omega}^b_{b/n} = \boldsymbol{m}^b_g \tag{3.18}
$$

since $\boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega}^b_{b/n})\boldsymbol{\omega}^b_{b/n} = (\boldsymbol{I}_g\boldsymbol{\omega}^b_{b/n}) \times \boldsymbol{\omega}^b_{b/n}$. This expression is also referred to as *Euler's equations.*

Definition 3.1 (Inertia Matrix)
The inertia matrix $\boldsymbol{I}_g \in \mathbb{R}^{3\times 3}$ *about* CG *is defined as*

$$
\boldsymbol{I}_g := \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}, \qquad \boldsymbol{I}_g = \boldsymbol{I}_g^\top > 0 \tag{3.19}
$$

where I_x, I_y *and* I_z *are the* moments of inertia *about the* x_b, y_b *and* z_b *axes, and* $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$ *and* $I_{yz} = I_{zy}$ *are the* products of inertia *defined as*

$$\begin{aligned} I_x &= \int_V (y^2 + z^2)\, \rho_m \mathrm{d}V; \qquad & I_{xy} &= \int_V xy\, \rho_m \mathrm{d}V = \int_V yx\, \rho_m \mathrm{d}V = I_{yx} \\ I_y &= \int_V (x^2 + z^2)\, \rho_m \mathrm{d}V; \qquad & I_{xz} &= \int_V xz\, \rho_m \mathrm{d}V = \int_V zx\, \rho_m \mathrm{d}V = I_{zx} \\ I_z &= \int_V (x^2 + y^2)\, \rho_m \mathrm{d}V; \qquad & I_{yz} &= \int_V yz\, \rho_m \mathrm{d}V = \int_V zy\, \rho_m \mathrm{d}V = I_{zy} \end{aligned}$$

3.1.3 Equations of Motion about CG

The Newton–Euler equations (3.16) and (3.18) can be represented in matrix form according to

$$\boldsymbol{M}_{RB}^{CG} \begin{bmatrix} \dot{\boldsymbol{v}}_{g/n}^b \\ \dot{\boldsymbol{\omega}}_{b/n}^b \end{bmatrix} + \boldsymbol{C}_{RB}^{CG} \begin{bmatrix} \boldsymbol{v}_{g/n}^b \\ \boldsymbol{\omega}_{b/n}^b \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_g^b \\ \boldsymbol{m}_g^b \end{bmatrix} \tag{3.20}$$

or

$$\underbrace{\begin{bmatrix} m\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_g \end{bmatrix}}_{\boldsymbol{M}_{RB}^{CG}} \begin{bmatrix} \dot{\boldsymbol{v}}_{g/n}^b \\ \dot{\boldsymbol{\omega}}_{b/n}^b \end{bmatrix} + \underbrace{\begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b) & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & -\boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega}_{b/n}^b) \end{bmatrix}}_{\boldsymbol{C}_{RB}^{CG}} \begin{bmatrix} \boldsymbol{v}_{g/n}^b \\ \boldsymbol{\omega}_{b/n}^b \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_g^b \\ \boldsymbol{m}_g^b \end{bmatrix} \tag{3.21}$$

3.2 Newton–Euler Equations of Motion about CO

For marine craft it is desirable to derive the equations of motion for an arbitrary origin CO to take advantage of the craft's geometric properties. Since the hydrodynamic forces and moments often are computed in CO, Newton's laws will be formulated in CO as well.

