Homework 1, General IMS Fall 2018

Prof. Dr. Francesco Maurelli, Jacobs University Bremen

Handed out 17.09.2018, due 24.09.2018 23:59:59

Please use the moodle system to upload your homework as pdf (`moodle.jacobs-university.de`). The system will shut down at the deadline and homeworks will not be accepted at a later time. No other form of submission is allowed. Homeworks are not compulsory but highly recommended. You will be able to choose if counting them for your grade (up to 50%), or if taking a longer and more comprehensive final exam.
Use of LaTeXis recommended, though not compulsory. Scans of hand-written papers are accepted as long as the writing is clearly understandable.

1 Vectors

1. (8 points) Consider the following vectors:
   $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}; v = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$
   (a) (1 point) Calculate the norm of u and of v
   (b) (1 point) Calculate the sum of the two vectors
   (c) (1 point) Show the result geometrically
   (d) (1 point) Calculate the difference $v - u$
   (e) (1 point) Calculate $7v - 5u$
   (f) (1 point) Calculate the dot product
   (g) (1 point) Calculate the cross product
   (h) (1 point) Are u and v linearly independent? Why?
2. (4 points) Suppose that $u \in \mathbb{R}^3$ is a vector which lies in the first quadrant of the xy−plane and has length 3 and that $v \in \mathbb{R}^3$ is a vector that lies along the positive z-axis and has length 5.
   (a) (1 point) Calculate $\|u \times v\|$
   (b) (1 point) The x-coordinate of $u \times v$ is ... 0 (choose <, >, or =, and motivate the answer)
   (c) (1 point) The y-coordinate of $u \times v$ is ... 0 (choose <, >, or =, and motivate the answer)
   (d) (1 point) The z-coordinate of $u \times v$ is ... 0 (choose <, >, or =, and motivate the answer)
3. (4 points) Suppose that u and v are vectors in $\mathbb{R}^3$, both of length $2\sqrt{2}$ and that the length of $u - v$ is also $2\sqrt{2}$.
   (a) (2 points) Calculate $\|u + v\|$
   (b) (2 points) Calculate the angle between u and v

2 Matrices

1. (3 points) Consider the following matrices:

   $A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -3 & 2 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}; C = \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}$

   Calculate, if possible:

   (a) (1 point) $A + B$

   (b) (1 point) $A + C$

   (c) (1 point) $2C + 3I_2$

2. (6 points) Consider the following matrices:

   $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$

   Calculate, if possible:

   (a) (1 point) A^T

   (b) (1 point) $A + B$

   (c) (1 point) $A^T + B$

   (d) (1 point) AB

   (e) (1 point) AB

   (f) (1 point) BA

3. (6 points) Consider the following matrices:

   $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$

   Calculate, if possible:

   (a) (1 point) A^T

   (b) (1 point) $A + B$

   (c) (1 point) $A^T + B$

   (d) (1 point) AB

   (e) (1 point) AB

   (f) (1 point) BA

4. (3 points) Consider the following matrix:

   $A = \begin{bmatrix} 1 & -1 & 3 \end{bmatrix}$

   Calculate, if possible:

   (a) (1 point) A^2

   (b) (1 point) $A^T A$

   (c) (1 point) AA^T

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1) Vectors

Consider the following vectors $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $v = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

a.) Calculate the norm of u and of v

$\vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ norm $\vec{u} = \sqrt{1^2 + (-3)^2}$; $\vec{u} = \sqrt{1+9}$; $\vec{u} = \sqrt{10}$

$\vec{u} = 3.162227$

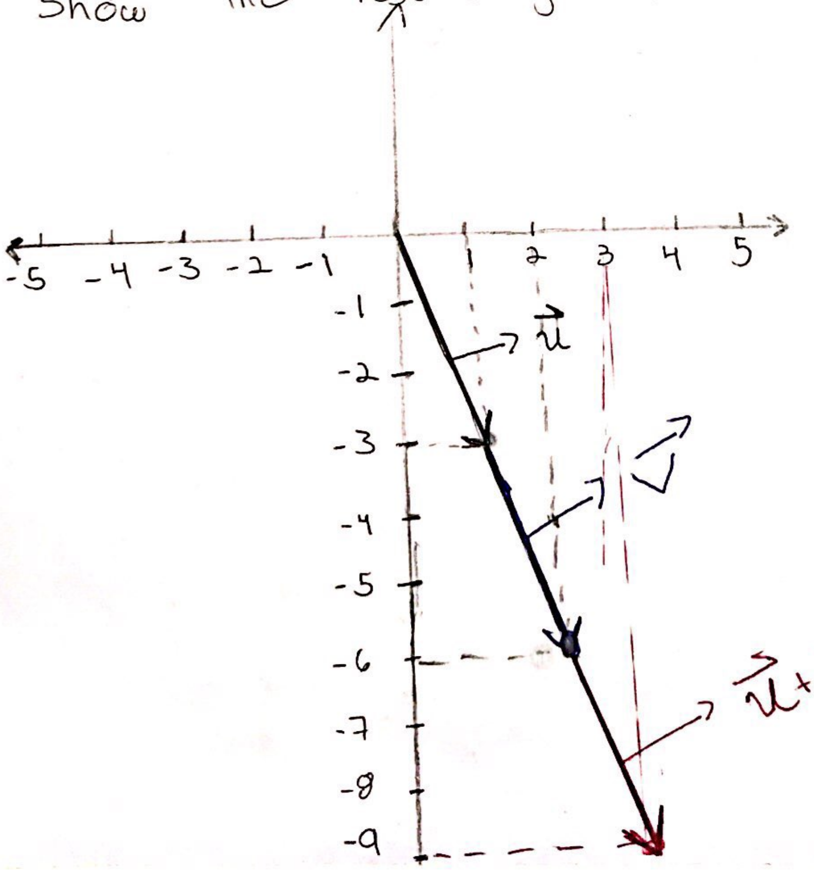
$\vec{v} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ norm $v = \sqrt{2^2 + (-6)^2}$; $v = \sqrt{4+36}$ $\vec{v} = \sqrt{40}$

$\vec{v} = 6.32455532$

b.) Calculate the sum of the two vectors

$\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ $\vec{u} + \vec{v} = \boxed{\begin{bmatrix} 3 \\ -9 \end{bmatrix}}$ $(3, -9)$

c.) Show the result geometrically



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d.) Calculat de difference $\vec{v} - \vec{u}$

$$v - u = \begin{bmatrix} 2 \\ -6 \end{bmatrix} - \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad \vec{v} - \vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

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e.) Calculate $7v - 5u$

$$\underbrace{7\vec{v}} - \underbrace{5\vec{u}} = 7\begin{bmatrix} 2 \\ -6 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ -42 \end{bmatrix} - \begin{bmatrix} 5 \\ -15 \end{bmatrix}$$

$$\begin{bmatrix} 14 \\ -42 \end{bmatrix} - \begin{bmatrix} 5 \\ -15 \end{bmatrix} = \begin{bmatrix} 9 \\ -27 \end{bmatrix} = 7\vec{v} - 5\vec{u}$$

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f.) Calculate the dot product

$$\begin{array}{cc} u & v \\ \begin{bmatrix} 1 \\ -3 \end{bmatrix} & \begin{bmatrix} 2 \\ -6 \end{bmatrix} \end{array}$$

$u \cdot v = 1(2) + (-3)(-6)$

$u \cdot v = 2 + (18)$

$\vec{u} \cdot \vec{v} = 20$

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g.) calculate the cross product

[redacted]

$$\begin{array}{cc} u & v \\ \begin{bmatrix} 1 \\ -3 \end{bmatrix} & \begin{bmatrix} 2 \\ -6 \end{bmatrix} \end{array} = \begin{bmatrix} 1(-6) - (2)(-3) \\ -3(2) - (-6)(1) \end{bmatrix} = \begin{bmatrix} -6 - (-6) \\ -6 - (-6) \end{bmatrix}$$

$$u \times v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

h.) Are u and v linearly independent? why?

