Physics for Scientists and Engineers,
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Chapter 4:
Motion in Two Dimensions

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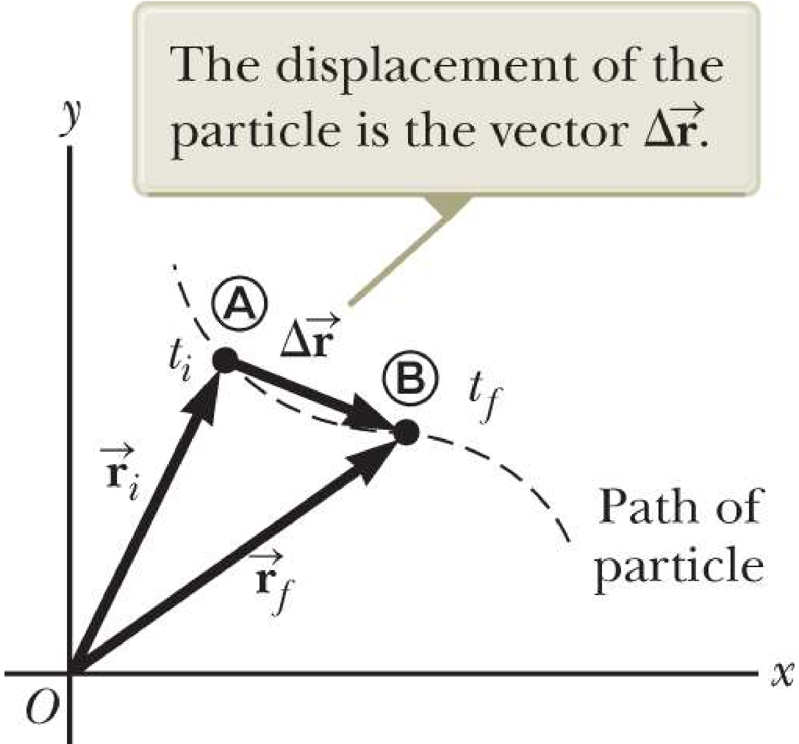
4.1. Position Vector

- One dimension: single numerical value describes particle's position
- Two dimensions: position indicated by **position vector r**
  - Drawn from origin of coordinate system to location of particle in *xy* plane (figure)
- At time $t_i \rightarrow$ particle at point A:
  - Described by position vector $\mathbf{r}_i$
- At later time $t_f \rightarrow$ at point B:
  - Described by position vector $\mathbf{r}_f$
- Path followed by particle from A to B not necessarily straight line



- As particle moves from A to B in time $\Delta t = t_f - t_i \rightarrow$ position vector changes from $\mathbf{r}_i$ to $\mathbf{r}_f$
  - Displacement is vector $\rightarrow$ displacement of particle difference between final and initial positions
- Define displacement vector Δr for particle: difference between final position vector and initial position vector:

$$\Delta\vec{\mathbf{r}} \equiv \vec{\mathbf{r}}_f - \vec{\mathbf{r}}_i$$

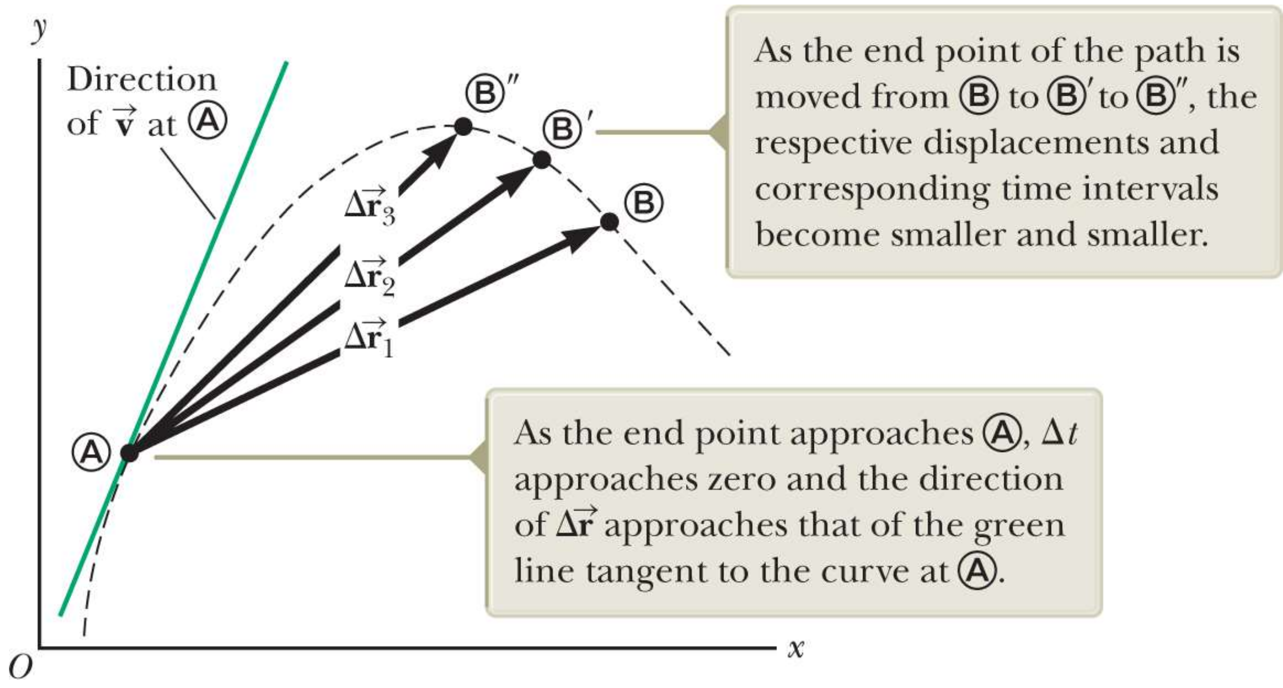
  - Direction of Δ**r** indicated in figure
  - Magnitude of Δ**r** *less* than distance traveled along curved path followed by particle.

4.1. Velocity Vector

- Often useful to quantify motion by looking at displacement divided by time interval during which that displacement occurs → rate of change of position
- Two-dimensional (or three-dimensional) kinematics is similar to one-dimensional kinematics
  - Must now use full vector notation rather than positive and negative signs to indicate direction of motion

- **Average velocity $\mathbf{v}_{avg}$** of particle during time interval Δt → displacement of particle divided by time interval:

$$\vec{v}_{avg} \equiv \frac{\Delta\vec{r}}{\Delta t}$$



4.1. Velocity Vector

- Because displacement vector quantity and time interval positive scalar quantity: average velocity vector quantity directed along $\Delta\mathbf{r}$
- Average velocity between points *independent of the path* taken
  - Average velocity proportional to displacement → depends only on initial and final position vectors, not on path taken
- Consider motion of particle between two points in *xy* plane (figure of slide 3)
  - Dashed curve shows path of particle
- As time interval becomes smaller and smaller (as B is moved to B′ and then to B″ etc.): direction of displacement approaches green line tangent to path at A
  - **Instantaneous velocity v** defined as limit of average velocity $\Delta\mathbf{r}/\Delta t$ as $\Delta t \to 0$:

$$\vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

  - Instantaneous velocity = derivative of position vector with respect to time
  - <u>Direction</u> of instantaneous velocity vector at any point in particle's path <u>along line tangent to path at that point</u> and <u>in direction of motion</u>
  - Magnitude of instantaneous velocity vector of a particle, $v = |\vec{v}|$, is the *speed* of the particle (scalar)

4.1. Acceleration Vector

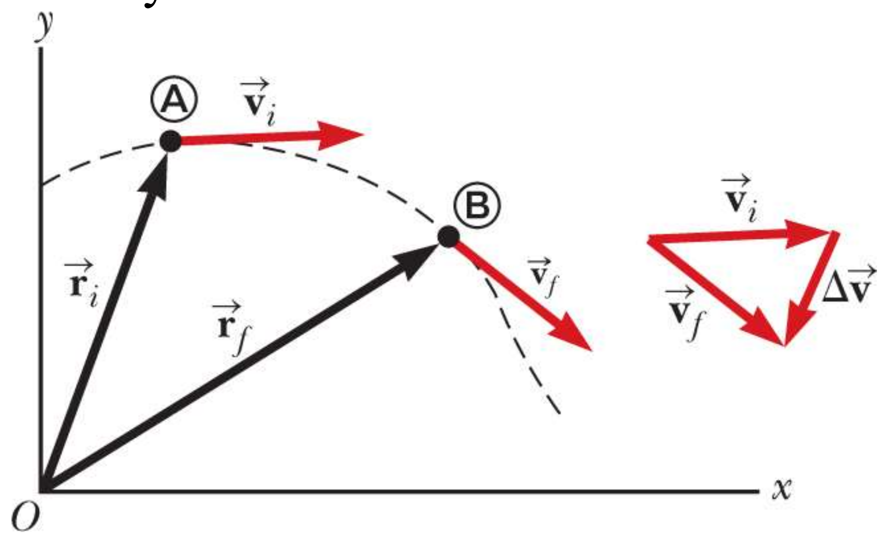
- As particle moves from one point to another along some path → instantaneous velocity vector changes from $\mathbf{v}_i$ at time t_i to $\mathbf{v}_f$ at time t_f
    - Knowing velocity at these points: we can determine average acceleration of particle
- **Average acceleration $\mathbf{a}_{avg}$**: change in instantaneous velocity vector $\Delta\mathbf{v}$ divided by time interval Δt during which that change occurs:

$$\vec{a}_{avg} \equiv \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

    - $\mathbf{a}_{avg}$ = ratio of vector quantity $\Delta\mathbf{v}$ and positive scalar quantity Δt:
    - Average acceleration = vector quantity directed along $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$

- **Instantaneous acceleration**: limiting value of ratio $\Delta\mathbf{v}/\Delta t$ as $\Delta t \to 0$:
    - Instantaneous acceleration = time derivative of velocity vector

$$\vec{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$



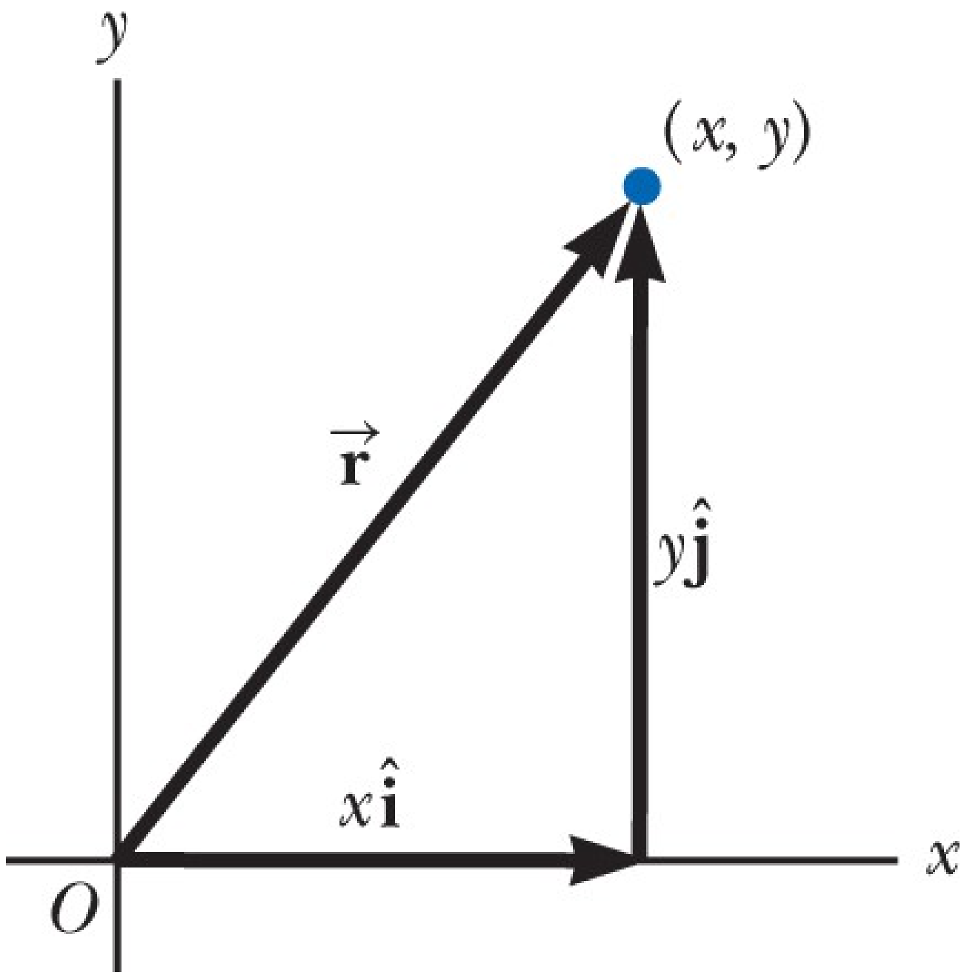
4.2. Two-Dimensional Motion with Constant Acceleration