In order to do this, we will start with the equations of motion about CG and transform these expressions to CO using a coordinate transformation. The needed coordinate transformation is derived from (3.14). Moreover,

$$\begin{aligned} \boldsymbol{v}_{g/n}^b &= \boldsymbol{v}_{b/n}^b + \boldsymbol{\omega}_{b/n}^b \times \boldsymbol{r}_g^b \\ &= \boldsymbol{v}_{b/n}^b - \boldsymbol{r}_g^b \times \boldsymbol{\omega}_{b/n}^b \\ &= \boldsymbol{v}_{b/n}^b + \boldsymbol{S}^\top(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{b/n}^b \end{aligned} \tag{3.22}$$

From this it follows that

$$\begin{bmatrix} \boldsymbol{v}_{g/n}^b \\ \boldsymbol{\omega}_{b/n}^b \end{bmatrix} = \boldsymbol{H}(\boldsymbol{r}_g^b) \begin{bmatrix} \boldsymbol{v}_{b/n}^b \\ \boldsymbol{\omega}_{b/n}^b \end{bmatrix} \tag{3.23}$$

where $\boldsymbol{r}_g^b = [x_g, y_g, z_g]^\top$ and $\boldsymbol{H}(\boldsymbol{r}_g^b) \in \mathbb{R}^{3\times3}$ is a transformation matrix:

$$\boldsymbol{H}(\boldsymbol{r}_g^b) := \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{S}^\top(\boldsymbol{r}_g^b) \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix}, \qquad \boldsymbol{H}^\top(\boldsymbol{r}_g^b) = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{S}(\boldsymbol{r}_g^b) & \boldsymbol{I}_{3\times3} \end{bmatrix} \tag{3.24}$$

Notice that angular velocity is unchanged during this transformation. The next step is to transform (3.20) from CG to CO using (3.23). This gives

$$\boldsymbol{H}^\top(\boldsymbol{r}_g^b)\boldsymbol{M}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_g^b) \begin{bmatrix} \dot{\boldsymbol{v}}_{b/n}^b \\ \dot{\boldsymbol{\omega}}_{b/n}^b \end{bmatrix} + \boldsymbol{H}^\top(\boldsymbol{r}_g^b)\boldsymbol{C}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_g^b) \begin{bmatrix} \boldsymbol{v}_{b/n}^b \\ \boldsymbol{\omega}_{b/n}^b \end{bmatrix} = \boldsymbol{H}^\top(\boldsymbol{r}_g^b) \begin{bmatrix} \boldsymbol{f}_g^b \\ \boldsymbol{m}_g^b \end{bmatrix} \tag{3.25}$$

We now define two new matrices in CO according to

$$\boldsymbol{M}_{RB}^{CO} := \boldsymbol{H}^{\top}(\boldsymbol{r}_g^b)\boldsymbol{M}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_g^b) \tag{3.26}$$

$$\boldsymbol{C}_{RB}^{CO} := \boldsymbol{H}^{\top}(\boldsymbol{r}_g^b)\boldsymbol{C}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_g^b) \tag{3.27}$$

Expanding these expressions gives

$$\boldsymbol{M}_{RB}^{CO} = \begin{bmatrix} m\boldsymbol{I}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b) & \boldsymbol{I}_g - m\boldsymbol{S}^2(\boldsymbol{r}_g^b) \end{bmatrix} \tag{3.28}$$

$$\boldsymbol{C}_{RB}^{CO} = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b) & -m\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b) & -\boldsymbol{S}\left(\left(\boldsymbol{I}_g - m\boldsymbol{S}^2(\boldsymbol{r}_g^b)\right)\boldsymbol{\omega}_{b/n}^b\right) \end{bmatrix} \tag{3.29}$$

where we have used the fact that

$$m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{S}^{\top}(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{b/n}^b - \boldsymbol{S}(\boldsymbol{I}_g\boldsymbol{\omega}_{b/n}^b)\boldsymbol{\omega}_{b/n}^b \equiv \boldsymbol{S}\left(\left(\boldsymbol{I}_g - m\boldsymbol{S}^2(\boldsymbol{r}_g^b)\right)\boldsymbol{\omega}_{b/n}^b\right)\boldsymbol{\omega}_{b/n}^b \tag{3.30}$$

3.2.1 Translational Motion about CO

From the first row in (3.25) with matrices (3.28) and (3.29) it is seen that the translational motion about CO satisfies

$$m[\dot{\boldsymbol{v}}_{b/n}^b + \boldsymbol{S}^{\top}(\boldsymbol{r}_g^b)\dot{\boldsymbol{\omega}}_{b/n}^b + \boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{v}_{b/n}^b + \boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{S}^{\top}(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{b/n}^b] = \boldsymbol{f}_g^b \tag{3.31}$$

