DS 289: Numerical Solution of Differential Equations

Assignment 1

Instuctor: Konduri Aditya
TA: Shubham Goswami

Due date: 27 Feb 2022
Total points: 100

Please follow the below instructions in preparing the solutions:

1. Provide solutions in the same order as questions.
2. All the codes should be in C/C++/Fortran. Use Matlab or Python for plotting graphs.
3. The report should be in a PDF format with the necessary steps, plots, explanations and discussions.
4. Compile all the solutions, including graphs, into a single PDF file.
5. Create a separate folder for each question that involves a code. Provide the code, input and output files, which are used to obtain the solution, in the folder.
6. For submission, create a single ZIP file that includes the code folders and the report. Name the ZIP file as *DS289_A1_firstname_lastname.zip* (your first and last names) and upload into MS Teams assignment portal.
7. All the codes will be scrutinizied for plagirism. Do not copy the codes from others or internet.

Questions

1. (a) For a function $u(x)$, derive a second order accurate finite difference approximation to compute du/dx at a point i on a non-uniform grid. The scheme should be centered around i. **(10 points)**
(b) Derive a second order accurate finite difference approximation to compute the cross derivative $\partial^2 u/\partial x \partial y$ of a function $u(x, y)$ on a uniform grid. Obtain the leading order terms in the truncation error.

2. Consider the initial value problem over the interval $t = 0$ to 20 where $y(0) = 1$. **(30 points)**

$$\frac{dy}{dt} = yt^2 - 1.1y \tag{1}$$

Solve this ODE:

(a) analytically and plot the solution.
(b) Obtain the numerical solution for the step sizes = 0.5, 0.25, 0.125, 0.0625 and 0.03125 using:
   i. Euler's explicit method.
   ii. Second-order Adams-Bashforth.
   iii. Fourth-order RK method.
(c) Get the maximum and average errors for each step size using the analytical solution and obtain the error graph demonstrating the order of accuracy for each method. Note: use separate graphs for maximum and average error to compare all the three methods.
(d) Discuss the results.

3. The motion of a damped spring-mass system (Fig. 1) is described by the following ordinary differential equation: **(20 points)**

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{2}$$

where x = displacement from equilibrium position (m), t = time (s), m = 20-kg mass, and c = the damping coefficient (N . s/m). The damping coefficient c takes on three values of 5 (under- damped), 40 (critically

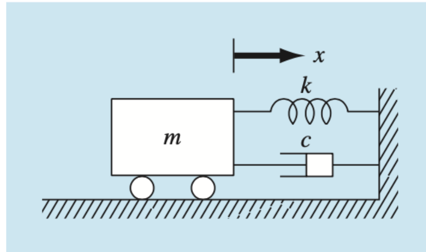

Figure 1

damped), and 200 (overdamped). The spring constant $k = 20$ N/m. The initial velocity is zero, and the initial displacement $x = 1$ m. Solve this equation using RK4 method over the time period $0 \leq t \leq 15s$. With a graph show the convergence of results for reducing step size (use $c = 5$). Plot the displacement versus time for each of the three values of the damping coefficient on the same curve. Discuss the results.

4. Given **(20 points)**

$$\frac{dy}{dx} = -2,00,000y + 2,00,000e^{-x} - e^{-x} \tag{3}$$

(a) Estimate the step-size required to maintain stability using the explicit Euler method. (b) If $y(0) = 0$, use the implicit Euler to obtain a solution from $x = 0$ to 2 using a step size of 0.1. Plot the solution.

5. Solve the ODE using finite difference methods that describe the temperature distribution in a circular rod with internal heat source S **(20 points)**

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + S = 0 \tag{4}$$

over the range $0 \leq r \leq 1$, with the boundary conditions $T(r = 1) = 1$ and dT/dr at $(r = 0) = 0$.

(a) Solve this equation using the second order central difference scheme and find temperature distribution along the radial direction for S = 100, 500, 1000, 1500. Use a grid size of 1024 points.

(b) Plot the temperature distribution for each case and discuss the results.

(c) Compare these plots for different S and find maximum S such that the peak temperature in the domain does not exceed 100.