Spherically symmetric bubble collapse problem

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Objective

The present work aims to solve the bubble collapse problem in one dimension, assuming spherical symmetry and compute the far-field pressure using the Kirchhoff solver. We do the convergence study for the Kirchhoff integral and see if the chosen surface is in the linear acoustic regime.

Governing equation

The Compressible Euler equations assuming spherical symmetry for the mulitphase system can be written as

$$(\rho)_t + (\rho u)_r = \frac{-2\rho u}{r}, \tag{1}$$

$$(\rho u)_t + (\rho u^2 + p)_r = \frac{-2\rho u^2}{r}, \tag{2}$$

$$(E)_t + ((E+p)u)_r = \frac{-2(E+p)u}{r}, \tag{3}$$

$$(\phi)_t + (\phi u)_r = \phi u_r, \tag{4}$$

where the equation of state for the mixture is given by

$$p = (\gamma - 1)(E - \rho u^2/2) - \gamma P^{\infty}, \tag{5}$$

$$\gamma = 1 + \frac{(\gamma_w - 1)(\gamma_a - 1)}{(1-\phi)(\gamma_w - 1) + \phi(\gamma_a - 1)}, \tag{6}$$

$$P^{\infty} = \frac{\gamma - 1}{\gamma}\Big(\phi \frac{\gamma_w P_w^{\infty}}{\gamma_w - 1} + (1-\phi)\frac{\gamma_a P_a^{\infty}}{\gamma_a - 1}\Big). \tag{7}$$

Initial condition

An air bubble of radius R is placed in a water medium, assuming the fluid to be stationary. The initial condition for the primitive variables is given by

$$\rho = \frac{(\rho_a + \rho_w)}{2} + \frac{(\rho_w - \rho_a)}{2}\tanh\Big(\frac{r-R}{\epsilon h}\Big), \tag{8}$$

$$u = 0, \tag{9}$$

$$p = \frac{(p_a + p_w)}{2} + \frac{(p_w - p_a)}{2}\tanh\Big(\frac{r-R}{\epsilon h}\Big), \tag{10}$$

$$\phi = \frac{(\phi_a + \phi_w)}{2} + \frac{(\phi_w - \phi_a)}{2}\tanh\Big(\frac{r-R}{\epsilon h}\Big). \tag{11}$$

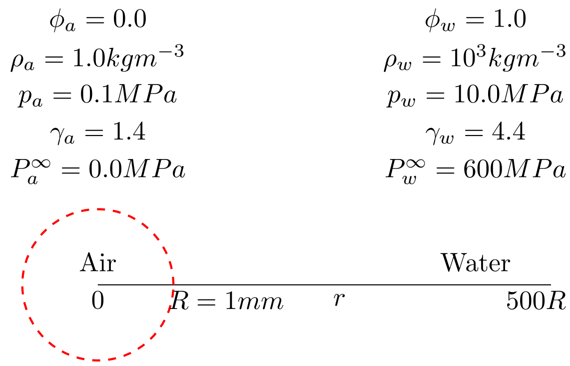

Figure 1: Schematic of air bubble in water medium with the initial conditions.

Boundary condition

We enforce homogeneous Neumann boundary condition at both ends of the domain $\frac{\partial(*)}{\partial r}\Big|_0 = \frac{\partial(*)}{\partial r}\Big|_L = 0$.