Since the translational motion is independent of the attack point of the external force $\boldsymbol{f}_g^b = \boldsymbol{f}_b^b$ it follows that

$$m[\dot{\boldsymbol{v}}_{b/n}^b + \boldsymbol{S}(\dot{\boldsymbol{\omega}}_{b/n}^b)\boldsymbol{r}_g^b + \boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{v}_{b/n}^b + \boldsymbol{S}^2(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{r}_g^b] = \boldsymbol{f}_b^b \tag{3.32}$$

where we have exploited the fact that $\boldsymbol{S}^{\top}(\boldsymbol{a})\boldsymbol{b} = -\boldsymbol{S}(\boldsymbol{a})\boldsymbol{b} = \boldsymbol{S}(\boldsymbol{b})\boldsymbol{a}$. An alternative representation of (3.32) using vector cross-products is

$$m[\dot{\boldsymbol{v}}_{b/n}^b + \dot{\boldsymbol{\omega}}_{b/n}^b \times \boldsymbol{r}_g^b + \boldsymbol{\omega}_{b/n}^b \times \boldsymbol{v}_{b/n}^b + \boldsymbol{\omega}_{b/n}^b \times (\boldsymbol{\omega}_{b/n}^b \times \boldsymbol{r}_g^b)] = \boldsymbol{f}_b^b \tag{3.33}$$

3.2.2 Rotational Motion about CO

In order to express the rotational motion (attitude dynamics) about CO we will make use of the parallel-axes theorem that transforms the inertia matrix to an arbitrarily point.

Theorem 3.1 (Parallel-Axes Theorem or Huygens–Steiner Theorem)
The inertia matrix $\boldsymbol{I}_b = \boldsymbol{I}_b^{\top} \in \mathbb{R}^{3\times3}$ about an arbitrary origin o_b is given by

$$\boldsymbol{I}_b = \boldsymbol{I}_g - m\boldsymbol{S}^2(\boldsymbol{r}_g^b) = \boldsymbol{I}_g - m\left(\boldsymbol{r}_g^b(\boldsymbol{r}_g^b)^{\top} - (\boldsymbol{r}_g^b)^{\top}\boldsymbol{r}_g^b\,\boldsymbol{I}_{3\times3}\right) \tag{3.34}$$

where $\boldsymbol{I}_g = \boldsymbol{I}_g^{\top} \in \mathbb{R}^{3\times3}$ is the inertia matrix about the body's center of gravity.

Proof. See Egeland and Gravdahl (2002).

The lower-right elements in (3.28) and (3.29) can be reformulated by using the parallel-axes theorem. For instance,

$$\begin{aligned} \boldsymbol{I}_g + m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}^\top(\boldsymbol{r}_g^b) &= \boldsymbol{I}_g - m\boldsymbol{S}^2(\boldsymbol{r}_g^b) \\ &= \boldsymbol{I}_b \end{aligned} \tag{3.35}$$

while the quadratic term in (3.29) satisfies (follows from the Jacobi identity)

$$\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{S}^\top(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{b/n}^b = -\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{S}^2(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{b/n}^b \tag{3.36}$$

such that

$$m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{S}^\top(\boldsymbol{r}_g^b)\boldsymbol{\omega}_{b/n}^b + \boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{I}_g\boldsymbol{\omega}_{b/n}^b = \boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{I}_b\boldsymbol{\omega}_{b/n}^b \tag{3.37}$$

Consequently, the rotational motion about CO is given by the last row in (3.25):

$$\boldsymbol{I}_b\dot{\boldsymbol{\omega}}_{b/n}^b + \boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{I}_b\boldsymbol{\omega}_{b/n}^b + m\boldsymbol{S}(\boldsymbol{r}_g^b)\dot{\boldsymbol{v}}_{b/n}^b + m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\omega}_{b/n}^b)\boldsymbol{v}_{b/n}^b = \boldsymbol{m}_b^b \tag{3.38}$$

where the moment about CO is

$$\begin{aligned} \boldsymbol{m}_b^b &= \boldsymbol{m}_g^b + \boldsymbol{r}_g^b \times \boldsymbol{f}_g^b \\ &= \boldsymbol{m}_g^b + \boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{f}_g^b \end{aligned} \tag{3.39}$$

