Computational study of far-field acoustic emission by collapsing bubbles

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1 Kirchhoff formulation

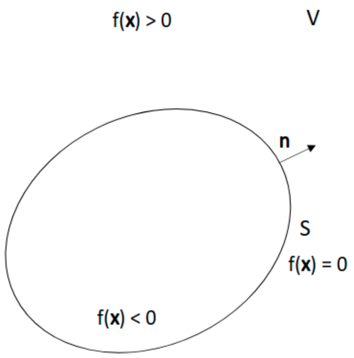

Figure 1: Stationary Kirchhoff surface S encloses sound source

In this section, we derive the Kirchhoff formula for a stationary control surface (Farassat et al. 1988). We chose a control surface S that encloses all the acoustic sources (1), and the pressure perturbations p satisfies the homogeneous wave equation

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p = 0 \qquad \text{in } V. \tag{1}$$

The control surface S is defined by $f(\mathbf{x}) = 0$, $f(\mathbf{x}) > 0$ for $\mathbf{x}$ in V and $f(\mathbf{x}) < 0$ for $\mathbf{x}$ inside surface S. We scale the function f such that $\nabla f = \mathbf{n}$. Then the Heaviside function of $f(\mathbf{x})$ is

$$H(f) = \begin{cases} 1, & \text{for } \mathbf{x} \text{ in V.} \\ 0, & \text{for } \mathbf{x} \text{ inside S.} \end{cases} \tag{2}$$

The gradient of the Heaviside function is given by

$$\nabla H(f) = \delta(f)\mathbf{n}. \tag{3}$$

We define the pressure p as a generalized function $pH(f)$ (Ffowcs Williams et al. 1969) where

$$pH(f) = \begin{cases} p, & \text{for } \mathbf{x} \text{ in V.} \\ 0, & \text{for } \mathbf{x} \text{ inside S.} \end{cases} \tag{4}$$

The generalized pressure $pH(f)$ is defined everywhere in space, unlike p defined only in V. We will derive the acoustic wave equation for the generalised pressure. Using (3), the gradient of pH is

$$\nabla(pH) = \nabla pH + p\delta(f)\mathbf{n}. \tag{5}$$

Therefor the Laplacian is given by

$$\nabla^2(pH) = \nabla^2 pH + \frac{\partial p}{\partial n}\delta(f) + \nabla.(p\delta(f)\mathbf{n}). \tag{6}$$

The partial derivative in time is

$$\frac{\partial^2}{\partial t^2}(pH) = \frac{\partial^2 p}{\partial t^2}H. \tag{7}$$

We premultiply (7) with $1/c_0^2$ and subtract (6) to obtain the acoustic wave equation in generalised pressure

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)pH = H\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p - \frac{\partial p}{\partial n}\delta(f) - \nabla.(p\delta(f)\mathbf{n}), \tag{8}$$

or

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)pH = -\frac{\partial p}{\partial n}\delta(f) - \nabla.(p\mathbf{n}\delta(f)). \tag{9}$$

The right side of the equation (9) is non-zero only at surface S, as it contains $\delta(f)$. The acoustic wave equation (9) in generalized variables is valid in the entire unbounded space. Therefore we can use free-space Green's function to solve the equation. The Green's function is the solution of wave equation for an impulsive point source $\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$ placed at point $\mathbf{y}$ and time τ

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)G(\mathbf{x}, t; \mathbf{y}, \tau) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau). \tag{10}$$

The Green's function for the acoustic wave operator (Howe 2003) in three dimensions is

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\delta\left(t - \tau - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{11}$$

The solution for arbirtary source can be obtained by multiplying $s(\mathbf{y}, \tau)$ in (10)

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)s(\mathbf{y}, \tau)G(\mathbf{x}, t; \mathbf{y}, \tau) = s(\mathbf{y}, \tau)\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau), \tag{12}$$

Integrating both sides

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\int s(\mathbf{y}, \tau)G(\mathbf{x}, t; \mathbf{y}, \tau)d\mathbf{y}d\tau = \int s(\mathbf{y}, \tau)\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)d\mathbf{y}d\tau, \tag{13}$$

and using the properties of delta function, we get the solution for the acoustic wave equation with source $s(\mathbf{x},t)$

