Implementation of Kirchhoff acoustic solver

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November 12, 2021

1 Mathematical formulation

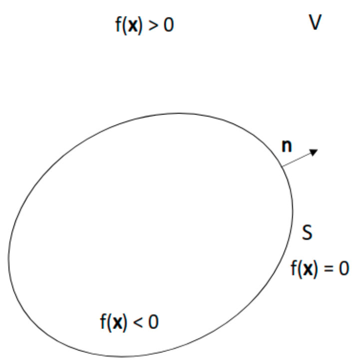

Figure 1: Stationary Kirchhoff surface S encloses sound source

In this section, we derive the Kirchhoff formula for a stationary control surface (Farassat et al. 1988). We chose a control surface S that encloses all the acoustic sources (1), and the pressure perturbations p satisfies the homogeneous wave equation

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p = 0 \qquad \text{in } V. \tag{1}$$

The control surface S is defined by $f(\mathbf{x}) = 0$, $f(\mathbf{x}) > 0$ for $\mathbf{x}$ in V and $f(\mathbf{x}) < 0$ for $\mathbf{x}$ inside surface S. We scale the function f such that $\nabla f = \mathbf{n}$. Then the Heaviside function of $f(\mathbf{x})$ is

$$H(f) = \begin{cases} 1, & \text{for } \mathbf{x} \text{ in V.} \\ 0, & \text{for } \mathbf{x} \text{ inside S.} \end{cases} \tag{2}$$

The gradient of the Heaviside function is given by

$$\nabla H(f) = \delta(f)\mathbf{n}. \tag{3}$$

We define the pressure p as a generalized function $pH(f)$ (Ffowcs Williams et al. 1969) where

$$pH(f) = \begin{cases} p, & \text{for } \mathbf{x} \text{ in V.} \\ 0, & \text{for } \mathbf{x} \text{ inside S.} \end{cases} \tag{4}$$

The generalized pressure $pH(f)$ is defined everywhere in space, unlike p defined only in V. We will derive the acoustic wave equation for the generalised pressure. Using (3), the gradient of pH is

$$\nabla(pH) = \nabla pH + p\delta(f)\mathbf{n}. \tag{5}$$

Therefor the Laplacian is given by

$$\nabla^2(pH) = \nabla^2 pH + \frac{\partial p}{\partial n}\delta(f) + \nabla.(p\delta(f)\mathbf{n}). \tag{6}$$

The partial derivative in time is

$$\frac{\partial^2}{\partial t^2}(pH) = \frac{\partial^2 p}{\partial t^2}H. \tag{7}$$

We premultiply (7) with $1/c_0^2$ and subtract (6) to obtain the acoustic wave equation in generalised pressure

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)pH = H\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p - \frac{\partial p}{\partial n}\delta(f) - \nabla.(p\delta(f)\mathbf{n}), \tag{8}$$

or

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)pH = -\frac{\partial p}{\partial n}\delta(f) - \nabla.(p\mathbf{n}\delta(f)). \tag{9}$$

The right side of the equation (9) is non-zero only at surface S, as it contains $\delta(f)$. The acoustic wave equation (9) in generalized variables is valid in the entire unbounded space. Therefore we can use free-space Green's function to solve the equation. The Green's function is the solution of wave equation for an impulsive point source $\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$ placed at point $\mathbf{y}$ and time τ

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)G(\mathbf{x},t;\mathbf{y},\tau) = \delta(\mathbf{x}-\mathbf{y})\delta(t-\tau). \tag{10}$$

The Green's function for the acoustic wave operator (Howe 2003) in three dimensions is

$$G(\mathbf{x},t;\mathbf{y},\tau) = \frac{\delta\left(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{11}$$

The solution for arbirtary source can be obtained by multiplying $s(\mathbf{y},\tau)$ in (10)

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)s(\mathbf{y},\tau)G(\mathbf{x},t;\mathbf{y},\tau) = s(\mathbf{y},\tau)\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau), \tag{12}$$

