Computation of far-field acoustic pressure from bubble collapse using the Keller-Miksis model

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Keller-Miksis model

In this work we compute the far-field acoustic pressure from bubble collapse using the Keller-Miksis model (Keller and Miksis 1980, Keller and Kolodner 1956).

Governing equation

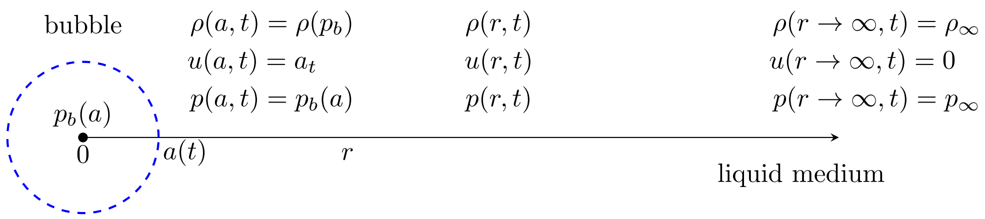


Figure 1: Schematic of a spherical air bubble in liquid medium.

A spherical air bubble is placed in an infinite inviscid compressible liquid medium as shown in figure. Assuming spherical symmetry, the mass and momentum conservation equation for the liquid medium is given by

$$\rho_t + \rho_r u + \rho u_r + \frac{2\rho u}{r} = 0, \tag{1}$$

$$u_t + u u_r + \frac{p_r}{\rho} = 0, \tag{2}$$

$$p = p(\rho). \tag{3}$$

Assuming the existence of velocity potential ϕ, the mass and the momentum conservation equation can be rewritten as

$$\rho_t + \rho_r \phi_r + \rho \phi_{rr} + \frac{2\rho\phi_r}{r} = 0, \tag{4}$$

$$\phi_{rt} + \phi_r \phi_{rr} + \frac{p_r}{\rho} = 0. \tag{5}$$

We assume the pressure inside the bubble is uniform p_b. Assuming adiabatic expansion, the bubble pressure is given as a function of radius a,

$$p_b(a) = ka^{-3\gamma}. \tag{6}$$

Where, k is the constant computed from initial radius and pressure and γ is the specific heat ratio of the gas. Neglecting surface tension and viscous effects the liquid pressure at bubble wall is same as the pressure inside the bubble. Using continuity, the bubble wall velocity is same as the fluid velocity at bubble radius:

$$p(a,t) = p_b(a), \qquad \phi_r(a,t) = a_t. \tag{7}$$

We can solve the equations $(3)-(6)$ for variables a, ρ, ϕ, p given initial conditions.

Simplifications using the Keller-Miksis model

To solve these equations, we integrate (5) with respect to r from r to infinity.

$$-\phi_t - \frac{1}{2}\phi_r^2 + \int_p^{p_\infty} \rho^{-1} dp = 0. \tag{8}$$

We assume ϕ tends to zero at $r = \infty$. We differentiate the above equation with respect to t and obtain

$$-\phi_{tt} - \phi_r\phi_{rt} + c^2\frac{\rho_t}{\rho} = 0. \tag{9}$$

Using (4) and replacing ρ_t in (9) we obtain,

$$\frac{1}{c^2}\phi_{tt} - \phi_{rr} - \frac{2\phi_r}{r} = \frac{\rho_r\phi_r}{\rho} - \frac{\phi_r\phi_{rt}}{c^2}. \tag{10}$$

We omit the right handside in (10) by assuming c is large and ρ_r is small for nearly incompressible fluid. And obtain the wave equation for ϕ

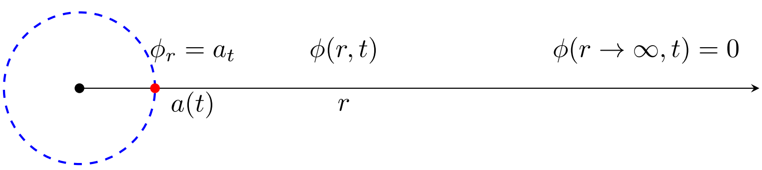
$$\frac{1}{c^2}\phi_{tt} - \phi_{rr} - \frac{2\phi_r}{r} = 0. \tag{11}$$

Assuming $\rho =$ constant in (8) we obtain an equation for pressure

$$p(r,t) = p_\infty - \rho\Big(\phi_t + \frac{1}{2}\phi_r^2\Big). \tag{12}$$

Thus the equations (4) and (5) are simplified to (11) and (12).

Simplified equations



We have to solve the following set of equation for $\phi(r,t)$ and $a(t)$:

$$\frac{1}{c^2}(r\phi)_{tt} - (r\phi)_{rr} = 0 \qquad r > a(t), \tag{13}$$

$$p_b(a(t)) - p_\infty = \rho_\infty\Big(\phi_t + \frac{1}{2}\phi_r^2\Big) \qquad r = a(t), \tag{14}$$

$$\phi_r = a_t \qquad r = a(t), \tag{15}$$

$$\phi = 0 \qquad r = \infty, \tag{16}$$

$$\phi(r,0) = 0 \qquad r > a(t), \tag{17}$$

$$\phi_t(r,0) = 0 \qquad r > a(t), \tag{18}$$

$$a(0) = a_0, \tag{19}$$

$$a_t(0) = a_{t0}. \tag{20}$$

Solution

A nonlinear second-order ordinary differential equation for $a(t)$ can be obatined by using the solution of wave equation $\phi = f(t - (r - a_0)/c)/r$

$$\Big(1 - \frac{\dot{a}}{c}\Big)a\ddot{a} + \frac{3}{2}\Big(1 - \frac{\dot{a}}{3c}\Big)\dot{a}^2 = \frac{1}{\rho_\infty}\Big(1 + \frac{\dot{a}}{c} + \frac{a}{c}\frac{d}{dt}\Big)(p_b - p_\infty). \tag{21}$$