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p(\mathbf{x},t) = s(\mathbf{x},t), \tag{14}$$

where,

$$p(\mathbf{x},t) = \int s(\mathbf{y},\tau)G(\mathbf{x},t;\mathbf{y},\tau)d\mathbf{y}d\tau. \tag{15}$$

We can use the above relation to solve the acoustic wave equation (9)

$$\begin{aligned}(pH)(\mathbf{x},t) =& -\frac{1}{4\pi}\int \frac{\partial p}{\partial n}\delta(f)\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau \\ &-\frac{1}{4\pi}\int \nabla_{\mathbf{y}}.(p\mathbf{n}\delta(f))\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau.\end{aligned} \tag{16}$$

$$\begin{aligned}(pH)(\mathbf{x},t) =& -\frac{1}{4\pi}\int \frac{\partial p}{\partial n}\delta(f)\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau \\ &-\frac{1}{4\pi}\nabla_{\mathbf{x}}.\int (p\mathbf{n}\delta(f))\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau.\end{aligned} \tag{17}$$

We use the following property to convert volume integral to surface integral (Farassat et al. 1988).

$$\int \phi(\mathbf{y})\delta(f)\mathbf{n}d\mathbf{y} = \int_S \phi(\mathbf{y})\mathbf{n}dS \tag{18}$$

We use the above property to convert volume integral to surface integral on S

$$\begin{aligned}(pH)(\mathbf{x},t) =& -\frac{1}{4\pi}\int \frac{\partial p}{\partial n}\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}dSd\tau \\ &-\frac{1}{4\pi}\nabla_{\mathbf{x}}.\int (p\mathbf{n})\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}dSd\tau.\end{aligned} \tag{19}$$

Using the property of delta function we obtain

$$\begin{aligned}(pH)(\mathbf{x},t) =& -\frac{1}{4\pi}\int_S \left[\frac{\partial p}{\partial n}\right]\frac{dS}{|\mathbf{x}-\mathbf{y}|} \\ &-\frac{1}{4\pi}\nabla_{\mathbf{x}}.\int_S [p]\mathbf{n}\frac{dS}{|\mathbf{x}-\mathbf{y}|}.\end{aligned} \tag{20}$$

The square bracket implies the functions are computed at the retarded time i.e, $[p] = p(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{c})$. Simplifying the equation further, we obtain the **Kirchhoff formula** for a stationary control surface (Farassat et al. 1988, Jamaluddin et al. 2011).

$$(pH)(\mathbf{x},t) = -\frac{1}{4\pi}\int_S \left[\frac{p}{r^2}\frac{\partial r}{\partial n} - \frac{1}{r}\frac{\partial p}{\partial n} + \frac{1}{cr}\frac{\partial r}{\partial n}\frac{\partial p}{\partial \tau}\right]_\tau dS. \tag{21}$$

Where, $r = |\mathbf{x}-\mathbf{y}|$, the square bracket again implies the functions are computed at the retarded time $\tau = t - r/c$.

2 Ffowcs William–Hawkings formulation

Unlike the Kirchhoff equation the FW-H formulation is derived directly from the conservation of mass and momentum. Therefore, it can be applied to an arbitrary surface whether or not the disturbance propagation is linear outside the control surface S. In this section, we derive the FW-H equation for a stationary control surface S from the mass and momentum conservation equation. The mass and momentum conservation equations are written using the Lighthill's analogy (Lighthill 1952)

$$\frac{\partial(\rho-\rho_0)}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}=0, \tag{22}$$

$$\frac{\partial(\rho u_i)}{\partial t}+c_0^2\frac{\partial(\rho-\rho_0)}{\partial x_i}=-\frac{\partial T_{ij}}{\partial x_j}. \tag{23}$$

where

$$T_{ij}=\rho u_i u_j+p_{ij}-c_0^2(\rho-\rho_0)\delta_{ij}, \tag{24}$$

is the Lighthill stress tensor and $p_{ij}=(p-p_0)\delta_{ij}-\sigma_{ij}$ is the compressive stress tensor. Using the Heaviside function we can write down the state variables ρ, p and $\mathbf{u}$ as generalised functions $\rho H(f)$, $pH(f)$ and $\mathbf{u}H(f)$.