- Consider two-dimensional motion with constant acceleration

- Consider a particle located in *xy* plane at position (*x* , *y*) (figure)
- Point can be specified by position vector **r** → in unit-vector form:

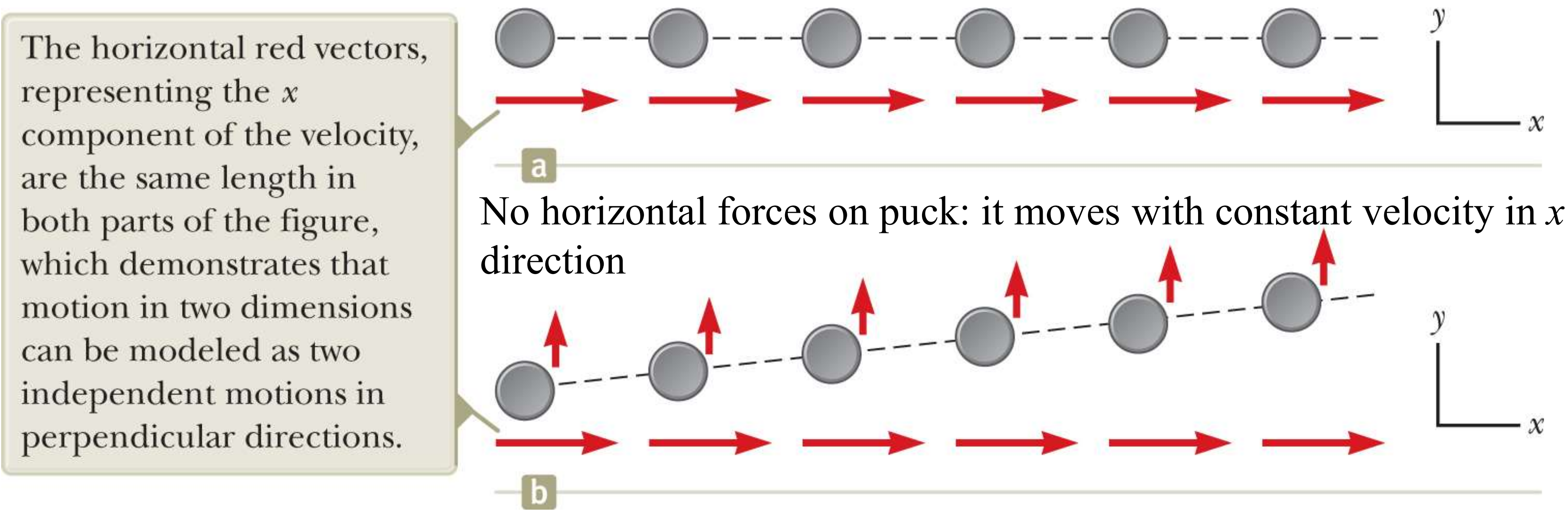
$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

  - *x*, *y*, and **r** change with time as particle moves
  - Unit vectors **i** and **j** remain constant



4.2. Two-Dimensional Motion with Constant Acceleration

Motion in two dimensions can be modeled as two *independent* motions in each of the two perpendicular directions associated with the *x* and *y* axes.



Any influence in *y* direction does not affect motion in *x* direction and vice versa

4.2. Two-Dimensional Motion with Constant Acceleration

- If position vector of particle is known → velocity of particle

$$\vec{\mathrm{v}} = \frac{d\vec{\mathrm{r}}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x\hat{\imath} + v_y\hat{\jmath}$$

$$v_{xf} = v_{xi} + a_x t \text{ and } v_{yf} = v_{yi} + a_y t$$

- Assume acceleration of particle constant:
  - Components a_x and a_y also constants
- Model particle as particle under constant acceleration independently in each of the two directions:

$$\vec{\mathrm{v}}_f = (v_{xi} + a_x t)\hat{\imath} + \left(v_{yi} + a_y t\right)\hat{\jmath} = \left(v_{xi}\hat{\imath} + v_{yi}\hat{\jmath}\right) + \left(a_x\hat{\imath} + a_y\hat{\jmath}\right)t$$

$$\vec{\mathrm{v}}_f = \vec{\mathrm{v}}_i + \vec{\mathrm{a}}t \text{ (for constant } \vec{\mathrm{a}})$$

- Velocity of particle at some time t = vector sum of initial velocity $\mathbf{v}_i$ at time $t = 0$ and additional velocity $\mathbf{a}t$ acquired at time t as result of constant acceleration

4.2. Two-Dimensional Motion with Constant Acceleration

- x and y coordinates of particle under constant acceleration

$$x_f = x_i + v_{xi}t + \frac{1}{2}a_x t^2 \quad y_f = y_i + v_{yi}t + \frac{1}{2}a_y t^2$$

$$\vec{\mathrm{r}}_f = \left(x_i + v_{xi}t + \frac{1}{2}a_x t^2\right)\hat{\imath} + \left(y_i + v_{yi}t + \frac{1}{2}a_y t^2\right)\hat{\jmath}$$
$$= (x_i\hat{\imath} + y_i\hat{\jmath}) + (v_{xi}\hat{\imath} + v_{yi}\hat{\jmath})t + \frac{1}{2}(a_x\hat{\imath} + a_y\hat{\jmath})t^2$$

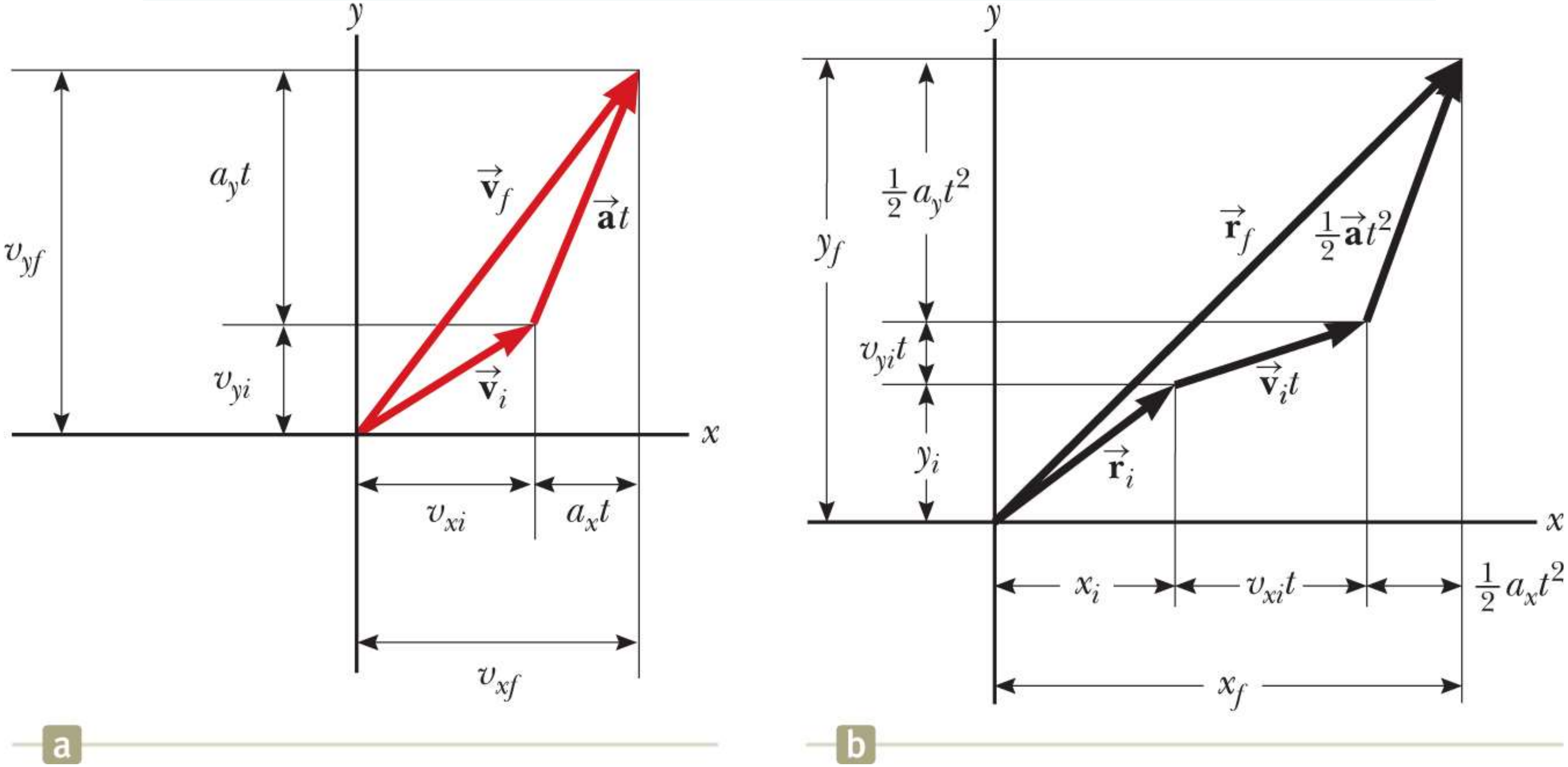
- Position vector $\mathbf{r}_f$ of particle = vector sum of original position $\mathbf{r}_i$, displacement $\mathbf{v}_i t$ arising from the initial velocity of the particle, and displacement ½ $\mathbf{a}t^2$ resulting from constant acceleration of particle:

$$\vec{\boldsymbol{r}}_f = \vec{\boldsymbol{r}}_i + \vec{\boldsymbol{v}}_i \boldsymbol{t} + \frac{1}{2}\vec{\boldsymbol{a}}\boldsymbol{t}^2 \quad \textbf{(for constant } \vec{\boldsymbol{a}}\textbf{)}$$

4.2. Two-Dimensional Motion with Constant Acceleration

- Equations: mathematical representation of two-dimensional version of particle under constant acceleration model:

$$\left.\begin{aligned} \vec{v}_f &= \vec{v}_i + \vec{a}t \\ \vec{r}_f &= \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2 \end{aligned}\right\} \text{(for constant } \vec{a}\text{)}$$



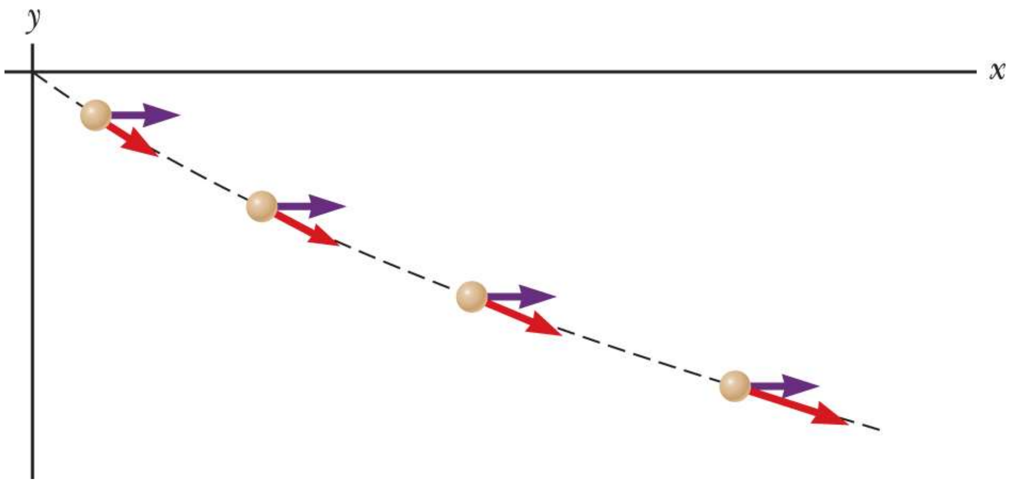
Graphical representations in figure: Components of the position and velocity vectors illustrated.