Scaling

Let $\bar{p}$ be the pressure scale, $\bar{\rho}$ be the density scale and $\bar{x}$ be the length scale. Then

$$p = \bar{p}p_s, \quad \rho = \bar{\rho}\rho_s, \quad x = \bar{l}x_s. \tag{12}$$

Where p_s, ρ_s and x_s are the scaled pressure, density and length. We can obtain the velocity and time scale from the above scales.

$$\frac{M}{LT^2} = \bar{p}\frac{M_s}{L_sT_s^2}, \quad \frac{M}{L^3} = \bar{\rho}\frac{M_s}{L_s^3}. \tag{13}$$

From the above relations we can obtain,

$$\frac{L}{T} = \sqrt{\frac{\bar{p}}{\bar{\rho}}}\frac{L_s}{T_s} \tag{14}$$

Therefore the velocity and time scale is given by

$$v = \sqrt{\frac{\bar{p}}{\bar{\rho}}}v_s, \quad t = \bar{l}\sqrt{\frac{\bar{\rho}}{\bar{p}}}t_s. \tag{15}$$

For the bubble problem we use the following scales,

$$\bar{p} = 10^5Pa, \quad \bar{\rho} = 1000kgm^{-3}, \quad \bar{l} = 10^{-3}m. \tag{16}$$

The scaled initial conditions are

Numerical method

We use the the minmod reconstruction for the spatial discretization and LLF path conservative Riemann solver for the flux computation. The time integration is done using the SSPRK22 method. The cell size is chosen $h = 0.002$ to resolve the bubble with 500 cells per bubble radius and the time step is computed based on CFL $= 0.8$.

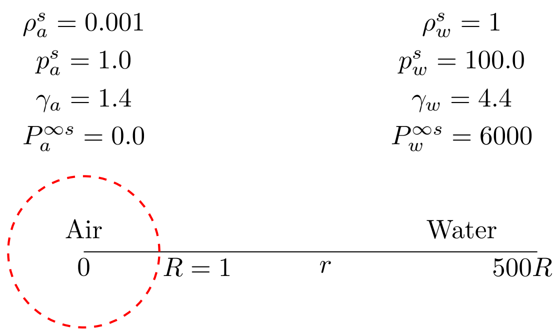

Figure 2: Schematic of air bubble in water medium with the scaled initial conditions.

Keller-Miksis model

We compare the bubble radius from the numerical solution of the multiphase Euler equation with the Keller-Miksis model. The Keller-Miksis model is an ordinary differential equation describing the time evolution of the bubble radius

$$\left(1 - \frac{\dot{R}}{c_\infty}\right) R\ddot{R} + \frac{3}{2}\left(1 - \frac{\dot{R}}{3c_\infty}\right)\dot{R}^2 = \frac{1}{\rho_\infty}\left(1 + \frac{\dot{R}}{c_\infty} + \frac{R}{c_\infty}\frac{d}{dt}\right)(p - p_\infty). \quad (17)$$

Where, p is the bubble wall pressure given by

$$p = p_\infty \left(\frac{R_0}{R}\right)^{3\kappa}, \quad \kappa = 1.4 \quad (18)$$

Convergence of Kirchhoff integral

We compute the Kirchhoff integral

$$p'(r,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{p'}{r'^2}\frac{\partial r'}{\partial n} - \frac{1}{r'}\frac{\partial p'}{\partial n} + \frac{1}{cr'}\frac{\partial r'}{\partial n}\frac{\partial p'}{\partial \tau}\right]_\tau R^2 \sin\theta d\theta d\phi \quad (19)$$

using the mid-point rule and obtain the acoustic pressure at the observer point.

