Machine Learning – Brett Bernstein

# Recitation 4: Subgradients

## Intro Question

1. When stating a convex optimization problem in standard form we write

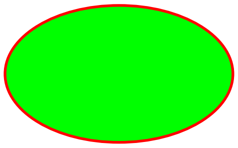
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0 \quad \text{for all } i = 1, \ldots, n. \end{array}$$

   where $f_0, f_1, \ldots, f_n$ are convex. Why don't we use $\geq$ or $=$ instead of $\leq$?

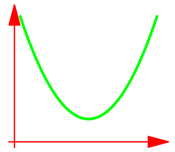
## More on Convexity and Review of Duality

Recall that a set $S \subseteq \mathbb{R}^d$ is convex if for any $x, y \in S$ and $\theta \in (0,1)$ we have $(1-\theta)x+\theta y \in S$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if for any $x, y \in \mathbb{R}^d$ and $\theta \in (0,1)$ we have $f((1-\theta)x+\theta y) \leq (1-\theta)f(x) + \theta f(y)$.

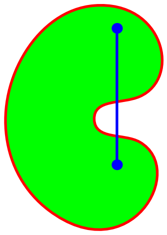
Convex Set



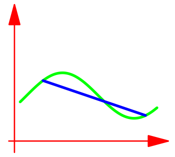
Convex Function



Non-convex Set



Non-convex Function

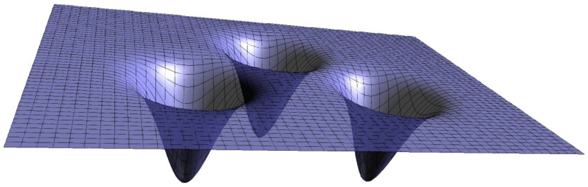


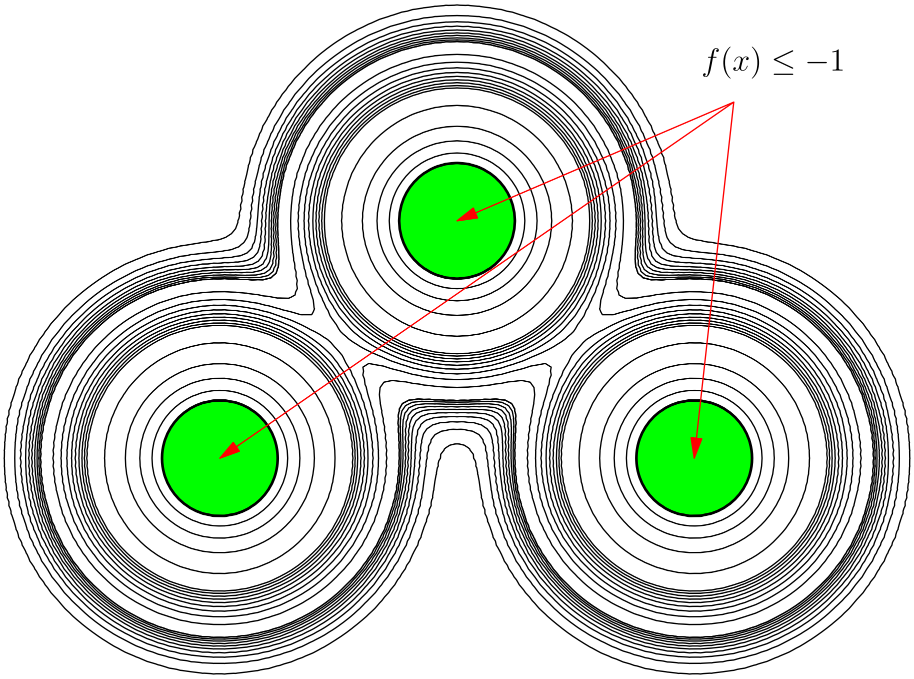
For a function $f : \mathbb{R}^d \to \mathbb{R}$, a level set (or contour line) corresponding to the value $c$ is given by the set of all points $x \in \mathbb{R}^d$ where $f(x) = c$:

$$f^{-1}\{c\} = \{x \in \mathbb{R}^d \mid f(x) = c\}.$$

Analogously, the sublevel set for the value $c$ is the set of all points $x \in \mathbb{R}^d$ where $f(x) \leq c$:

$$f^{-1}(-\infty, c] = \{x \in \mathbb{R}^d \mid f(x) \leq c\}.$$





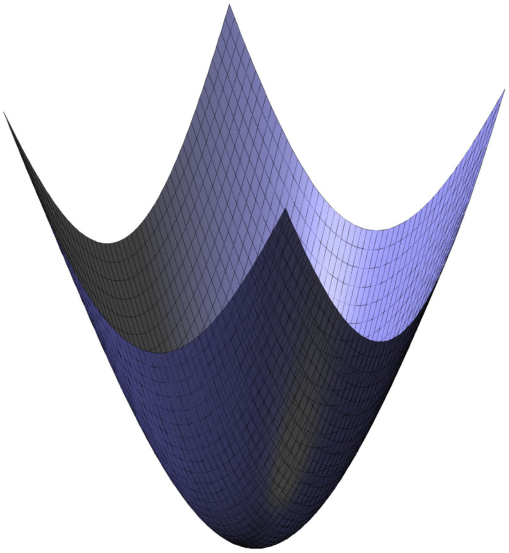
Above is a non-convex function, the contour plot, and the sublevel set where $f(x) \leq -1$. When $f$ is convex, we can say something nice about these sets.

