# Fourier Analysis


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Advance Image Analysis, ChallengeII


## Introduction

Fourier analysis can be described as the study of the way signals may be represented as a combination of other simpler sine and cosine signals. Got its name after Jean Baptiste Joseph Fourier, a French mathematician and physicist.
So, Fourier Transform is the process of decomposing the signal into a combination of sines and cosines and it can be described as decomposing the signal into its frequencies.
Fourier analysis has many applications and is used in image processing as we can define an image as 1D or 2D signal.

## Transform Types

. Fourier analysis consist of four types of transforms.Discrete Time Fourier Transform, Discrete Time Fourier Series, Continuous Time Fourier Transform, Continuous Time Fourier Series. We have Fourier series to represent periodic signals and Fourier transform for non periodic signals.

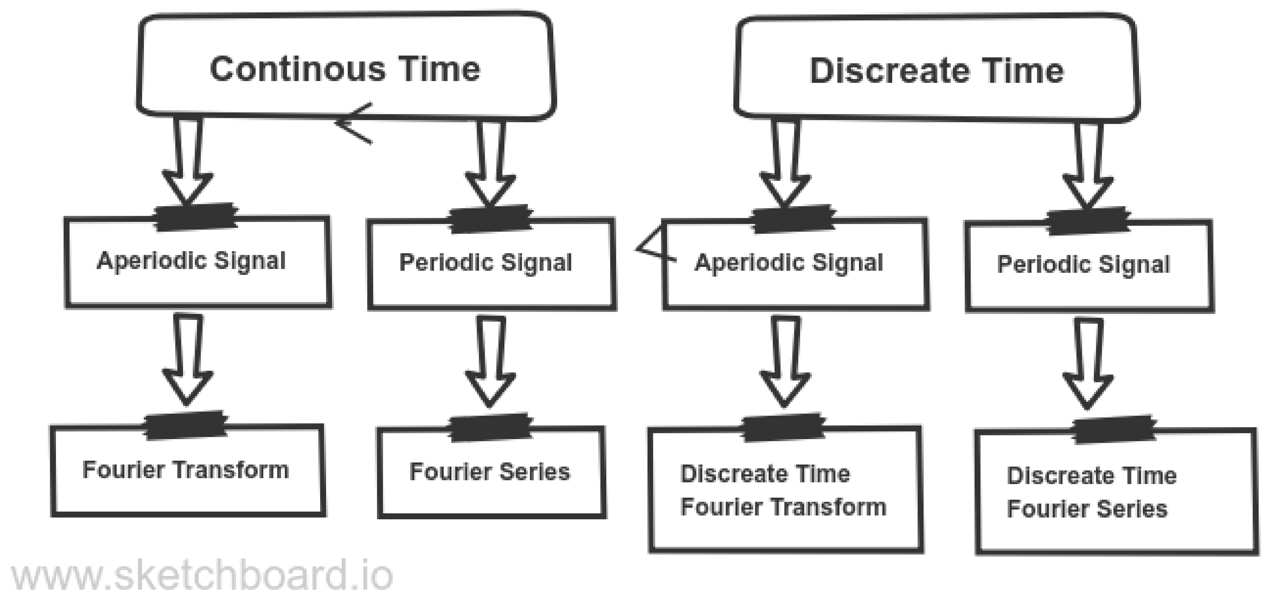


**Figura 1:** Fourier Analysis

### 0.1. Continuous Time Periodic Signals

A periodic signal is a function that repeats itself after a specific period in time $T_0$ from $\infty$ to $-\infty$. Continuous signals are functions of real independent variables that defines a set of continuous values. like functions of time. A periodic continuous signal can be defined Mathematically as:

$$f(t) = f(t + nT0) \;\; n \in Z \tag{1}$$

Where $T_0$ is the fundamental period which means the smallest positive value of cycle or period, and $n$ is an integer number.
The frequency of a periodic signal is opposite to the period where the period is a duration when a cycle is complete. Frequency is how many cycles is achieved in a given time. Mathematically can be described:

$$f_0 = \frac{1}{T_0} \tag{2}$$

An example of a periodic signal is a sine signal.

$$f(t) = Asin(wt + T) \tag{3}$$

Where $A$ is the amplitude of the signal, $w$ is the frequency, and $T$ is the shift in the input or phase.

