DS 289: Numerical Solution of Differential Equations

Assignment 2

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Due date: 10 April 2022
Total points: 100

Please follow the below instructions in preparing the solutions:

1. Provide solutions in the same order as questions.
2. All the codes should be in C/C++/Fortran. Use Matlab or Python for plotting graphs.
3. The report should be in a PDF format with the necessary steps, plots, explanations and discussions.
4. Compile all the solutions, including graphs, into a single PDF file.
5. Create a separate folder for each question that involves a code. Provide the code, input and output files, which are used to obtain the solution, in the folder.
6. For submission, create a single ZIP file that includes the code folders and the report. Name the ZIP file as *DS289_A1_firstname_lastname.zip* (your first and last names) and upload into MS Teams assignment portal.
7. All the codes will be scrutinizied for plagirism. Do not copy the codes from others or internet.
8. **Late submission penalty is 10 points per day.**

Questions

1. Consider a rectangular plate $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}$ with the heat conduction equation
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$
The boundary conditions are: $T(0, y) = 30$, $T(1, y) = 60$, $\partial T/\partial y(x, 0) = 0$, and $\partial T/\partial y(x, 2) = (T(x, 2) - 60)$. (a) Approximate the equation using second order central difference and obtain the algebraic equation. Use a 128×256 grid. (b) Write the structure of the coefficient matrix. (c) Solve the algebraic system using a direct solver. Standard linear solver libraries can be used. (d) Plot contours of the solution. **(20 points)**

2. Show that a two level scheme with $\theta \geq 0.5$ (as described in class) for time derivative and a second order central difference for space derivative results in an unconditionally stable method for diffusion equation. **(10 points)**

3. (a) Use the explicit Euler and second order central difference schemes to discretise the diffusion equation ($u_t = \alpha u_{xx}$), and obtain the modified equation. (b) Comment on the dissipative and dispersive errors. **(10 points)**

4. (a) Derive the analytical solution for the equation in question 3 (use $\alpha = 1$). The initial condition is $u(x, 0) = sin(2x)$. The domain size is 2π with a periodic boundary condition.

   (b) Perform numerical experiments with a constant r_d of $1/2$ and $1/6(= 0.166667)$, and grid sizes $N = \{32, 64, 128, 256\}$. Here, r_d is the stability parameter. In each simulation, evolve the solution to an end time $t_{end} = 0.4$.

   (c) Use the analytical solution to compute the average of absolute error ($E(N)$) in each simulation at $t = t_{end}$.

   (d) In a graph with logarithmic scale, plot N vs $E(N)$ for $r_d = 1/2$ and $1/6$. Obtain the order of accuracy for the two r_d cases.

   (e) Explain the abnormal order of accuracy observed in $r_d = 1/6$ case. Hint: analyse the modified equation. **(25 points)**

5. (a) Perform numerical experiments with implicit Euler scheme. Use the same equation and parameters provided in questions 3 and 4. Solver the linear system at each time step using the Jacobi method (tolerence $= 10^{-4}$; do not use any libraries).

   (b) In a N vs $E(N)$ graph, compare the errors with explicit method for $r_d = 1/2$. Provide an explanation for your observations. **(20 points)**

6. The transient 1D heat conduction problem is modelled as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

   where $T(x,t)$ is the temperature, and α (a constant) is the thermal diffusivity. This equation is approximated using

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^{n-k} - 2T_i^n + T_{i-1}^{n-k}}{\Delta x^2}$$

   where $T_i^n = T(x_i, t_n)$, Δx is the grid spacing, Δt is the time step, and k is a constant ($\neq 0$). (a) Find the expression for truncation error. (b) What is the order of accuracy? (c) Is this a consistent finite difference approximation? **(15 points)**

Note: Do not hesitate of contact me if you any questions or doubts.