Example 4.1: Motion in a Plane

A particle moves in the *xy* plane, starting from the origin at $t = 0$ with an initial velocity having an x component of 20 m/s and a y component of –15 m/s. The particle experiences an acceleration in the x direction, given by $a_x = 4.0$ m/s^2.

(A) Determine the total velocity vector at any later time.

Categorize Because the initial velocity has components in both the x and y directions, we categorize this problem as one involving a particle moving in two dimensions. Because the particle only has an x component of acceleration, we model it as a *particle under constant acceleration* in the x direction and a *particle under constant velocity* in the y direction.



Example 4.1: Motion in a Plane

Analyze To begin the mathematical analysis, we set:

$$v_{xi} = 20 \text{ m/s},\ v_{yi} = -15 \text{ m/s},\ a_x = 4.0 \text{ m/s},\ a_y = 0$$

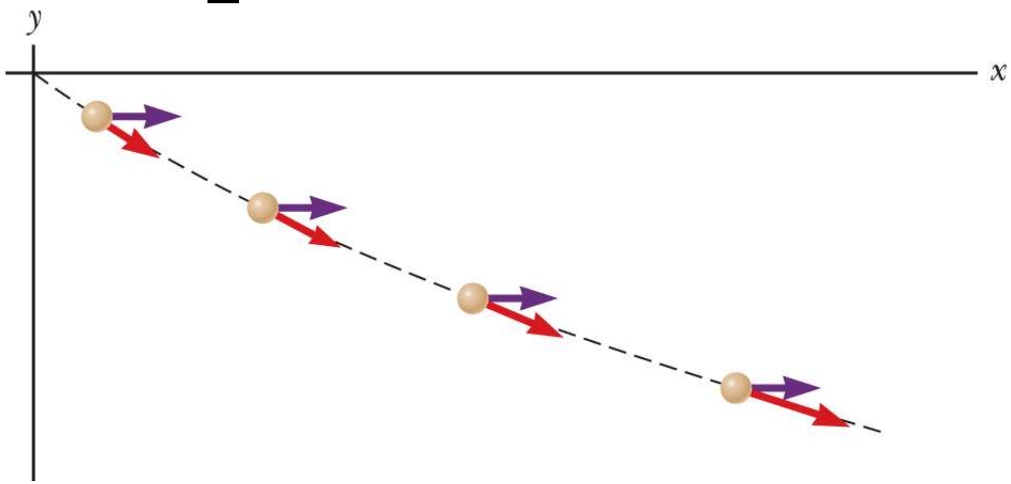
Write the equation for the velocity vector:

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = \left(v_{xi} + a_x t\right)\hat{\mathbf{i}} + \left(v_{yi} + a_y t\right)\hat{\mathbf{j}}$$

Substitute numerical values in metric units:

$$\vec{\mathbf{v}}_f = \left[20 + (4.0)t\right]\hat{\mathbf{i}} + \left[-15 + (0)t\right]\hat{\mathbf{j}}$$

$$\vec{\mathbf{v}}_f = \boxed{\left[20 + (4.0)t\right]\hat{\mathbf{i}} - 15\hat{\mathbf{j}}}$$



Finalize Notice from this expression that the x component of velocity increases in time while the y component remains constant; this result is consistent with our prediction.

Example 4.1: Motion in a Plane

(B) Calculate the velocity and speed of the particle at t = 5.0 s and the angle the velocity vector makes with the x axis.

$$\vec{\mathbf{v}}_f = \left[\left(20 + 4.0(5.0)\right)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}\right] = \boxed{\left(40\hat{\mathbf{i}} - 15\hat{\mathbf{j}}\right)\ \text{m/s}}$$

$$\mathbf{v}_f = \left|\vec{\mathbf{v}}_f\right| = \sqrt{v_{xf}^{\ 2} + v_{yf}^{\ 2}} = \sqrt{(40)^2 + (-15)^2} = \boxed{43\ \text{m/s}}$$

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \text{m/s}}{40\ \text{m/s}}\right) = -21°$$

Finalize The negative sign for the angle θ indicates that the velocity vector is directed at an angle of 21° below the positive x axis.

Example 4.1: Motion in a Plane

(C) Determine the *x* and *y* coordinates of the particle at any time *t* and its position vector at this time.

Analyze: Use the equation for the position vector:

$$\vec{r}_f = \vec{r}_i + \vec{v}_i t + \frac{1}{2}\vec{a}t^2 = \left(r_{xi} + v_{xi}t + \frac{1}{2}a_x t^2\right)\hat{\imath} + \left(r_{yi} + v_{yi}t + \frac{1}{2}a_y t^2\right)\hat{\jmath}$$

Substitute numerical values in metric units:

$$\vec{\mathbf{r}}_f = \left[0 + (20)t + \frac{1}{2}(4.0)t^2\right]\hat{\mathbf{i}} + \left[0 + (-15)t + \frac{1}{2}(0)t^2\right]\hat{\mathbf{j}}$$

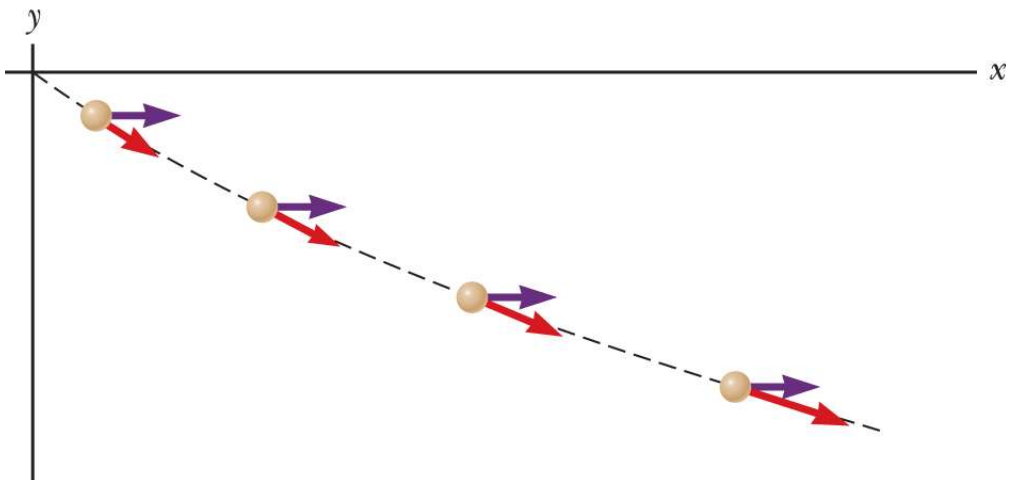
Simplify:

$$\vec{\mathbf{r}}_f = \boxed{\left(20t + 2.0t^2\right)\hat{\mathbf{i}} - 15t\hat{\mathbf{j}}}$$

Example 4.1:
Motion in a Plane

What if we wait a very long time and then observe the motion of the particle? How would we describe the motion of the particle for large values of the time?

$$\vec{\mathbf{v}}_f = \left[20 + (4.0)t\right]\hat{\mathbf{i}} - 15\hat{\mathbf{j}}$$



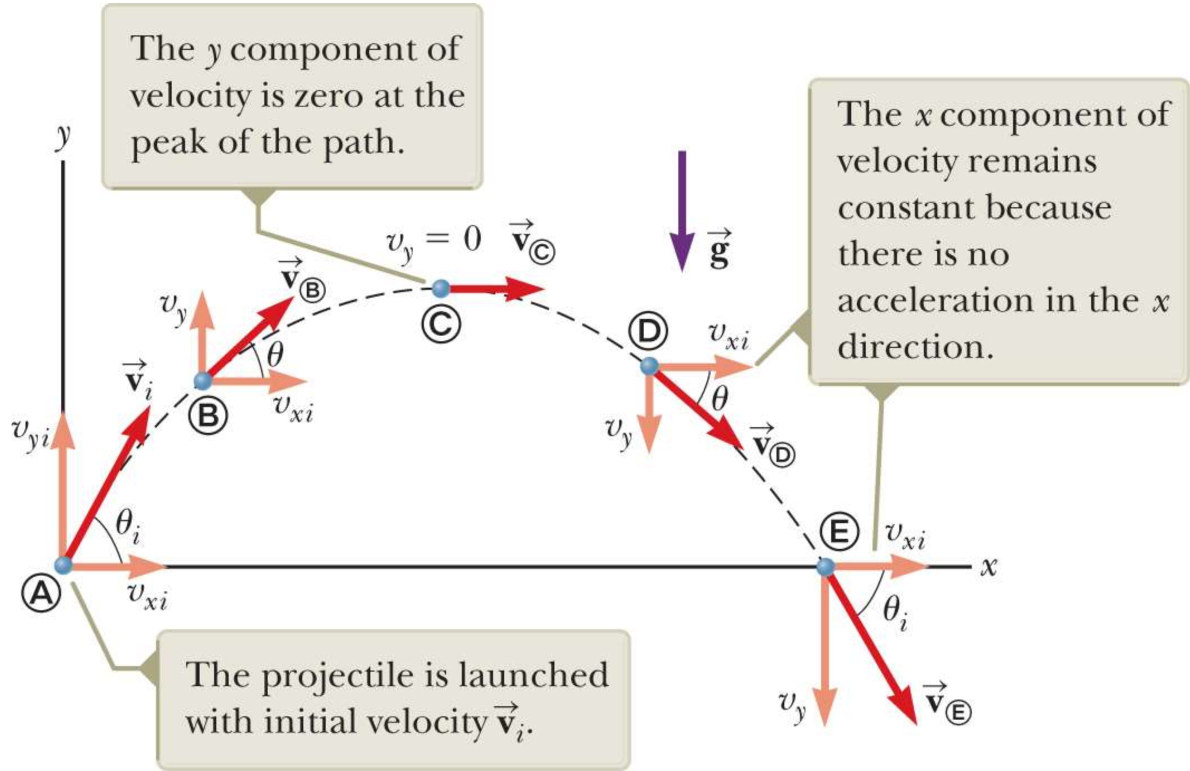
The y component of the velocity remains constant while the x component grows linearly with t. Therefore, when t is very large, the x component of the velocity will be much larger than the y component, suggesting that the velocity vector becomes more and more parallel to the x axis.

Path becomes more and more parallel to *x* axis.

