Kirchhoff formula for spherically symmetric acoustic problem

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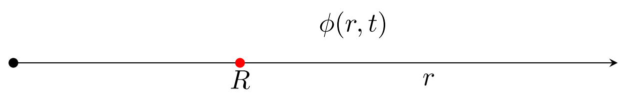
Formulation



In this work we derive the Kirchhoff integral formula for spherically symmetric acoustic problems. We assume the wave propogates linearly in the region $r > R$ and write down the spherically symmetric wave equation

$$\frac{1}{c^2}(\phi)_{tt} - (\phi)_{rr} - \frac{2\phi_r}{r} = 0 \quad r > R. \tag{1}$$

We define a generalised variable $\phi H(r-R)$ using the Heaviside function:

$$\phi H(r-R) = \begin{cases} \phi, & \text{for } r > R. \\ 0, & \text{for } r < R \text{ .} \end{cases} \tag{2}$$

The wave equation in generalised variable is given by

$$\frac{1}{c^2}(\phi H(r-R))_{tt} - (\phi H(r-R))_{rr} - \frac{2(\phi H(r-R))_r}{r} = s(r,t) \quad r > 0. \tag{3}$$

Where $s(r,t)$ is the source term given by

$$s(r,t) = -\phi_r \delta(r-R) - (\phi\delta(r-R))_r - \frac{2\phi\delta(r-R)}{r}. \tag{4}$$

Unlike (1), the generalised wave equation (3) is valid over the entire space. So we can use the free space Green's function to solve the equation. The Green's function for the wave operator in (1) is given by

$$G(r,t;r',t') = \frac{1}{4\pi|r-r'|}\delta\Big(t - t' - \frac{|r-r'|}{c}\Big). \tag{5}$$

We can solve the wave equation (3) by convolution of source and the Green's function.

$$\phi(r,t)H(r-R) = \int_{t'}\int_{r'} s(r',t')G(r,t;r',t')dr'dt' \tag{6}$$

$$
\begin{aligned}
\phi H(r-R) = &-\frac{1}{4\pi}\int_{t'}\int_{r'} \phi_{r'}\delta(r'-R)\frac{\delta\Big(t-t'-\frac{|r-r'|}{c}\Big)}{|r-r'|}dr'dt' \\
&-\frac{1}{4\pi}\int_{t'}\int_{r'} (\phi\delta(r'-R))_{r'}\frac{\delta\Big(t-t'-\frac{|r-r'|}{c}\Big)}{|r-r'|}dr'dt' \\
&-\frac{1}{4\pi}\int_{t'}\int_{r'} \frac{2\phi\delta(r'-R)}{r'}\frac{\delta\Big(t-t'-\frac{|r-r'|}{c}\Big)}{|r-r'|}dr'dt'.
\end{aligned} \tag{7}
$$

We simplify the above equations using the delta function property

$$
\phi H(r-R) = -\frac{\phi_r(R,\tau)}{4\pi|r-R|} - \frac{1}{4\pi}\Big(\frac{\phi(R,\tau)}{|r-R|}\Big)_r - \frac{1}{4\pi}\frac{2\phi(R,\tau)}{R|r-R|}. \tag{8}
$$

Where $\tau = t - \frac{|r-R|}{c}$ is the retarded time. Simplifying further we obtain

$$
\phi(r,t) = -\frac{1}{4\pi(r-R)}\Big[\phi_r(R,\tau) - \frac{1}{c}\phi_t(R,\tau) - \frac{\phi(R,\tau)}{(r-R)} + \frac{2\phi(R,\tau)}{R}\Big] \qquad r > R. \tag{9}
$$