Integrating both sides

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\int s(\mathbf{y},\tau)G(\mathbf{x},t;\mathbf{y},\tau)d\mathbf{y}d\tau = \int s(\mathbf{y},\tau)\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)d\mathbf{y}d\tau, \tag{13}$$

and using the properties of delta function, we get the solution for the acoustic wave equation with source $s(\mathbf{x},t)$

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p(\mathbf{x},t) = s(\mathbf{x},t), \tag{14}$$

where,

$$p(\mathbf{x},t) = \int s(\mathbf{y},\tau)G(\mathbf{x},t;\mathbf{y},\tau)d\mathbf{y}d\tau. \tag{15}$$

We can use the above relation to solve the acoustic wave equation (9)

$$\begin{aligned}(pH)(\mathbf{x},t) = &-\frac{1}{4\pi}\int \frac{\partial p}{\partial n}\delta(f)\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau \\ &-\frac{1}{4\pi}\int \nabla_{\mathbf{y}}.(p\mathbf{n}\delta(f))\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau.\end{aligned} \tag{16}$$

Incomplete step!

$$\begin{aligned}(pH)(\mathbf{x},t) = &-\frac{1}{4\pi}\int \frac{\partial p}{\partial n}\delta(f)\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau \\ &-\frac{1}{4\pi}\nabla_{\mathbf{x}}.\int (p\mathbf{n}\delta(f))\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}d\tau.\end{aligned} \tag{17}$$

We use the following property to convert volume integral to surface integral (Farassat et al. 1988). Incomplete step !

$$\int \phi(\mathbf{y})\delta(f)\mathbf{n}d\mathbf{y} = \int_S \phi(\mathbf{y})\mathbf{n}dS \tag{18}$$

We use the above property to convert volume integral to surface integral on S

$$\begin{aligned}(pH)(\mathbf{x},t) = &-\frac{1}{4\pi}\int \frac{\partial p}{\partial n}\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}dSd\tau \\ &-\frac{1}{4\pi}\nabla_{\mathbf{x}}.\int (p\mathbf{n})\frac{\delta\Big(t-\tau-\frac{|\mathbf{x}-\mathbf{y}|}{c_0}\Big)}{|\mathbf{x}-\mathbf{y}|}dSd\tau.\end{aligned} \tag{19}$$

Using the property of delta function we obtain

$$\begin{aligned}(pH)(\mathbf{x},t) = &-\frac{1}{4\pi}\int_S \left[\frac{\partial p}{\partial n}\right]\frac{dS}{|\mathbf{x}-\mathbf{y}|} \\ &-\frac{1}{4\pi}\nabla_{\mathbf{x}}.\int_S [p]\mathbf{n}\frac{dS}{|\mathbf{x}-\mathbf{y}|}.\end{aligned} \tag{20}$$

The square bracket implies the functions are computed at the retarded time i.e, $[p] = p(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{c})$. Simplifying the equation further, we obtain the **Kirchhoff formula** for a stationary control surface (Farassat et al. 1988, Jamaluddin et al. 2011).

$$(pH)(\mathbf{x},t) = -\frac{1}{4\pi}\int_S \left[\frac{p}{r^2}\frac{\partial r}{\partial n} - \frac{1}{r}\frac{\partial p}{\partial n} + \frac{1}{cr}\frac{\partial r}{\partial n}\frac{\partial p}{\partial \tau}\right]_\tau dS. \tag{21}$$

Where, $r = |\mathbf{x}-\mathbf{y}|$, the square bracket again implies the functions are computed at the retarded time $\tau = t - r/c$.