No, they are linearly dependent. Because the vector $\vec{v}$ is a multiplication of a scalar times vector $\vec{u}$. So, they are not linearly independent.

$$\vec{v} = 2\vec{u}$$

$$\begin{bmatrix} 2 \\ -6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$

Linearly dependent not Linearly independent.

2.) (4 points) Suppose that $u \in \mathbb{R}^3$ is a vector which lies in the first quadrant of the xy-plane and has length 3 and that $v \in \mathbb{R}^3$ is a vector that lies along the positive z-axis and has a length 5.

a.) Calculate $\|u \times v\|$

$\|u\| = 3$ $\quad \|3 \times 5\| = 15$

$\|v\| = 5$ $\quad \|\vec{u} \times \vec{v}\| = \|u\| \times \|v\|$

$$\|u \times v\| = 3 \times 5 = 15$$

b.) The x-coordinate of $u \times v$ is ..... 0 choose >, <, =

The x-coordinate of $u \times v$ is $= 0$; $u \times v = 0$

because $\begin{array}{cc} u & v \\ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \end{array}$ x times 0 is 0, we know that $\vec{u}$ has a value asigned in x and y but not in z and $\vec{v}$ in z but not in x and y.

c.) The y- coordinate of $u \times v$ is ... 0 choose <, >, =

the y-coordinate of $u \times v = 0$.

Because thanks to the statement we know that u is on the first quadrant on the xy-plane so he has value in x and y but not in z so we conclude: $\vec{u} = \begin{bmatrix} +x \\ +y \\ 0 \end{bmatrix}$ and $\vec{v}$ is on the positive z-axis so conclude: $\begin{bmatrix} 0 \\ 0 \\ +z \end{bmatrix}$

so $\begin{bmatrix} +x \\ +y \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ +z \end{bmatrix} = \begin{bmatrix} 0x \\ 0y \\ 0z \end{bmatrix}$ $\underline{0y = 0}$

d.) The z- coordinate of $u \times v$ is ....0 choose <, >, =

The z-coordinate of $u \times v = 0$

Because thanks to the statements we know that $\vec{u}$ is on the first quadrant on the xy-plane so it has a value in +x and +y axis but not in z axis so we conclude $\vec{u} = \begin{bmatrix} +x \\ +y \\ 0 \end{bmatrix}$ and $\vec{v}$ is on the positive z-axis so conclusion $\vec{z} = \begin{bmatrix} 0 \\ 0 \\ +z \end{bmatrix}$

$$\text{therefore} \begin{array}{cc} \vec{u} & \vec{v} \\ \begin{bmatrix} +x \\ +y \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \end{array} = \begin{bmatrix} 0x \\ 0y \\ 0z \end{bmatrix} \Rightarrow \underline{0z = 0}$$

3.) (4 points) Suppose that u and v are vectors in $\mathbb{R}^3$, both of length $2\sqrt{2}$ and that the length of $u-v$ is also that.

a.) calculate $\|u+v\|$

$\|\vec{u}\| = 2\sqrt{2} = \sqrt{8}$

$\|\vec{v}\| = 2\sqrt{2} = \sqrt{8}$

$\|u-v\| = 2\sqrt{2} = \sqrt{8}$

$\sqrt{8} = \sqrt{a^2+b^2+c^2}$

$\sqrt{4+4+0}$

$\sqrt{2^2+2^2+0^2}$

$\vec{u} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \|\vec{u}\| = \sqrt{8}$

$\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ $\|\vec{v}\| = \sqrt{8}$

$\|\vec{u} - \vec{v}\| = \left\| \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \right\| = \sqrt{8}$

$\vec{u} + \vec{v} = \left\| \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \right\| = \sqrt{2^2+4^2+2^2}$

$= \sqrt{24} = 2\sqrt{6}$

b.) 2 points Calculate the angle between $\vec{u}$ and $\vec{v}$

$\vec{u} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

$\underbrace{\vec{u} \cdot \vec{v}}_{4} = \underbrace{\|\vec{u}\| \, \|\vec{v}\|}_{2\sqrt{2} \cdot 2\sqrt{2} \,=\, 8} \cos\theta$

$\vec{u} \cdot \vec{v} = (2 \cdot 0 + 2 \cdot 2 + 2 \cdot 0)$

$\boxed{\vec{u} \cdot \vec{v} = 4}$

$4 = 8 \cos\theta$

$\theta = \cos^{-1}\left(\frac{4}{8}\right)$

$\theta = 60^\circ$

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2 Matrices

1.) (3 points) Consider the following matrices

$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -3 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix}$; $C \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}$

calculate if possible

a.) A + B

$$\begin{bmatrix} 3 & -1 & 0 \\ 2 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

b.) A + C

$$\overset{A}{\begin{bmatrix} 3 & -1 & 0 \\ 2 & -3 & 2 \end{bmatrix}} + \overset{C}{\begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}} =$$ It's not posible! they have different dimensions.

c.) $2C + 3I_2$

there is no matrix named I_2

2.) (6 points) Consider the following matrices

$A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$ 2×4 2×4

calculate if posible

a.) (1 point) $A^T = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 1 & 0 \end{bmatrix}$

b.) A + B It's not posible! They have different dimencions c.) $A^T + B$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 6 & -1 \\ 5 & -4 & 3 & 1 \end{bmatrix}$$

d.) AB

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (1\cdot1)+(3\cdot2) & 1(0)+3(-2) & 1(3)+(3(2)) & 1(0)+3(1) \\ 2(1)+(-2)(2) & 2(0)+(-2)(-2) & 2(3)+(-2)2 & 2(0)+(-2)(1) \\ 3(1)+1(2) & 3(0)+1(-2) & 3(3)+2(1) & 3(0)+1(1) \\ -1(1)+0(2) & -1(0)+0(-2) & -1(3)+0(2) & -1(0)+0(1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -6 & 9 & 3 \\ -2 & 4 & 2 & -2 \\ 5 & -2 & 11 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix}$$

e.) AB

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 0-6 & 3+6 & 0+3 \\ 2-4 & 0+4 & 6-4 & 0-2 \\ 3+2 & 0-2 & 9+2 & 0+1 \\ -1+0 & 0+0 & -3+0 & 0+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -6 & 9 & 3 \\ -2 & 4 & 2 & -2 \\ 5 & -2 & 11 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix}$$

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f.) BA

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+0+9+0 & 3+0+3+0 \\ 2-4+6-1 & 6+4+2+0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 10 & 6 \\ 3 & 12 \end{bmatrix}$$

3.) (6 points) Consider the following

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix}$$

calculate if posible

a.) $A^T = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 1 & 0 \end{bmatrix}$

b.) $A + B$ = It's not posible they have diffcient dimensions.

c.) $A^T + B$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 6 & -1 \\ 5 & -4 & 3 & 1 \end{bmatrix}$$

d.) AB

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 2 & -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 & 3 \\ -2 & 4 & 2 & -2 \\ 5 & -2 & 11 & 1 \\ -1 & 0 & -3 & 0 \end{bmatrix}$$

