RIS Practice Exam solutions

Jacobs University

12 April, 2019

Linear Algebra and Quaternions

Question 1

-a
We know that the area of a parallelogram S = B * H where B is the base and H is the height, and assuming B to be v_1
$S = v_1 \cdot v_2 \cdot sin(\pi/6)$

$40 = 10 \cdot v_2 \cdot 0.5$

$v_2 = 8$

-b

Solution 1:

Using the vector formula: $v_1 \cdot v_2 = a_1(b_1) + a_2(b_2) + a_3(b_3)$

$v_1 = [10 \quad 0 \quad 0]^\intercal$

$v_2 = [8 \cdot cos(\pi/6) \quad 8 \cdot sin(\pi/6) \quad 0]^\intercal$

$v_2 = [4\sqrt{3} \quad 4 \quad 0]^\intercal$

$v_1 \cdot v_2 = 10 \cdot 4\sqrt{3} = 40\sqrt{3}$

Solution 2:

Using the following formula: $v_1 \cdot v_2 = |v_1||v_2|.cos(\theta)$

$v_1 \cdot v_2 = 8 \cdot 10 \cdot cos(\pi/6)$

$v_1 \cdot v_2 = 80 \cdot 0.5\sqrt{3} = 40\sqrt{3}$

-c
Remember that the result of the cross product is a vector that is perpendicular to the plane defined by the two vectors used in the cross product.

Solution 1:

We have:
$v_1 = [10 \quad 0 \quad 0]^\intercal$

$v_2 = [4\sqrt{3} \quad 4 \quad 0]^\intercal$

Using the vector formula:
$v_1 \times v_1 = [a_2(b_3) - a_3(b_2), \;\; a_3(b_1) - a_1(b_3), \;\; a_1(b_2) - a_2(b_1)]$

$v_1 \times v_2 = [0, \;\; 0, \;\; 40]$

Solution 2:

Using the magnitude of both vectors and sin of the angle in between:

$v_1 \times v_2 = |v_1| \cdot |v_2| \cdot sin(\pi/6) \cdot n$

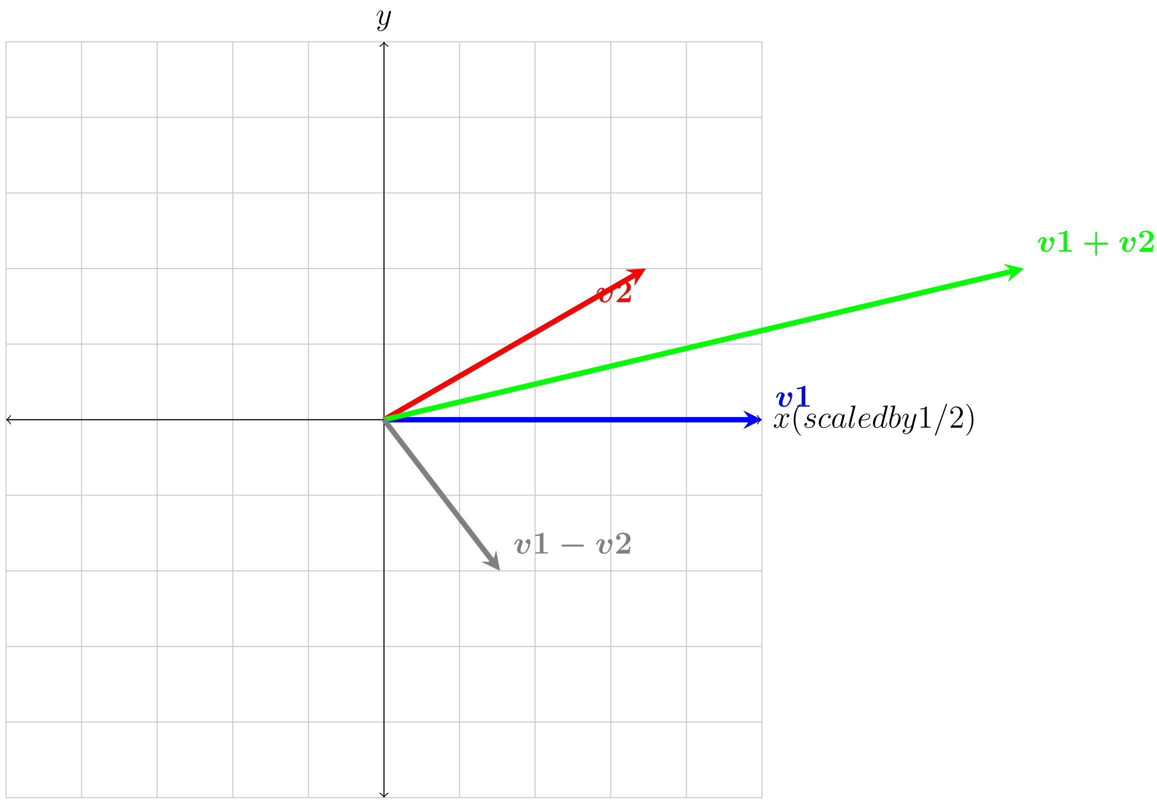
Where: $n = [0 \;\; 0 \;\; 1]^\intercal$