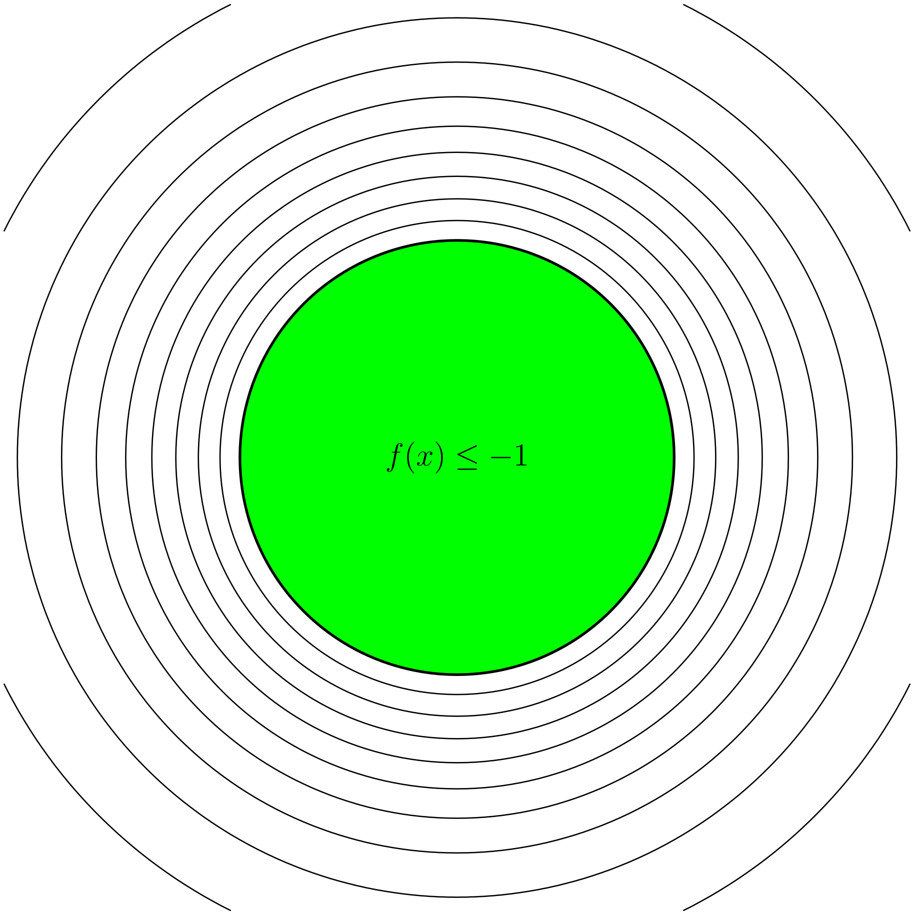
**Theorem 1.** *If $f : \mathbb{R}^d \to \mathbb{R}$ is convex then the sublevel sets are convex.*

*Proof.* Fix a sublevel set $S = \{x \in \mathbb{R}^d \mid f(x) \leq c\}$ for some fixed $c \in \mathbb{R}$. If $x, y \in S$ and $\theta \in (0, 1)$ then we have

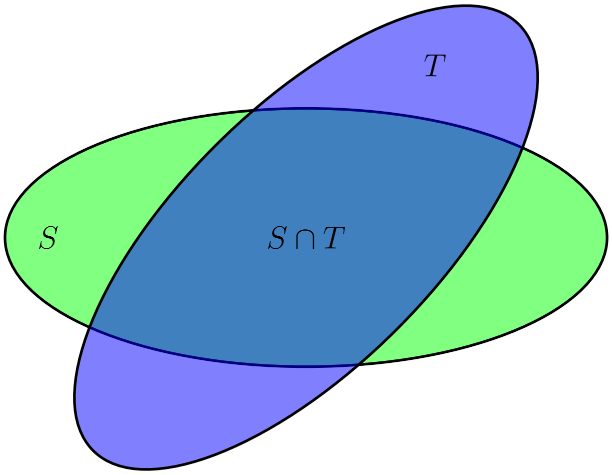
$$f((1-\theta)x + \theta y) \leq (1-\theta)f(x) + \theta f(y) \leq (1-\theta)c + \theta c = c.$$

