Exercise 1.)

$$X = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}_{3\times 2} \qquad y = \begin{bmatrix} 1 \\ 0.5 \\ 3 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

So,

$$(X^T X)^{-1} = \frac{1}{3 \cdot 2} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix}$$

So,

$$\beta = (X^T X)^{-1} X^T y$$

$$= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} . \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 3 \end{bmatrix}.$$

$$= \begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4.5 \\ 4 \end{bmatrix}.$$

$$= \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}.$$

$$\therefore \hat{y}(x) = \begin{bmatrix} 1 & x \end{bmatrix} . \begin{bmatrix} 1.5 \\ 2 \end{bmatrix}$$

$$= 1.5 + 2x.$$

Now, for quadratic model

$$X = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 2 \\ 6 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\det(X^T X) = 3 \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + 2 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= 3 \cdot 4 + 2 \cdot -4$$

$$= 4.$$

$$\mathrm{Adj}(X^T X) = \begin{bmatrix} \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 6 \end{bmatrix}$$

$$(X^T X)^{-1} = \frac{1}{4} \begin{bmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 3/2 \end{bmatrix}.$$

So

$$\beta = (X^T X)^{-1} X^T \cdot y$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 8 \end{bmatrix}.$$

$$= \begin{bmatrix} -2 \\ 1/2 \\ 6 \end{bmatrix}.$$

So,

$$\hat{y}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} -2 \\ 1/2 \\ 6 \end{bmatrix}$$

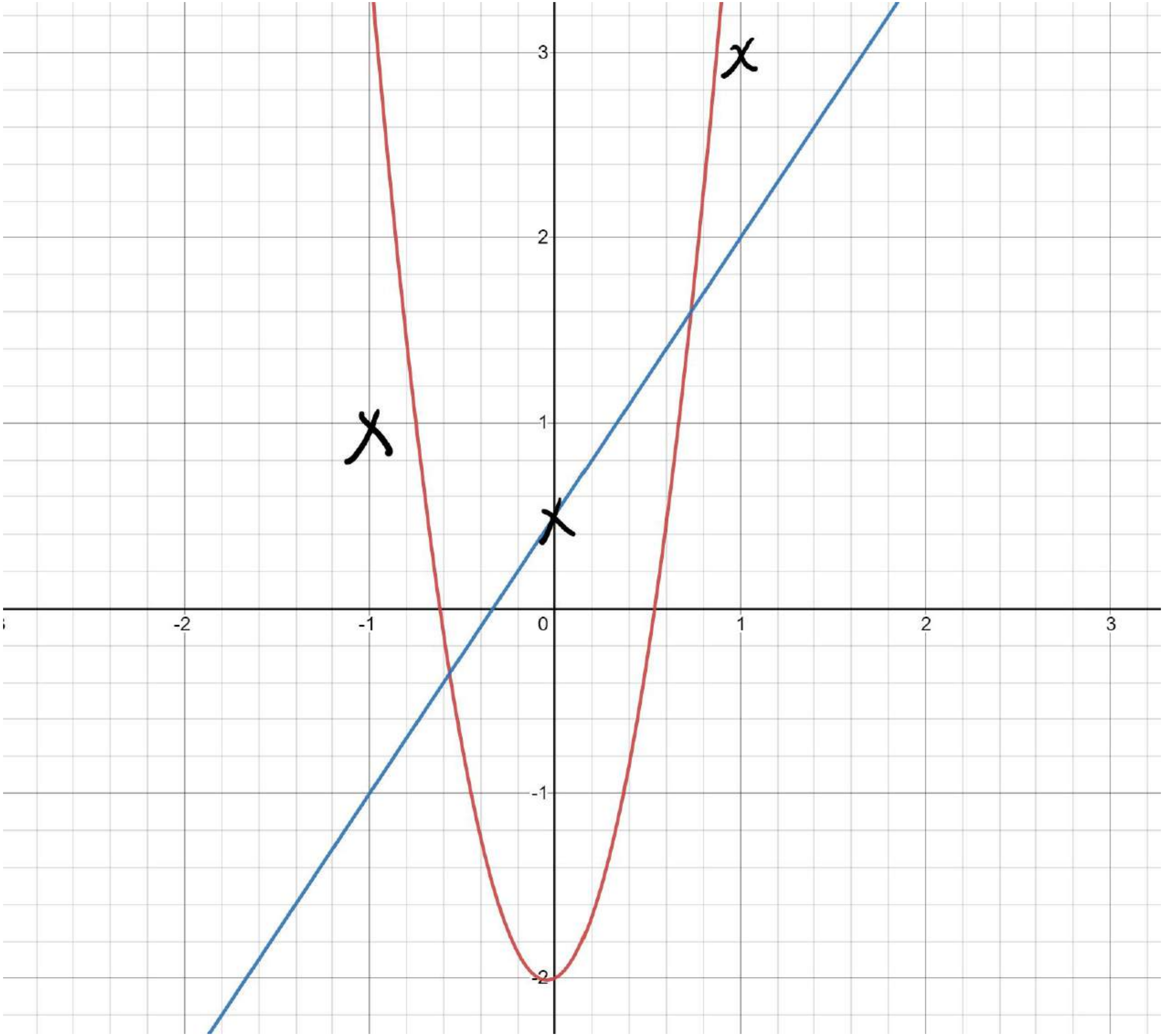
$$= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} -2 \\ 1/2 \\ 6 \end{bmatrix}$$