$|v_1| = 10$

$|v_2| = 8$

$v_1 \times v_2 = 10 \cdot 8 \cdot 0.5 \cdot n = 40n = [0 \;\; 0 \;\; 40]^\intercal$

-d



Question 2

Answer:

a- $v_2 = 8/3$

b- $v_1 \cdot v_2 = 40\sqrt{3}$

c- $v_1 \times v_2 = [0, \ 0, \ 40]$

Question 3

For a rotation of 60 degrees around the y-axis:
$q = [cos(60/2), \ 0, \ sin(60/2), \ 0]$

$q = [\sqrt{3}/2, \ 0, \ 1/2, \ 0]$

$q^* = [\sqrt{3}/2, \ 0, \ -1/2, \ 0]$

And point $(1, 1, 1)$:
$p = [1, \ 1, \ 1]^\intercal$

The equation for rotating a point p with quaternion q is $p' = qpq^*$. Please remember that if you want to express the point in a new reference frame, where the quaternion represents the transformation between the frames, the equation is different, and is $p' = q^*pq$. In the first case you are rotating the point, in the second case you are rotating the reference frame. The two results are symmetric: rotating a point with angle α is equal to rotate the reference frame of $-\alpha$.

Solution 1

You can consider the point a vector and express it as a linear combination of the orthonormal vectors i, j, k. The point $p = [1,\ 1,\ 1]^T$ can be expressed as $p = i + j + k$. Remember that $i^2 = j^2 = k^2 = ijk = -1$, and that $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. Therefore, you can express the rotation as:

$p' = qpq^* = (\frac{\sqrt{3}}{2} + \frac{1}{2}j)(i + j + k)(\frac{\sqrt{3}}{2} - \frac{1}{2}j)$

$p' = (\frac{\sqrt{3}}{2} + \frac{1}{2}j)(\frac{\sqrt{3}}{2}i - \frac{1}{2}k + \frac{\sqrt{3}}{2}j + \frac{1}{2} + \frac{\sqrt{3}}{2}k + \frac{1}{2}i)$

$p' = (\frac{\sqrt{3}}{2} + \frac{1}{2}j)(\frac{1}{2} + \frac{\sqrt{3}+1}{2}i + \frac{\sqrt{3}}{2}j + \frac{\sqrt{3}-1}{2}k)$

$p' = \frac{\sqrt{3}}{4} + \frac{3+\sqrt{3}}{4}i + \frac{3}{4}j + \frac{3-\sqrt{3}}{4}k + \frac{1}{4}j - \frac{1+\sqrt{3}}{4}k - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}-1}{4}i$

$p' = \frac{3+\sqrt{3}+\sqrt{3}-1}{4}i + j + \frac{3-\sqrt{3}-1-\sqrt{3}}{4}k$

$p' = \frac{\sqrt{3}+1}{2}i + j + \frac{1-\sqrt{3}}{2}k$

Therefore, $p' = [\frac{\sqrt{3}+1}{2},\ 1,\ \frac{1-\sqrt{3}}{2}]^T$

Solution 2

We can use the general formula for multiplication between quaternions:
$V_1 \times V_2 = S_1S_2 - V_1 \cdot V_2, S_1V_2 + S_2V_1 + V_1 \times V_2$
Knowing that $p' = qpq^*$, we can apply two multiplications, one after the other. Because of the associative property of multiplication, we can choose if performing qp first and multiply for q^*, or start with pq^* and multiply q with the result. In this example we will show the latter one:

$p = i + j + k \Rightarrow S_p = 0; V_p = [1, 1, 1]^T$

$q = \frac{\sqrt{3}}{2} + \frac{1}{2}j \Rightarrow S_q = \frac{\sqrt{3}}{2}; V_q = [0, \frac{1}{2}, 0]^T$

$q^* = \frac{\sqrt{3}}{2} - \frac{1}{2}j \Rightarrow S_{q*} = \frac{\sqrt{3}}{2}; V_{q*} = [0, -\frac{1}{2}, 0]^T$

$$pq^* = \frac{1}{2}, [0, 0, 0]^T + [\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]^T + \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -1/2 & 0 \end{vmatrix} = \frac{1}{2}, [\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]^T + [\frac{1}{2}, 0, -\frac{1}{2}]^T$$

$pq^* = (\frac{1}{2}, \frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}-1}{2})$