$\square$

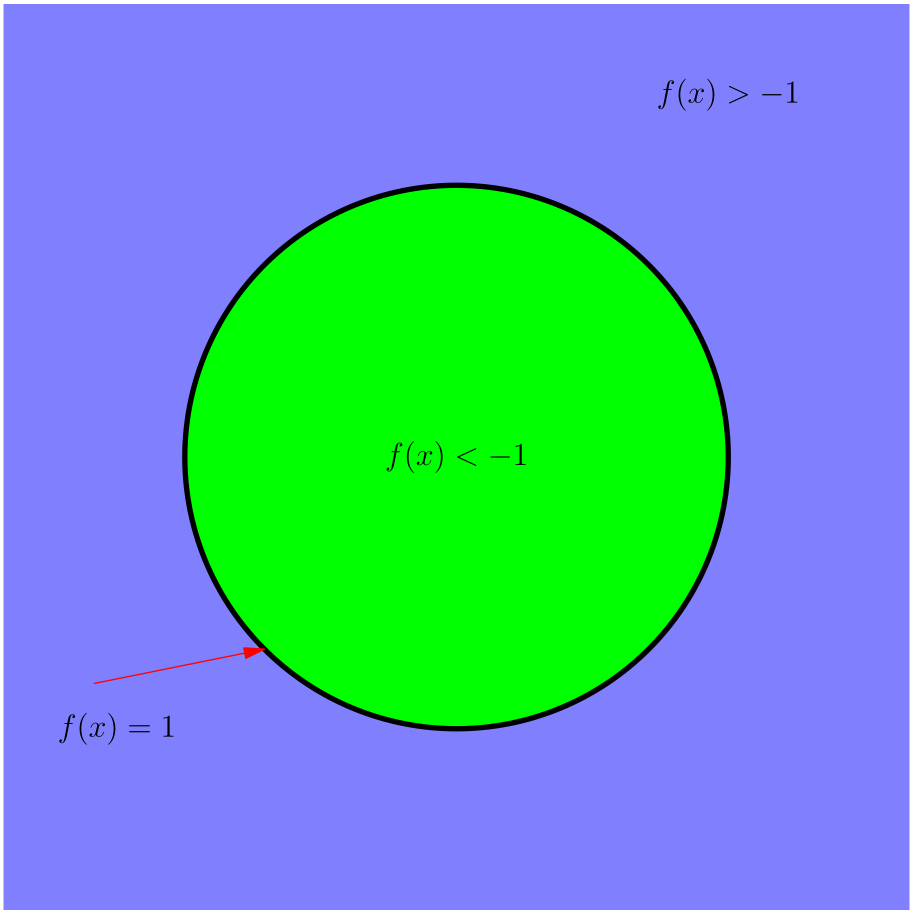




In the concept check questions we will show that the intersection of convex sets is convex.



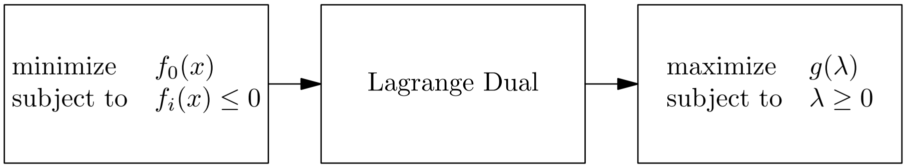
This proves that having a bunch of conditions of the form $f_i(x) \leq 0$ where the $f_i$ are convex gives us a convex feasible set. While the sublevel sets are convex, a convex function need not have convex level sets. Furthermore, sets of the form $\{x \in \mathbb{R}^d \mid f(x) \geq c\}$ also need not be convex (called superlevel sets).

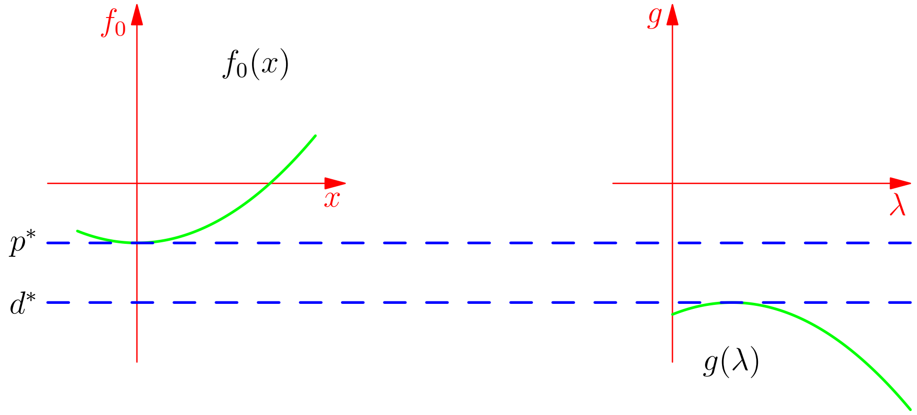


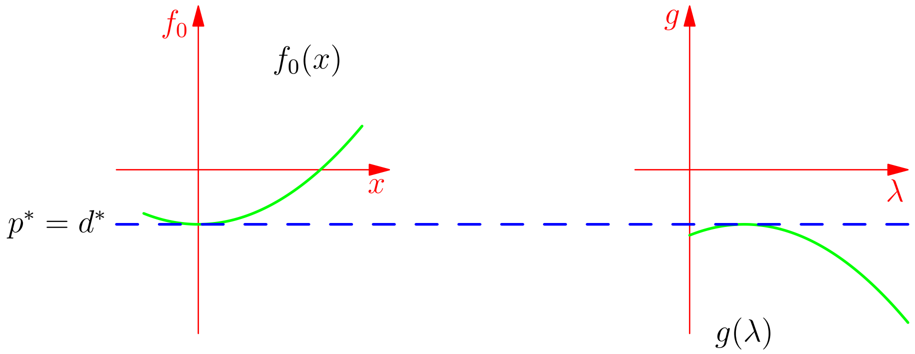
This brings us to the question, why do we care about convexity? Here are some reasons.

1. If $f : \mathbb{R}^d \to \mathbb{R}$ is convex, then local minima are global minima.
2. Given a point $x \in \mathbb{R}^d$ and a closed convex set $S$, there is a unique point of $S$ that is closest to $x$ (called the projection of $x$ onto $S$).
3. A pair of disjoint convex sets can be separated by a hyperplane (used to prove Slater's condition for strong duality).

We also discussed duality as seen below. Lagrange duality let's us change our optimization problem into a new problem with potentially simpler constraints. Moreover, the Lagrange dual optimal value $d^*$ will always be less than the primal optimal value $p^*$ (called weak duality). If we satisfy certain conditions (Slater) we get strong duality ($p^* = d^*$). Using the strong duality relationship we can derive interesting relations between the primal and dual solutions (e.g., complementary slackness).