Equation (3.38) can be written in cross-product form as

$$\boldsymbol{I}_b\dot{\boldsymbol{\omega}}_{b/n}^b + \boldsymbol{\omega}_{b/n}^b \times \boldsymbol{I}_b\boldsymbol{\omega}_{b/n}^b + m\boldsymbol{r}_g^b \times (\dot{\boldsymbol{v}}_{b/n}^b + \boldsymbol{\omega}_{b/n}^b \times \boldsymbol{v}_{b/n}^b) = \boldsymbol{m}_b^b \tag{3.40}$$

3.3 Rigid-Body Equations of Motion

In the previous sections it was shown how the rigid-body kinetics can be derived by applying *Newtonian* mechanics. In this section, useful properties of the equations of motion are discussed and it is also demonstrated how these properties considerably simplify the representation of the nonlinear equations of motion.

3.3.1 Nonlinear 6 DOF Rigid-Body Equations of Motion

Equations (3.33) and (3.40) are usually written in component form according to the SNAME (1950) notation by defining:

$\boldsymbol{f}_b^b = [X, Y, Z]^\top$ - force through o_b expressed in $\{b\}$
$\boldsymbol{m}_b^b = [K, M, N]^\top$ - moment about o_b expressed in $\{b\}$
$\boldsymbol{v}_{b/n}^b = [u, v, w]^\top$ - linear velocity of o_b relative o_n expressed in $\{b\}$
$\boldsymbol{\omega}_{b/n}^b = [p, q, r]^\top$ - angular velocity of $\{b\}$ relative to $\{n\}$ expressed in $\{b\}$
$\boldsymbol{r}_g^b = [x_g, y_g, z_g]^\top$ - vector from o_b to CG expressed in $\{b\}$

Applying this notation, (3.33) and (3.40) become

$$
\begin{aligned}
m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] &= X \\
m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] &= Y \\
m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] &= Z \\
I_x\dot{p} + (I_z - I_y)qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy} & \\
+m\left[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)\right] &= K \\
I_y\dot{q} + (I_x - I_z)rp - (\dot{p} + qr)I_{xy} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz} & \\
+m\left[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)\right] &= M \\
I_z\dot{r} + (I_y - I_x)pq - (\dot{q} + rp)I_{yz} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx} & \\
+m\left[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)\right] &= N
\end{aligned}
\tag{3.41}
$$

The first three equations represent the translational motion, while the last three equations represent the rotational motion.

Vectorial Representation

The rigid-body kinetics (3.41) can be expressed in a vectorial setting as (Fossen, 1991)

$$
\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{3.42}
$$

where $\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ is the generalized velocity vector expressed in $\{b\}$ and $\boldsymbol{\tau}_{RB} = [X, Y, Z, K, M, N]^\top$ is a generalized vector of external forces and moments.

Property 3.1 (Rigid-Body System Inertia Matrix $\boldsymbol{M}_{RB}$)
The representation of the rigid-body system inertia matrix $\boldsymbol{M}_{RB}$ is unique and satisfies

$$
\boldsymbol{M}_{RB} = \boldsymbol{M}_{RB}^\top > 0, \qquad \dot{\boldsymbol{M}}_{RB} = \boldsymbol{0}_{6\times 6} \tag{3.43}
$$

where

$$
\begin{aligned}
\boldsymbol{M}_{RB} &= \begin{bmatrix} m\boldsymbol{I}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b) & \boldsymbol{I}_b \end{bmatrix} \\
&= \begin{bmatrix}
m & 0 & 0 & 0 & mz_g & -my_g \\
0 & m & 0 & -mz_g & 0 & mx_g \\
0 & 0 & m & my_g & -mx_g & 0 \\
0 & -mz_g & my_g & I_x & -I_{xy} & -I_{xz} \\
mz_g & 0 & -mx_g & -I_{yx} & I_y & -I_{yz} \\
-my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z
\end{bmatrix}
\end{aligned}
\tag{3.44}
$$