4.3. Projectile Motion

- Anyone who has observed a football in motion has observed projectile motion

- **Projectile motion**: simple to analyze if we make two assumptions:
  1. free-fall acceleration constant over range of motion and directed downward (i.e., $a_x = 0$, $a_y = -g$)
  2. effect of air resistance negligible
- With these assumptions: path of projectile (its *trajectory*) *always* parabola (figure)
  - **We use these assumptions throughout this chapter**



4.3. Projectile Motion

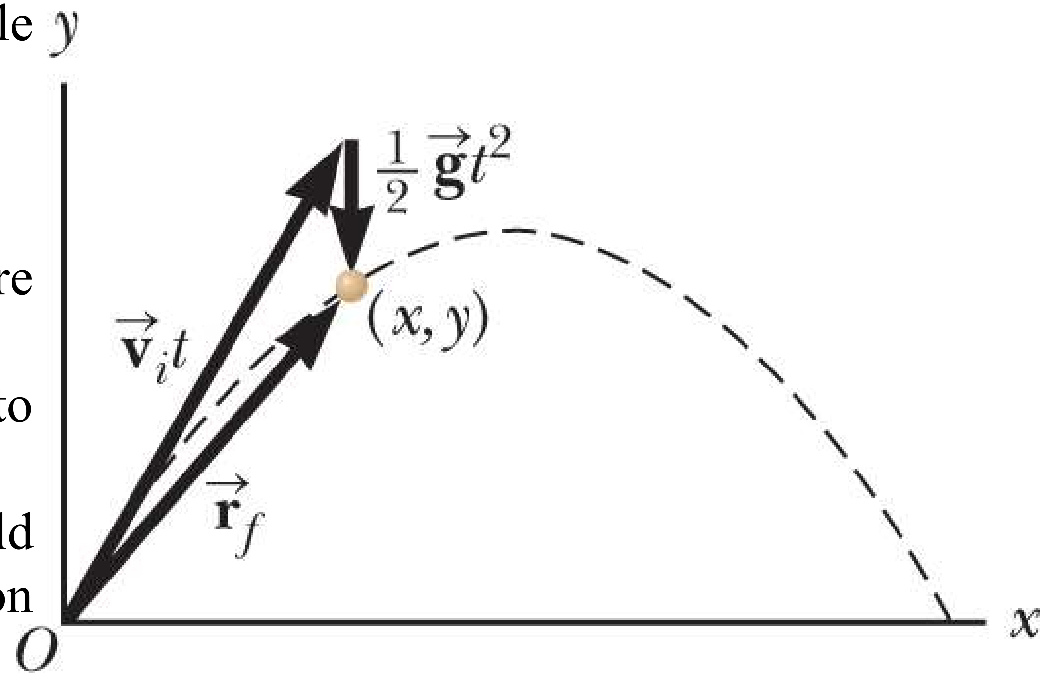
- For acceleration being that due to gravity, **a** = **g**:

$$\vec{r}_f = \vec{r}_i + \vec{v}_f t + \frac{1}{2}\vec{g}t^2$$

  - Where initial x and y components of velocity of projectile:

$$v_{xi} = v_i \cos\theta_i \quad v_{yi} = v_i \sin\theta_i$$

- Figure: pictorial representation of projectile launched from origin ($\mathbf{r}_i = 0$)
- Final position: superposition of:
  - Initial position $\mathbf{r}_i$
  - $\mathbf{v}_i t$ (displacement if no acceleration were present)
  - ½ $\mathbf{g}t^2$ that arises from its acceleration due to gravity
- If no gravitational acceleration: particle would continue to move along straight path in direction of $\mathbf{v}_i$



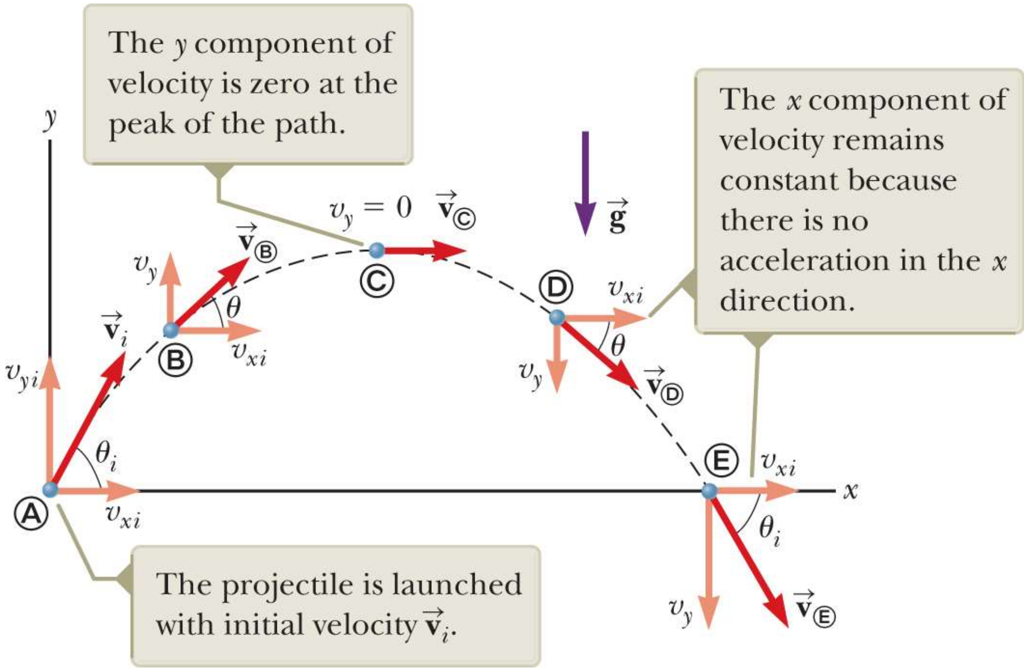
- Vertical distance ½ $\mathbf{g}t^2$ through which particle "falls" off straight-line path = same distance that object dropped from rest would fall during same time interval

4.3. Projectile Motion

- Projectile motion: acceleration $a_x = 0$ in x direction and constant acceleration $a_y = -g$ in y direction
- When solving projectile motion problems → use two analysis models:
  1. Particle under constant velocity in the horizontal direction: $x_f = x_i + v_{xi}t$
  2. Particle under constant acceleration in vertical direction (with x changed to y and $a_y = -g$):

$$v_{yf} = v_{yi} - gt \quad ; \quad v_{y,\text{avg}} = \frac{v_{yi}+v_{yf}}{2}$$
$$y_f = y_i + \frac{1}{2}(v_{yi} + v_{yf})t \quad ; \quad y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$$
$$v_{yf}^{\,2} = v_{yi}^{\,2} - 2g(y_f - y_i)$$

- Horizontal and vertical components of projectile's motion completely independent
  - Can be handled separately:
    - With time t as common variable for both components



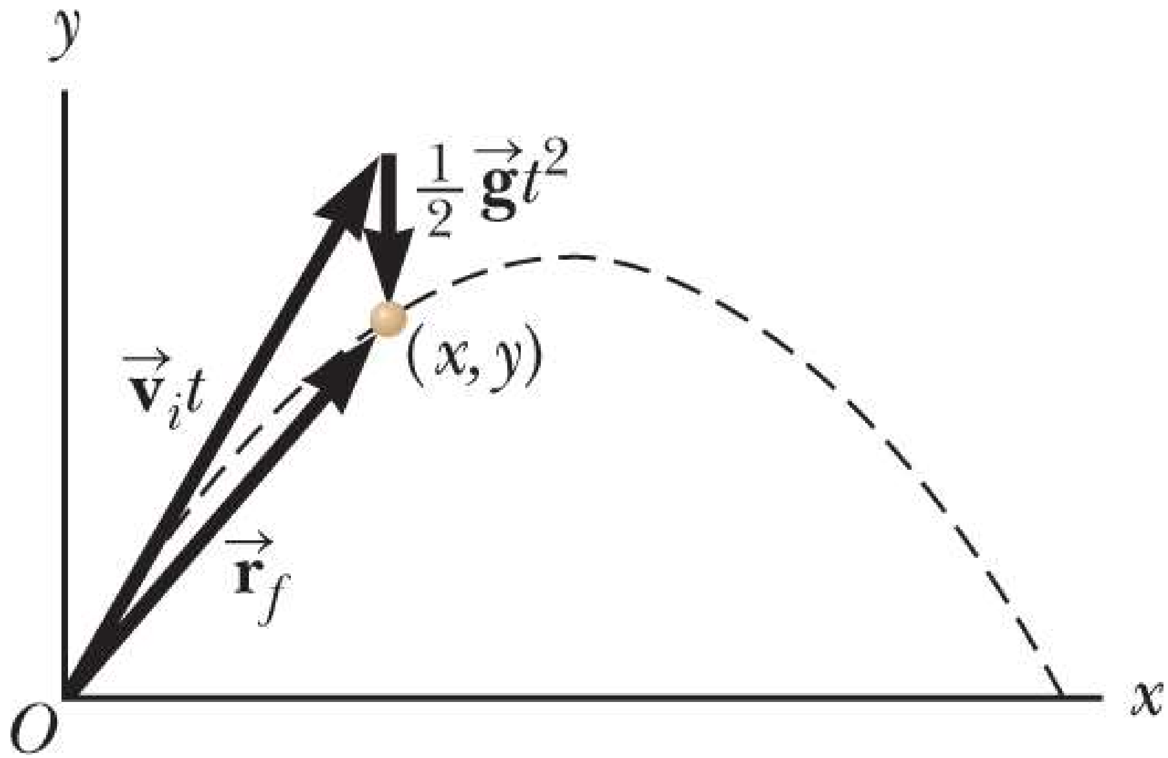
Quick Quiz 4.2 Part I

As a projectile thrown at an upward angle moves in its parabolic path (such as in the figure), at what point along its path are the velocity and acceleration vectors for the projectile perpendicular to each other?

(a) nowhere

(b) the highest point

(c) the launch point



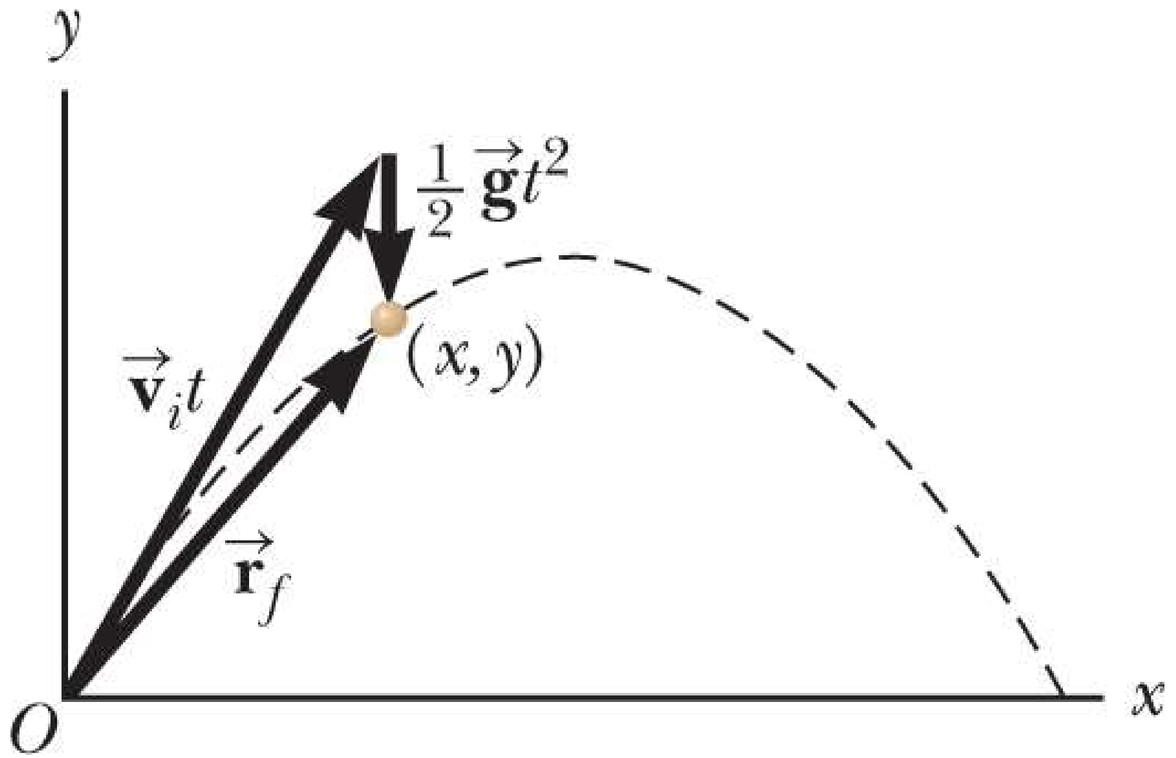
Quick Quiz 4.2 Part II

At what point are the velocity and acceleration vectors for the projectile parallel to each other?