## Gradients and Subgradients

### Definitions and Basic Properties

Recall that for differentiable $f : \mathbb{R}^d \to \mathbb{R}$ we can write the linear approximation

$$f(x + v) \approx f(x) + \nabla f(x)^T v,$$

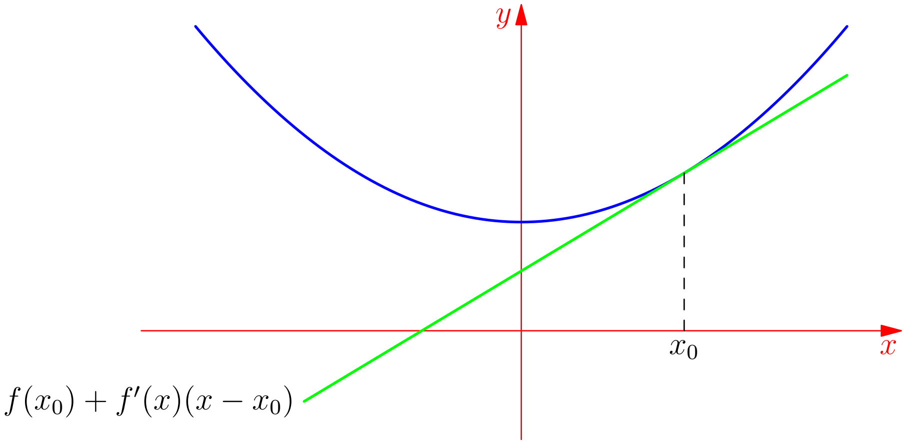
when $v$ is small. We can use gradients to characterize convexity.

**Theorem 2.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be differentiable. Then $f$ is convex iff*

$$f(x + v) \geq f(x) + \nabla f(x)^T v$$

*hold for all $x, v \in \mathbb{R}^d$.*

In words, this says that the approximating tangent line (or hyperplane in higher dimensions) is a global underestimator (lies entirely below the function).



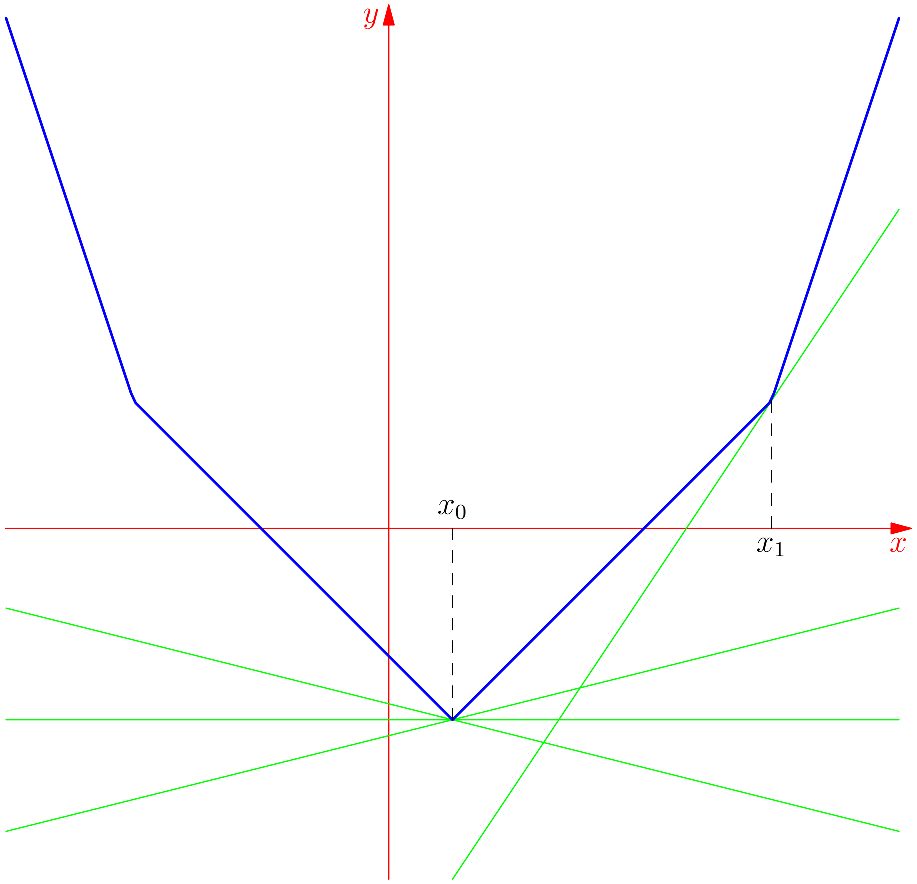
Even if $f$ is not differentiable at $x$, we can still look for vectors satisfying a similar relationship.

**Definition 3** (Subgradient, Subdifferential, Subdifferentiable)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$. We say that $g \in \mathbb{R}^d$ is a *subgradient* of $f$ at $x \in \mathbb{R}^d$ if

$$f(x + v) \geq f(x) + g^T v$$

for all $v \in \mathbb{R}^d$. The *subdifferential* $\partial f(x)$ is the set of all subgradients of $f$ at $x$. We say that $f$ is *subdifferentiable* at $x$ if $\partial f(x) \neq \emptyset$ (i.e., if there is at least one subgradient).

Below are subgradients drawn at $x_0$ and $x_1$.