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e) AB

$$\begin{bmatrix}1 & 3\\ 2 & -2\\ 3 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 3 & 0\\ 2 & -2 & 2 & 1\end{bmatrix} = \boxed{\begin{bmatrix}7 & -6 & 9 & 3\\ -2 & 4 & 2 & -2\\ 5 & -2 & 11 & 1\\ -1 & 0 & -3 & 0\end{bmatrix}}$$

f.) BA

$$\begin{bmatrix}1 & 0 & 3 & 0\\ 2 & -2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & -2\\ 3 & 1\\ -1 & 0\end{bmatrix} = \boxed{\begin{bmatrix}10 & 6\\ 3 & 12\end{bmatrix}}$$

4.) 3 Points Consider the following motrix

$A = [1, -1\ 3]$

calculate if posible

a.) $A^2 \rightarrow$ It's not possible, there dimensions are different

b.) $A^T A$ $\begin{bmatrix}1\\ -1\\ 3\end{bmatrix}[1, -1, 3] = \boxed{\begin{bmatrix}1 & -1 & 3\\ -1 & +1 & -3\\ 3 & -3 & 9\end{bmatrix}}$

c) AA^T $[1, -1, 3]\begin{bmatrix}1\\ -1\\ 3\end{bmatrix}$ $1+1+9=11$ $=$ $=$ $[11] =$ AA^T

Homework 2, General IMS Fall 2018

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Handed out 2018.10.08, due 2018.10.15 23:59:59

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Use of LaTeXis recommended, though not compulsory. Scans of hand-written papers are accepted as long as the writing is clearly understandable.

1 Matrices - 6 points

1. Using rotation matrices, prove that 2D rotation is commutative but 3D rotation is not.
2. Consider a 2D point P expressed in polar coordinates $[\rho, \theta]$. Define the rotation matrix M for a rotation of ϕ about the origin and the new position P' after the rotation.
3. Referring to the previous problem, use geometrical considerations to calculate the point P', instead of a Rotation Matrix
4. Write a matrix M that mirrors points about the yz-plane
5. Construct a matrix to rotate -30deg about the y-axis.
6. Calculate determinant and trace of the following matrix:
   $A = \begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix}$

2 Quaternions - 6 points

1. Construct a quaternion to rotate 60deg about the z-axis
2. Calculate the magnitude of the quaternion from the previous question
3. Calculate the conjugate of the same quaternion
4. Calculate angle of rotation and axis vector for the following quaternion: $[0.5(0, 0, \sqrt{3}/2)]$
5. Express in both Rotation matrixes and quaternions the composition of two rotations on the z axis, the first of an angle θ and the second of an angle ϕ.
6. Calculate the multiplication between the quaternions $q' = (2 - i + j + 3k)$ and $q'' = (-1 + i + 4j - 2k)$

3 Robot motion - 2 points

Let's consider a mobile robot that moves in the 2D space. It starts at (0,0). It will move along the x axis for 5 meters. Then he/she/it will rotate about the origin of 45 degrees. Then he will move of a vector $(-2.5\sqrt{2}, -2.5\sqrt{2})^T$. Calculate the final transformation matrix as a combination of the three individual transformations, and show the robot path in a graph.

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Homework 2, General IMS

1) Matrices - 6 points

1) Using rotation matrices, prove that 2D rotation is commutative but 3D rotation is not.

proof that 2D rotation is commutative:

Lets considered we have a vector $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$ and we rotate it first 90° and then 180°. ex:

$$P_1' = \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ & 0 \\ \sin 180^\circ & \cos 180^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}}$$

$$P_1' = \begin{bmatrix} \cos 180^\circ & -\sin 180^\circ & 0 \\ \sin 180^\circ & \cos 180^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 5 \\ 1 \end{bmatrix}$$

$$P_1' = \begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix} = P_1' \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

Now, let's consider the same vector, but the first rotation now is going to be 180° and then 90°.

$$P_2' = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ & 0 \\ \sin 180^\circ & \cos 180^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}}$$

$$P_2' = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -8 \\ 1 \end{bmatrix}$$

$$P_2' = \begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix} = P_2' = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

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as you can see p_1' and p_2' are the same

$$p_1' = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \qquad p_2' = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

so, we can see that 2D rotation is commutative.

If we do it more general let $90° = \alpha$ and $180° = \beta$ then:

$$p_1' = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}} =$$

$$\begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x\cos\alpha - y\sin\alpha \\ x\sin\alpha + y\cos\alpha \\ 1 \end{bmatrix}$$

$$p_1' = \begin{bmatrix} x\cos\alpha\cos\beta - y\sin\alpha\cos\beta - x\sin\alpha\sin\beta - y\cos\alpha\sin\beta \\ x\cos\alpha\sin\beta - y\sin\alpha\sin\beta + x\sin\alpha\cos\beta + y\cos\alpha\cos\beta \\ 1 \end{bmatrix}$$

$$p_2' = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$p_2' = \begin{bmatrix} x\cos\alpha\cos\beta - y\sin\alpha\cos\beta - x\sin\alpha\sin\beta - y\cos\alpha\sin\beta \\ x\cos\alpha\sin\beta - y\sin\alpha\sin\beta + x\sin\alpha\cos\beta + y\cos\alpha\cos\beta \\ 1 \end{bmatrix}$$

as we can se $p_1' = p_2'$ so we can say that they are commutative

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Proof that 3D rotation is not commutative

let considerd we have a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and we rotate it in the x axis by α, and in the y-axis by β and at last to the z-axis by η

so this is the formula

$$\begin{array}{ccccc} & \text{z-axis Rotation} & \text{y axis rotation} & \text{x-axis rotation} & \\ P_1' = & \begin{bmatrix} \cos\eta & -\sin\eta & 0 & 0 \\ \sin\eta & \cos\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \end{array}$$

$$P_1' = \begin{bmatrix} \cos\eta & -\sin\eta & 0 & 0 \\ \sin\eta & \cos\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y\cos\alpha - z\sin\alpha \\ y\sin\alpha + z\cos\alpha \\ 1 \end{bmatrix}$$

$$P_1' = \begin{bmatrix} \cos\eta & -\sin\eta & 0 & 0 \\ \sin\eta & \cos\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x\cos\beta + \sin\beta(y\cos\alpha - z\sin\alpha) \\ y\cos\alpha - z\sin\alpha \\ -x\sin\beta + \cos\beta(y\sin\alpha + z\cos\alpha) \\ 1 \end{bmatrix}$$

$$P_1' = \begin{bmatrix} \cos\eta\,[x\cos\beta(1) + \sin\beta(y\cos\alpha - z\sin\alpha)] - \sin\eta\,[y\cos\alpha - z\sin\alpha] \\ \sin\eta\,[x\cos\beta + \sin\beta(y\cos\alpha - z\sin\alpha)] + \cos\eta(y\cos\alpha - z\sin\alpha) \\ -x\sin\beta + \cos\beta(y\sin\alpha + z\cos\alpha) \\ 1 \end{bmatrix}$$

$$P_1' = \begin{bmatrix} x\cos\beta\cos\eta + y\cos\alpha\sin\beta\cos\eta - z\sin\alpha\sin\beta\cos\eta - y\cos\alpha\sin\eta - z\sin\alpha\sin\eta \\ x\cos\beta\sin\eta + y\cos\alpha\sin\beta\sin\eta - z\sin\alpha\sin\beta\sin\eta + y\cos\alpha\cos\eta - z\sin\alpha\cos\eta \\ -x\sin\beta + y\sin\alpha\cos\beta + z\cos\alpha\cos\beta \\ 1 \end{bmatrix}$$