(a) nowhere

(b) the highest point

(c) the launch point

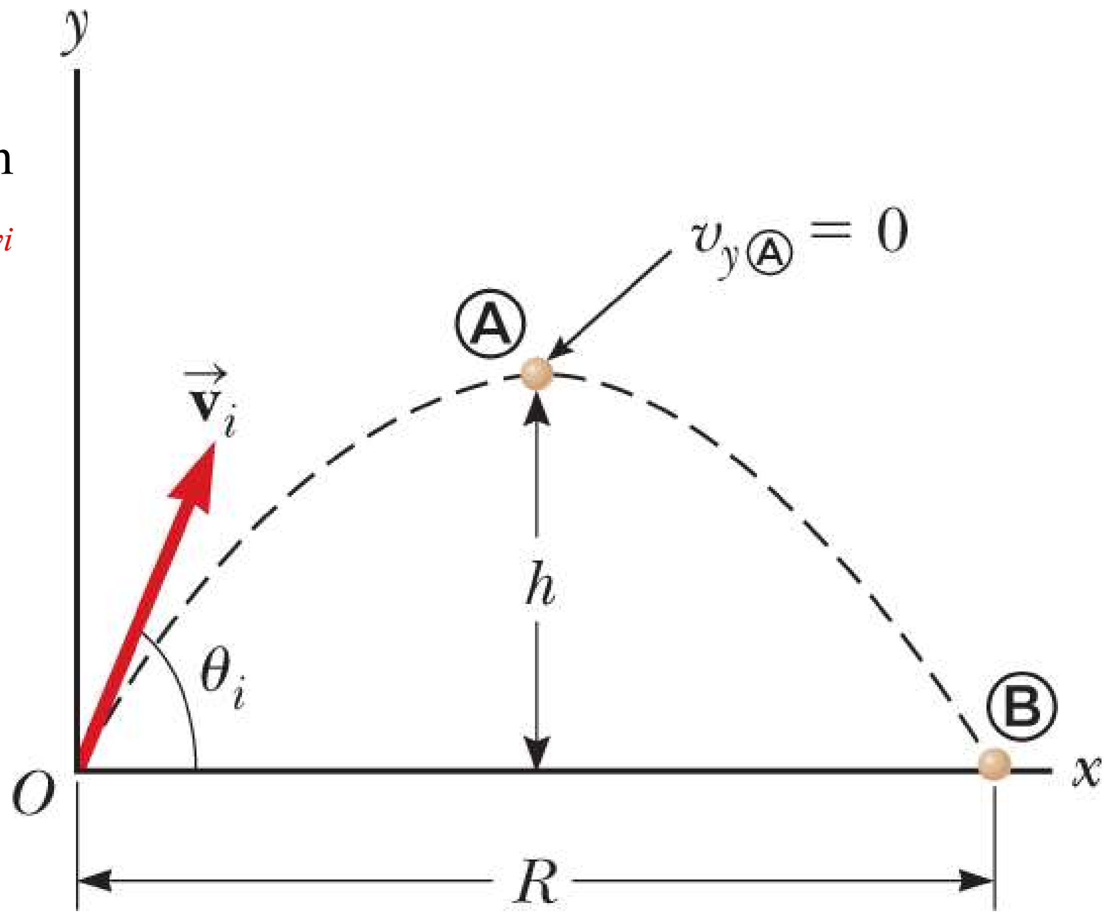


Horizontal Range and Maximum Height of a Projectile

- Two points especially interesting to analyze:
  - Peak **point A**, coordinates $(R/2, h)$
  - **Point B**, coordinates $(R, 0)$
    - ***R* = *horizontal range* of projectile** and ***h* = *maximum height***

- Let us find h and R mathematically in terms of v_i, θ_i, and g?

- <u>Special case of projectile motion</u>:
  - Assume projectile launched from origin at $t_i = 0$ with positive v_{yi} component (figure)
  - Returns to *same horizontal level*

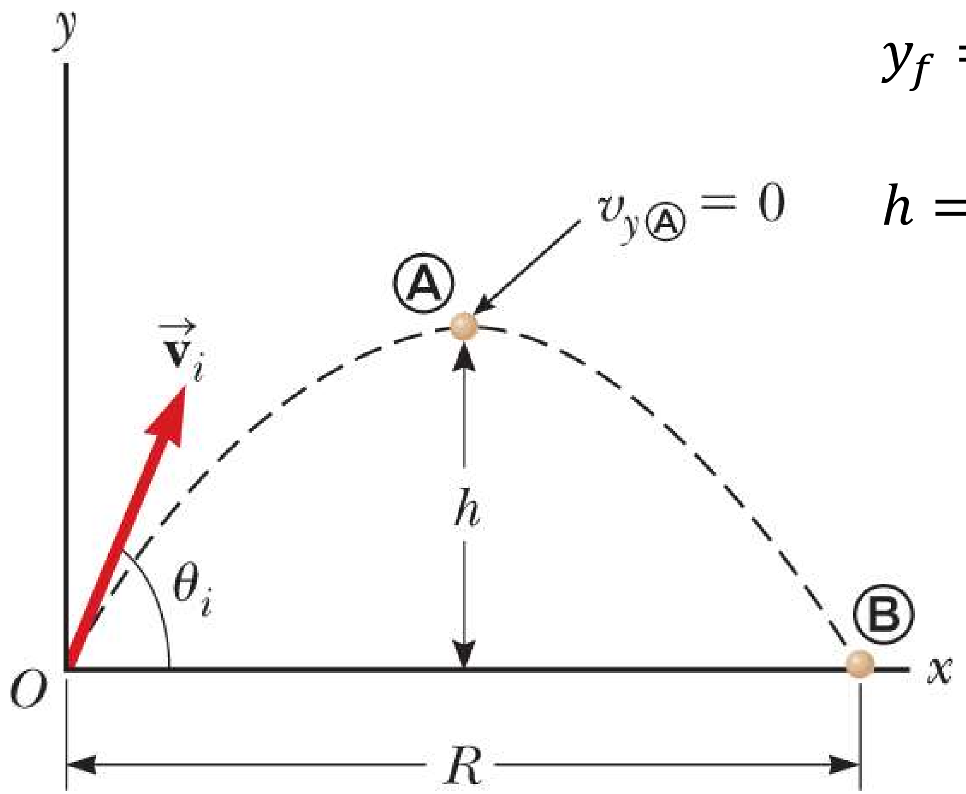


Height of a Projectile

- Determine h by noting that at peak $v_{yA} = 0$. From particle under constant acceleration model, determine time t_A at which projectile reaches peak:

$$v_{yf} = v_{yi} - gt \Rightarrow 0 = v_i \sin\theta_i - gt_A \Rightarrow t_A = \frac{v_i \sin\theta_i}{g}$$

- Substituting for t_A and replacing $y_f = y_A$ with h:



$$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2 \Rightarrow$$

$$h = (v_i \sin\theta_i)\frac{v_i \sin\theta_i}{g} - \frac{1}{2}g\left(\frac{v_i \sin\theta_i}{g}\right)^2$$

$$h = \frac{{v_i}^2 \sin^2\theta_i}{2g}$$

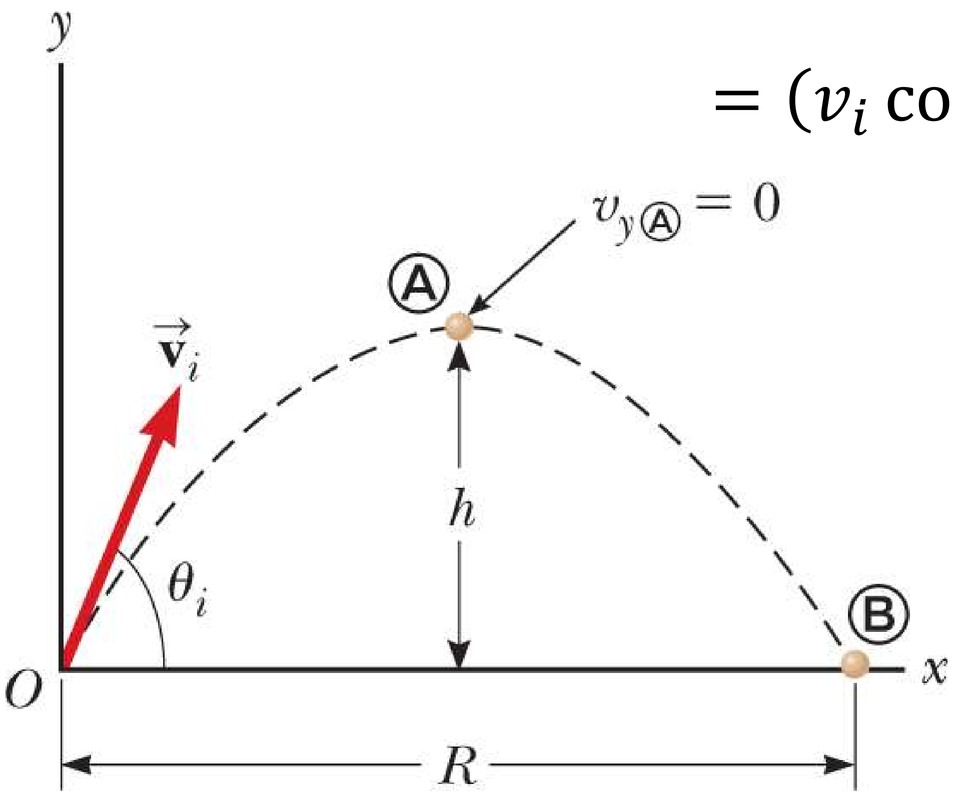
$$hmax = \frac{{v_i}^2}{2g} \; when\ \theta = 90^o$$