$$\rho H(f)=\begin{cases}\rho, & \text{for } \mathbf{x} \text{ in V.}\\ 0, & \text{otherwise.}\end{cases} \tag{25}$$

We multiply the Heaviside function (2) with equations (22) and (23) to obtain the mass and momentum conservation equations in the generalised state variables

$$H\frac{\partial(\rho-\rho_0)}{\partial t}+H\frac{\partial(\rho u_i)}{\partial x_i}=0, \tag{26}$$

$$H\frac{\partial(\rho u_i)}{\partial t}+Hc_0^2\frac{\partial(\rho-\rho_0)}{\partial x_i}=-H\frac{\partial T_{ij}}{\partial x_j}, \tag{27}$$

Using the product rule we obtain the generalized mass and momentum conservation equation

$$\frac{\partial H(\rho-\rho_0)}{\partial t}+\frac{\partial(H\rho u_i)}{\partial x_i}=\rho u_i\frac{\partial H}{\partial x_i}. \tag{28}$$

$$\frac{\partial(\rho u_i H)}{\partial t}+\frac{\partial(Hc_0^2(\rho-\rho_0))}{\partial x_i}=-\frac{\partial(HT_{ij})}{\partial x_j}+(\rho u_i u_j+(p-p_0)\delta_{ij}-\sigma_{ij})\frac{\partial H}{\partial x_j}. \tag{29}$$

Taking the time derivative of mass conservation equation

$$\frac{\partial^2 H(\rho-\rho_0)}{\partial^2 t}+\frac{\partial^2(H\rho u_i)}{\partial x_i\partial t}=\frac{\partial}{\partial t}\left(\rho u_i\frac{\partial H}{\partial x_i}\right), \tag{30}$$

and the spatial derivative of momentum conservation equation

$$\begin{aligned}\frac{\partial^2(H\rho u_i)}{\partial x_i\partial t}+\frac{\partial^2(Hc_0^2(\rho-\rho_0))}{\partial^2 x_i}=&-\frac{\partial^2(HT_{ij})}{\partial x_i\partial x_j}+\\ &\frac{\partial}{\partial x_i}\Big((\rho u_i u_j+(p-p_0)\delta_{ij}-\sigma_{ij})\frac{\partial H}{\partial x_j}\Big).\end{aligned} \tag{31}$$

And subtracting the two equations we obtain a wave equation with source terms,

$$
\begin{aligned}
&\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)[Hc_0^2(\rho-\rho_0)] \\
&=\frac{\partial^2(HT_{ij})}{\partial x_i\partial x_j}-\frac{\partial}{\partial x_i}\Big((\rho u_i u_j+(p-p_0)\delta_{ij}-\sigma_{ij})\frac{\partial H}{\partial x_j}\Big)+\frac{\partial}{\partial t}\left(\rho u_i\frac{\partial H}{\partial x_i}\right)
\end{aligned}
\tag{32}
$$

We can use the Green's function technique 15 to obtain the FW-H integral equation

$$
\begin{aligned}
Hc_0^2(\rho-\rho_0)=&\frac{1}{4\pi}\frac{\partial^2}{\partial x_i\partial x_j}\int_V\frac{[T_{ij}]}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y} \\
&-\frac{1}{4\pi}\frac{\partial}{\partial x_i}\oint_S[\rho u_i u_j+(p-p_0)\delta_{ij}-\sigma_{ij}]\frac{ds_j}{|\mathbf{x}-\mathbf{y}|} \\
&+\frac{1}{4\pi}\frac{\partial}{\partial t}\oint_S[\rho u_j]\frac{ds_j}{|\mathbf{x}-\mathbf{y}|}.
\end{aligned}
\tag{33}
$$