Now, let's consider the same vector, but we change the order of the rotations

$$
P_2' = \underbrace{\begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{y-axis rotation}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{x-axis rotation}} \underbrace{\begin{bmatrix} \cos\eta & -\sin\eta & 0 & 0 \\ \sin\eta & \cos\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{z-axis rotation}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

$$
P_2' = [\text{y-axis rotation}] \cdot [\text{x-axis rotation}] \begin{bmatrix} x\cos\eta - y\sin\eta \\ x\sin\eta + y\cos\eta \\ z \\ 1 \end{bmatrix}
$$

$$
P_2' = [\text{y-axis rotation}] \begin{bmatrix} x\cos\eta - y\sin\eta \\ x\sin\eta\cos\alpha + y\cos\eta\cos\alpha - z\sin\alpha \\ x\sin\alpha\sin\eta + y\cos\eta\sin\alpha + z\cos\alpha \\ 1 \end{bmatrix}
$$

$$
P_2' = \begin{bmatrix} x\cos\eta\cos\beta - y\sin\eta\cos\beta + x\sin\alpha\sin\eta\sin\beta + y\cos\eta\sin\alpha\sin\beta + z\cos\alpha\sin\beta \\ x\sin\eta\cos\alpha + y\cos\eta\cos\alpha - z\sin\alpha \\ + x\sin\alpha\sin\eta\cos\beta + y\cos\eta\sin\alpha\cos\beta + z\cos\alpha\cos\beta - x\cos\eta\sin\beta + y\sin\eta\sin\beta \\ 1 \end{bmatrix}
$$

As we can se P_1' and P_2' are different so they are not commutative. This is a proof that 3D rotations aren't commutative.

2.) Considered a 2D point P expessed in polar coordinates (p, θ). Define the rotation matrix M for a rotation of $\varnothing$ about the origin and the new position p' after the rotation.

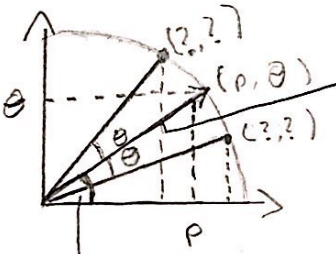
vector = $\begin{bmatrix} p \\ \theta \end{bmatrix}$ angle = $\varnothing$

$$p' = \begin{bmatrix} \cos\varnothing & -\sin\varnothing & 0 \\ \sin\varnothing & \cos\varnothing & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ 1 \end{bmatrix}$$

$$p' = \begin{bmatrix} p\cos\varnothing - \theta\sin\varnothing \\ p\sin\varnothing + \theta\cos\varnothing \\ 1 \end{bmatrix}$$

this is the matrix for the rotation of $\varnothing$

3.) Referring to the previous problem, use geometry to solvet witrot a rotation matrix



The length of this point can be calculate by

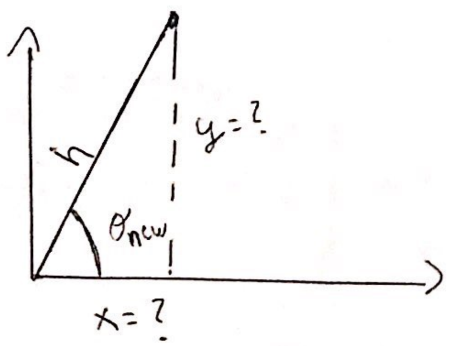
$$h = \sqrt{p^2 + \theta^2}$$

The angle of the point (p,θ) can be calculate by $\theta' = \tan^{-1} \frac{\theta}{p}$

now if we add the angle of (p,θ) and the angle we want to rotate we obtained

$$\text{total angle} = \varnothing + \theta' = \theta_{new} = \varnothing + \tan^{-1}\frac{\theta}{p}$$

then our new point



we know that $h = \sqrt{p^2 + \theta^2}$

to calculate y we use : $\sin\theta_{new} = \frac{y}{h}$

to calculate x we use: $\cos\theta_{new} = \frac{x}{h}$

we know that $\theta_{new} = \varnothing + \tan^{-1}\frac{\theta}{p}$

So we do the next to calculate y & x

$\sin \theta_{new} = \frac{y}{h}$ $\qquad$ $\cos \theta_{new} = \frac{x}{h}$

$y = h \sin \theta_{new}$ $\qquad$ $x = h \cos \theta_{new}$

If we substitute we get.

$y = \left(\sqrt{p^2 + \theta^2}\right) \sin\left(\phi + \tan^{-1}\frac{\theta}{p}\right)$ $\qquad$ $x = \left(\sqrt{p^2 + \theta^2}\right) \cos\left(\phi + \tan^{-1}\frac{\theta}{p}\right)$

to prove that is correct we use the $(1,1)$ and $\phi = 30$.
if you plug in this in my equation you get

$x = 0.366025$

$y = 1.366025$

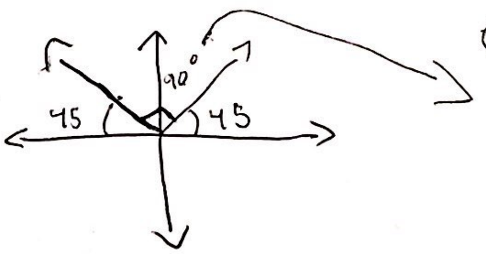
and if you plug in the same values in the rotation matrix
we get:

$x = 0.366025$ $\qquad$ $y = 1.366025$

As you can see they are the same, so it's proof.

5) Write a matrix M that mirrors points about the yz-plane

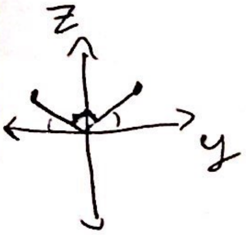
In order for a point to be in its mirror position
we use the angle $\theta' = 180 - 2\theta$. proof:



$\theta' = 180 - 2(45°)$

$\theta' = 90$

So we are going to use θ' in order to rotate.
If we considere the yz-plane as a 2D then θ'



$$p' = \begin{bmatrix} \cos\theta' & -\sin\theta' & 0 \\ \sin\theta' & \cos\theta' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y\cos\overbrace{(180°-2\theta)}^{\theta'} - z\sin(180°-2\theta) \\ y\sin(180°-2\theta) + z\cos(180°-2\theta) \\ 1 \end{bmatrix}$$

5.1 Construct a matrix to rotate $-30°$ dg about the y-axis

$$M = \begin{bmatrix} \cos -30° & 0 & \sin -30° & 0 \\ 0 & 1 & 0 & 0 \\ -\sin -30° & 0 & \cos -30° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$M = \begin{bmatrix} x\cos(-30°) + z\sin(-30°) \\ y \\ -x\sin(-30°) + z\cos(-30°) \\ 1 \end{bmatrix} \quad M = \begin{bmatrix} x\cos(-30°) - 0.5z \\ y \\ 0.5x + z\cos(-30°) \\ 1 \end{bmatrix}$$