Facts about subgradients (proven in the concept check exercises).

1. If $f$ is convex and differentiable at $x$ then $\partial f(x) = \{\nabla f(x)\}$.

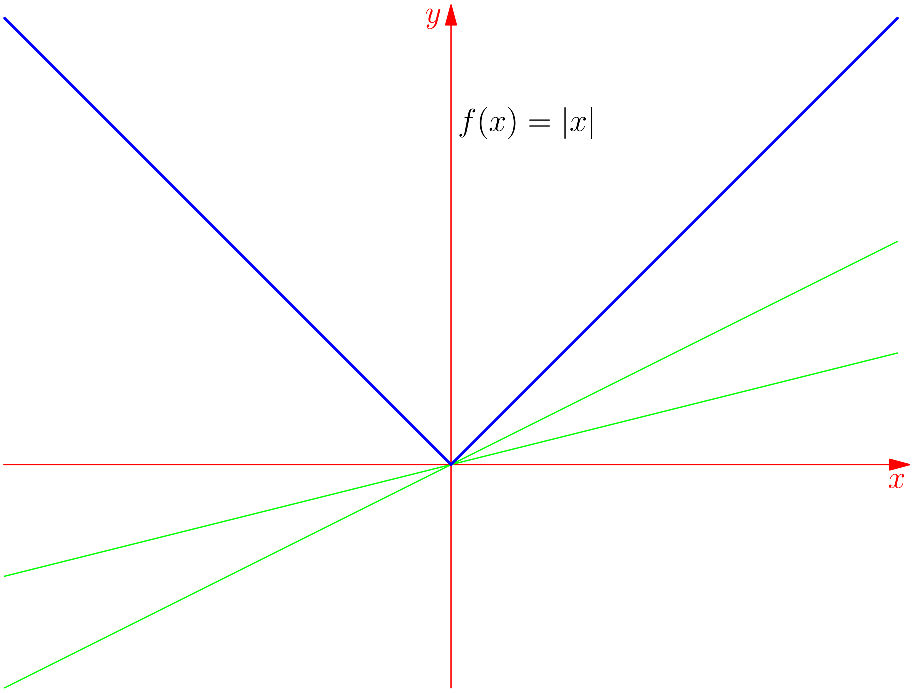
2. If $f$ is convex then $\partial f(x) \neq \emptyset$ for all $x$.

3. The subdifferential $\partial f(x)$ is a convex set. Thus the subdifferential can contain 0, 1, or infinitely many elements.

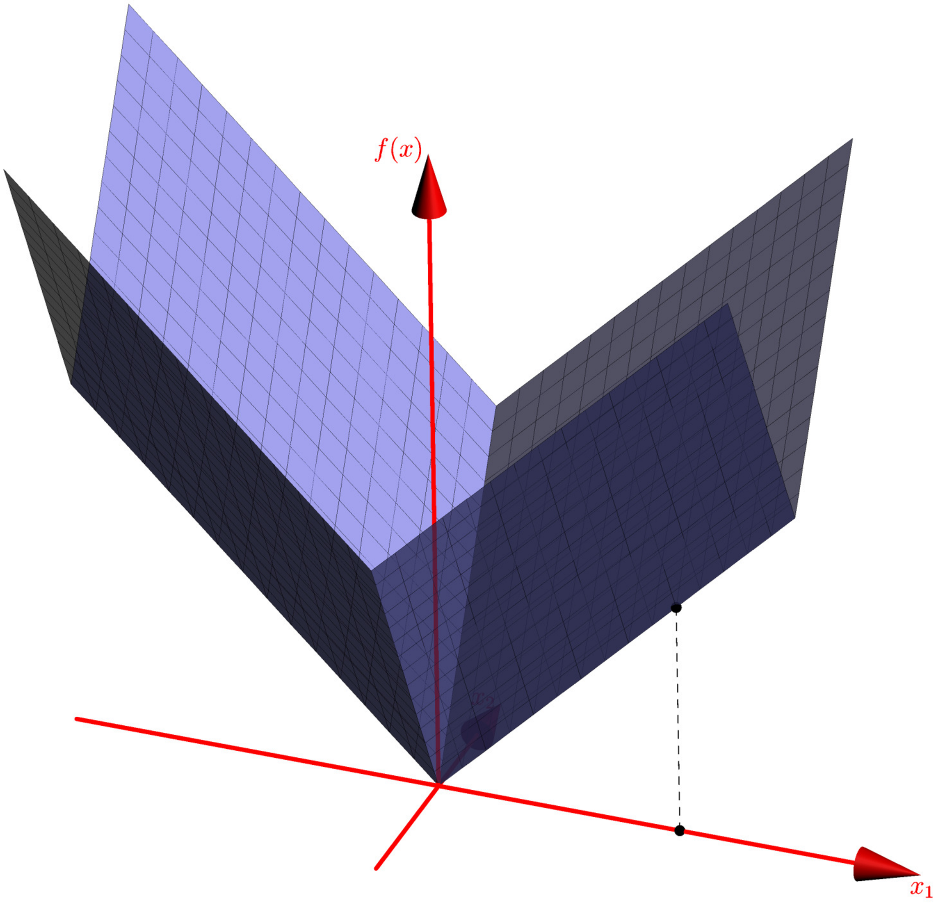
4. If the zero vector is a subgradient of $f$ at $x$, then $x$ is a global minimum.