Horizontal Range of a Projectile

- From symmetry: projectile covers upward part of trajectory to top in exactly same time interval as it requires to come back to ground from topmost point
- Range R = horizontal position of projectile at time twice time at which it reaches its peak:
    - At time $t_B = 2t_A$
- Using particle under constant velocity model
    - With $v_{xi} = v_{xB} = v_i \cos \theta_i$
    - Setting $x_B = R$ at $t = 2t_A$: $x_f = x_i + v_{xi}t \Rightarrow R = v_{xi}t_B = (v_i \cos\theta)2t_A$

$$= (v_i \cos\theta_i)\frac{2v_i \sin\theta_i}{g} = \frac{2{v_i}^2 \sin\theta_i \cos\theta_i}{g}$$

$$2\sin\theta_i \cos\theta_i = \sin 2\,\theta_i$$

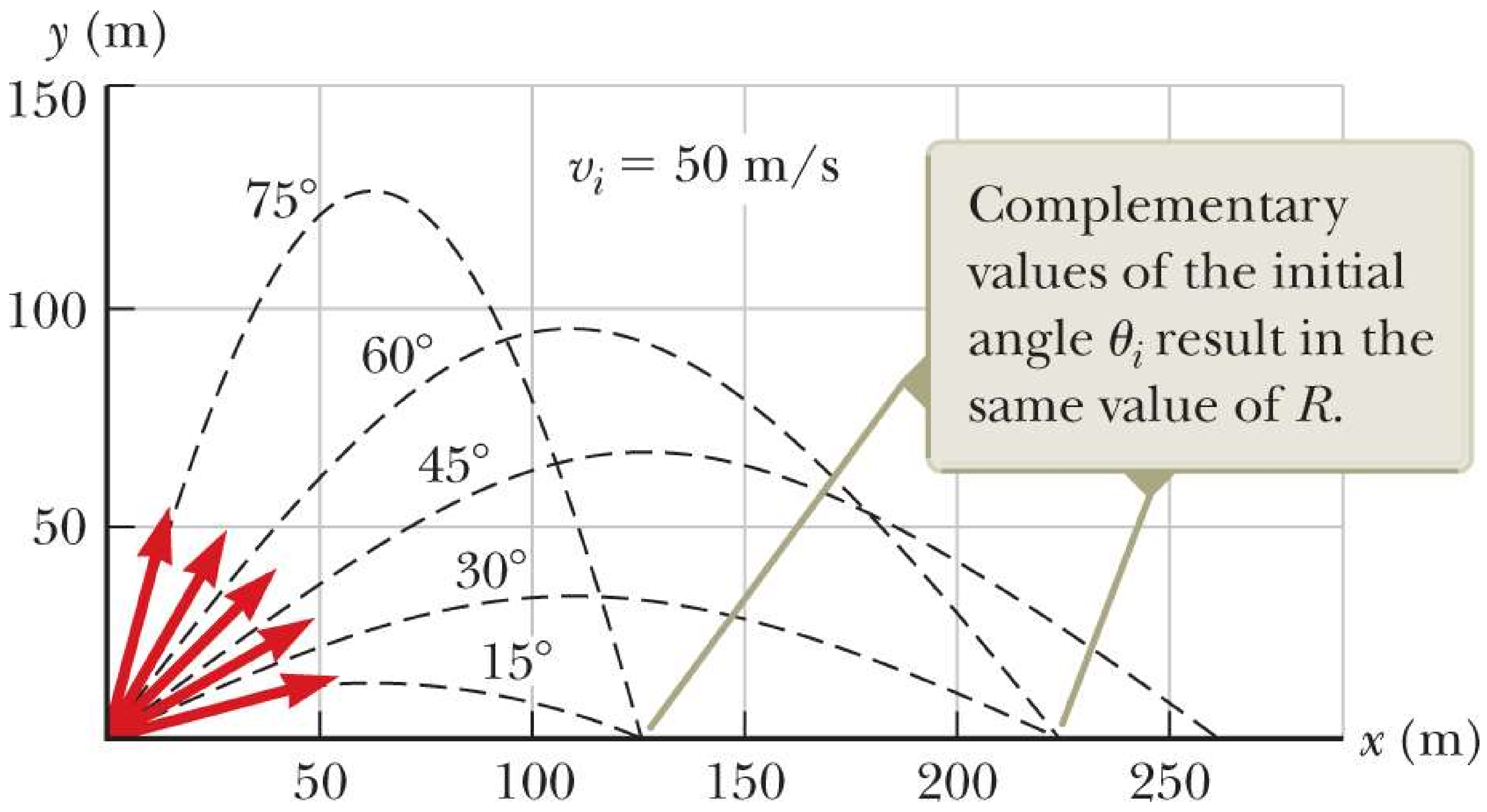


$$R = \frac{{v_i}^2 \sin 2\,\theta_i}{g}$$

- Note: maximum value of $\sin 2\theta_i = 1$, when $2\theta_i = 90°$.
    - R is maximum when $\theta_i = 45°$

Trajectories of a Projectile

- Figure shows various trajectories for projectile with given initial speed, launched at different angles:



- For any θ_i other than 45° → point with coordinates (R, 0) can be reached by using either of two complementary values of θ_i

Problem-Solving Strategy: Projectile Motion

1. Conceptualize

- Think about what is going on physically in problem.
- Establish mental representation by imagining the projectile moving along its trajectory.

2. Categorize

- Confirm that problem involves particle in free fall and that air resistance is neglected.
- Select coordinate system with *x* in horizontal direction and *y* in vertical direction.
  - Use particle under constant velocity model for *x* component of motion.
  - Use particle under constant acceleration model for *y* direction.

- In special case of projectile returning to same level from which it was launched, use:

$$h = \frac{{v_i}^2 \sin^2 \theta_i}{2g} \qquad R = \frac{{v_i}^2 \sin 2\,\theta_i}{g}$$

Problem-Solving Strategy: Projectile Motion

3. Analyze

- If initial velocity vector given: resolve into *x* and *y* components.
- Select appropriate equation(s) from particle under constant acceleration model for vertical motion
  - Use these along with the top equation for horizontal motion to solve for the unknown(s).

$$x_f = x_i + v_x t$$
$$v_{yf} = v_{yi} - gt$$
$$v_{\mathrm{y,avg}} = \frac{v_{yi} + v_{yf}}{2}$$

$$y_f = y_i + \frac{1}{2}\left(v_{yi} + v_{yf}\right)t$$
$$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$$
$${v_{yf}}^2 = {v_{yi}}^2 - 2g\left(y_f - y_i\right)$$

4. Finalize

- Once you have determined result:
  - check to see if answers consistent with mental and pictorial representations and results are realistic.

Example 4.2: The Long Jump

A long jumper leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.

(A) How far does he jump in the horizontal direction?

Conceptualize We model the long jumper as a particle and conceptualize his motion as equivalent to that of a simple projectile.

Analyze We find the range of the jumper using the range equation:

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

$$= \frac{(11.0 \text{ m/s})^2 \sin(2 \cdot 20.0°)}{9.80 \text{ m/s}^2}$$

$$= \boxed{7.94 \text{ m}}$$

Example 4.2: The Long Jump

(B) What is the maximum height reached?

Analyze We find the maximum height reached using the equation for the height:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g} = \frac{(11.0\text{ m/s})^2 (\sin 20.0°)^2}{2(9.80\text{ m/s}^2)}$$

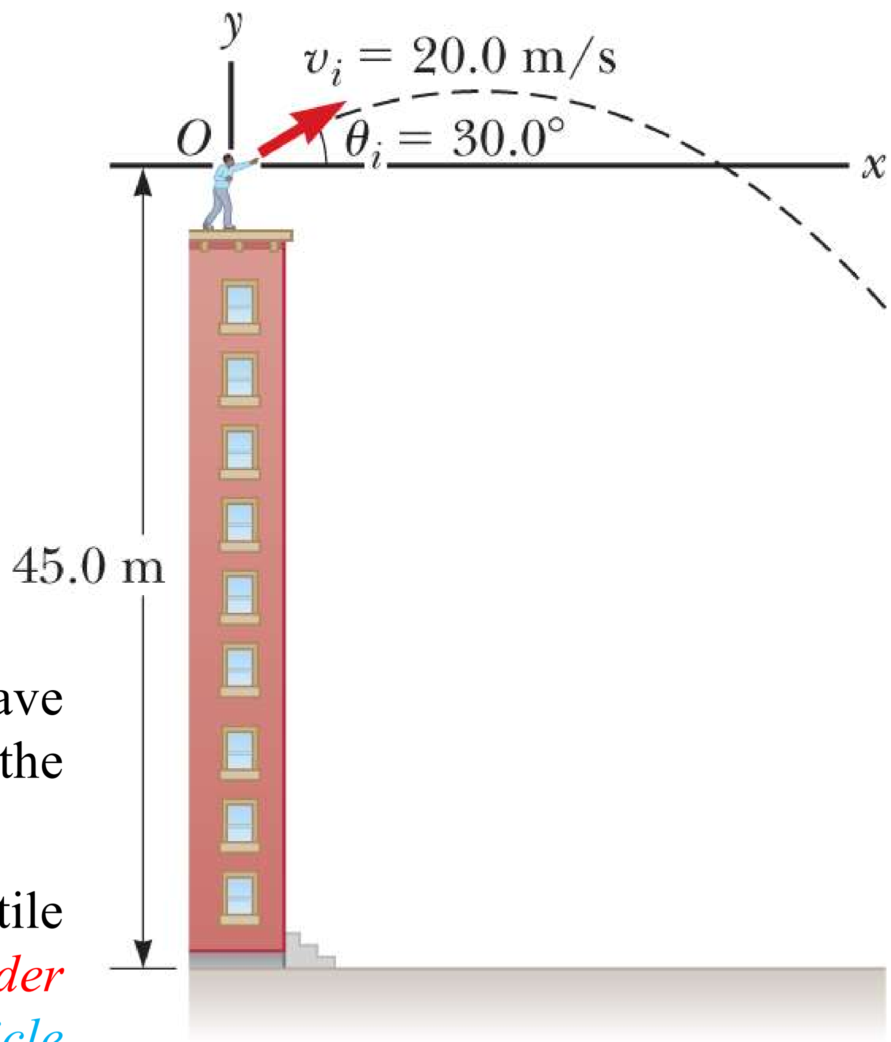
$$= \boxed{0.722\text{ m}}$$

Finalize Try finding the answers to parts (A) and (B) using the general method. The results should agree. Treating the long jumper as a particle is an oversimplification. Nevertheless, the values obtained are consistent with experience in sports. We can model a complicated system such as a long jumper as a particle and still obtain reasonable results.

Example 4.4: That's Quite an Arm

A stone is thrown from the top of a building upward at an angle of 30.0° to the horizontal with an initial speed of 20.0 m/s as shown in the figure. The height from which the stone is thrown is 45.0 m above the ground.

(A) How long does it take the stone to reach the ground?



Conceptualize Study the figure, in which we have indicated the trajectory and various parameters of the motion of the stone.

Categorize We categorize this problem as a projectile motion problem. The stone is modeled as a *particle under constant acceleration* in the *y* direction and a *particle under constant velocity* in the *x* direction.

Example 4.4: That's Quite an Arm

Analyze We have the information $x_i = y_i = 0$, $y_f = -45.0$ m, $a_y = -g$, and $v_i = 20.0$ m/s (the numerical value of y_f is negative because we have chosen the point of the throw as the origin).

Let us find the initial x and y components of the stone's velocity using the equations:

$$v_{xi} = v_i \cos\theta_i = (20.0 \text{ m/s}) \cos 30° = 17.3 \text{ m/s}$$
$$v_{yi} = v_i \sin\theta_i = (20.0 \text{ m/s}) \sin 30° = 10.0 \text{ m/s}$$

Express the vertical position of the stone from the particle under constant acceleration model:

$$y_f = y_i + v_{yi}t - \frac{1}{2}gt^2$$

Substitute numerical values: $$-45.0 \text{ m} = 0 + (10.0 \text{ m/s})t + \frac{1}{2}\left(-9.80 \text{ m/s}^2\right)t^2$$

Solve the quadratic equation for t: $$t = \boxed{4.22 \text{ s}}$$

Example 4.4:
That's Quite an Arm

(B) What is the speed of the stone just before it strikes the ground?

Analyze Use the velocity equation in the particle under constant acceleration model to obtain the *y* component of the velocity of the stone just before it strikes the ground

$$v_{yf} = v_{yi} - gt$$

Substitute numerical values, using $t = 4.22$ s:

$$v_{yf} = 10.0 \text{ m/s} + (-9.80 \text{ m/s}^2)(4.22 \text{ s}) = -31.3 \text{ m/s}$$

Use this component with the horizontal component $v_{xf} = v_{xi} = 17.3$ m/s to find the speed of the stone at $t = 4.22$ s:

$$v_f = \sqrt{v_{xf}{}^2 + v_{yf}{}^2}$$

$$= \sqrt{(17.3 \text{ m/s})^2 + (-31.3 \text{ m/s})^2} = \boxed{35.8 \text{ m/s}}$$

Finalize Is it reasonable that the *y* component of the final velocity is negative? Is it reasonable that the final speed is larger than the initial speed of 20.0 m/s?

Example 4.4: That's Quite an Arm

What if a horizontal wind is blowing in the same direction as the stone is thrown and it causes the stone to have a horizontal acceleration component $a_x = 0.500\ \text{m/s}^2$? Which part of this example, (A) or (B), will have a different answer?