$qpq^* = q(pq^*) = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}, [\frac{\sqrt{3}(\sqrt{3}+1)}{4}, \frac{3}{4}, \frac{\sqrt{3}(\sqrt{3}-1)}{4}]^T + [0, \frac{1}{4}, 0]^T + \begin{vmatrix} i & j & k \\ 0 & 1/2 & 0 \\ (\sqrt{3}+1)/2 & \sqrt{3}/2 & (\sqrt{3}-1)/2 \end{vmatrix}$

$qpq^* = 0, [\frac{\sqrt{3}(\sqrt{3}+1)}{4}, 1, \frac{\sqrt{3}(\sqrt{3}-1)}{4}]^T + [\frac{\sqrt{3}-1}{4}, 0, -\frac{\sqrt{3}+1}{4}]^T$

$qpq^* = (0, \frac{3+\sqrt{3}+\sqrt{3}-1)}{4}, 1, \frac{3-\sqrt{3}-\sqrt{3}-1)}{4}) = (0, \frac{\sqrt{3}+1}{2}, 1, \frac{1-\sqrt{3}}{2})$

Therefore, $p' = [\frac{\sqrt{3}+1}{2},\ 1,\ \frac{1-\sqrt{3}}{2}]^\intercal$

Question 4

Answer:
$q = [0.966,\ 0,\ 0.259,\ 0]$

$q^* = [-0.966,\ 0,\ 0.259,\ 0]$

$qp^{new} = [\frac{\sqrt{3}+1}{2},\ 1,\ \frac{\sqrt{3}-1}{2}]^\intercal$

Question 5

Answer:
$q = [\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}},\ 0]$

$q^* = [-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}},\ 0]$

$qp^{new} = [0,\ 1,\ 1]^\intercal$

Question 6

Answer:
$q = [\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}},\ 0]$

$q^* = [-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}},\ 0]$

$qp^{new} = [0,\ -1,\ 1]^\intercal$

Question 7

Answer:
$q_{result} = -4 + 2i + 2j - 6k$

Question 8

Answer:
$q_{result} = -1 - 3i + 8j$

URDF

Question 1

The fixed parts have (⇒)in front of them:

```
<robot name=”Robot1”/ >
    <link name=”link1”/ >
    <link name=”link2”/ >
    <link name=”link3”/ >
    <link name=”link4”/ >
    <joint name=”joint1” >
        <parent link=”link1”/ >
        <child link=”link3”/ >
    </joint >
    <joint name=”joint2” >
        <parent link=”link1”/ >
        <child link=”link2”/ >
    </joint >
    <joint name=”joint3” >
        <parent link=”link3”/ >
        ⇒< child  link=”link4”/ >
    </joint >
</robot >
```

Question 2

```
<robot name=”Robot1” >
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    <link name=”link2”/ >
    <link name=”link3”/ >
    <link name=”link4”/ >
    <joint name=”joint1” >
        <parent link=”link1”/ >
        <child link=”link3”/ >
    ⇒</joint >
    <joint name=”joint2” >
        <parent link=”link1”/ >
        <child link=”link2”/ >
    ⇒</joint >
    <joint name=”joint3” >
        <parent link=”link2”/ >
        ⇒<child link=”link4”/ >
    ⇒</joint >
```

```
</robot >
```

Question 3

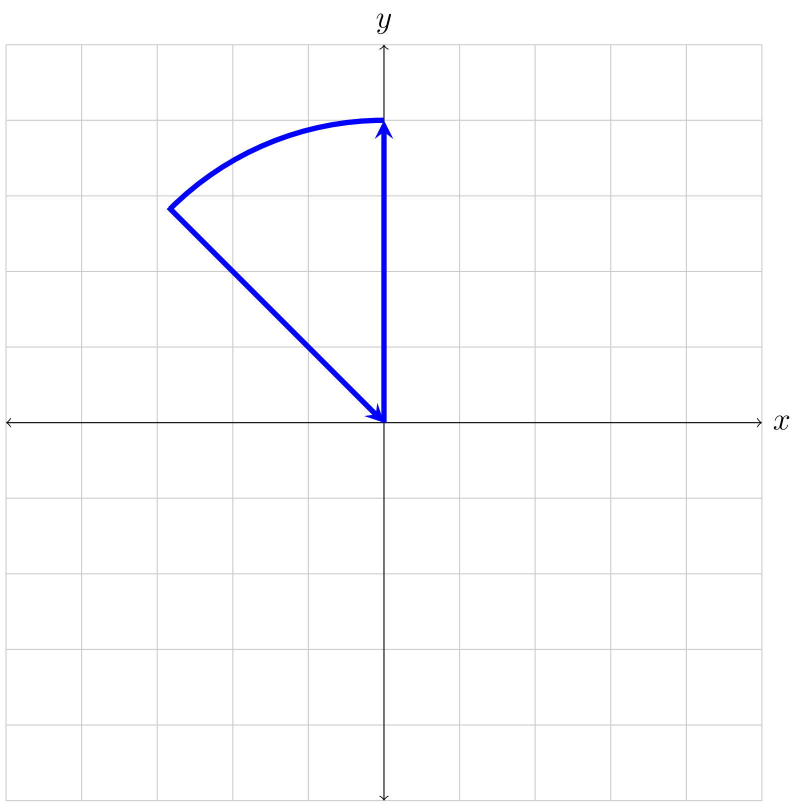
```
<robot name=”Robot1” >
    <link name=”link1”/ >
    <link name=”link2”/ >
    <link name=”link3”/ >
    <link name=”link4”/ >
    <joint name=”joint1” >
        <parent link=”link1”/ >
        <child link=”link3”/ >
    ⇒</joint >
    <joint name=”joint2” >
        <parent link=”link1”/ >
        <child link=”link2”/ >
    ⇒</joint >
    <joint name=”joint3” >
        <parent link=”link3”/ >
        <child link=”link4”/ >
    ⇒</joint >
</robot >
```