So,

$$\hat{y}(x) = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T \begin{bmatrix} -2 \\ 1/2 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} -2 \\ 1/2 \\ 6 \end{bmatrix}$$

$$= -2 + \frac{x}{2} + 6x^2$$


3
2
1
-1
-2
-2
-1
0
1
2
3

Exercise 2.

Now

$$A = \begin{bmatrix} e^{-\frac{\|-2+2\|^2}{1}} & e^{-\frac{\|-2-0\|^2}{1}} & e^{-\frac{\|-2-2\|^2}{1}} \\ e^{-\frac{\|0+2\|^2}{1}} & e^{-\frac{\|0-0\|^2}{1}} & e^{-\frac{\|0-1\|^2}{1}} \\ e^{-\frac{\|1+2\|^2}{1}} & e^{-\frac{\|1+0\|^2}{1}} & e^{-\frac{\|1-1\|^2}{1}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & e^{-4} & e^{-9} \\ e^{-4} & 1 & e^{-1} \\ e^{-9} & e^{-1} & 1 \end{bmatrix}$$

$$AB = y$$

or, $\begin{bmatrix} 1 & e^{-4} & e^{-9} \\ e^{-4} & 1 & e^{-1} \\ e^{-9} & e^{-1} & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$$\det(A) = 1 \begin{vmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{vmatrix} - e^{-4} \begin{vmatrix} e^{-4} & e^{-1} \\ e^{-9} & 1 \end{vmatrix} + e^{-9} \begin{vmatrix} e^{-4} & 1 \\ e^{-9} & e^{-1} \end{vmatrix}$$

$$= 1 - e^{-2} - e^{-4}\left(e^{-4} - e^{-10}\right) + e^{-9}\left(e^{-5} - e^{-9}\right)$$

$$= 1 - e^{-2} - e^{-8} + e^{-14} + e^{-14} + e^{-18}$$

$$= 0.8643309$$

Now,

$$\operatorname{Adj}(A) = \begin{bmatrix} \begin{vmatrix} 1 & e^{-1} \\ e^{-1} & 1 \end{vmatrix} & -\begin{vmatrix} e^{-4} & e^{-1} \\ e^{-9} & 1 \end{vmatrix} & \begin{vmatrix} e^{-4} & 1 \\ e^{-9} & e^{-1} \end{vmatrix} \\ -\begin{vmatrix} e^{-4} & e^{-9} \\ e^{-1} & 1 \end{vmatrix} & \begin{vmatrix} 1 & e^{-9} \\ e^{-9} & 1 \end{vmatrix} & -\begin{vmatrix} 1 & e^{-4} \\ e^{-9} & e^{-1} \end{vmatrix} \\ \begin{vmatrix} e^{-4} & e^{-9} \\ 1 & e^{-1} \end{vmatrix} & -\begin{vmatrix} 1 & e^{-9} \\ e^{-4} & e^{-1} \end{vmatrix} & \begin{vmatrix} 1 & e^{-4} \\ e^{-4} & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - e^{-2} & -(e^{-4} - e^{-10}) & (e^{-5} - e^{-9}) \\ -(e^{-4} - e^{-10}) & (1 - e^{-18}) & -(e^{-1} - e^{-13}) \\ (e^{-5} - e^{-9}) & -(e^{-1} - e^{-13}) & (1 - e^{-8}) \end{bmatrix}$$

$$= \begin{bmatrix} 0.864665 & -0.0182702 & 6.67453 \times 10^{-3} \\ -0.0182702 & 0.9999999 & -0.367877 \\ 6.614537 \times 10^{-3} & -0.3678772 & 0.999664 \end{bmatrix}$$