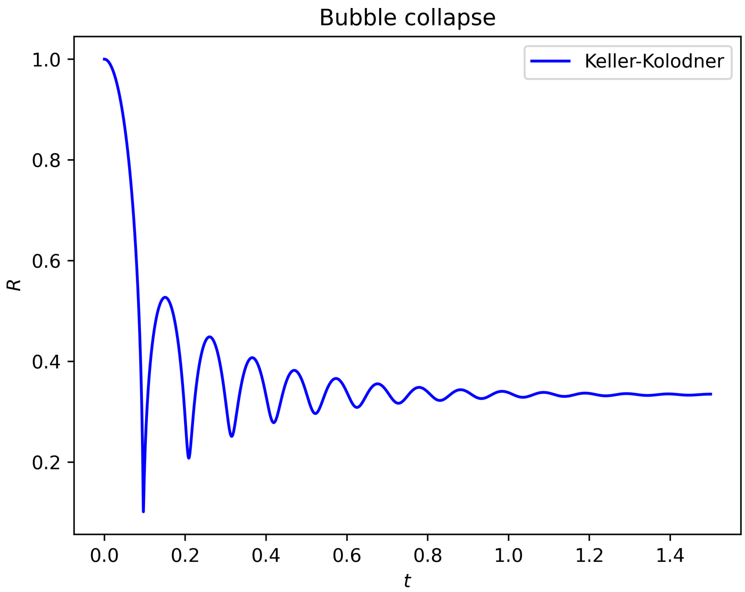

Figure 3: Time evolution of the bubble radius obtained from solving the multiphase Euler equation compared with the Keller-Miksis model.

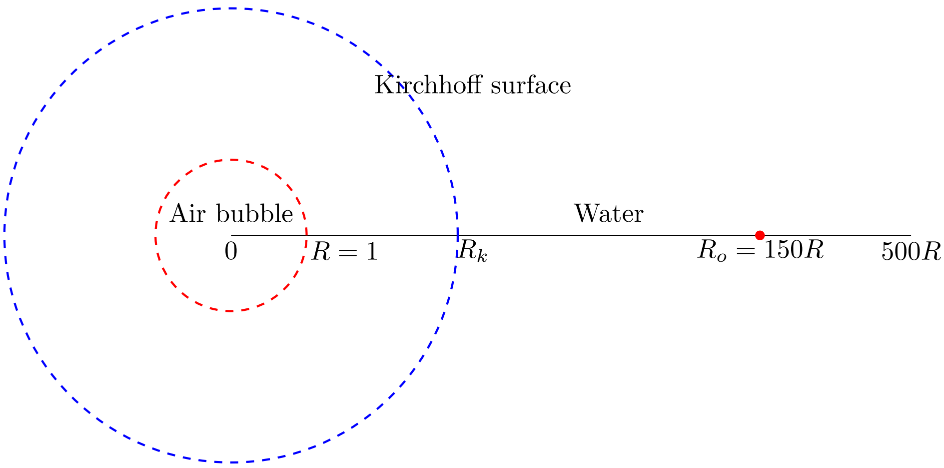

Figure 4: Schematic of Kirchhoff control surface and far-field observer point

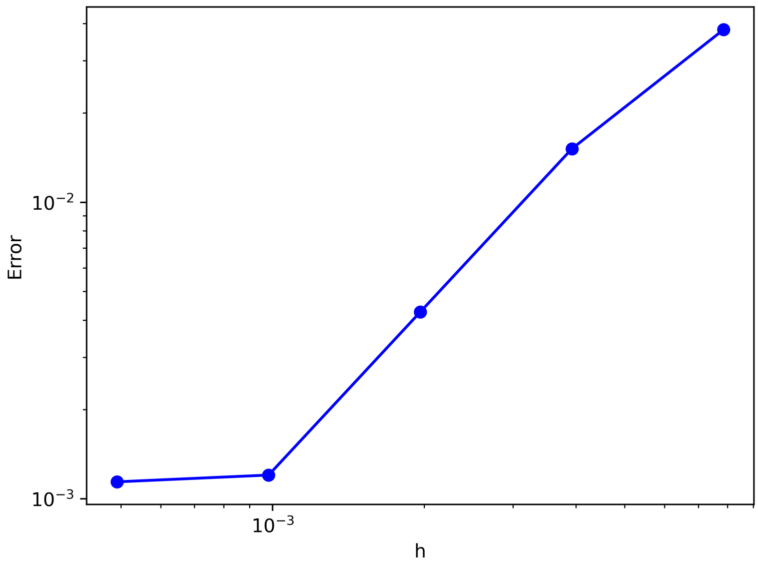

Figure 5: Convergence of Kirchhoff solution