ME 257: Problem set 3

Rules: Work out yourself. No handwaving.
Post on piazza if you have questions. Have fun.

1. Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, where $x_1 = 1/6$ and $x_2 = 1/2$ be a partition of the interval $I = [0, 1]$ into three subintervals. Let $V_{h,0}$ be the space of continuous piecewise linear functions that vanish at the end points $x = 0, 1$ for the model problem $u'' + 1 = 0$ with homogeneous boundary conditions.
   (i) Compute the stiffness matrix K.
   (ii) Compute the load vector F.
   (iii) Solve the linear system for the degrees of freedom.
   (iv) Plot the computed and exact solutions.

2. Compute the stiffness matrix for the problem

$$\begin{aligned} u'' + f &= 0 \qquad \text{on } x \in [0, 1] \\ u'(0) = u'(1) &= 0 \end{aligned}$$

   on a uniform mesh consisting of two elements. Is the matrix invertible? Why/Why not?

3. Consider the problem

$$-u'' + u = f, \qquad x \in [0, 1]$$

   satisfying homogeneous boundary conditions at both ends.
   (i) Choose a suitable finite element space V_h.
   (ii) Formulate the Galerkin approximation.
   (iii) Derive the discrete system of equations.

4. We have seen that the weak form for a model elliptic problem has the form:

$$\text{Find } u \in V_0 \text{ such that } a(u, v) = F(v) \text{ for all } v \in V_0,$$

   where $a(\cdot, \cdot)$ and $F(\cdot)$ are bilinear and linear forms, respectively. Here, we will further assume that $a(\cdot, \cdot)$ is symmetric, i.e., $a(v, w) = a(w, v)$.

   It turns out that the solution to the weak form can also be identified by minimizing the functional

$$\Pi[w] = \frac{1}{2} a(w, w) - F(w).$$

   Prove that if u is a solution to the weak problem, then necessarily, $\Pi[u] \leq \Pi[v]$ for all v in V_0. Clearly identify where in your proof you used linearity of $a(\cdot, \cdot)$ and $F(\cdot)$, and symmetry of $a(\cdot, \cdot)$.

   Hint: Try showing that $\Pi[u + v] \geq \Pi[u]$. Then argue why you have proved the required result.