Sheet 4

$\int_{\mathbb{R}} f(x)\,\delta(x-x_0)\,\delta x = f(x_0)$

Ex 1

$Y := g(X)$ with $g(x) = x^4$

Claim that, $f(x) = x^3$.

(a) Expected (squared) prediction error of f

$$EPE(f) = E\left(L_2(Y, f(X))\right)$$

$$= \iint (y - f(x))^2 \, p(x,y)\, dx\, dy.$$

$$p(x,y) = P_X(x)\,\delta(y - x^4)$$

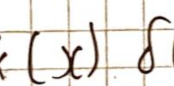
$$EPE(f) = \iint (y - f(x))^2 \, P_X(x)\, \delta(y - x^4)\, dx\, dy$$

$$= \iint (y - x^3)^2 \, P_X(x)\, \delta(y - x^4)\, \delta x\, \delta y$$

$$= \iint (y - x^3)^2 \, \delta(y - x^4)\, P_X(x)\, dy\, dx$$

$$= \int (x^4 - x^3)^2 \, P_X(x)\, dx \,.$$

$X \sim \mathcal{U}[-1, 1)$



$P_X(x) \cdot 2 = 1$

$P_X(x) = 1/2$

$\therefore P_X(x) = \frac{1}{2}$

$$= \frac{1}{2} \cdot \int_{-1}^{1} (x^4 - x^3)^2\, dx$$

$$= \frac{1}{2} \cdot \int_{-1}^{1} (x^4 - x^3)^2 \, dx$$

$$= \frac{1}{2} \int_{-1}^{1} x^8 - 2x^4 x^3 + x^6 \, dx$$

$$= \frac{1}{2} \left[\frac{x^9}{9} - \frac{2 \times x^8}{8} + \frac{x^7}{7} \right]_{-1}^{1}$$

$$= \frac{1}{2} \left[\left(\frac{1}{9} - \frac{2}{8} + \frac{1}{7} \right) - \left(\frac{-1}{9} - \frac{2}{8} - \frac{1}{7} \right) \right]$$

$$= 0.2639$$

(b) find the regrenor for $Y = g(x)$

~~Regrenor = $E(Y|X=x)$~~

$$p(y|x) = \frac{p(y,x)}{p_X(x)} = \frac{\delta(y - x^4) \, p_X(x)}{p_X(x)} = \delta(y - x^4)$$

$$E(Y|X=x) = \int y \, p(y|x) \, dy = \int y \, \delta(y - x^4) \, dy = x^4$$

$$E_X\left[E_{Y|X}\left[(f(X) - E[Y|X])(E[Y|X]-Y) \mid X\right]\right]$$

$$= \cancel{E_X\left[\{E_{Y|X}(f(x)) - E_{Y|X}[E[Y|X]]\}\{E[Y|X]-Y)\mid X\right]}$$

$$= \cancel{E_X\left[(E_{Y|X}(f(x) - E[Y|X]))(E[Y|X]-Y)\mid X\right]}$$

$$= E_X\left[E_{Y|X}\left(f(X)E[Y|X] - f(X)Y - (E[Y|X])^2 + E[Y|X]Y \mid X\right)\right]$$

$$= E_X\left[E_{Y|X}[f(X)E[Y|X]] - E_{Y|X}[f(X)Y] - E_{Y|X}[(E[Y|X])^2] + E_{Y|X}[E[Y|X]Y] \mid X\right]$$

Tower property:-

$$= E_X\left[\cancel{f(X)E_{Y|X}[Y]} - \cancel{f(X)E_{Y|X}[Y]} - \cancel{E[Y]E[Y]} + \cancel{E[Y]E[Y]} \mid X\right]$$

$$= E_X[0 \mid X]$$

$$= 0$$

Exercise 3

$T = \{((1,3)^T, 77), ((2,5)^T, 47), ((3,2)^T, 55), ((5,5)^T, 59),$
$((5,8)^T, 72), ((10,6)^T, 60)\}.$

Carry out KNN regression prediction for $x_1 = (8,5)^T$, $x_2 = (1,3)$ and $k=1$, $k=3$.

$k=1$

for $x_1 = (8,5)^T$, the nearest neighbour is $(10,6)^T$
with $k=1$ ~~$f(x_1) = (8,5)^T = 60$~~ $(8,5)^T$ will be mapped to 60.

for $x_2 = (1,3)$, the nearest neighbour is $(1,3)$ (in $k=1$ case)
It will be mapped to 77.

$k=3$

For $x_1 = (8,5)^T$, 3 nearest neighbours are $(10,6)^T$, $(5,5)^T$, $(5,8)^T$

Hence, it will be mapped to $\frac{60+59+72}{3} = 63.67$

for, $x_2 = (1,3)$, 3 nearest neighbours are $(1,3)^T$, $(2,5)^T$ and $(3,2)^T$.

It will be mapped to $\frac{77+47+55}{3} = 59.67$