(a) $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix}$

$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} x+2y & 2x+y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= x(x+2y) + y(2x+y)$

$= x^2 + 2xy + 2xy + y^2$

$= x^2 + 2xy + y^2 + 2xy$

$\cancel{x^2+2xy}$ $(x+y)^2 + 2xy \not> 0 \quad \forall_{x,y} \neq 0.$

Hence it is not positive definite.

(b) $N = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Computing

$\vec{x}^T N \vec{x}$

$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= \begin{bmatrix} 2x-y & -x+2y-z & -y+2z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$= x(2x-y) + (2y-x-z)\,y + (2z-y)\,z$$

$$= 2x^2 - xy + 2y^2 - xy - zy + 2z^2 - yz$$

$$= 2x^2 - 2xy + 2y^2 - 2zy + 2z^2$$

$$= 2(x^2 - xy + y^2 - zy + z^2)$$

$$= 2(x^2 - 2xy + y^2 + xy - zy + z^2)$$

$$= 2\{(x-y)^2 + xy - zy + z^2\}$$

$$= 2\{(x-y)^2 + xy - z(y-z)\}$$

$$= 2\left(\left(\tfrac{1}{\sqrt{2}}x\right)^2 - xy + \left(\tfrac{1}{\sqrt{2}}y\right)^2 + \left(\tfrac{1}{\sqrt{2}}y\right)^2 - zy + \left(\tfrac{1}{\sqrt{2}}z\right)^2 + \left(\tfrac{1}{\sqrt{2}}z\right)^2 + \left(\tfrac{1}{\sqrt{2}}x\right)^2\right)$$

$$= 2\left(\left(\tfrac{1}{\sqrt{2}}x - \tfrac{1}{\sqrt{2}}y\right)^2 + \left(\tfrac{1}{\sqrt{2}}y - \tfrac{1}{\sqrt{2}}z\right)^2 + \frac{z^2}{2} + \frac{x^2}{2}\right) = f(x)$$

Since all the terms are squared and are added, we can say that $\forall \vec{x} \neq \vec{0}$ $f(x) > 0$.

Hence it is positive definite

(b) $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$$\vec{\nabla} f(x,y) = \begin{pmatrix} \frac{\partial f(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} 6xy - 6x \\ 3x^2 + 3y^2 - 6y \end{pmatrix}$$

$$\begin{pmatrix} 6xy - 6x \\ 3x^2 + 3y^2 - 6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$6xy - 6x = 0$

~~$6xy - 6x$~~

$6x(y-1) = 0$

$x = 0$ or $y = 1$.

$3x^2 + 3y^2 - 6y = 0$

when $x = 0$

$3y^2 - 6y = 0$

$3y(y-2) = 0$

$y = 0$ or, $y = 2$

when $y = 1$.

$3x^2 + 3 - 6 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$.

so,

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are the critical points.

Calculating the Hessian matrix,

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x \partial x} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y \partial y} \end{bmatrix} = \begin{bmatrix} 6y - 6 & 6x \\ 6x & 6y - 6 \end{bmatrix}$$

At $(0,0)$

$$H = \begin{bmatrix} 6y-6 & 6x \\ 6x & 6y-6 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -6x & -6y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= -6x^2 - 6y^2$$

$$= -6(x^2+y^2) < 0, \ \forall \vec{x} \neq 0$$

Hence it is negative definite

$\therefore$ $(0,0)$ is local maximum.

At $(0\ 2)$

$$H = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 6x & 6y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$6(x^2+y^2) > 0, \ \forall \vec{x} \neq 0$ (Positive definite)

$(0,2)$ is local minima

At $(1\ 1)$

$$H = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 6y & 6x \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 6xy + 6xy$$

$$= 12xy$$

At $(1\ -1)$

$$H = \begin{bmatrix} -12 & 6 \\ 6 & -12 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -12 & 6 \\ 6 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} -12x + 6y & 6x - 12y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= -12x^2 + 6yx + 6xy - 12y^2$$

$$= -12x^2 + 12xy - 12y^2$$

$$= -12(x^2 + xy + y^2)$$

$$= -12\left[\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 x^2 + \frac{1}{2}y^2\right] < 0, \ \forall xy \neq 0$$

Negative definite.

Hence local maxima.

Ex 2

$$T = \{((1,1)^T, 2), ((1,2)^T, 3), ((2,2)^T, 3), ((2,4)^T, 4)\}.$$

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}, \quad y = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

we further evaluate

$$\therefore X^T X = \cancel{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & 9 \\ 6 & 10 & 15 \\ 9 & 13 & 25 \end{bmatrix}$$

$$X^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 19 \\ 30 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & 9 \\ 6 & 10 & 15 \\ 9 & 13 & 25 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 19 \\ 30 \end{bmatrix}$$

$4\beta_1 + 6\beta_2 + 9\beta_3 = 12$

$6\beta_1 + 10\beta_2 + 15\beta_3 = 19$

$9\beta_1 + 13\beta_2 + 25\beta_3 = 30.$

$$\begin{aligned} 12\beta_1 + 18\beta_2 + 27\beta_3 &= 36 \\ 12\beta_1 + 20\beta_2 + 30\beta_3 &= 38 \\ - \quad - \quad - \quad & - \\ \hline -2\beta_2 - 3\beta_3 &= -2 \end{aligned}$$

$2\beta_2 + 3\beta_3 = 2$

$$\begin{aligned} 18\beta_1 + 30\beta_2 + 45\beta_3 &= 57 \\ 18\beta_1 + 26\beta_2 + 50\beta_3 &= 60 \\ - \quad - \quad - \quad & - \\ \hline 4\beta_2 - 5\beta_3 &= -3 \end{aligned}$$

$\cancel{2\beta_2 + 3\beta_3 = 2}$

$\cancel{4\beta}$

$$\begin{aligned} 4\beta_2 + 6\beta_3 &= 4 \\ 4\beta_2 - 5\beta_3 &= -3 \\ \hline - \quad + \quad & + \\ \hline 11\beta_3 &= 7 \end{aligned}$$

$\beta_3 = 7/11$

$2\beta_2 + 3 \times \frac{7}{11} = 2$

$\beta_2 = \frac{1}{22}$

$4\beta_1 + 6 \times \frac{1}{22} + 9 \times \frac{7}{11} = 12$

$\beta_1 = 3/2.$

$\beta_2 = 1/22$

$\beta_3 = 7/11$ //

Predictor

$$f(x) = \frac{3}{2} + \frac{1}{22}x_1 + \frac{7}{11}x_2$$

At $\vec{x} = (1.5, 1.5)$

$$f(x) = \frac{3}{2} + \frac{1}{22} \times \frac{3}{2} + \frac{7}{11} \times \frac{3}{2}$$

$$= 2.522$$

Ex 3