6.1 Calculate determinant and trace of

$$\begin{bmatrix} 3 & -1 \\ 2 & -3 \end{bmatrix}$$

trace $= 3 - 3 = 0$

determinant $[3 \times (-3)] - (2(-1)) = -7$

trace $= 0$

determinant $= -7$

Kelan Fernando Garcia Osorio

Quaternions

1.) construct a quaternion that rotates $60°$ in z-axis

formula for quaternions:

$$q = s + xi + yj + zk$$ s, x, y, z are scalars

so, saying this we know

$$q = \cos(\theta/2) + \sin(\theta/2)xi + \sin(\theta/2)yj + \sin(\theta/2)zk$$

since we only rotate $60°$ in z-axis our formula is:

$$q = [(\cos(\tfrac{60}{2})), (0i + 0j + \sin(60/2)k)]$$

$$q = [(0.866025403), (+0i + 0j + 0.5k)]$$

2.) Calculate the magnitude of the quaternion
to calculate the magnitude we use:

$$||q|| = \sqrt{s^2 + xi^2 + yj^2 + zk^2} \rightarrow \sqrt{\sum_{i=1}^{n} v_i^2}$$

therefore we use:

$$||q|| = \sqrt{0.866025403^2 + 0^2 + 0^2 + 0.5^2}$$

$$||q|| = \sqrt{1}$$ $$|q| = 1$$

3.) Calculate the conjugate of the same quaternion

q -> quaternion q^* -> conjugate $q^* = [s, -v]$

$$q = [0.866025403, (-0i - 0j - 0.5k)]$$

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4.) Calculate the angle of rotation and axis vector for the following quatenion: $[0.5(0,0,\frac{\sqrt{3}}{2})]$

$$q = [\overset{s}{0.5}(0,0,\overset{z}{\frac{\sqrt{3}}{2}})] \; ; \; q = [\overset{s}{\cos(\frac{\theta}{2})}(0,0,\overset{z}{\sin\frac{\theta}{2}})]$$

s: $\cos\frac{\theta}{2} = 0.5$

$\theta/2 = \cos^{-1} 0.5$

$\theta = 2(\cos^{-1} 0.5)$

$\theta = 120°$

z: $\sin\frac{\theta}{2} = \frac{\sqrt{3}}{2}$

$\theta = 2[\sin^{-1}\frac{\sqrt{3}}{2}]$

$\theta = 120°$

the angle is $120°$

the axis vector =

$x = \frac{qx}{s} = \frac{0}{0.5} = 0$

$y = \frac{qy}{s} = \frac{0}{0.5} = 0$

$z = \frac{qz}{s} = \frac{\sqrt{3}/2}{0.5} = 1.732050808$ = z-axis

axis $= (0,0,1.732)$

5.) Rotatin in z-axis by matrix

at angle θ at angle ϕ

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If we rotate first at θ and then at ϕ both in the z-axis we get:

$$M = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{M = \begin{bmatrix} x\cos\theta\cos\phi - y\sin\theta\cos\phi - x\sin\theta\sin\phi - y\cos\theta\sin\phi \\ x\cos\theta\sin\theta - y\sin\theta\sin\phi + x\sin\theta\cos\phi + y\cos\theta\cos\phi \\ z \\ 1 \end{bmatrix}}$$

Now by quaternions

first rotation by θ in z-axis

$$q = \left[\left(\cos\frac{\theta}{2}\right), \left(xi + yJ + \sin\frac{\theta}{2}\right)\right]$$

Now we rotate it by ϕ and we get

$$q = \left[\left(\cos\frac{\theta}{2}\right)\left(\cos\frac{\phi}{2}\right), \left(xi + yJ + \sin\frac{\theta}{2}\left(\sin\frac{\phi}{2}\right)\right)\right.$$

$$\boxed{q = \left[\left(\cos\frac{\theta}{2}\right)\left(\cos\frac{\phi}{2}\right), \left(xi + yJ + \left(\sin\frac{\theta}{2}\right)\left(\sin\frac{\phi}{2}\right)\right)\right]}$$

6) Multiplicate this quaternions $q' = (2 - i + j + 3k)$; $q'' = (-1 + i + 4j - 2k)$

$q' \cdot q'' \quad [s_1 s_2 - v_1 \cdot v_2 \;,\; s_1 v_2 + s_2 v_2 + v_1 \times v_2]$

$q' \cdot q'' = [-2 - (-1 + 4 - 6)] \;,\; 2(i + 4j - 2k) + (-1)(-i + j + 3k) + v_1 \times v_2$

Cross product of $v_1 \times v_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} = -14i + 1J + 5k$

$q' \cdot q'' \quad (1, \; 2i + 8j - 4k + i - j - 3k - 14i + J - 5k)$

$q' \cdot q'' = [1 \, (-11i + 8j - 12k)]$

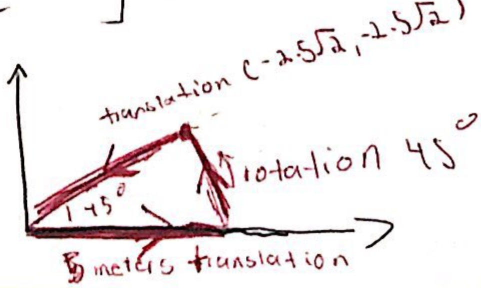
3.)

$$P = \overset{\text{transition}}{\begin{bmatrix} 1 & 6 & -2.5\sqrt{2} \\ 0 & 1 & -2.5\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{rotation in } x\text{-axis at } 45^\circ}{\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 6 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{transition}}{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{\text{origin}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$[\text{transition}] \begin{bmatrix} 3.535533906 \\ 3.535533909 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$ It goes to the origin



Homework 3, General IMS Fall 2018

Prof. Dr. Francesco Maurelli, Jacobs University Bremen

Handed out 2018.11.27, due 2018.12.02 23:55:00

Please use the moodle system to upload your homework as pdf (`moodle.jacobs-university.de`). The system will shut down at the deadline and homeworks will **not** be accepted at a later time (submissions will be accepted until midnight). No other form of submission is allowed. Homeworks are not compulsory but highly recommended. You will be able to choose if counting them for your grade (up to 50%).
Use of LaTeXis recommended, though not compulsory. Scans of hand-written papers are accepted as long as the writing is **clearly** understandable.

1 Cylindrical coordinates

1. (2 points) Convert $(2, \frac{\pi}{6}, 4)$, $(\sqrt{18}, -\frac{\pi}{4}, -7)$ in cylindrical coordinates to rectangular coordinates
2. (4 points) Consider a cube centered at the origin, with edges parallel to the x, y, z axes and with edge length of 2, find out the cylindrical coordinates of the vertices
3. (1 point) M is a point with rectangular coordinates $x = y = 0, z = 1$, N is a point with cylindrical coordinates r, θ, z, find out the Euclidean distance from M to N
4. (1 point) Describe cylindrical equation $z = r$ geometrically

   (Hint: convert the equation to rectangular first)

2 Spherical coordinates

1. (3 points) Convert $(3, \pi/6, \pi/4)$ and $(3, \pi/6, 3\pi/4)$ in spherical coordinates to rectangular coordinates, and compute the Euclidean distance between these two points
2. (1 point) Convert $(1, \sqrt{3}, 2)$ in rectangular coordinates to spherical coordinates
3. (1 point) Convert $(\sqrt{2}, \frac{\pi}{4}, \sqrt{6})$ in cylindrical coordinates to spherical coordinates
4. (1 point) Describe spherical equation $r = 1$ geometrically
5. (2 points) Convert spherical equation $tan(\phi)sin(\theta) = 1$ to rectangular form and describe the meaning

3 System of Forces

1. (3 points) Consider $F_A = 20i - 10j$ kN applied on point A, whose vector from the origin is $r_A = 10i + 5k$ m and $F_B = -10i + 10k$ kN applied on point B, whose vector from the origin is $r_B = 20j$ m. Calculate the resultant moment about the origin.

4 URDF

1. (3 points) Write a URDF file for a humanoid robot.