Motion

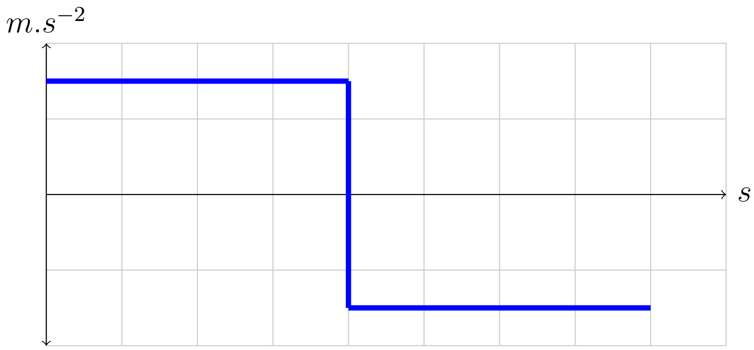
Question 1

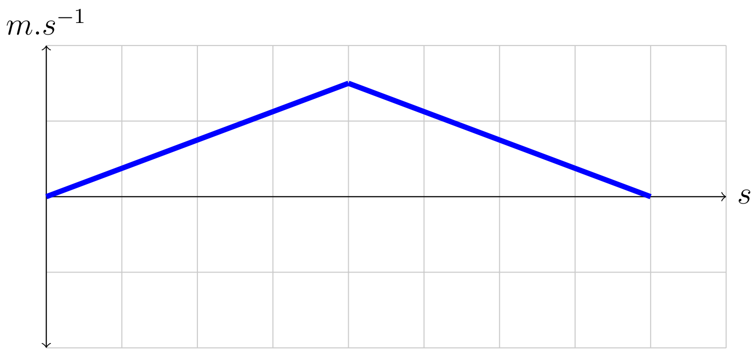
a. Remember that the order of transformations starts from right to left:

$$\begin{bmatrix}1 & 0 & 3\sqrt{2}\\0 & 1 & -3\sqrt{2}\\0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}cos(45) & -sin(45) & 0\\sin(45) & cos(45) & 0\\0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & 0 & 0\\0 & 1 & 6\\0 & 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0 & 3\sqrt{2}\\0 & 1 & -3\sqrt{2}\\0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}cos(45) & -sin(45) & -6sin(45)\\sin(45) & cos(45) & 6cos(45)\\0 & 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}cos(45) & -sin(45) & -6sin(45)+3\sqrt{2}\\sin(45) & cos(45) & 6cos(45)-3\sqrt{2}\\0 & 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -6\frac{1}{\sqrt{2}}+3\sqrt{2}\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 6\frac{1}{\sqrt{2}}-3\sqrt{2}\\0 & 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\0 & 0 & 1\end{bmatrix}$$



b.
Remember, a bang bang motion is defined by it's two states; acceleration and deceleration. The two states splits the time and distance in half. Also they share the same maximum speed
$s = 1/2at^2$
$3 = 1/2a16$
$3 = 8a$
$a = 3/8\ m.s^{-2}$
$v = at$
$= 3/8 * 4$
$= 3/2\ m.s^{-1}$