Recall that the motions in the *x* and *y* directions are independent. Therefore, the horizontal wind cannot affect the vertical motion. The vertical motion determines the time of the projectile in the air, so the answer to part (A) does not change:

Time does not change.

The wind causes the horizontal velocity component to increase with time, so the final speed will be larger in part (B):

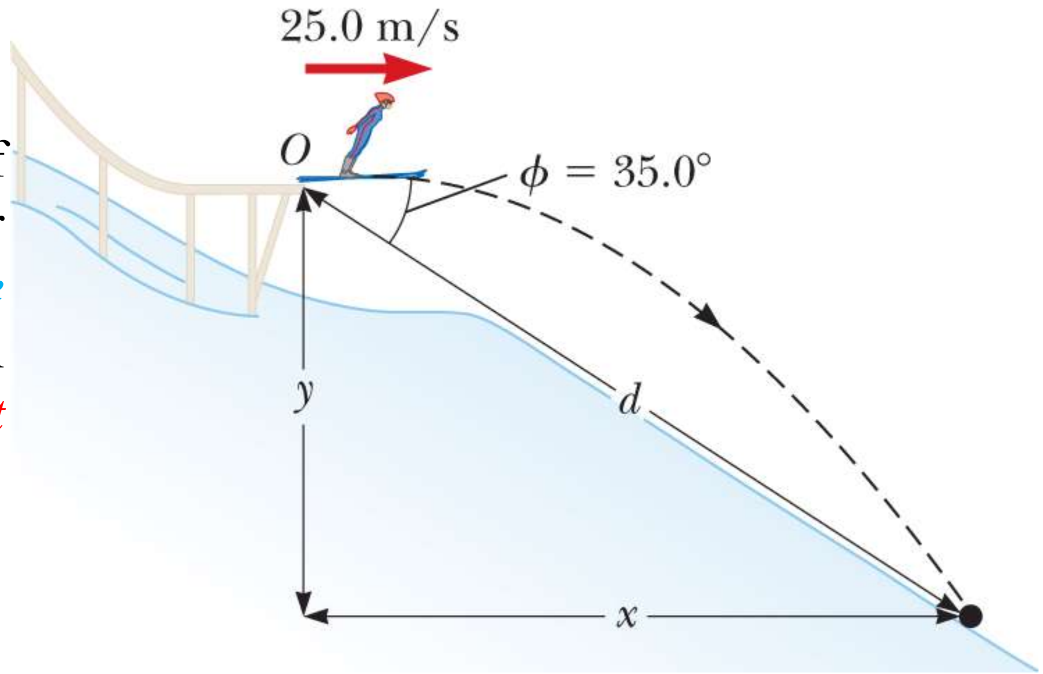
$$a_x = 0.500\ \text{m/s}^2 \rightarrow v_{xf} = 19.4\ \text{m/s} \text{ and } v_f = 36.9\ \text{m/s}$$

Final speed will be larger.

Example 4.5:
The End of the Ski Jump

A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s as shown in the figure. The landing incline below her falls off with a slope of 35.0°. Where does she land on the incline?

Categorize We categorize the problem as one of a particle in projectile motion. As with other projectile motion problems, we use the *particle under constant velocity* model for the horizontal motion and the *particle under constant acceleration* model for the vertical motion.



Example 4.5: The End of the Ski Jump

Analyze It is convenient to select the beginning of the jump as the origin. The initial velocity components are v_{xi} = 25.0 m/s and v_{yi} = 0. From the right triangle in the figure, we see that the jumper's x and y coordinates at the landing point are given by: (1) $x_f = d\cos\phi$ and (2) $y_f = 2d\sin\phi$.

$$(1)\ x_f = v_{xi}t \ \rightarrow (2)\ d\cos\phi = v_{xi}t$$

$$(3)\ y_f = v_{yi}t - \frac{1}{2}gt^2 \ \rightarrow (4)\ -d\sin\phi = -\frac{1}{2}gt^2$$

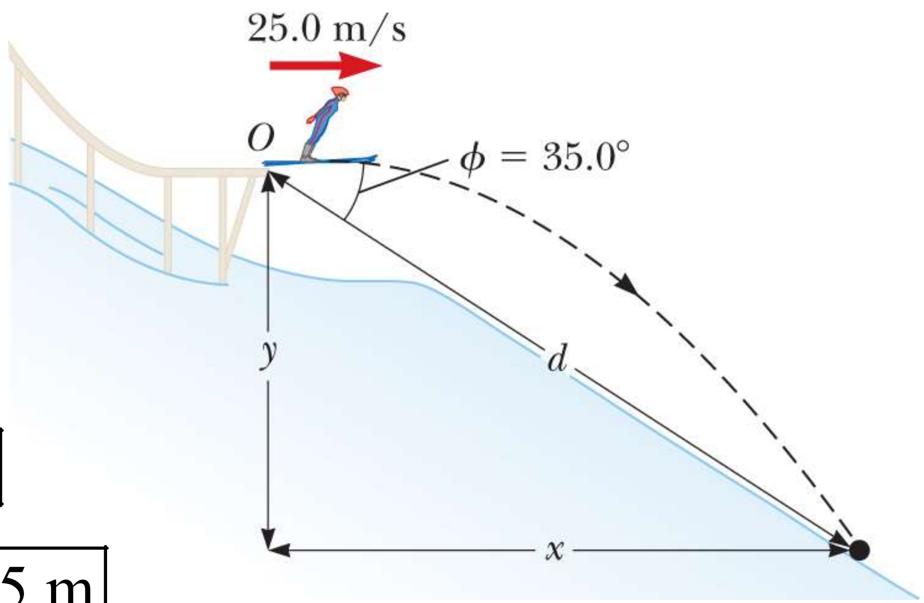
Solve Equation (2) for t and substitute the result into Equ. (4): $-d\sin\emptyset = -\frac{1}{2}g\left(\frac{d\cos\phi}{v_{xi}}\right)^2$

Solve for d and substitute numerical values:

$$d = \frac{2{v_{xi}}^2\sin\phi}{g\cos^2\phi} = \frac{2(25.0\ \mathrm{m/s})^2\sin 35.0°}{(9.80\ \mathrm{m/s^2})\cos^2 35.0°} = 109\ \mathrm{m}$$

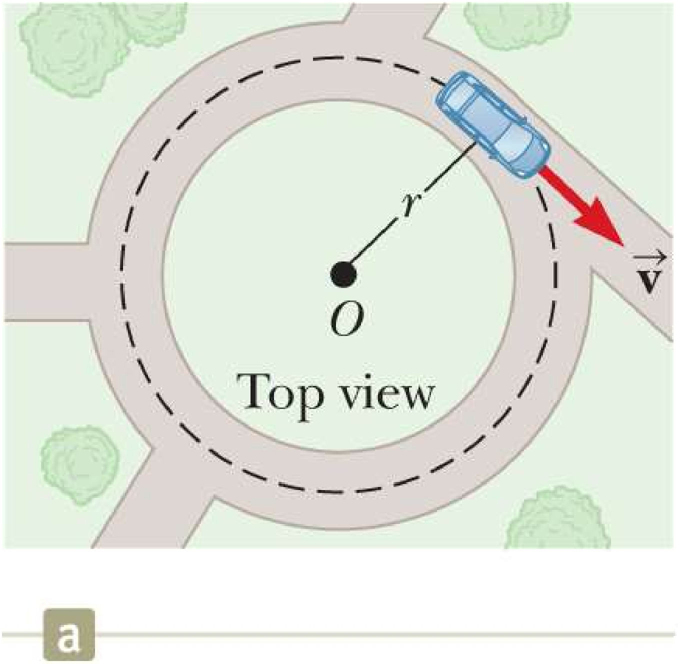
$$x_f = d\cos\phi = (109\ \mathrm{m})\cos 35.0° = \boxed{89.3\ \mathrm{m}}$$

$$y_f = -d\sin\phi = -(109\ \mathrm{m})\sin 35.0° = \boxed{-62.5\ \mathrm{m}}$$



4.4. Analysis Model: Particle in Uniform Circular Motion

- Figure shows car moving in circular path → **circular motion**
- If car moving with *constant speed v* → **uniform circular motion**
- Analysis model: **particle in uniform circular motion**



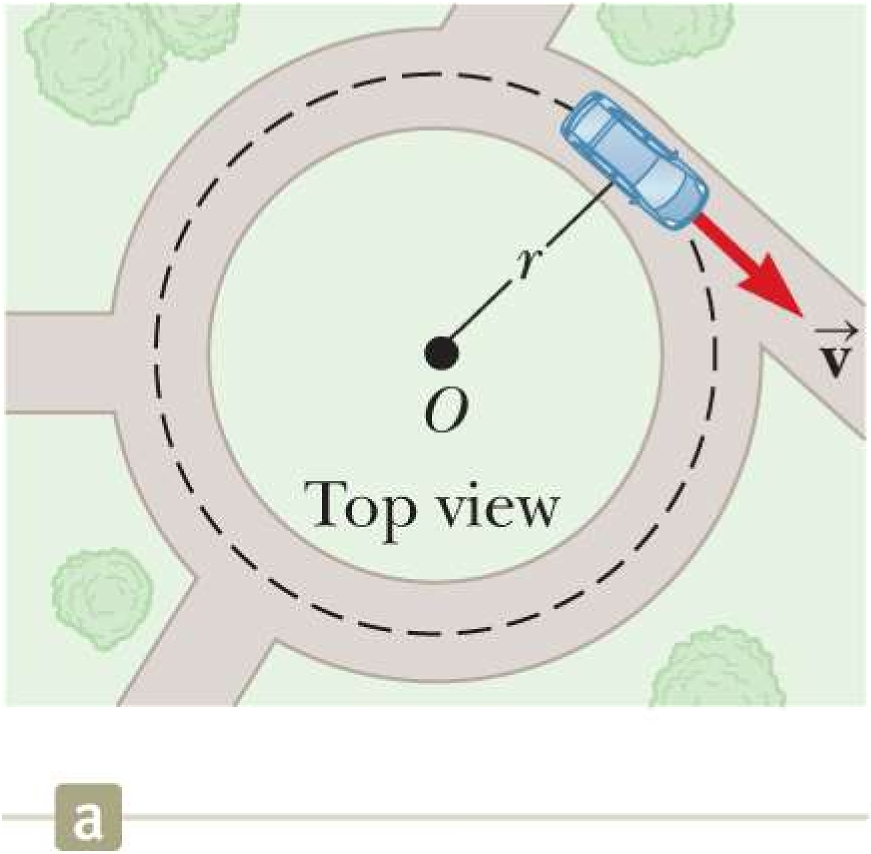
- Even though object moves at constant speed in circular path, *it still has an **acceleration***.
- Why? Consider defining equation for acceleration:

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt}$$

- Acceleration depends on change in *velocity*
- *V*elocity is vector quantity → acceleration can occur in two ways:
  - Change in *magnitude* of velocity
  - Change in *direction* of velocity
- Object moving with constant speed in circular path: change in direction
- Constant-magnitude velocity vector always tangent to path of object and perpendicular to radius of circular path. But the direction of velocity vector *always changing*

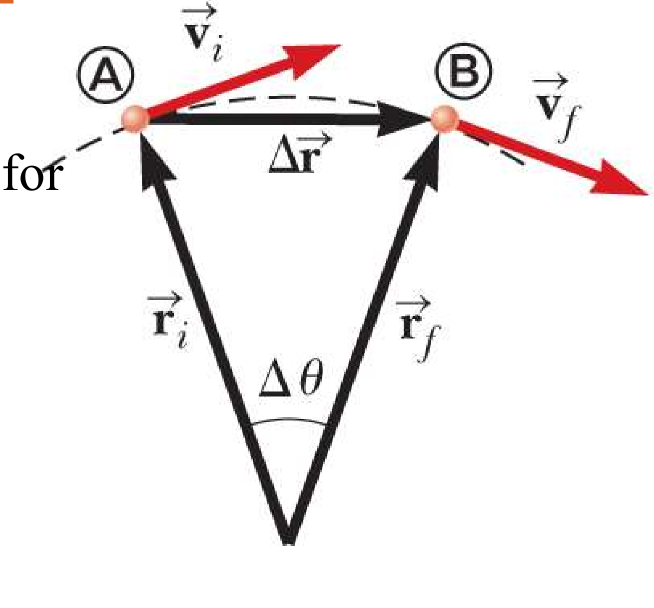
4.4. Analysis Model: Particle in Uniform Circular Motion