2 Results

We compute the sound waves emitted by a monopole source using the Kirchhoff solver. The acoustic wave equation for a monopole source placed at a point $\mathbf{x} = 0$ is

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p(\mathbf{x}, t) = -q(t)\delta(\mathbf{x}), \tag{22}$$

where $q(t)$ is the time dependent source function. The solution can be obtained by substituting the free space Green's function (11) in (15)

$$p(\mathbf{x}, t) = \int s(\mathbf{y}, \tau)\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}\delta\left(t - \tau - \frac{|\mathbf{x} - \mathbf{y}|}{c_0}\right)d\mathbf{y}d\tau, \tag{23}$$

and using the property of delta function we obtain,

$$p(\mathbf{x}, t) = \frac{1}{4\pi}\int \frac{s\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{|\mathbf{x} - \mathbf{y}|}d\mathbf{y}. \tag{24}$$

The pressure at any point $\mathbf{x}$ and time t is a linear superposition of contributions from all the sources located at $\mathbf{y}$, which radiated at the earliest times $t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}$. The integral formula (24) is called the retarded potential. Substituting $s(\mathbf{y}, \tau) = -q(t)\delta(\mathbf{x})$ in the above equation we obtain

$$p(\mathbf{x}, t) = -\frac{1}{4\pi}\int \frac{\delta(\mathbf{y})q(t - \frac{|\mathbf{x}-\mathbf{y}|}{c_0})}{|\mathbf{x} - \mathbf{y}|}d\mathbf{y}, \tag{25}$$

using the property of delta function, the pressure radiated by a monopole source is given by

$$p(\mathbf{x}, t) = -\frac{1}{4\pi}\frac{q(t - \frac{r}{c_0})}{r}. \tag{26}$$

where $r = |\mathbf{x}|$.

2.1 Monopole test case

We chose the monopole of strength

$$q(t) = 2(t - t0)f_0^2\exp(-f_0^2(t - t_0)^2). \tag{27}$$

where $f_0 = 100$ is the dominant frequency and $t_0 = \frac{4}{f_0}$. We enclose the monopole source using a cuboidal Kirchhoff surface whose diagonally opposite points are $p_1 = (-1.0, -1.0, -1.0)$ and $p_2 = (1.0, 1.0, 1.0)$. The surface is embedded in a cuboidal domain of size $[-5.0, 5.0] \times [-5.0, 5.0] \times [-5.0, 5.0]$. The domain is discretized into structured grid of size $h = 0.1$ and the Kirchhoff surface is discretized into square cells of size $h = 0.1$. The pressure and its derivatives are interpolated from cell centers to quadrature points on Kirchhoff surface using fourth-order WENO polynomial. The Kirchhoff Integral (21) is computed using the two-point Gauss quadrature formula. The exact and numerical pressure are evaluated at observer point $xo = (3.0, 3.0, 3.0)$ and plotted against the function of time.

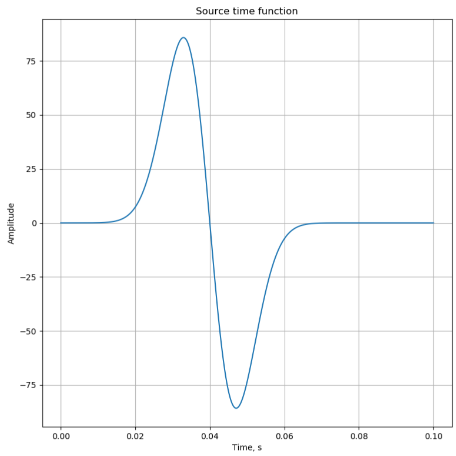

Figure 2: Monopole source as a function of time

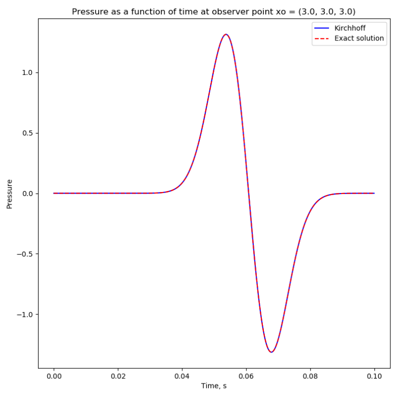

Figure 3: The exact and numerical pressure as a function of time at observer point $xo = (3.0, 3.0, 3.0)$.The L_∞ error is 3.11e-4.

References

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