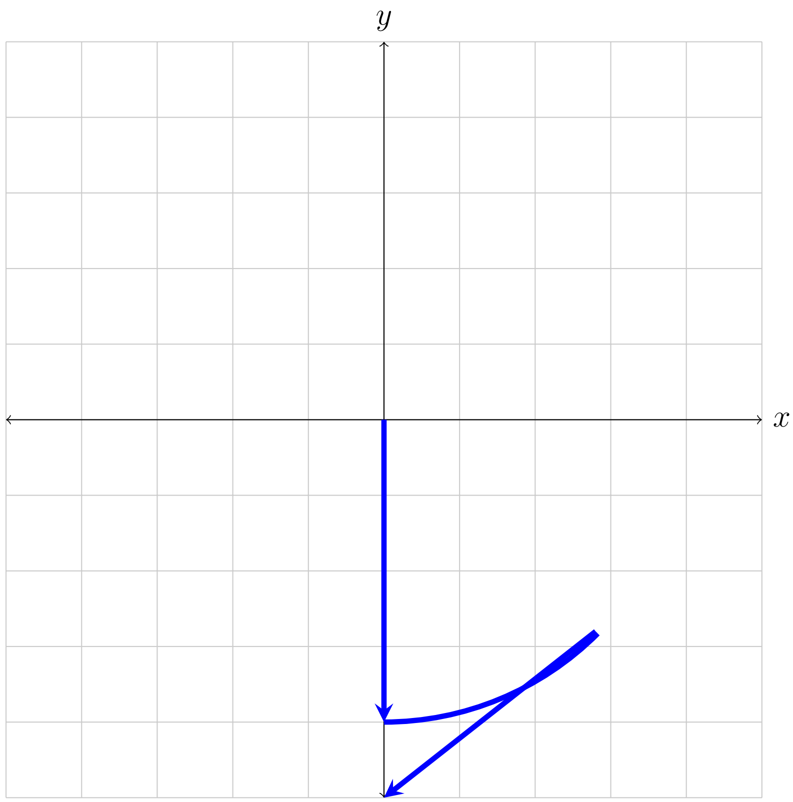




Question 2

a.
Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -6\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

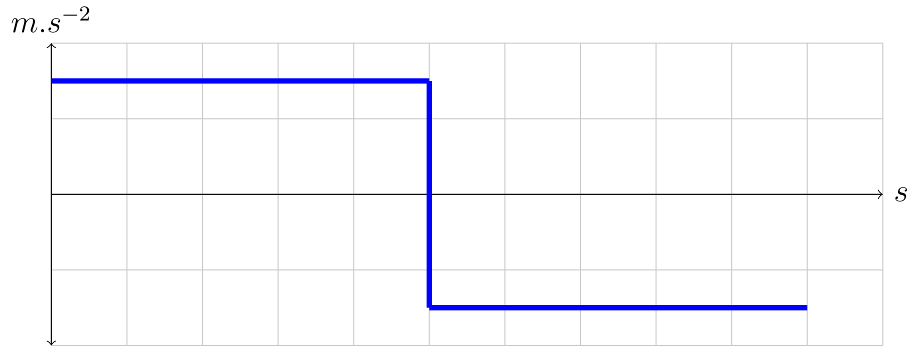


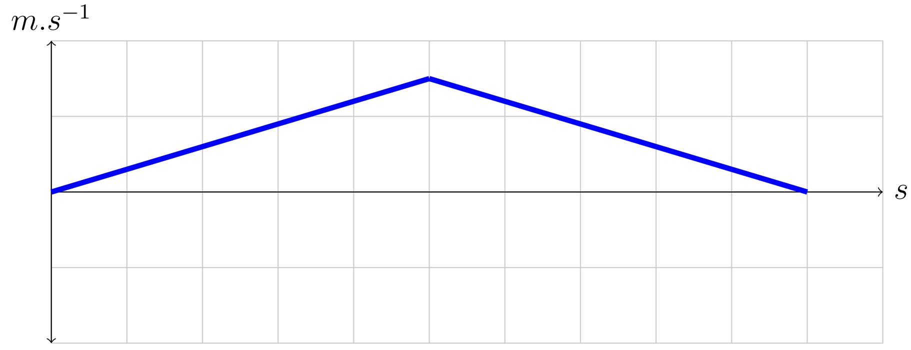
b.

Answer:

$a = 6/25\ m.s^{-2}$

$v = 6/5\ m.s^{-1}$

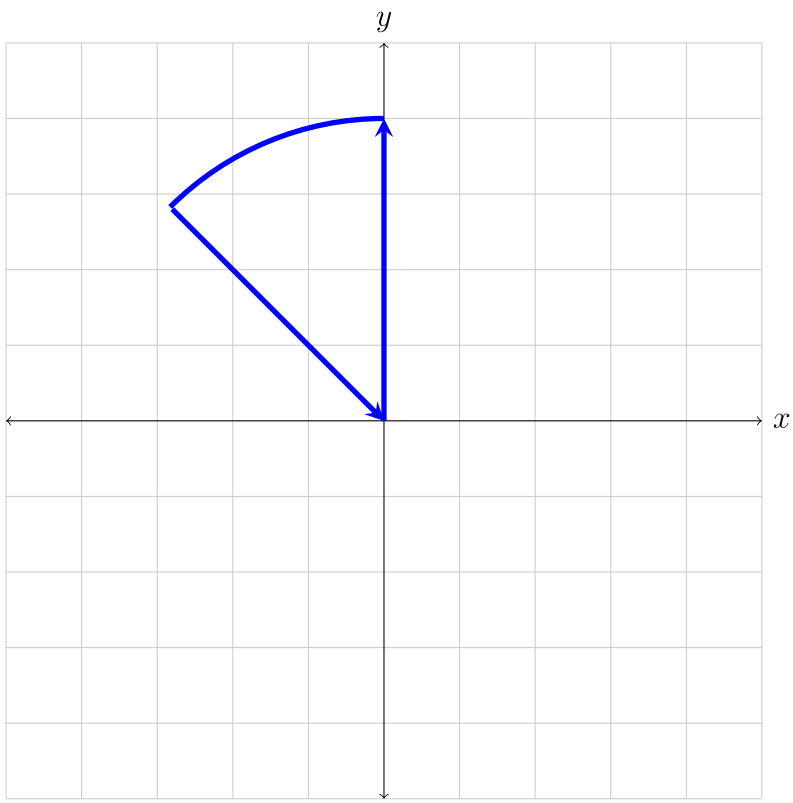




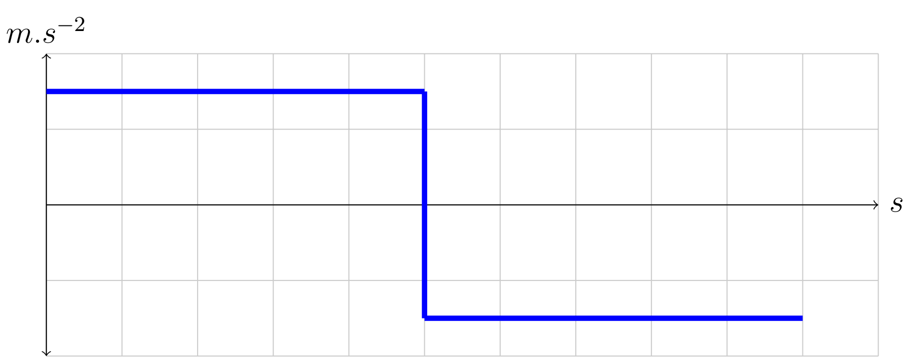
Question 3

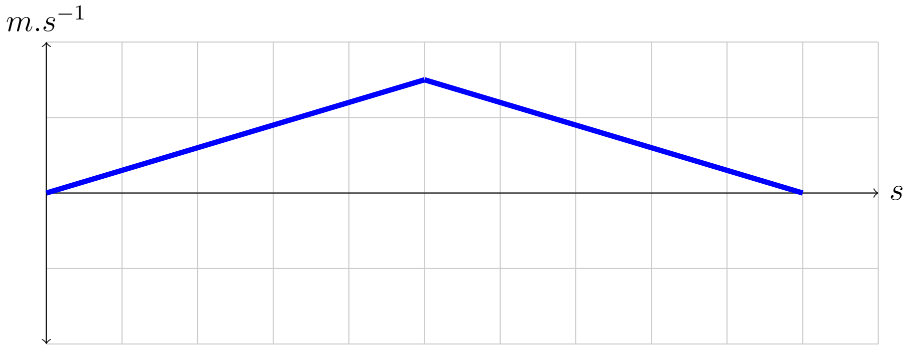
a.
Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