Here, $\boldsymbol{I}_{3\times3}$ is the identity matrix, $\boldsymbol{I}_b = \boldsymbol{I}_b^\top > 0$ is the inertia matrix according to Definition 3.1 and $\boldsymbol{S}(\boldsymbol{r}_g^b)$ is a skew-symmetric matrix according to Definition 2.2.

Matlab

The rigid-body system inertia matrix $\boldsymbol{M}_{RB}$ can be computed in Matlab as

```
r_g = [10 0 1]'          % location of the CG with respect to CO
nu  = [10 0 1 0 0 1]'    % velocity vector
I_g = 10000*eye(3)       % inertia tensor
m   = 1000;              % mass

% rigid-body system inertia matrix
MRB = [ m*eye(3)        -m*Smtrx(r_g)
        m*Smtrx(r_g)    I_g           ]
```

which produces the numerical result

$$\boldsymbol{M}_{RB} = \begin{bmatrix} 1000 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 1000 & 0 & -1000 & 0 & 10000 \\ 0 & 0 & 1000 & 0 & -10000 & 0 \\ 0 & -1000 & 0 & 10000 & 0 & 0 \\ 1000 & 0 & -10000 & 0 & 10000 & 0 \\ 0 & 10000 & 0 & 0 & 0 & 10000 \end{bmatrix}$$

The matrix $\boldsymbol{C}_{RB}$ in (3.42) represents the Coriolis vector term $\boldsymbol{\omega}_{b/n}^b \times \boldsymbol{v}_{b/n}^b$ and the centripetal vector term $\boldsymbol{\omega}_{b/n}^b \times (\boldsymbol{\omega}_{b/n}^b \times \boldsymbol{r}_g^b)$. Contrary to the representation of $\boldsymbol{M}_{RB}$, it is possible to find a large number of representations for the matrix $\boldsymbol{C}_{RB}$.

Theorem 3.2 (Coriolis–Centripetal Matrix from System Inertia Matrix)
Let $\boldsymbol{M}$ be a 6×6 system inertia matrix defined as

$$\boldsymbol{M} = \boldsymbol{M}^\top = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{bmatrix} > 0 \tag{3.45}$$

where $\boldsymbol{M}_{21} = \boldsymbol{M}_{12}^\top$. Then the Coriolis–centripetal matrix can always be parameterized such that $\boldsymbol{C}(\boldsymbol{\nu}) = -\boldsymbol{C}^\top(\boldsymbol{\nu})$ by choosing (Sagatun and Fossen, 1991)

$$\boldsymbol{C}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -\boldsymbol{S}(\boldsymbol{M}_{11}\boldsymbol{\nu}_1 + \boldsymbol{M}_{12}\boldsymbol{\nu}_2) \\ -\boldsymbol{S}(\boldsymbol{M}_{11}\boldsymbol{\nu}_1 + \boldsymbol{M}_{12}\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{M}_{21}\boldsymbol{\nu}_1 + \boldsymbol{M}_{22}\boldsymbol{\nu}_2) \end{bmatrix} \tag{3.46}$$

where $\boldsymbol{\nu}_1 := \boldsymbol{v}_{b/n}^b = [u, v, w]^\top$, $\boldsymbol{\nu}_2 := \boldsymbol{\omega}_{b/n}^b = [p, q, r]^\top$ and $\boldsymbol{S}$ is the cross-product operator according to Definition 2.2.