#### 0.1.1. Continuous Time Fourier Series

The Fourier series represents periodic, continuous-time signals as a weighted sum of continuous-time sinusoids, or we can say a linear combination of harmonically complex exponentials.

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j\pi\omega_0 t} \tag{4}$$

Where $w_0 = \frac{2\pi}{T_0}$ is the fundamental frequency and $a_n$ is the Fourier coefficient and can be computed using the following formula:

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt \tag{5}$$

Let's have a **square wave** as an example:

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leqslant x < 0 \\ 1 & \text{if } 0 \leqslant x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x) \tag{6}$$

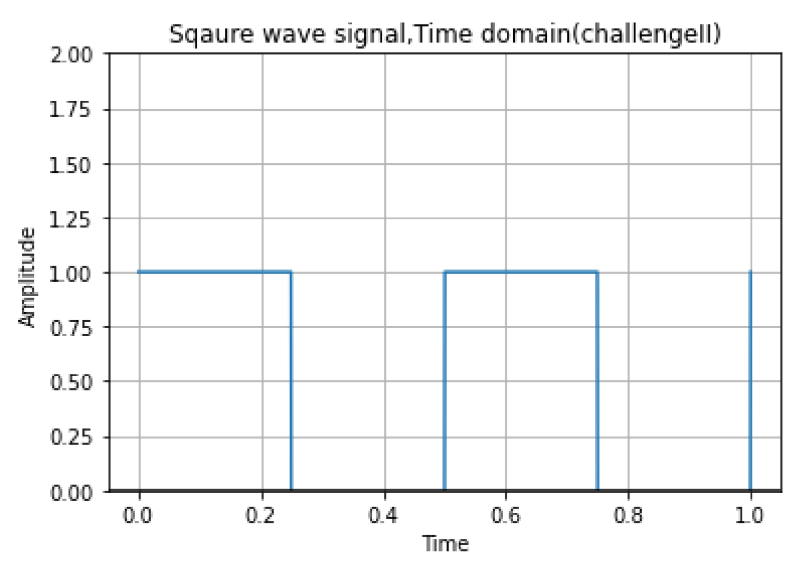


**Figura 2:** Square Wave Time Domain

To find the coefficient of this periodic square wave signal we use equation (5) where the fundamental frequency $T = 2\pi$ so we have :

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0} 0dx + \frac{1}{2\pi}\int_0^{\pi} 1dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2} \tag{7}$$

This is for the first coefficient where $n = 0$, and we keep finding the coefficient for $0 \leq n$, following equation (5) and after finding all coefficient, we substitute in equation (4) which is a combination of sine waves and Fourier coefficients. So after applying the Fourier series we have the The resulting frequency domain representation which is discrete and non-periodic :

$$\frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \frac{2}{7\pi}\sin 7x + ... \tag{8}$$