b.
Answer:
$a = 1/2\ m.s^{-2}$
$v = 2\ m.s^{-1}$



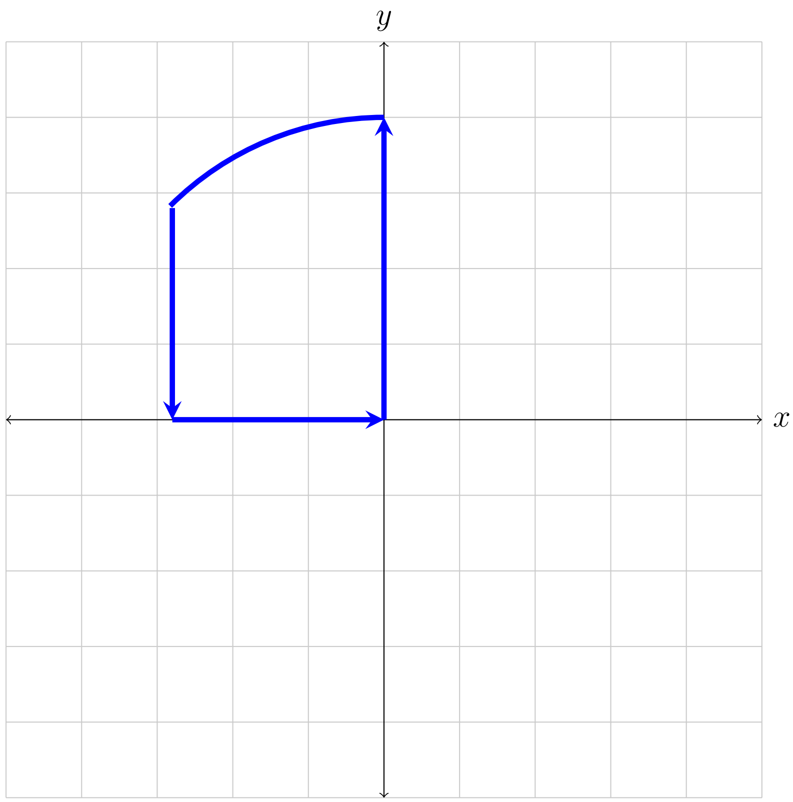


Question 4

a.
Answer:
$\vec{BC} \cdot \vec{CD} = 0$
$\vec{BC} \times \vec{CD} = [0,\ 0,\ 72]^\intercal$
b.
Answer:
$\vec{OA} + \vec{CD} = [6\sqrt{2},\ 12,\ 0]^\intercal$
$\vec{OA} - \vec{CD} = [-6\sqrt{2},\ 12,\ 0]^\intercal$
c.
-The cross product are not commutative. Because the the operator uses the sine of the angle in between the two vectors and this angle is usually measured counterclockwise by convention.
-The dot product on the other hand is commutative. Because it uses the cosine of the angle in between the two vectors $\Rightarrow cos(-\theta) = cos(\theta)$
-While the sum of vectors is commutative as it uses simple linear operators.
-The subtraction of two vector is not commutative, as the order effects the direction of the resulting vector.

d.
Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}) & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



e.

$a = 1m.s^{-2}$

$d = 12m$

$d = 0.5 \cdot a \cdot t^2$

$t = \sqrt{12 \cdot 2/1}$

$t = \sqrt{24} = 2\sqrt{6}s$

$v = at$

$v = 1 \cdot 2\sqrt{6} = 2\sqrt{6}\ m.s^{-1}$

f.

$a_c = 0\ m.s^{-2}$

$a_t = v^2/r$

where $r = 12$

$\Rightarrow a_t = 24/12 = 2m/s^2$

g.

$d_1 = \Delta T \cdot v_1/2$

$d_1 = 3 \cdot 1/2 = 1.5m$

$d_2 = d_1 \cdot v_2/v_1$

$d_2 = 1.5m$

The robot leaves the BC path 1.5 meters before reaching the C and arrives to CD 1.5 meters away from C.

h.

$O = (0, 0, 0) \quad A = (0, 12, 0)$

$$B = \begin{bmatrix} cos(45) & -sin(45) & 0 \\ sin(45) & cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [0,\ 12,\ 0]^\intercal =$$

$= \ [-12sin(45),\ 12cos(45),\ 0]^\intercal$
$= \ [-6\sqrt{2},\ 6\sqrt{2}, 0]^\intercal$
To polar coordinates:
$O = [0,\ 0]$
$A = [\frac{\pi}{2},\ 12]$
$B = [tan^{-1}(By/Bx) + \pi,\ \sqrt{Bx^2 + By^2}]$
$B = [\frac{3\pi}{2},\ 12]$
To spherical coordinates:
$O = [0,\ 0,\ 0]$
$A = [\sqrt{12^2},\ tan^{-1}(Ay/Ax),\ cos^{-1}(0/\sqrt{12^2})]$
$A = [12,\ 90,\ 90]$
$B = [\sqrt{(-6\sqrt{2})^2 + (6\sqrt{2})^2},\ tan^{-1}(-6\sqrt{2}/6\sqrt{2}) + 180,\ cos^{-1}(0/\sqrt{(-6\sqrt{2})^2 + (6\sqrt{2})^2})]$
$B = [12,\ 135,\ 90]$
$B = [12,\ \frac{3\pi}{2},\ \frac{\pi}{2}]$

Question 5

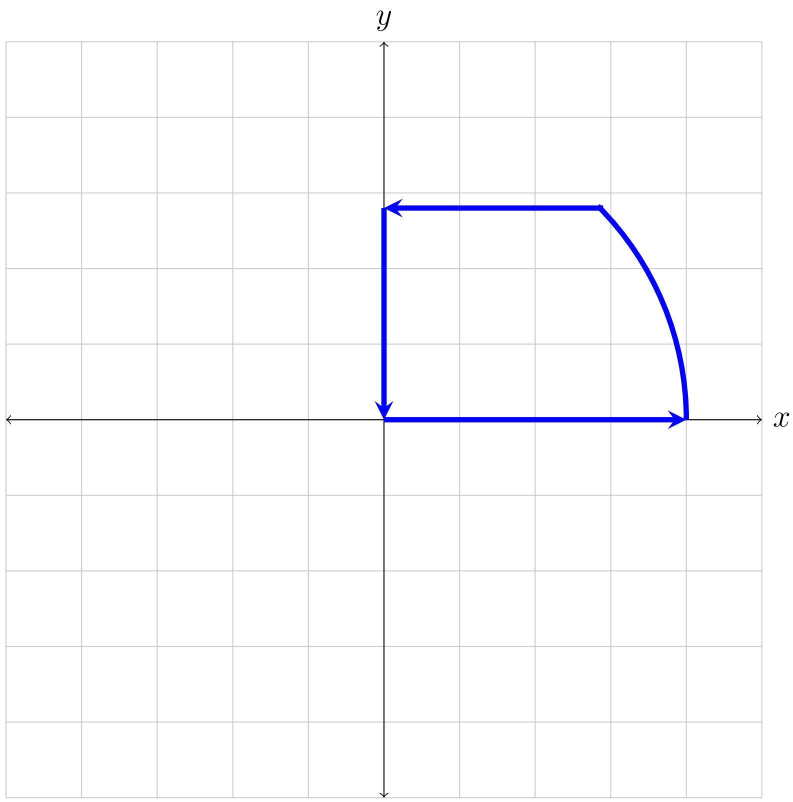
a.
Answer:
$\vec{BC} \cdot \vec{CD} = 0$
$\vec{BC} \times \vec{CD} = [0,\ 0,\ -72]^\intercal$