Where, p is the bubble wall pressure given by

$$p_b = p_0\Big(\frac{a_0}{a}\Big)^{3\gamma}, \quad \gamma = 1.4. \tag{22}$$

The far-field pressure is given by

$$p(r,t) - p_\infty = \rho_\infty\Big(-\frac{f'}{r} - \frac{f^2}{2r^4} - \frac{1}{2c}\Big(\frac{f'^2}{cr^2} + \frac{2ff'}{r^3}\Big)\Big). \tag{23}$$

Where

$$f = -a^2 a_t + \frac{a^2}{c}\Big(\frac{a_t^2}{2} + \frac{(p_b - p_\infty)}{\rho_\infty}\Big) \tag{24}$$

and

$$f' = -a\Big(\frac{a_t^2}{2} + \frac{(p_b - p_\infty)}{\rho_\infty}\Big). \tag{25}$$

Where a is evaluated at retarded time $\tau \approx t - r/c$.

Results

We solve the evolution of bubble radius for the following simulation parameters We scale the pressure, density and length using the following scales

$$\bar{p} = 10^5 Pa, \quad \bar{\rho} = 1000 kgm^{-3}, \quad \bar{l} = 10^{-3} m. \tag{26}$$

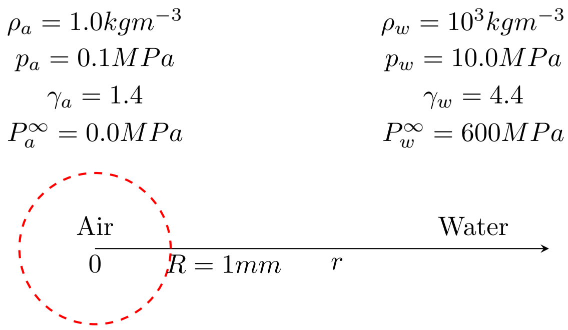


Figure 2: Schematic of air bubble in water medium with the initial conditions.

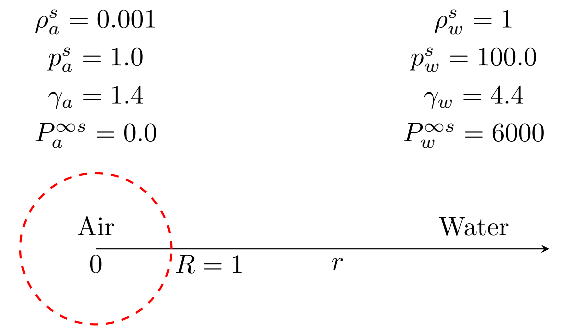


Figure 3: Schematic of air bubble in water medium with the scaled simulation parameters.

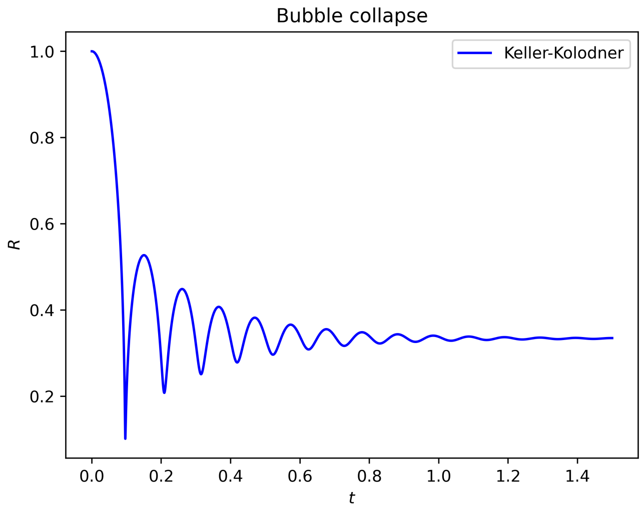


Figure 4: Time evolution of the bubble radius.

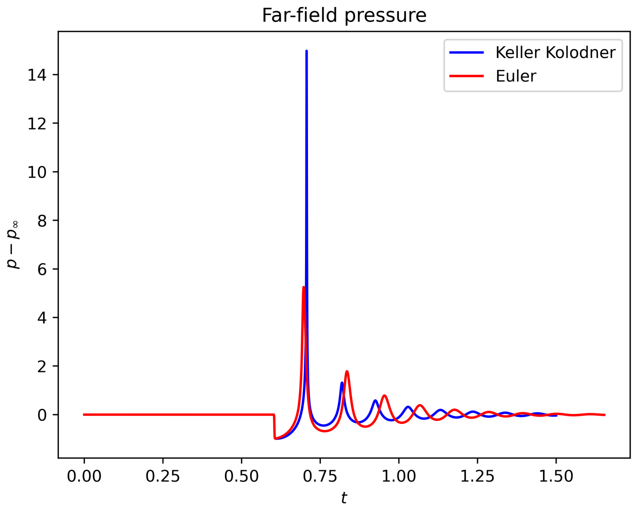


Figure 5: The far-field pressure is computed at the observer point $r = 100R$ using the Keller-Kolodner model and compared with the numerical solution of the compressible multiphase Euler equations.

References

Keller, Joseph B and Ignace I Kolodner (1956). "Damping of underwater explosion bubble oscillations". In: *Journal of applied physics* 27.10, pp. 1152–1161.

Keller, Joseph B and Michael Miksis (1980). "Bubble oscillations of large amplitude". In: *The Journal of the Acoustical Society of America* 68.2, pp. 628–633.