Proof. The kinetic energy T is written in the quadratic form:

$$T = \frac{1}{2}\boldsymbol{\nu}^\top \boldsymbol{M}\boldsymbol{\nu}, \quad \boldsymbol{M} = \boldsymbol{M}^\top > 0 \tag{3.47}$$

Expanding this expression yields

$$T = \frac{1}{2}\left(\boldsymbol{\nu}_1^\top \boldsymbol{M}_{11}\boldsymbol{\nu}_1 + \boldsymbol{\nu}_1^\top \boldsymbol{M}_{12}\boldsymbol{\nu}_2 + \boldsymbol{\nu}_2^\top \boldsymbol{M}_{21}\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2^\top \boldsymbol{M}_{22}\boldsymbol{\nu}_2\right) \tag{3.48}$$

where $\boldsymbol{M}_{12} = \boldsymbol{M}_{21}^\top$ and $\boldsymbol{M}_{21} = \boldsymbol{M}_{12}^\top$. This gives

$$\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{M}_{11}\boldsymbol{\nu}_1 + \boldsymbol{M}_{12}\boldsymbol{\nu}_2 \tag{3.49}$$

$$\frac{\partial T}{\partial \boldsymbol{\nu}_2} = \boldsymbol{M}_{21}\boldsymbol{\nu}_1 + \boldsymbol{M}_{22}\boldsymbol{\nu}_2 \tag{3.50}$$

Using Kirchhoff's equations (Kirchhoff, 1869):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_1}\right) + \boldsymbol{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{\tau}_1 \tag{3.51}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \boldsymbol{\nu}_2}\right) + \boldsymbol{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_2} + \boldsymbol{S}(\boldsymbol{\nu}_1)\frac{\partial T}{\partial \boldsymbol{\nu}_1} = \boldsymbol{\tau}_2 \tag{3.52}$$

where $\boldsymbol{S}$ is the skew-symmetric cross-product operator in Definition 2.2, it is seen that there are some terms dependent on acceleration, that is $(\mathrm{d}/\mathrm{d}t)(\partial T/\partial \boldsymbol{\nu}_1)$ and $(\mathrm{d}/\mathrm{d}t)(\partial T/\partial \boldsymbol{\nu}_2)$. The remaining terms are due to Coriolis–centripetal forces. Consequently,

$$\boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} := \begin{bmatrix} \boldsymbol{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_1} \\ \boldsymbol{S}(\boldsymbol{\nu}_2)\frac{\partial T}{\partial \boldsymbol{\nu}_2} + \boldsymbol{S}(\boldsymbol{\nu}_1)\frac{\partial T}{\partial \boldsymbol{\nu}_1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -\boldsymbol{S}(\frac{\partial T}{\partial \boldsymbol{\nu}_1}) \\ -\boldsymbol{S}(\frac{\partial T}{\partial \boldsymbol{\nu}_1}) & -\boldsymbol{S}(\frac{\partial T}{\partial \boldsymbol{\nu}_2}) \end{bmatrix} \begin{bmatrix} \boldsymbol{\nu}_1 \\ \boldsymbol{\nu}_2 \end{bmatrix} \tag{3.53}$$

which after substitution of (3.49) and (3.50) gives (3.46); see Sagatun and Fossen (1991) for the original proof of this theorem.

We next state some useful properties of the Coriolis and centripetal matrix $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$.

Property 3.2 (Rigid-Body Coriolis and Centripetal Matrix $\boldsymbol{C}_{RB}$)
According to Theorem 3.2 the rigid-body Coriolis and centripetal matrix $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ can always be represented such that $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ is skew-symmetric. Moreover,

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = -\boldsymbol{C}_{RB}^\top(\boldsymbol{\nu}), \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6 \tag{3.54}$$

The skew-symmetric property is very useful when designing a nonlinear motion control system since the quadratic form $\boldsymbol{\nu}^\top \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv 0$. This is exploited in energy-based designs where Lyapunov functions play a key role. The same property is also used in nonlinear observer design. There exist several parametrizations that satisfy Property 3.2. Two of them are presented below:

1. **Lagrangian parametrization:** Application of Theorem 3.2 with $\boldsymbol{M} = \boldsymbol{M}_{RB}$ yields the following expression:

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_g^b) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_g^b) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_g^b) - \boldsymbol{S}(\boldsymbol{I}_b\boldsymbol{\nu}_2) \end{bmatrix} \tag{3.55}$$

which can be rewritten according to (Fossen and Fjellstad, 1995)

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_g^b) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) + m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_b\boldsymbol{\nu}_2) \end{bmatrix} \tag{3.56}$$

In order to ensure that $\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = -\boldsymbol{C}_{RB}^{\top}(\boldsymbol{\nu})$, it is necessary to use $\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_1 = \boldsymbol{0}$ and include $\boldsymbol{S}(\boldsymbol{\nu}_1)$ in $C_{RB}^{\{21\}}$.

2. **Velocity-independent parametrizations:** By using the cross-product property $\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2 = -\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{\nu}_1$, it is possible to move $\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2$ from $C_{RB}^{\{12\}}$ to $C_{RB}^{\{11\}}$ in (3.55). This gives an expression for $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ that is independent of linear velocity $\boldsymbol{\nu}_1$ (Fossen and Fjellstad, 1995):

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_g^b) \\ m\boldsymbol{S}(\boldsymbol{r}_g^b)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_b\boldsymbol{\nu}_2) \end{bmatrix} \tag{3.57}$$

Notice that this expression is similar to (3.29) which was derived using Newton–Euler equations.

Remark 3.1.
Expression (3.57) is useful when ocean currents enter the equations of motion. The main reason for this is that $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ does not depend on linear velocity $\boldsymbol{\nu}_1$ (uses only angular velocity $\boldsymbol{\nu}_2$ and lever arm $\boldsymbol{r}_g^b$). This can be further exploited when considering a marine craft exposed to irrotational *ocean currents. According to Property 8.1 in Section 8.3:*

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv \boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r \tag{3.58}$$

if the relative velocity vector $\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$ is defined such that only linear current velocities are used:

$$\boldsymbol{\nu} := [u_c, v_c, w_c, 0, 0, 0]^{\top} \tag{3.59}$$

Since the ocean current (3.59) is assumed to be irrotational, Equation (3.58) can be proven using parametrization (3.57). The details are outlined in Section 8.3.

Component Form

To illustrate the complexity of 6 DOF modeling, the rigid-body Coriolis and centripetal terms in expression (3.55) are expanded according to give (Fossen, 1991)

$$
\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \left[\begin{array}{cccccc}
0 & 0 & 0 & m(y_g q + z_g r) & -m(x_g q - w) & -m(x_g r + v) \\
0 & 0 & 0 & -m(y_g p + w) & m(z_g r + x_g p) & -m(y_g r - u) \\
0 & 0 & 0 & -m(z_g p - v) & -m(z_g q + u) & m(x_g p + y_g q) \\
-m(y_g q + z_g r) & m(y_g p + w) & m(z_g p - v) & 0 & -I_{yz}q - I_{xz}p + I_z r & I_{yz}r + I_{xy}p - I_y q \\
m(x_g q - w) & -m(z_g r + x_g p) & m(z_g q + u) & I_{yz}q + I_{xz}p - I_z r & 0 & -I_{xz}r - I_{xy}q + I_x p \\
m(x_g r + v) & m(y_g r - u) & -m(x_g p + y_g q) & -I_{yz}r - I_{xy}p + I_y q & I_{xz}r + I_{xy}q - I_x p & 0
\end{array}\right] \tag{3.60}
$$

Matlab

Theorem 3.2 is implemented in the Matlab MSS toolbox in the function `m2c.m`. The following example demonstrates how $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ can be computed numerically using the command

```
% rigid-body system inertia matrix
MRB = [1000*eye(3)  zeros(3,3)
        zeros(3,3)  10000*eye(3)];

% rigid-body Coriolis and centripetal matrix
nu  = [10 1 1 1 2 3]'
CRB = m2c(MRB,nu)
```

which produces the numerical result

$$
\boldsymbol{C}_{RB} = \left[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1000 & -1000 \\
0 & 0 & 0 & -1000 & 0 & 10000 \\
0 & 0 & 0 & 1000 & -10000 & 0 \\
0 & 1000 & -1000 & 0 & 30000 & -20000 \\
-1000 & 0 & 10000 & -30000 & 0 & 10000 \\
1000 & -10000 & 0 & 20000 & -10000 & 0
\end{array}\right]
$$