3 Validation of Kirchhoff solver

3.1 Acoustic monopole

We compute the sound waves emitted by a monopole source using the Kirchhoff solver. The acoustic wave equation for a monopole source placed at a point $\mathbf{x}=0$ is

$$
\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)p(\mathbf{x},t)=-q(t)\delta(\mathbf{x}),
\tag{34}
$$

where $q(t)$ is the time dependent source function. The solution can be obtained by substituting the free space Green's function (11) in (15)

$$
p(\mathbf{x},t)=\int s(\mathbf{y},\tau)\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\delta\left(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)d\mathbf{y}d\tau,
\tag{35}
$$

and using the property of delta function we obtain,

$$
p(\mathbf{x},t)=\frac{1}{4\pi}\int\frac{s\Big(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}.
\tag{36}
$$

The pressure at any point $\mathbf{x}$ and time t is a linear superposition of contributions from all the sources located at $\mathbf{y}$, which radiated at the earliest times $t-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}$. The integral formula (36) is called the retarded potential. Substituting $s(\mathbf{y},\tau)=-q(t)\delta(\mathbf{x})$ in the above equation we obtain

$$
p(\mathbf{x},t)=-\frac{1}{4\pi}\int\frac{\delta(\mathbf{y})q(t-\frac{|\mathbf{x}-\mathbf{y}|}{c_0})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y},
\tag{37}
$$

using the property of delta function, the pressure radiated by a monopole source is given by

$$
p(\mathbf{x},t)=-\frac{1}{4\pi}\frac{q(t-\frac{r}{c_0})}{r}.
\tag{38}
$$

where $r = |\mathbf{x}|$. We chose the monopole of strength

$$q(t) = 2(t - t0)f_0^2 \exp(-f_0^2(t - t_0)^2). \tag{39}$$

where $f_0 = 100$ is the dominant frequency and $t_0 = \frac{4}{f_0}$. We enclose the monopole

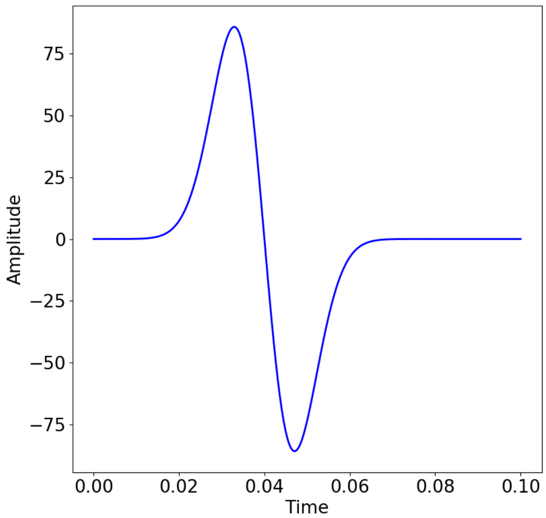

Figure 2: Monopole source as a function of time

source using a cuboidal Kirchhoff surface whose diagonally opposite points are $p_1 = (-1.0, -1.0, -1.0)$ and $p_2 = (1.0, 1.0, 1.0)$. The surface is embedded in a cuboidal domain of size $[-5.0, 5.0] \times [-5.0, 5.0] \times [-5.0, 5.0]$. The domain is discretized into structured grid of size $h = 0.1$ and the Kirchhoff surface is discretized into square cells of size $h = 0.1$. The pressure and its derivatives are interpolated from cell centers to quadrature points on Kirchhoff surface using fourth-order WENO polynomial. The Kirchhoff Integral (21) is computed using the two-point Gauss quadrature formula. The exact and numerical pressure are evaluated at observer point $xo = (3.0, 3.0, 3.0)$ and plotted against the function of time.