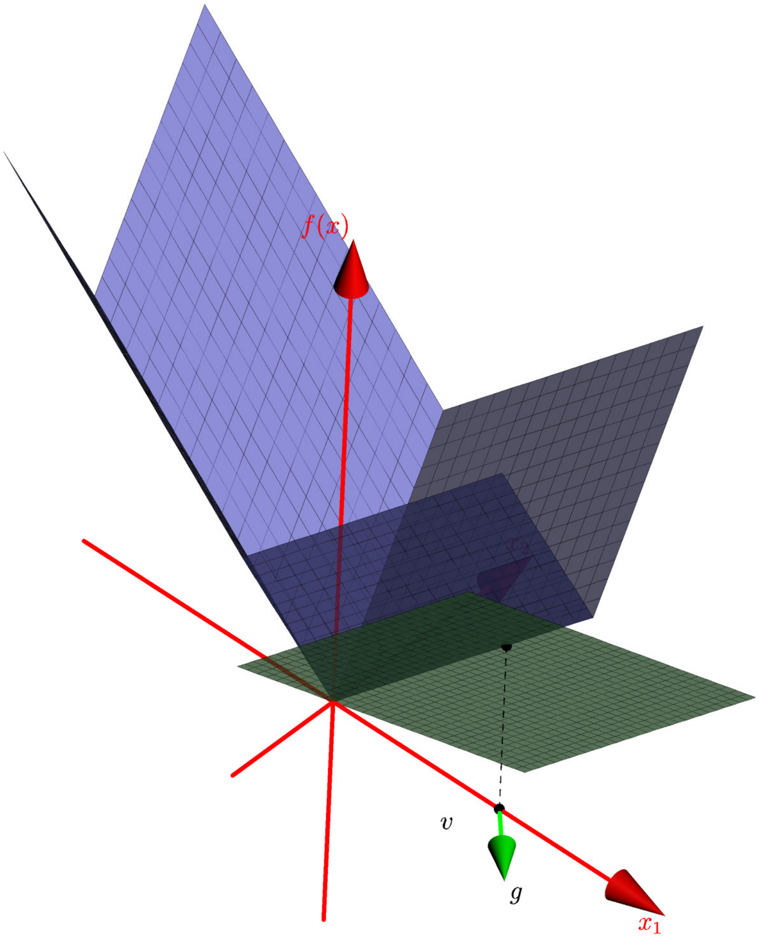
5. If $g$ is a subgradient of $f$ at $x$, then $(g, -1)$ is orthogonal to the underestimating hyperplane $\{(x+v, f(x)+g^T v) \mid v \in \mathbb{R}^d\}$ at $(x, f(x))$.

Consider $f(x) = |x|$ depicted below with some underestimating linear approximations.