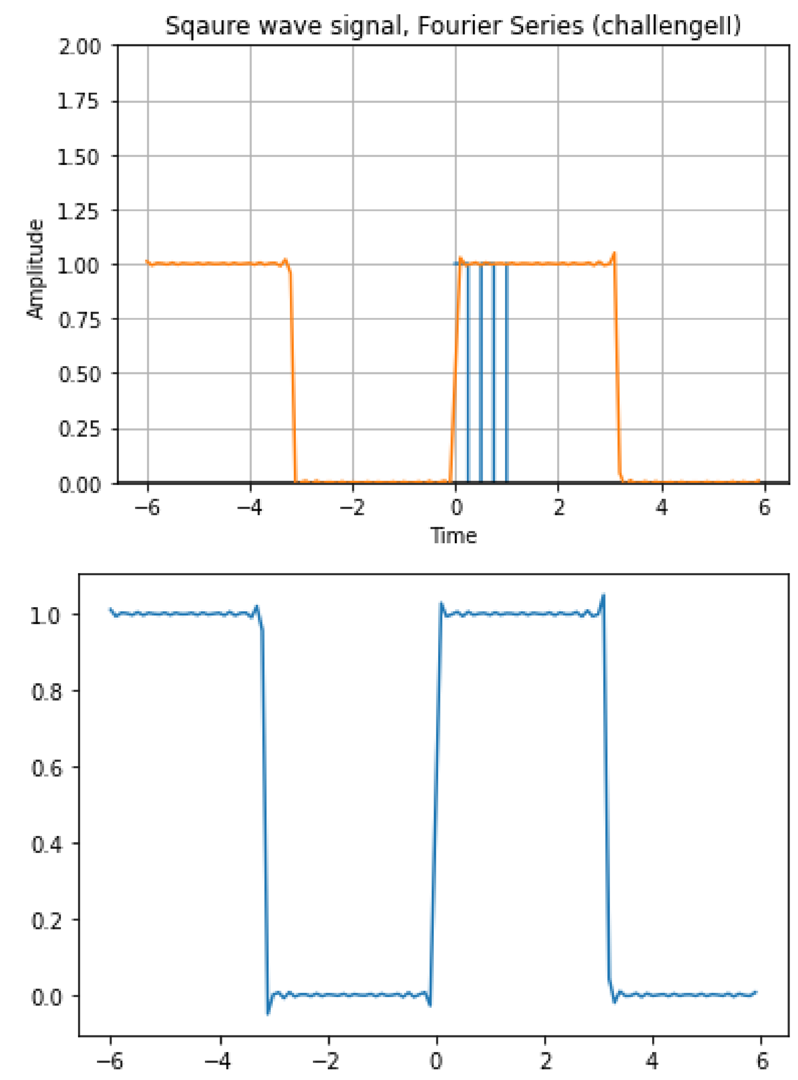


**Figura 3:** Square Wave Frequency Domain

### 0.2. Continuous Time Aperiodic Signals

A non periodic signal is a signal where it doesn't have a fixed frequency and is spread with different frequency values over a continuous time. and the Fourier representation take the form of an integral rather than a sum which is called Fourier Transform.

#### 0.2.1. Continuous Time Fourier Transform

$$F(\boldsymbol{w}) = \int_{-\infty} f(t)e^{-iwt}dt \tag{9}$$

Where $f(t)$ is a non periodic continuous time signal, and $F(jw)$ is a continuous function of frequency $w$. let's take this non periodic signal as an example:

$$f(t) = \begin{cases} e^-(at) \text{ if } t \geq 0 \\ 0 \text{ otherwise} \end{cases} \tag{10}$$

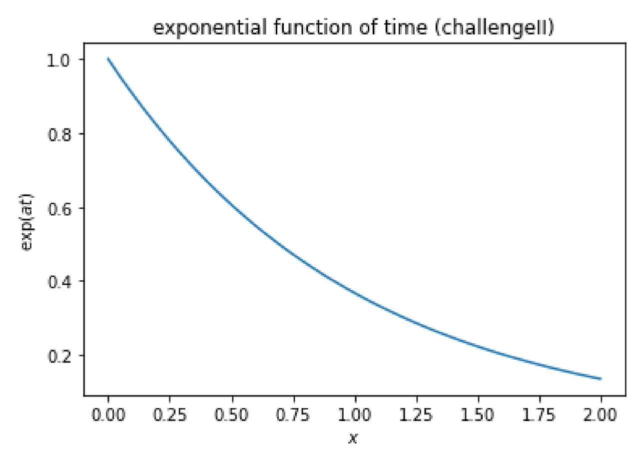


**Figura 4:** Exp.Signal Time Domain

To find the Fourier transform of this signal, we apply the signal on equation (9) but the integration will be from $0$ to $\infty$ because we don't have values below $0$. and The resulting frequency domain representation is continuous and non-periodic

$$\begin{aligned} F(w) &= \int_{-\infty}^{\infty} f(t)e^{-(iwt)}\mathrm{d}t \\ &= \int_0^{\infty} e^{-(\alpha t)}e^{-(iwt)}\mathrm{d}t \\ &= \int_0^{\infty} e^{(-t)(a+iw)}\mathrm{d}t \\ &= 0 - \frac{-1}{\alpha + i\Omega} \\ F(w) &= \frac{1}{\alpha + iw} \end{aligned} \tag{11}$$

