Numerical integration of Kirchhoff integral formula and convergence

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Abstract

In this report, we have computed the Kirchhoff integral using the two-point Gaussian quadrature formula and showed fourth-order convergence. The function values are evaluated at quadrature points and emission time without numerical interpolation.

Kirchhoff integral formula

The Kirchhoff integral equation for a stationary control surface is given by

$$p'(\mathbf{x},t) = \frac{1}{4\pi}\int_S \Big[\frac{p'}{r'^2}\frac{\partial r'}{\partial n} - \frac{1}{r'}\frac{\partial p'}{\partial n} + \frac{1}{cr'}\frac{\partial r'}{\partial n}\frac{\partial p'}{\partial \tau}\Big]_\tau dS. \tag{1}$$

Where, p' is the acoustic pressure satisfying the wave equation outside the control surface $\mathbf{S}$, c is the speed of sound at ambient conditions and n is the normal. The integrands are evaluated at the emission time $\tau = t - \mathbf{r}'/c$ and $\mathbf{r'} = |\mathbf{x} - \mathbf{y}|$ is the distance between observer and source. We compute the integral on a spherical surface. Rewriting the integral using the spherical coordinates, we obtain

$$p'(r,\theta,\phi,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \Big[\frac{p'}{r'^2}\frac{\partial r'}{\partial n} - \frac{1}{r'}\frac{\partial p'}{\partial n} + \frac{1}{cr'}\frac{\partial r'}{\partial n}\frac{\partial p'}{\partial \tau}\Big]_\tau R^2 \sin\theta d\theta d\phi \tag{2}$$

The above integral is numerically computed using the two-point Gaussian quadrature formula

$$p'(r,\theta,\phi,t) = \frac{1}{4\pi}\sum_i^{N_\theta}\sum_j^{N_\phi}\sum_q^{N_q} \Big[\frac{p'}{r'^2}\frac{\partial r'}{\partial n} - \frac{1}{r'}\frac{\partial p'}{\partial n} + \frac{1}{cr'}\frac{\partial r'}{\partial n}\frac{\partial p'}{\partial \tau}\Big]_\tau R^2 \sin\theta\Big|_q w_q \tag{3}$$

Results

We compute the Kirchhoff integral for a pressure emitted by a monopole source $p'(\mathbf{x},t) = -\frac{1}{4\pi}\frac{q(t-\frac{r}{c_0})}{r}$. We choose a monopole of strength $q(t) = 2(t - t0)f_0^2 \exp(-f_0^2(t - t_0)^2)$, where $f_0 = 100$ is the dominant frequency and $t_0 = \frac{4}{f_0}$. Acoustic pressure is computed at observer point $(x_0, y_0, z_0) = (3.0, 3.0, 3.0)$ using the Kirchhoff method and compared with the analytical solution. We chose a sphere of radius $R = 1.0$.

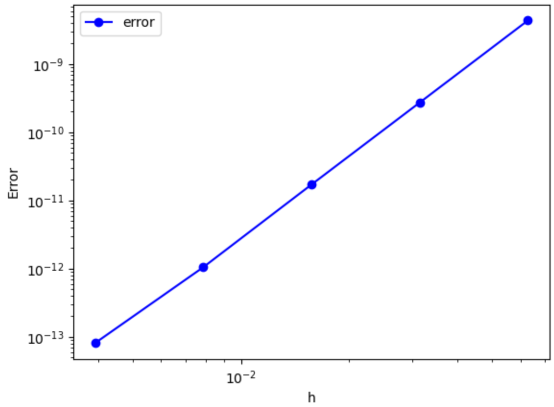

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Figure 1: We have obtained fourth-order convergence for the Kirchhoff integral computed using the two-point Gaussian quadrature

conclusion

We have shown the fourth-order convergence of the Kirchhoff integral formula when computed using the two-point Gaussian quadrature formula as expected. We haven't shown the Kirchhoff integral's convergence using the WENO polynomial interpolation.