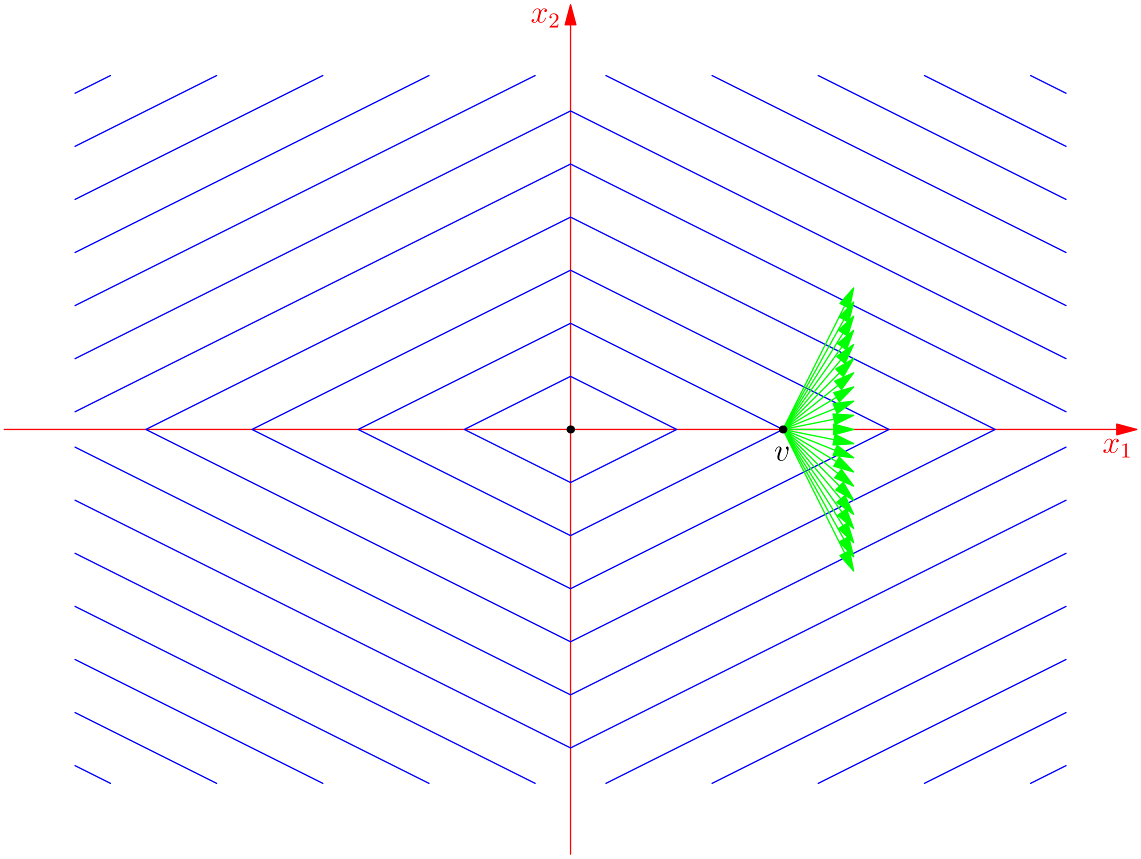


For $x \neq 0$ we have $\partial f(x) = \mathrm{sgn}(x)$ since the function is convex and differentiable. At $x = 0$ we have $\partial f(x) = [-1, 1]$ since any slope between $-1$ and 1 will give an underestimating line. Note that the subgradients are **numbers** here since $f : \mathbb{R} \to \mathbb{R}$. Next we compute $\partial f(3, 0)$ where $f(x_1, x_2) = |x_1| + 2|x_2|$. The first coordinate of any subgradient must be 1 due to the $|x_1|$ part. The second coordinate can have any value between $-2$ and 2 to keep the hyperplane under the function.


$f(x)$
$x_2$
$x_1$


$f(x)$
$x_2$
$v$
$g$
$x_1$

$$\partial f(3,0) = \{(1,b)^T \mid b \in [-2,2]\}$$



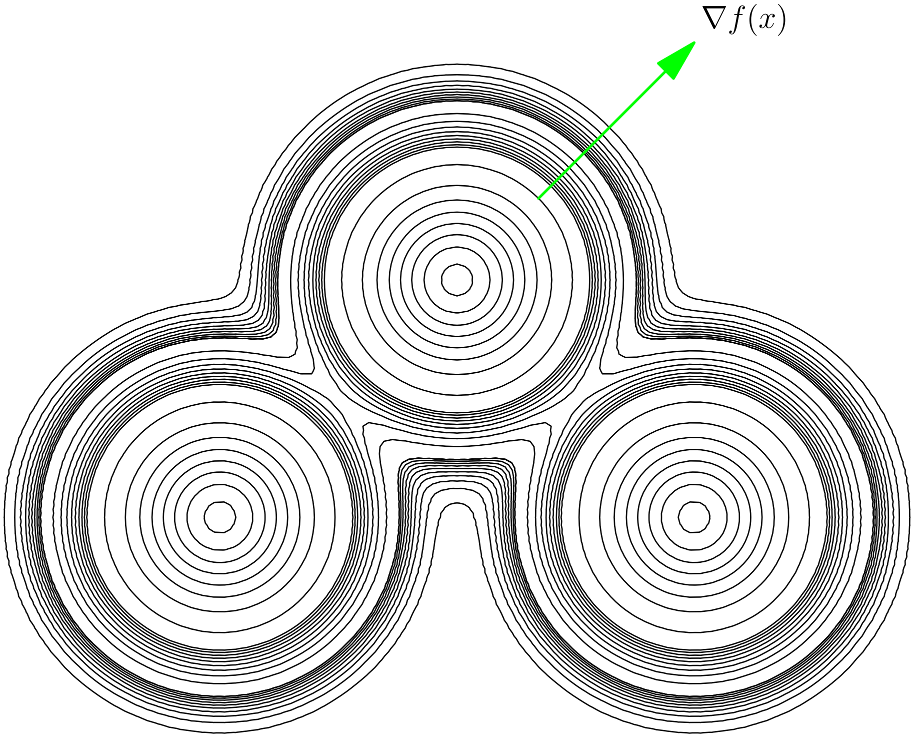
### Contour Lines and Descent Directions

We can also look at the relationship between gradients and contour lines. Remember that for a function $f : \mathbb{R}^d \to \mathbb{R}$, the graph lies in $\mathbb{R}^{d+1}$ but the contour plot, level sets, gradients, and subgradients all live in $\mathbb{R}^d$. This is often a point of confusion. If $f : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable and $x_0 \in \mathbb{R}^d$ with $\nabla f(x_0) \neq 0$ then $\nabla f(x_0)$ is normal to the level set $S = \{x \in \mathbb{R}^d \mid f(x) = f(x_0)\}$.

*Proof sketch.* Let $\gamma : (-1,1) \to S$ be differentiable path lying in $S$ with $\gamma(0) = x_0$ (think of $\gamma$ as describing a particle moving along the contour $S$). Then $f(\gamma(t)) = f(x_0)$ for all $t \in (-1,1)$ so that $\frac{d}{dt} f(\gamma(t)) = 0$. Thus we have