Kelan Fernando Garcia Osorio

Homework 3, General IMS Fall 2018

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1.) Cylindrical coordinates

1.1.) Convert $(2, \frac{\pi}{6}, 4)$, $(\sqrt{18}, -\frac{\pi}{4}, -7)$ in cylindrical coordinates to rectangular coordinates.

$r \quad \theta \quad z$

$(2, \frac{\pi}{6}, 4)$

cylindrical coordinate

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$x = 2\cos\frac{\pi}{6} = 2(\frac{\sqrt{3}}{2}) = \sqrt{3}$
$y = 2\sin\frac{\pi}{6} = 2(\frac{1}{2}) = 1$
$z = 4 = 4$

$= (\sqrt{3}, 1, 4)$

rectangular coordinate

$(\sqrt{18}, -\frac{\pi}{4}, -7)$

cylindrical coordinate

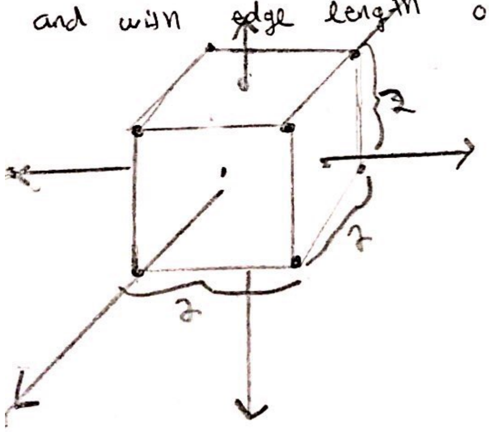
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$x = \sqrt{18}\cos -\frac{\pi}{4} = \sqrt{18} \cdot \frac{\sqrt{2}}{2} = 3$
$y = \sqrt{18}\sin -\frac{\pi}{4} = \sqrt{18} \cdot -\frac{\sqrt{2}}{2} = -3$
$z = -7$

$= (3, -3, -7)$

rectangular coordinate

1.2.) Consider a cube centered at the origin, with edges parallel to the x, y, z axes and with edge length of 2, find out the cylindrical coordinates of the vertices



8 edges

rectangular coordinates

$(1,1,1)$; $(-1,1,1)$; $(1,-1,1)$; $(-1,-1,1)$
$(1,1,-1)$; $(1,-1,-1)$; $(-1,1,-1)$; $(-1,-1,-1)$

to cylindrical coordinates

$r^2 = x^2 + y^2$ → since all x^2 and y^2 is 1 each, all of them have the same r

$r = \sqrt{1^2 + 1^2}$; $r = \sqrt{2}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}$

$(1,1,1)$	$(-1,1,1)$	$(1,-1,1)$	$(-1,-1,1)$	$(1,1,-1)$	$(1,-1,-1)$	$(-1,1,-1)$	$(-1,-1,-1)$
$\theta = \tan^{-1} 1$	$\theta = \tan^{-1} -1$	$\theta = \tan^{-1} -1$	$\theta = \tan^{-1} 1$	$\theta = \tan^{-1} 1$	$\theta = \tan^{-1} -1$	$\theta = \tan^{-1} -1$	$\theta = \tan^{-1} 1$
$= 45°$	$= -45° + 180 = 135°$	$= -45°$	$= 45° + 180 = 225°$	$= 45°$	$= -45°$	$= -45° + 180 = 135°$	$= 45° + 180 = 225°$

$z = Z$

so then we have

rectangular coordinates		cylindrical coordinates
$(1,1,1)$	$=$	$(\sqrt{2}, 45°, 1)$
$(-1,1,1)$	$=$	$(\sqrt{2}, 135°, 1)$
$(1,-1,1)$	$=$	$(\sqrt{2}, -45°, 1)$
$(-1,-1,1)$	$=$	$(\sqrt{2}, 225°, 1)$
$(1,1,-1)$	$=$	$(\sqrt{2}, 45°, -1)$
$(1,-1,-1)$	$=$	$(\sqrt{2}, -45°, -1)$
$(-1,1,-1)$	$=$	$(\sqrt{2}, 135°, -1)$
$(-1,-1,-1)$	$=$	$(\sqrt{2}, 225°, -1)$

1.3.1 M is a point with rectangular coordinates $x=y=0$, $z=1$, N is a point with cylindrical coordinates r, θ, z find out the Euclidean distance from M to N

$x \quad y \quad z$

$(0 \quad 0 \quad 1)$

cylindrical coordinates

(r, θ, z)

$z = 1$

$x_1 = 0$

$y_1 = 0$

$z_1 = 1$

$x_2 = r\cos\theta$

$y_2 = r\sin\theta$

$z_2 = Z$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(r\cos\theta - 0)^2 + (r\sin\theta - 0)^2 + (z_2 - 1)^2}$$

$$d = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + (z_2 - 1)^2}$$

$$d = \sqrt{r^2(\cos^2\theta + \sin^2\theta) + (z_2 - 1)^2}$$

$$z_1 = z_2$$

$$d = \sqrt{r^2(1) + (1-1)^2}$$

$$z = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$d = \sqrt{r^2}$$

$d = r$ only if $z_1 = z_2 = 1$, but if not, $d = \sqrt{r^2 + (z_2 - 1)^2}$

1.4.1 Describe cylindrical equation $z = r$ geometrically

$r^2 = x^2 + y^2$ $\qquad (r, \theta, z)$

Since we want $z = r$ then

$z = r = \sqrt{x^2 + y^2}$ $\rightarrow$ $(z, \theta z)$ or (r, θ, r)

then we have

$$z = \sqrt{x^2 + y^2}$$

$$z^2 = x^2 + y^2$$

we know

$x = r\cos\theta$

$y = r\sin\theta$

$z = r$ $\rightarrow$ $x = z\cos\theta$, $y = z\sin\theta$

$$z^2 = x^2 + y^2$$

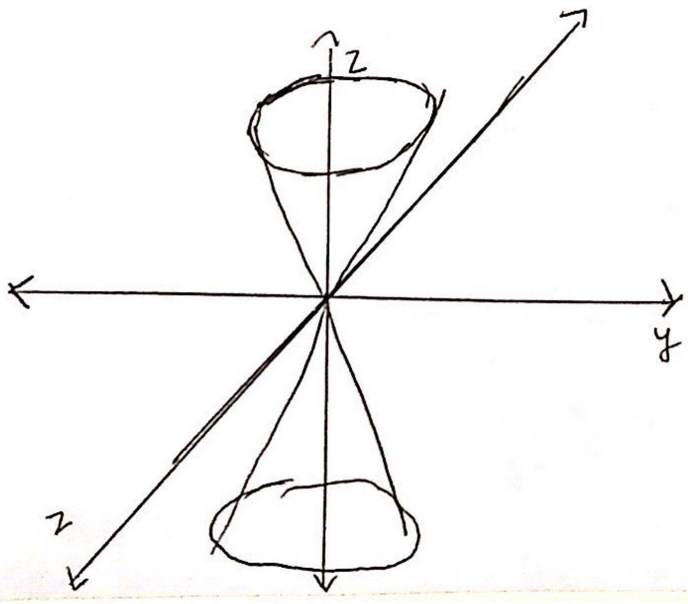
wi

if $z = 0$ then $0^2 = x^2 + y^2$

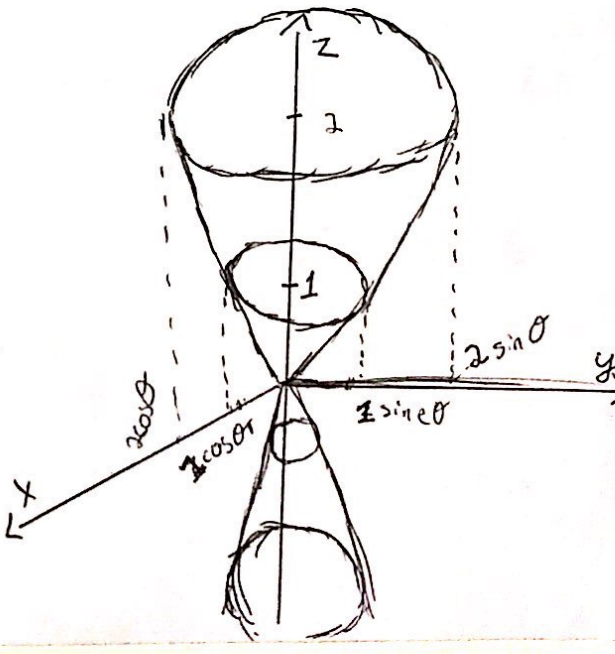
if $z = 1$ then $1^2 = x^2 + y^2$ -> formula of a circle