3.3.2 *Linearized 6 DOF Rigid-Body Equations of Motion*

The rigid-body equations of motion (3.42) can be linearized about $\boldsymbol{\nu}_0 = [U, 0, 0, 0, 0, 0]^\top$ for a marine craft moving at forward speed U. This gives

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}^{*}_{RB}\boldsymbol{\nu} = \boldsymbol{\tau}_{RB} \tag{3.61}$$

where

$$\boldsymbol{C}^{*}_{RB} = \boldsymbol{M}_{RB}\boldsymbol{L}U \tag{3.62}$$

and $\boldsymbol{L}$ is a selection matrix

$$\boldsymbol{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.63}$$

The linearized Coriolis and centripetal forces are recognized as

$$\boldsymbol{f}_c = \boldsymbol{C}^{*}_{RB}\boldsymbol{\nu} = \begin{bmatrix} 0 \\ mUr \\ -mUq \\ -my_gUq - mz_gUr \\ mx_gUq \\ mx_gUr \end{bmatrix} \tag{3.64}$$

Simplified 6 DOF Rigid-Body Equations of Motion

The rigid-body equations of motion can be simplified by choosing the origin of the body-fixed coordinate system according to the following criteria:

1. **Origin CO coincides with the CG:** This implies that $\boldsymbol{r}^b_g = [0, 0, 0]^\top$, $\boldsymbol{I}_b = \boldsymbol{I}_g$ (see Theorem 3.1), and

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_g \end{bmatrix} \tag{3.65}$$

   A further simplification is obtained when the body axes (x_b, y_b, z_b) coincide with the principal axes of inertia. This implies that $\boldsymbol{I}_g = \mathrm{diag}\{I^{cg}_x, I^{cg}_y, I^{cg}_z\}$.
2. **Translation of the origin CO such that $\boldsymbol{I}_b$ becomes diagonal:** It is often convenient to let the body axes coincide with the principal axes of inertia or the longitudinal, lateral and normal symmetry axes of the craft. The origin of the body-fixed coordinate system can then be chosen such that the inertia matrix of the body-fixed coordinate system will be diagonal, that is $\boldsymbol{I}_b = \mathrm{diag}\{I_x, I_y, I_z\}$, by applying

the parallel-axes theorem; see Theorem 3.1. Expanding (3.34) with $\boldsymbol{I}_b = \mathrm{diag}\{I_x, I_y, I_z\}$ and $\boldsymbol{I}_g$ as a full matrix yields the following set of equations:

$$\begin{aligned} I_x &= I_x^{cg} + m(y_g^2 + z_g^2) \\ I_y &= I_y^{cg} + m(x_g^2 + z_g^2) \\ I_z &= I_z^{cg} + m(x_g^2 + y_g^2) \end{aligned} \tag{3.66}$$

where x_g, y_g and z_g must be chosen such that

$$\begin{aligned} m I_{yz}^{cg} x_g^2 &= -I_{xy}^{cg} I_{xz}^{cg} \\ m I_{xz}^{cg} y_g^2 &= -I_{xy}^{cg} I_{yz}^{cg} \\ m I_{xy}^{cg} z_g^2 &= -I_{xz}^{cg} I_{yz}^{cg} \end{aligned} \tag{3.67}$$

are satisfied. For this case, (3.41) reduces to

$$\begin{aligned} m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] &= X \\ m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] &= Y \\ m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] &= Z \\ I_x\dot{p} + (I_z - I_y)qr + m\left[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)\right] &= K \\ I_y\dot{q} + (I_x - I_z)rp + m\left[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)\right] &= M \\ I_z\dot{r} + (I_y - I_x)pq + m\left[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)\right] &= N \end{aligned} \tag{3.68}$$