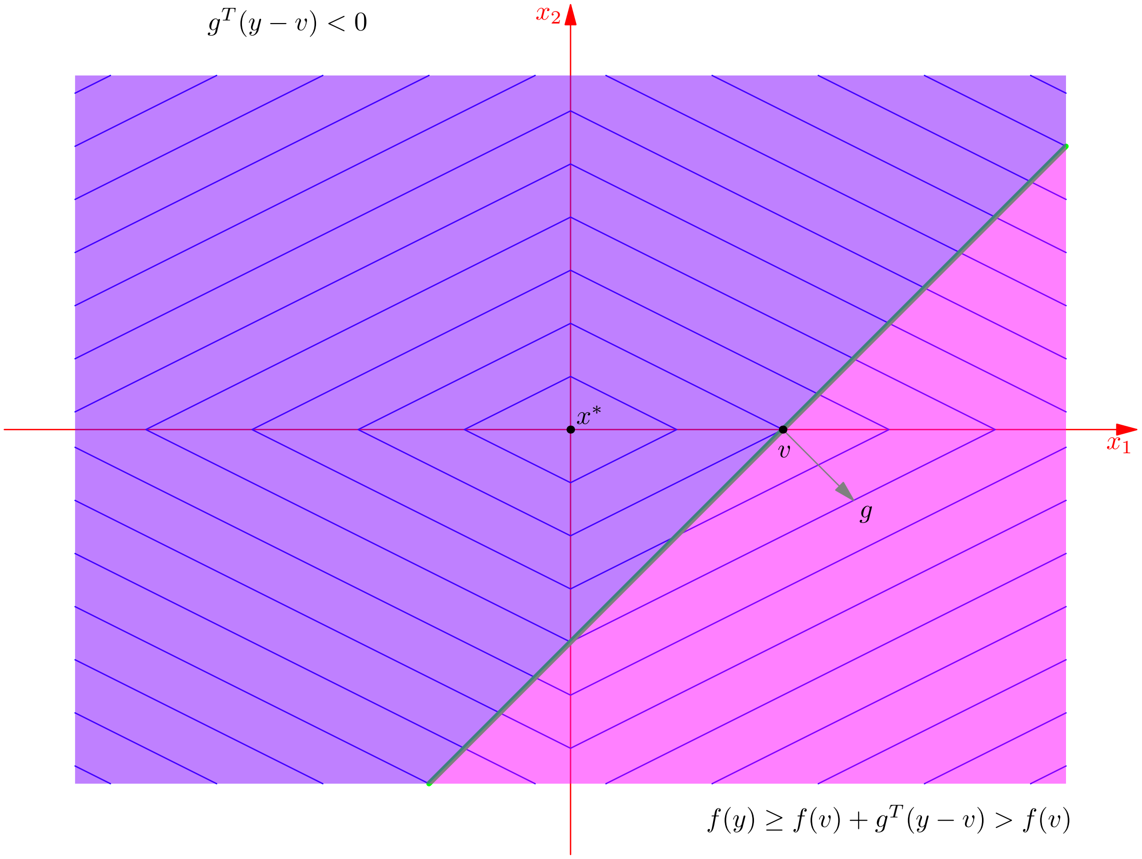
$$0 = \frac{d}{dt} f(\gamma(0)) = \nabla f(x_0)^T \gamma'(0),$$

so $\nabla f(x_0)$ is orthogonal to $\gamma'(0)$ (i.e., the gradient is orthogonal to the velocity vector of the particle $\gamma$ that is tangent to the contour line at $x_0$). As $\gamma$ is arbitrary, the result follows.



Now let's handle the non-differentiable case. Let $f : \mathbb{R}^d \to \mathbb{R}$ have subgradient $g$ at $x_0$. The hyperplane $H$ orthogonal to $g$ at $x_0$ must *support* the level set $S = \{x \in \mathbb{R}^d \mid f(x) = f(x_0)\}$. That is, $H$ passes through $x_0$ and all of $S$ lies on one side of $H$ (the side containing $-g$). This is immediate since any point $y$ lying strictly on the side containing $g$ must have

$$f(y) \geq f(x) + g^T(y - x) > f(x).$$



Even though points on the $g$ side of $H$ have larger $f$-values than $f(x_0)$, it is not true that points on the $-g$ side have smaller $f$-values. In other words, if $g$ is a subgradient it may be true that $-g$ is not a descent direction (this is the case above). Using the same logic we obtain the following theorem.

**Theorem 4.** *Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is convex, let $x_0 \in \mathbb{R}^d$ not be a minimizer, let $g$ be a subgradient of $f$ at $x_0$, and suppose $x_* \in \mathbb{R}^d$ is a minimizer of $f$. Then for sufficiently small $t > 0$*

$$\|x_* - (x_0 - tg)\|_2 < \|x_* - x_0\|_2.$$

*In other words, stepping in the direction of a negative subgradient brings us closer to a minimizer.*

In fact, we can just choose $t$ in the interval

$$t \in \left(0, \frac{2(f(x_0) - f(x^*))}{\|g\|_2^2}\right),$$

but since we usually don't know $f(x^*)$ this is of limited use.

This theorem suggests the following algorithm called Subgradient Descent.

1. Let $x^{(0)}$ denote the initial point.

2. For $k = 1, 2, \ldots$

    (a) Assign $x^{(k)} = x^{(k-1)} - \alpha_k g$, where $g \in \partial f(x^{(k-1)})$ and $\alpha_k$ is the step size.

    (b) Set $f^{(k)}_{\text{best}} = \min_{i=1,\ldots,k} f(x^{(i)})$. (Used since this isn't a descent method.)

Unfortuntely, there aren't any good stopping conditions worth mentioning. Recall that $f$ is called Lipschitz with constant $L$ if

$$|f(x) - f(y)| \leq L\|x - y\|$$

for all $x, y$.

**Theorem 5.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and Lipschitz with constant $G$, and let $x^*$ be a minimizer. For a fixed step size $t$, the subgradient method satisfies:*

$$\lim_{k\to\infty} f(x^{(k)}_{best}) \leq f(x^*) + G^2 t/2.$$

*For step sizes respecting the Robbins-Monro conditions,*

$$\lim_{k\to\infty} f(x^{(k)}_{best}) = f(x^*).$$

Subgradient descent can be fairly slow, with a provable convergence rate of $O(1/\epsilon^2)$ to achieve an error of order $\epsilon$. Recall that the nice case for (unaccelerated) gradient descent was $O(1/\epsilon)$.