Gradient Descent

Julia Kempe & David S. Rosenberg

CDS & NYU

February 5, 2019

Gradient Descent

Unconstrained Optimization

Setting

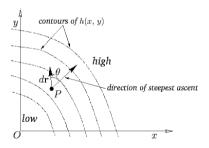
Objective function $f: \mathbb{R}^d \to \mathbb{R}$ is differentiable.

Want to find

$$x^* = \arg\min_{x \in \mathbf{R}^d} f(x)$$

The Gradient

- Let $f: \mathbb{R}^d \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}^d$.
- The gradient of f at the point x_0 , denoted $\nabla_x f(x_0)$, is the direction to move in for the fastest increase in f(x), when starting from x_0 .



Gradient Descent

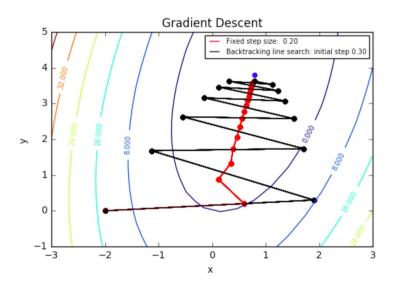
Gradient Descent

- Initialize x = 0
- repeat

•
$$x \leftarrow x - \underbrace{\eta}_{\text{step size}} \nabla f(x)$$

until stopping criterion satisfied

Gradient Descent Path



Gradient Descent: Step Size

- A fixed step size will work, eventually, as long as it's small enough (roughly details to come)
 - Too fast, may diverge
 - In practice, try several fixed step sizes
- Intuition on when to take big steps and when to take small steps?

Convergence Theorem for Fixed Step Size

Theorem

Suppose $f: \mathbb{R}^d \to \mathbb{R}$ is convex and differentiable, and ∇f is **Lipschitz continuous** with constant L > 0, i.e.

$$\|\nabla f(x) - \nabla f(x')\| \le L\|x - x'\|$$

for for any $x, x' \in \mathbf{R}^d$. Then gradient descent with fixed step size $\eta \leqslant 1/L$ converges. In particular,

$$f(x^{(k)}) - f(x^*) \le \frac{\|x^{(0)} - x^*\|^2}{2nk}.$$

Step Size: Practical Note

- Although a 1/L step-size guarantees convergence,
 it may be much slower than necessary.
- timay be much slower than necessary.
- May be worth trying larger step sizes as well.
- But math tells us, no need for anything smaller.

Gradient Descent: When to Stop?

- Wait until $\|\nabla f(x)\|_2 \le \varepsilon$, for some ε of your choosing.
 - (Recall $\nabla f(x) = 0$ at minimum.)
- For learning setting,
 - evalute performance on validation data as you go
 - stop when not improving, or getting worse