if $z = 2$ then $2^2 = x^2 + y^2$

Infinite cone is produced when $r = z$



$\Longrightarrow =$



2.1 Spherical coordinates

1.) convert $(3, \pi/4, \pi/4)$ and $(3, \frac{\pi}{6}, \frac{3\pi}{4})$ in spherical coordinates to rectangular coordinates, and compute the euclidean distance between this two

$\rho \quad \theta \quad \phi$

$\left(3 \quad \frac{\pi}{6} \quad \frac{\pi}{4}\right)$

$x = \rho \sin(\phi)\cos(\theta) \quad \rightarrow \quad x = 3\sin\frac{\pi}{4}\cos\frac{\pi}{6} = 1.837117307 = 3 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$

$y = \rho \sin(\phi) \sin(\theta) \qquad y = 3\sin\frac{\pi}{4}\sin\frac{\pi}{6} = 1.060660172 = 3\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$

$z = \rho\cos\phi \qquad z = \rho\cos\left(\frac{\pi}{4}\right) = 2.121320344 = 3\frac{\sqrt{2}}{2}$

$$\left(3, \frac{\pi}{6}, \frac{\pi}{4}\right) = \left(\frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{2}\right)$$

$\left(3, \frac{\pi}{6}, \frac{3\pi}{4}\right)$

$X = 3\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) = 3\,\frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} = \frac{3\sqrt{6}}{4}$

$y = 3\sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) = 3\,\frac{\sqrt{2}}{2}\,\frac{1}{2} = \frac{3\sqrt{2}}{4}$

$Z = 3\cos\left(\frac{3\pi}{4}\right) = -3\frac{\sqrt{2}}{2} = \frac{-3\sqrt{2}}{2}$

$$\left(3, \frac{\pi}{6}, \frac{\pi}{4}\right) = \left(\frac{3\sqrt{6}}{4}, \frac{3\sqrt{2}}{4}, -\frac{3\sqrt{2}}{2}\right)$$

$$d = \sqrt{\left(\frac{3\sqrt{6}}{4} - \frac{3\sqrt{6}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4} - \frac{3\sqrt{2}}{4}\right)^2 + \left(-\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\right)^2}$$

$$d = \sqrt{\left(-\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}\right)^2} = \boxed{d = 3\sqrt{2}}$$

so

$$\boxed{\begin{aligned} \left(3, \tfrac{\pi}{6}, \tfrac{\pi}{4}\right) &= \left(\tfrac{3\sqrt{6}}{4}, \tfrac{3\sqrt{2}}{4}, \tfrac{3\sqrt{2}}{2}\right) \\ \left(3, \tfrac{\pi}{6}, \tfrac{3\pi}{4}\right) &= \left(\tfrac{3\sqrt{6}}{4}, \tfrac{3\sqrt{2}}{4}, -\tfrac{3\sqrt{2}}{2}\right) \\ d &= 3\sqrt{2} \end{aligned}}$$

2.2 Convert $(1, \sqrt{3}, 2)$ in rectangular coordinates to spherical coordinates

$\rho = \sqrt{1^2 + \sqrt{3}^2 + 2^2}$

$\rho = \sqrt{1+3+4}$

$\rho = \sqrt{8}$

$\theta = \tan^{-1} \frac{\sqrt{3}}{1} \quad = \quad \theta = 60^\circ$

$\cos\varphi = \frac{z}{\rho} \quad = \quad \phi = \cos^{-1} \frac{2}{\sqrt{8}} = \cos^{-1} \frac{\sqrt{2}}{2} \quad = \quad \phi = 45^\circ$

$\boxed{= (\sqrt{8}, 60^\circ, 45^\circ)}$

2.3 Convert $(\sqrt{2}, \frac{\pi}{4}, \sqrt{6})$ in cylindrical coordinates to spherical coordinates

$(\sqrt{2}, \frac{\pi}{4}, \sqrt{6})$ cylindrical coordinate

$x = \sqrt{2} \cos \frac{\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{4}}{2} = 1$

$y = \sqrt{2} \sin \frac{\pi}{4} = \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{4}}{2} = 1$

$z = \sqrt{6}$

$(1, 1, \sqrt{6})$ rectangular coordinate

$\rho = \sqrt{1^2 + 1^2 + \sqrt{6}^2}$

$\rho = \sqrt{1+1+6}$

$\rho = \sqrt{8} \quad \rightarrow \quad \rho = 2\sqrt{2}$

$\theta = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}$

$\tan\phi = \frac{r}{z} \quad \phi = \tan^{-1} \frac{r}{z} \quad = \quad \phi = \tan^{-1} \frac{\sqrt{1^2+1^2}}{\sqrt{6}} = \tan^{-1} \frac{\sqrt{2}}{\sqrt{6}} = \tan^{-1} \frac{\sqrt{12}}{6}$

$\phi = 0.52359875$

$\boxed{\text{sperical coordinate} = (2\sqrt{2}, \frac{\pi}{4}, 0.524)}$

24 Describe spherical equation r=1 geometrically

(r, θ, z)

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$\theta = \theta$ $\qquad$ $\phi = \tan^{-1}\frac{1}{z}$

$p = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + z^2}$

or

$p = \sqrt{r^2(\cos^2\theta + \sin^2\theta) + z^2}$ $\qquad$ since $r = 1$

$\phi = \cos^{-1}\frac{z}{\sqrt{1+z^2}}$

$p = \sqrt{1+z^2}$

$\phi = \cos^{-1}\frac{z\sqrt{1+z^2}}{1+z^2}$

$p = \sqrt{1+z^2}$ $\quad$ $\theta = \theta$ $\quad$ $\phi = \cos^{-1}\frac{z\sqrt{1+z^2}}{1+z^2}$

when z increase p increase

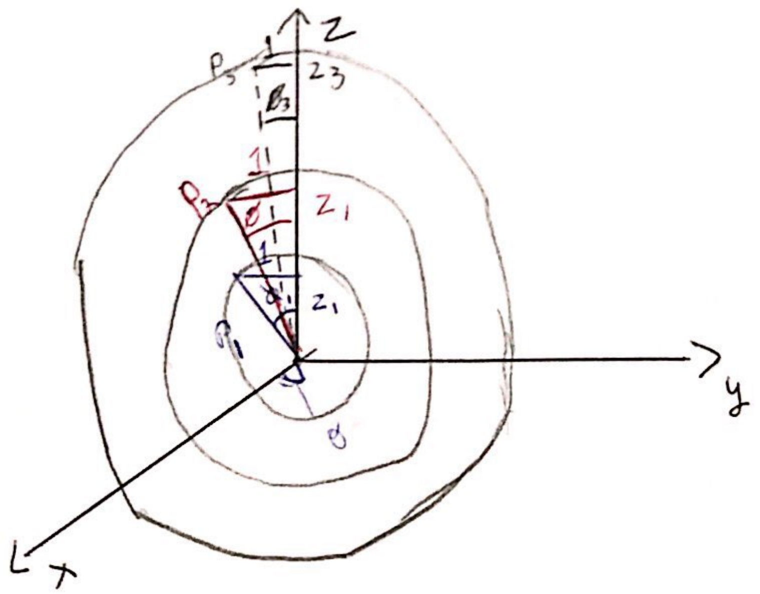
when z increase θ stays the same. $\theta = \theta$

when z increas ϕ decrease

so when z increase

$p = \uparrow$; $\theta = \theta$; $\phi = \downarrow$

graphically this means



when r=1 is allways going to be a sphere but of different shapes, when z increas, p increase and ϕ decrease