So,

$$\beta = A^{-1} y$$

$$= \frac{1}{0.8643309} \begin{bmatrix} 0.864865 & -0.0182702 & 6.67453 \times 10^{-3} \\ -0.0182702 & 0.9999999 & -0.367877 \\ 6.614537 \times 10^{-3} & -0.3678772 & 0.999664 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.000386 & -0.02113796 & 7.7221929 \times 10^{-3} \\ -0.02113796 & 1.15696419 & -0.42562055 \\ 7.7221929 \times 10^{-3} & -0.42562055 & 1.1565755 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.023164 \\ -0.893517 \\ 2.328595 \end{bmatrix}$$

Therefore, the model is

$$f(x) = 2.023164 \cdot K(x,-2) + -0.893517\, K(x,0) + 2.328595 \cdot K(x,2).$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ \vdots & & & \\ \vdots & & & \\ a_{N1} & \cdots & \cdots & a_{NN} \end{bmatrix}$$

~~Symmetric~~

$$[x_1 \ x_2 \ \cdots \ x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & & \\ \vdots & a_{32} & & \\ a_{N1} & \cdots & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$= \begin{bmatrix} x_1 a_{11} + x_2 a_{21} + x_3 a_{31} \cdots + x_N a_{N1} \\ x_1 a_{12} + x_2 a_{22} + x_3 a_{32} + x_4 a_{42} \cdots + x_n a_{N2} \\ \vdots \\ x_1 a_{1N} + x_2 a_{2N} + x_3 a_{3N} + \cdots \ x_N a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$= x_1^2 a_{11} + x_2^2 a_{21} + x_3^2 a_{31} \cdots \cdots + x_1^2 a_{12} + x_2^2 a_{22} \cdots$$

$$\cdots \cdots x_1^2 a_{1N} + x_2^2 a_{2N} \cdots + x_n^2 a_{NN}$$

Since all entries a_{ij} are positive and all x_j^2 are positive, all the values are all positive

Since the entries in A are calculated as $e^{-\frac{(x_i - x_j)^2}{\sigma^2}}$,

~~we~~ for a_{ij}, we see that $a_{ij} = a_{ji} = e^{-\frac{(x_j - x_i)^2}{\sigma^2}}$

$$= e^{-\frac{(x_i - x_j)^2}{\sigma^2}} \quad //$$

Hence it is positive definite and symmetric.

(b) $T = \{(-2,2)^T, (0,0)^T, (1,2)^T\}$, $\sigma = 1$, $\lambda = 1$.

$$A = \begin{bmatrix} e^{0} & e^{-4} & e^{-9} \\ e^{-4} & e^{0} & e^{-1} \\ e^{-9} & e^{-1} & e^{0} \end{bmatrix}$$

$$A + \lambda I = \begin{bmatrix} 2 & e^{-4} & e^{-9} \\ e^{-4} & 2 & e^{-1} \\ e^{-9} & e^{-1} & 2 \end{bmatrix}$$

$$\alpha = (A + \lambda I)^{-1} y$$

$$\operatorname{Det}(A + \lambda I) = 2 \cdot (4 - e^{-2}) - e^{-4}(2e^{-4} - e^{-10})$$
$$e^{-9}(e^{-5} - 2e^{-9})$$

$$= 7.7286$$

$$= \begin{bmatrix} 2 - e^{-2} & -(2e^{-4} - e^{-10}) & (2e^{-4} - e^{-10}) \\ -(2e^{-4} - e^{-10}) & +(4 - e^{-18}) & -(2e^{-1} - e^{-13}) \\ (e^{-5} - 2e^{-9}) & -(2e^{-1} - e^{-13}) & (4 - e^{-8}) \end{bmatrix} = \operatorname{Adj}(A)$$

$$(A + \lambda I)^{-1} = \frac{1}{7.7286} \operatorname{Adj}(A)$$

$$\therefore \alpha = \frac{1}{7.7286} \mathrm{Adj}(A)\, \vec{y}$$

$$= \begin{bmatrix} 1.0018 \\ \cancel{0.4920} \\ -0.199 \\ 1.03 \end{bmatrix}$$

$$= 1.0018 \cdot k(x,-2) - 0.199 \cdot k(x,0) + 1.03\, k(x,1)$$

$$= 1.0018 \cdot e^{-(x+2)^2} - 0.199 \cdot e^{-(x^2)} + 1.03\, e^{-(x-1)}$$