3.2 Kirchhoff solver coupled with compressible Euler equation solver

The goal of the test case is to validate the Kirchhoff solver by comparing the acoustic pressure computed using the Kirchhoff method and the flow solver at

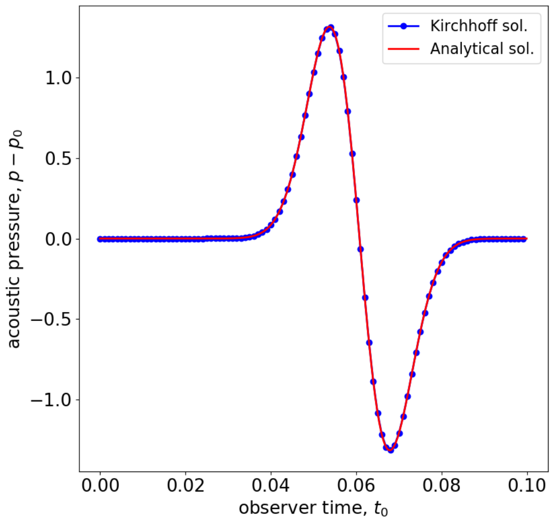

Figure 3: The exact and numerical pressure as a function of time at observer point $xo = (3.0, 3.0, 3.0)$.

the observer point. We solve the compressible Euler equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0, \tag{40}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla.(\rho \mathbf{u}\mathbf{u}) + \nabla P = 0, \tag{41}$$

$$\frac{\partial E}{\partial t} + \nabla.((E + P)\mathbf{u}) = 0, \tag{42}$$

$$p = (\gamma - 1)(E - \frac{1}{2}\rho \mathbf{u}^2). \tag{43}$$

for an initial condition

$$\rho = \rho_0 + \rho',$$
$$\mathbf{u} = 0,$$
$$p = p_0 + c_0^2 \rho'.$$

Where,

$$\rho' = \begin{cases} A \exp(-30r^3), & \text{if } r \leq .125 \\ 0, & \text{otherwise.} \end{cases}$$

A = 0.01 is the amplitude of density perturbation and r is the distance measured from center of the domain. The ratio of the specific heat of the gas is $\gamma = 1.4$. The mean density and pressure of the gas are $\rho_0 = 1.0$ and $p_0 = 1.0$. Then the speed of sound is given by $c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}} = 1.18321$. We apply the transmissive boundary conditions in all the boundaries. The compressible Euler equation is discretized using the finite-volume method. We use the fourth-order WENO polynomial for spatial reconstruction and SSPRK54 for temporal discretization. The flux is computed using LLF Riemann solver. We discretize a domain of size $[0, 1] \times [0, 1] \times [0, 1]$ using $200 \times 200 \times 200$ cells. A CFL number of 0.9 is chosen for time discretization. The simulation is carried out for time $T = 1.0$.

We create a Kirchhoff box surface whose diagonally opposite points are $p_1 = (0.195, 0.195, 0.195)$ and $p_2 = (0.8, 0.8, 0.8)$. The surface is discretized using the square cells of size $h = 0.005$. We use fourth-order WENO polynomial to interpolate pressure data from cell centers to quadrature points on Kirchhoff surface. We use linear interpolation in time to compute pressure at emission time. The Kirchhoff Integral is computed using the two-point Gauss quadrature formula. The acoustic pressure $p' = p - p_0$ is computed at observer point $xo = (0.8975, 0.8975, 0.8975)$ and the observer time ranges from $t_0 = [0, 1.0]$ with time-step $dt = 0.01$.

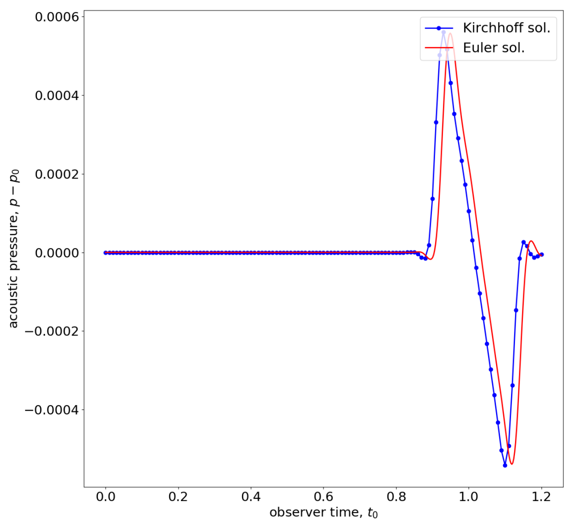

Figure 4: The acoustic pressure $p' = p - p_0$ is computed at observer point x_0 using Euler and Kirchhoff solver and plotted against observer time t_0.

References

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