b.
Answer:
$\vec{OA} + \vec{CD} = [12,\ -6\sqrt{2},\ 0]^\intercal$
$\vec{OA} - \vec{CD} = [12,\ 6\sqrt{2},\ 0]^\intercal$

c.
Answer: 4.c

d.
Answer:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



e.
Answer:
$t = \sqrt{24} = 2\sqrt{6}s$
$v = 2\sqrt{6}m.s^{-1}$

f.
Answer:
$a_c = 0m.s^{-2}$
$a_t = 24/12 = 2m/s^2$

g.
Answer:
$d_1 = 1.5m$
$d_2 = 1.5m$
The robot leaves the BC path 1.5 meters before reaching the C and arrives to CD 1.5 meters away from C.

h.
$O = (0, 0, 0)$
$A = (12, 0, 0)$
$B = [6\sqrt{2},\ 6\sqrt{2}, 0]^\intercal$

To polar coordinates:
$O = [0,\ 0]$
$A = [0,\ 12]$
$B = [\frac{\pi}{4},\ 12]$

To spherical coordinates:
$O = [0,\ 0,\ 0]$
$A = [12,\ 0,\ \frac{\pi}{2}]$
$B = [12,\ \frac{\pi}{4},\ \frac{\pi}{2}]$

Rigid Bodies

Question 1

$M_0 = F \cdot d \cdot sin(\theta) = 12 \cdot 3 = 36N \cdot m$

Question 2

The minimum torque will be generated from S because the angle $\theta = 0$
$\Rightarrow M0 = F \cdot r \cdot sin(0) = 0$ While the maximum torque will be generated from Q because the angle $\theta = 90$
$\Rightarrow M_0 = F \cdot r \cdot sin(90) = F \cdot r$

Question 3

$M = r \times F = r \cdot F \cdot sin(\theta) \cdot n$
$Where: \ n = [0,\ 0,\ 1]$
$\Rightarrow M \cdot r = M \cdot r \cdot cos(90) = 0$

Question 4

$M = \sum r \times F$
$M = 2 \cdot 5 \cdot sin(270) + 3 \cdot 10 \cdot sin(90) = -10 + 30 = 20N.m$

Question 5

$M = r \times F = r \cdot F \cdot sin(90) = 5 \cdot 10 \cdot 1 \cdot n = 50\vec{i}N.m$

Question 6

$$M = r \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 10 & 20 & 30 \end{vmatrix} = [2 \cdot 30 - 0 \cdot 20,\ 0 \cdot 10 - 1 \cdot 30,\ 1 \cdot 20 - 2 \cdot 10]$$
$= [60,\ -30,\ 0]$
$M_y = -30N.m$

Sensor

Question 1

Accuracy: Agreement of measured values with a given reference standard.
Repeatability: Capability of reproducing as output similar measured values over consecutive measurements of the same constant input quantity.

Robot Unit

Question 1

Mechanical unit:Rigid links connected through a rotational or a prismatic link (1 DOF). Example: car's door
Sensor unit:
a.Proprioceptive: sense the internal state of the machine. Example: thermal sensor.
b.Exteroceptive: sense the external world. Example: ultrasonic sensor.
Actuation unit: a mechanism to introduce motion. Example: car's engine.
Supervision unit: Task planning an control or artificial intelligence. Example: micro-controller.

Question 2

Mechanical unit:Rigid links connected through a rotational or a prismatic link (1 DOF). Example: Gripper
Sensor unit:
a.Proprioceptive: sense the internal state of the machine. Example: IMU.
b.Exteroceptive: sense the external world. Example: pressure sensor.
Actuation unit: a mechanism to introduce motion. Example: motor.
Supervision unit: Task planning an control or artificial intelligence. Example: micro-controller.

True or False

1 False
2 True
3 False
4 True
5 False
6 False
7 True
8 False
9 False
10 False
11 True
12 False
13 False
14 False
15 False
16 False
17 True
18 False