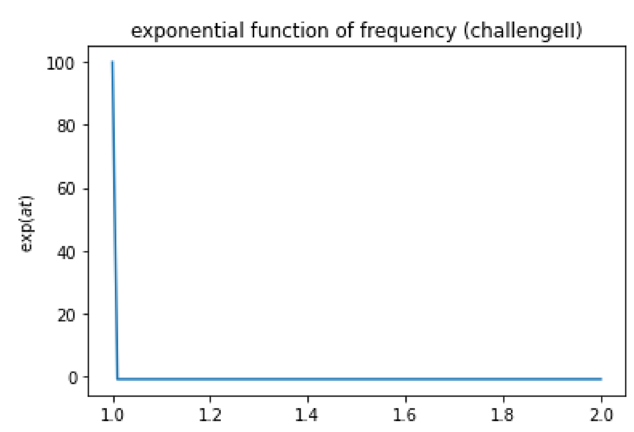


**Figura 5:** Exp.Signal Frequency Domain

## 0.3. Discrete Time Periodic Signals

Discrete Time Signals can be described as a function of a sequence of integer values, quantities and can be represented mathematically as

$$x = x(n) \tag{12}$$

Where $n$ is an integer number between $-\infty$ to $\infty$ and it can be obtained by sampling a continuous time signal. If the signal is discrete and periodic, then we can analyse the signal using Discrete Time Fourier Series.

### 0.3.1. Discrete Time Fourier Series

The discrete-time Fourier series is (DFTS) used to transform a discrete and periodic time domain signal into its frequency domain. The result frequency domain representation will be discrete and periodic.
The DTFS can be described Fourier representation that can be evaluated numerically by a computer or a digital system. and it is given by:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\Omega_0 bn} \tag{13}$$

where $\Omega_0 = \frac{2\pi}{N}$ and $N$ is the sampled period, $k = 0, 1, \cdots, N-1$.
The inverse of DTFT can be obtained by:

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{j\Omega_0 kn} \tag{14}$$

From Euler's formulas, we have:

$$\begin{aligned} e^{j\Omega_0 kn} &= \cos(\Omega_0 kn) + j\sin(\Omega_0 kn) \\ e^{-j\Omega_0 kn} &= \cos(\Omega_0 kn) - j\sin(\Omega_0 kn) \end{aligned} \tag{15}$$

And can be observed here that we have an imaginary part and a real part.

$$\begin{aligned} \mathrm{Re}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) &= \cos\left(\frac{2\pi kn}{N}\right) \\ \mathrm{Im}\left(\exp\left(\frac{-j2\pi kn}{N}\right)\right) &= -j\sin\left(\frac{2\pi kn}{N}\right) \end{aligned} \tag{16}$$

The magnitude and the phase can be described:

$$|X[k]| = \sqrt{\mathrm{Re}^2(X[k]) + \mathrm{Im}^2(X[k])} \tag{17}$$

$$\Phi_k = \mathrm{arctg}\frac{\mathrm{Im}(X[k])}{\mathrm{Re}(X[k])} \tag{18}$$

Let's have an example with this signal with a period $N = 5$

$$\tilde{x}[n] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & n = 3, 4 \end{cases} \tag{19}$$

Going back to equation (13) to find the Fourier coefficients

$$\begin{aligned} X]k] &= 1 + e^{-\frac{j2\pi k}{5}} + e^{-\frac{j4\pi k}{5}} \\ &= e^{-\frac{j2\pi k}{5}}\left(e^{\frac{j2\pi k}{5}} + 1 + e^{-\frac{j2\pi k}{5}}\right) \\ &= e^{-\frac{j2\pi k}{5}}\left[1 + 2\cos\left(\frac{2\pi k}{5}\right)\right] \end{aligned} \tag{20}$$

