ME 282, Jan-April 2020
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Assignment 3, Due Thursday, March 12

1. Consider the leap frog scheme for numerical integration of the ODE

$$\frac{dy}{dt} = f(y,t), \; y(0) = y_0. \tag{1}$$

Analyze the stability of this scheme and express it in the form of a stability diagram.

Now consider application of leap frog method with central in space discretization of

$$u_t = bu_{xx}, u(0,t) = u(1,t) = 0, \; u(x,0) = u_0(x) \tag{2}$$

constructed using uniform grid points in [0,1] and the standard three-point, second order, discretization to approximate u_{xx} (let $x_i = i/N, i = 0, 1, \ldots, N$). Derive a stability criterion for this method using the stability diagram obtained from the first part.

2. Determine the order of accuracy and discuss the stability of the following difference scheme

$$\left(1 + \frac{1}{12}\delta_x^{(2)}\right)\left(u_j^{n+1} - u_j^n\right) = \frac{b\Delta t}{2h^2}\delta_x^{(2)}\left(u_j^{n+1} + u_j^n\right) + \frac{\Delta t}{2}\left[f_j^{n+1} + \left(1 + \frac{1}{6}\delta_x^{(2)}\right)f_j^n\right] \tag{3}$$

for the solution of $u_t = bu_{xx} + f$, where $\delta_x^{(2)}u_j^n = u_{j+1}^n - 2u_j^n + u_{j-1}^n$.

3. Consider the DuFort-Frankel scheme for the diffusion equation $u_t = bu_{xx}$, where $b > 0$, on a uniform mesh,

$$u_j^{n+1} = u_j^{n-1} + \frac{2b\Delta t}{h^2}\left(u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n\right). \tag{4}$$

Determine the restrictions on time step Δt and grid spacing h required to ensure convergence.

4. Write a computer code to solve

$$\begin{aligned} &u_t = \nabla^2 u + \sin(2\pi x)\sin(2\pi y)\sin(2\pi t) \quad \text{for} \quad (x,y) \in (0,1)^2, \\ &u(x,0,t) = u(x,1,t) = 0, \\ &u(0,y,t) = u(1,y,t) = 0, \end{aligned} \tag{5}$$

using the Crank-Nicolson method. Use the matrix diagonalization method discussed in the class to solve the resulting linear system of equations. Plot the L_2 and L_∞ errors as a function of number of mesh nodes N ($h_x = h_y = 1/N$) and the time step size Δt. What rate of convergence do you observe in space and in time?