2.5 Convert spherical equation $\tan\phi \sin\theta = 1$ to rectangular form and describe the meaning

$$\tan\phi \sin\theta = 1$$

ϕ

$\tan\phi \sin\theta = 1$

$\tan\phi = \frac{1}{\sin\theta}$

$\phi = \tan^{-1}\left(\frac{1}{\sin\theta}\right)$

$\phi = \tan^{-1}(\csc\theta)$

θ

$\tan\phi \sin\theta = 1$

$\sin\theta = \frac{1}{\tan\phi}$

$\theta = \sin^{-1}\left(\frac{1}{\tan\phi}\right)$

$\theta = \sin^{-1}(\cot\phi)$

$$(\rho, \theta, \phi) \to (\rho, \sin^{-1}(\cot\phi), \tan^{-1}(\csc\theta))$$

$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$x = \rho\sin(\tan^{-1}(\csc\theta))\cos\theta \to \quad x = \rho\sin[\arctan(\csc\theta)]\cos\theta$

~~y =~~ $x = \rho\sin[\arctan(\csc\theta)]\cos[\arcsin(\cot\phi)]$

$y = \rho\sin[\arctan(\csc\theta)]\sin[\arcsin(\cot\phi)]$

$z = \rho\cos[\arctan(\csc\theta)]$

$\cos\phi = \frac{z}{\rho} \to \quad \phi = \cos^{-1}\frac{z}{\rho}$

$\cos^{-1}\frac{z}{\rho} = \tan^{-1}(\csc\theta)$

$\rho = z\sqrt{\csc^2(\theta)+1}$

so $\to$

$z = z\sqrt{\csc^2(\theta)+1}\cos[\arctan(\csc\theta)] \to 1 = \sqrt{\csc^2(\theta)+1}\cos[\arctan(\csc\theta)]$

$$z = z_1 \underbrace{\sqrt{\csc^2\theta + 1} \cdot \cos[\arctan(\csc\theta)]}_{=1}$$

So, since we know $\tan\phi \sin\theta = 1$ and $1 = \sqrt{\csc^2\theta + 1} \cdot \cos[\arctan(\csc\theta)]$ we can do this

$$\tan\phi \sin\theta = \sqrt{\csc^2\theta + 1} \cdot \cos[\arctan(\csc\theta)]$$

$$\phi = \arctan\left(\frac{\sqrt{\csc^2\theta + 1} \cdot \cos[\arctan(\csc\theta)]}{\sin\theta}\right)$$

Since now we know ϕ we can use the last formulas of

$x; y; z$ with only terms of θ

So this means that when we have the formula $\tan\phi \sin\theta = 1$ in sperical coordinates and we want to convert it to rectangular coordinates, we only need the value of θ and of z or P in order to find x, y, z.

3.) System of Forces

1.) consider $F_A = 20i - 10kj$ kN applied on point A, whose vector from origin is $r_A = 10i + 5k$ m and $F_B = -10i + 10k$ kN applied on point B, whose vector from the origin is $r_B = 20j$ m. calculate the resultant moment about the origin

$F_A = 20i - 10j$ kN

$r_A = 10i + 5k$ m

$F_B = -10i + 10k$ kN

$r_B = 20j$ m

remember:

$$\vec{M_o} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\begin{vmatrix} i & j & k \\ 10 & 0 & 5 \\ 20 & -10 & 0 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0 & 20 & 0 \\ -10 & 0 & 10 \end{vmatrix}$$

$$i[0 - 5(-10)] - j[0(10) - 5(20)] + k[-10(10) - 0(20)] + i[10(20) - 0] - j[0] + k[0 + 20(10)]$$

$$= 250i - 100j + 100k$$

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    <color rgb="250, 219, 216"/>
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     <origin rpy="0 0 0" xyz="0 1.9 0"/>
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    </visual>
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        <cylinder length="0.1" radius="0.05"/>
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       <origin rpy="0 0.1 0" xyz="0 1.7 0"/>
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  <joint name="head_to_neck" type="fixed">
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  <link name="torso">
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  <link name="leftupperarm">
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  <link name="leftlowerarm">
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```

```
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  <link name="lefthand">
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    </visual>
  </link>


  <link name="rightupperarm">
    <visual>
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       <origin rpy="0.03 0.3 0" xyz="1.5 1.59 0"/>
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  </link>

  <link name="rightlowerarm">
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      </geometry>
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  </link>

  <link name="righthand">
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        <origin rpy="0 0 0" xyz="1.5 1.04 0"/>
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        <origin rpy="-0.23 0.1 0" xyz="0.05 0.1 0"/>
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  </link>


  <joint name="neck_to_torso" type="fixed">
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    <child link="torso"/>
    <origin xyz="0 1.7 0"/>
  </joint>

  <joint name="torso_to_leftupperarm" type="fixed">
    <parent link="torso"/>
    <child link="leftupperarm"/>
    <origin xyz="-1.5 1.59 0"/>
  </joint>

  <joint name="leftupperarm_to_leftlowerarm" type="fixed">
    <parent link="leftupperarm"/>
    <child link="leftlowerarm"/>
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  <joint name="leftlowerarm_to_lefthand" type="fixed">
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    <child link="lefthand"/>
    <origin xyz="-1.5 1.04 0"/>
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  <joint name="torso_to_rightupperarm" type="fixed">
```

```
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      <child link="rightupperarm"/>
      <origin xyz="1.5 1.59 0"/>
    </joint>

    <joint name="rightupperarm_to_rightlowerarm" type="fixed">
      <parent link="rightupperarm"/>
      <child link="rightlowerarm"/>
      <origin xyz="1.5 1.29 0"/>
    </joint>

    <joint name="rightlowerarm_to_righthand" type="fixed">
      <parent link="rightlowerarm"/>
      <child link="righthand"/>
      <origin xyz="1.5 1.04 0"/>
    </joint>

    <joint name="torso_to_leftleg" type="fixed">
      <parent link="torso"/>
      <child link="leftleg"/>
      <origin xyz="-0.05 1 0"/>
    </joint>

    <joint name="leftleg_to_leftfeet" type="fixed">
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      <child link="leftfeet"/>
      <origin xyz="-0.05 0.1 0"/>
    </joint>

    <joint name="torso_to_rightleg" type="fixed">
      <parent link="torso"/>
      <child link="rightleg"/>
      <origin xyz="0.05 1 0"/>
    </joint>

    <joint name="rightleg_to_rightfeet" type="fixed">
      <parent link="rightleg"/>
      <child link="rightfeet"/>
      <origin xyz="0.5 0.1 0"/>
    </joint>


</robot>
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