So, the magnitude and the phase

$$|\tilde{X}[k]| = \left|1 + 2\cos\left(\frac{2\pi k}{5}\right)\right| \tag{21}$$

$$\angle(\tilde{X}[k]) = -\frac{2\pi k}{5} + \angle\left(1 + 2\cos\left(\frac{2\pi k}{5}\right)\right) \tag{22}$$

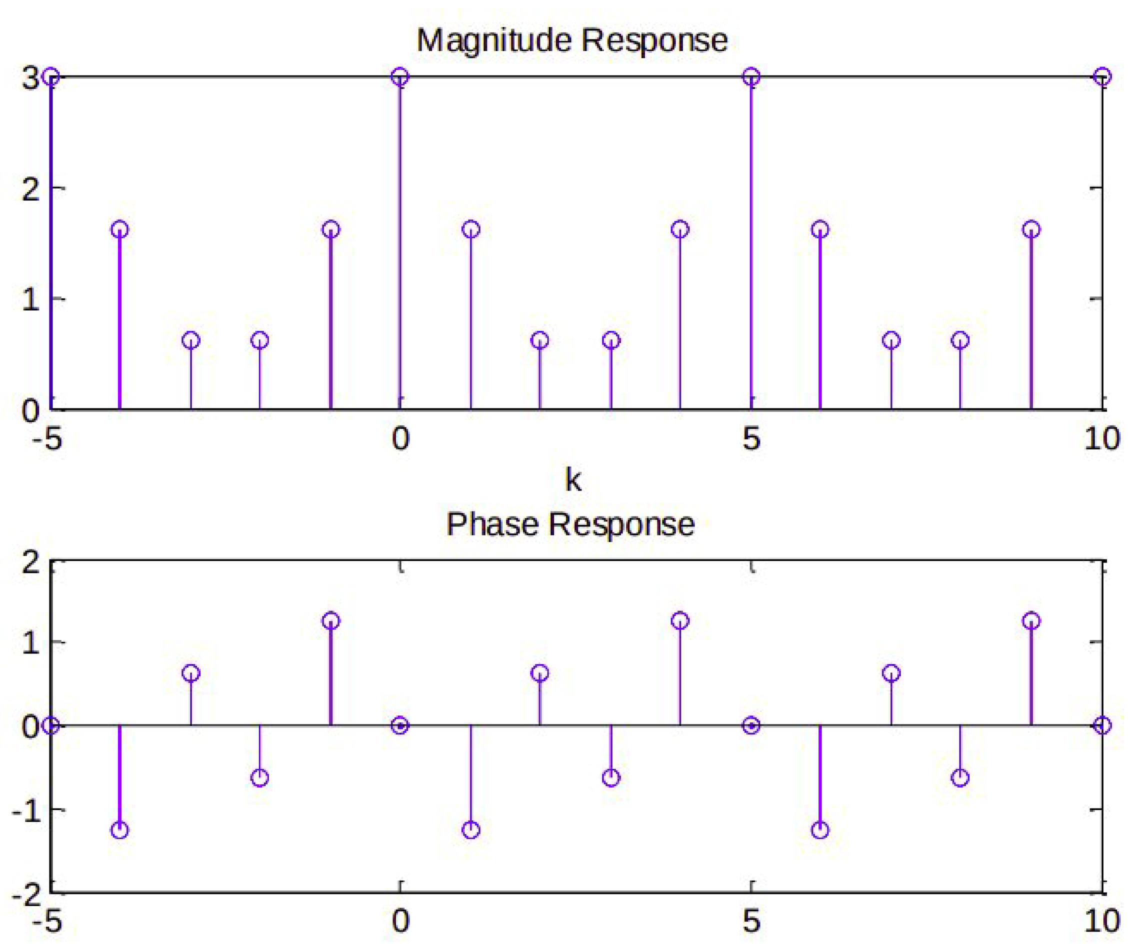


**Figura 6:** Magnitude and Phase [2]

## 0.4. Discreate Time Aperiodic Signals

### 0.4.1. Discrete Time Fourier Transform

The discrete-time Fourier transform is used to convert a discrete and non-periodic time-domain signal into the frequency domain. The resulting frequency domain representation from performing the discrete Fourier transform is continuous and periodic.
It is given by:

$$X\left(e^{j\hat{\omega}}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{23}$$

where $w$ is the discrete frequency. The frequency domain $X\left(e^{j\hat{\omega}}\right)$ is a continuous and periodic function as been said before.

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = X\left(e^{j(\omega+2k\pi)}\right) \tag{24}$$

with $2\pi$ period.
The inverse transform can be given by:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}d\omega \tag{25}$$

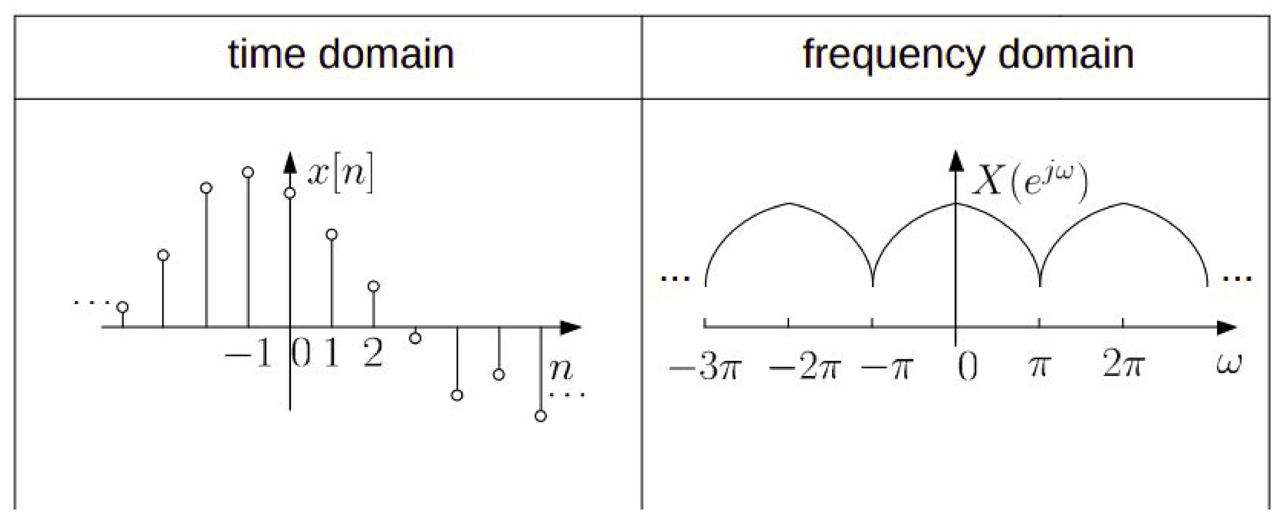


**Figura 7:** Time and Frequency domain (DTFT) [1]

let's find the DTFT for this signal

$$x[n] = a^n u[n], \quad |a| < 1 \tag{26}$$

Substitute $x[n]$ in equation (23), we will have

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n u[n]e^{-j\omega n} \tag{27}$$

we started from $0$ to $\infty$ because unit function has $0$ values for less than $0$ inputs, and because above $0$ we have the value of 1, we get this:

$$\sum_{n=0}^{\infty} a^n e^{-j\omega n} \tag{28}$$

Applying geometric sum, the DTFT of $x[n]$ is

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty}\left(ae^{-j\omega}\right)^n = \frac{1}{1 - ae^{-j\omega}} \tag{29}$$

## References


1. Discrete-Time Fourier Transform Lecture

2. Discrete Fourier Series Discrete Fourier Transform Lecture