- Acceleration vector in uniform circular motion always perpendicular to path
  - Always points toward center of circle
- If not true → would be component of acceleration parallel to path
  - therefore parallel to velocity vector
- This acceleration component would lead to change in speed of particle
  - But: particle moves with constant speed along path
- For *uniform* circular motion →
  - acceleration vector can only have component perpendicular to path:
    - toward center of circle



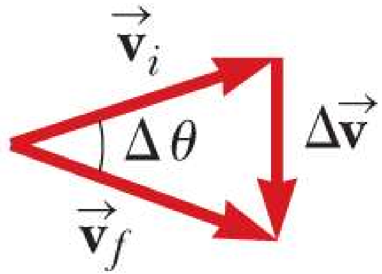
4.4. Analysis Model: Particle in Uniform Circular Motion

- Magnitude of acceleration of particle:
- Consider diagram of position and velocity vectors (figure (b))
  - Figure also shows vector representing change in position $\Delta\mathbf{r}$ for arbitrary time interval



b

- Particle follows circular path of radius r (dashed curve)
  - Particle at A at time t_i
    - velocity at that time = $\mathbf{v}_i$
  - At B at some later time t_f
    - velocity at that time = $\mathbf{v}_f$
- Assume $\mathbf{v}_i$ and $\mathbf{v}_f$ differ only in direction; their magnitudes are the same:
  - $v_i = v_f = v$ because *uniform* circular motion



c

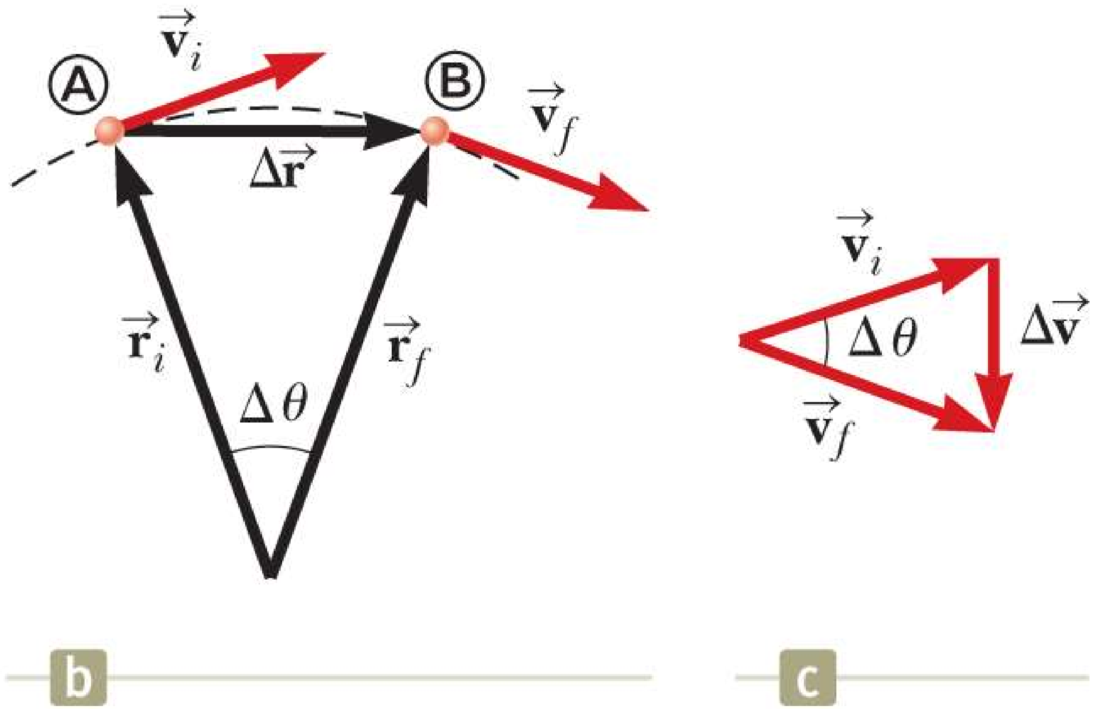
- Figure (c): velocity vectors redrawn tail to tail.
  - Vector $\Delta\mathbf{v}$ connects tips of vectors → vector addition $\mathbf{v}_f = \mathbf{v}_i + \Delta\mathbf{v}$
- Both figures: identify triangles:
- Figure (b): angle $\Delta\theta$ between two position vectors same as angle between velocity vectors in right figure →
  - velocity vector $\mathbf{v}$ is always perpendicular to position vector $\mathbf{r}$
  - Two triangles *similar*. Recall: two triangles similar if angle between any two sides same for both triangles and if ratio of lengths of these sides same

4.4. Analysis Model: Particle in Uniform Circular Motion

- Write relationship between lengths of sides for two triangles:

$$\frac{|\Delta\vec{\mathbf{v}}|}{v} = \frac{|\Delta\vec{\mathbf{r}}|}{r}$$

  - Where $v = v_i = v_f$ and $r = r_i = r_f$



- Solve for $|\Delta\mathbf{v}|$ and substitute into equation for $\mathbf{a}_{avg}$:

$$\left|\vec{\mathbf{a}}_{avg}\right| = \frac{|\Delta\vec{\mathbf{v}}|}{\Delta t} = \frac{v|\Delta\vec{\mathbf{r}}|}{r\Delta t}$$

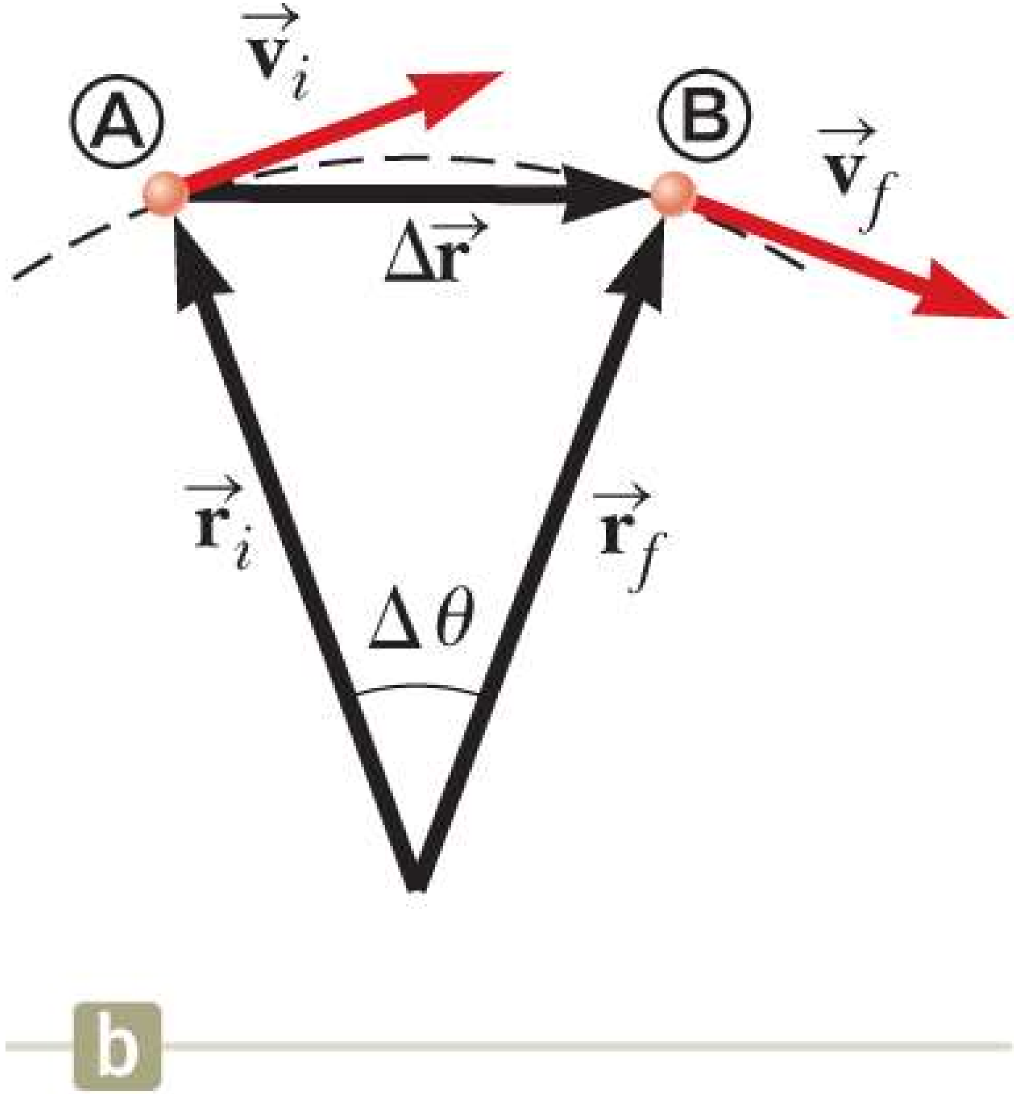
  - Equation gives magnitude of average acceleration over time interval for particle to move from A to B

4.4. Centripetal Acceleration

- Imagine points A and B in Figure become extremely close together
- As A and B approach each other:
    - $\Delta t \to 0$
    - $|\Delta \mathbf{r}| \to$ distance traveled by particle along circular path
    - Ratio $|\Delta \mathbf{r}|/\Delta t \to$ speed v
- Average acceleration becomes instantaneous acceleration at point A
- In limit $\Delta t \to 0$: magnitude of acceleration is:

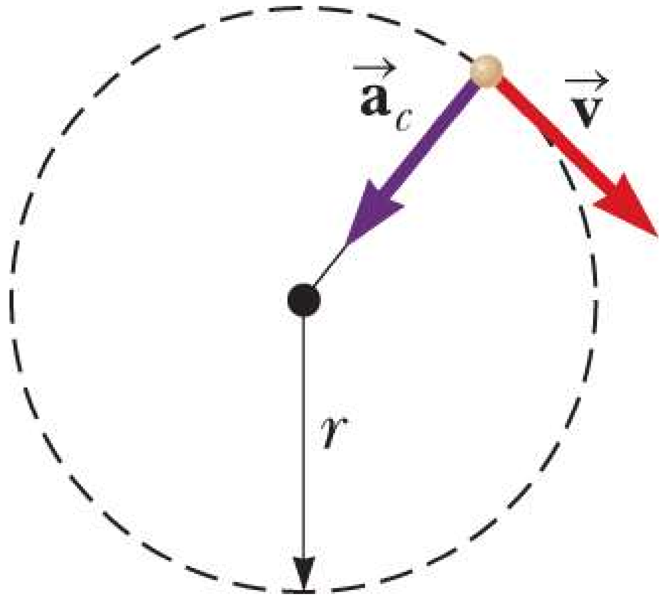
$$a_c = \frac{v^2}{r}$$

- Called **centripetal acceleration** (*centripetal* means *center-seeking*)
    - Subscript on acceleration symbol reminds us that acceleration centripetal



4.4. Particle in Uniform Circular Motion

- Describe motion of particle moving with constant speed in circle of radius r in terms of **period** T:
  - Defined as time interval required for one complete revolution of particle
  - In time interval T: particle moves distance of $2\pi r$ (circumference of particle's circular path)
- Because speed = circumference of circular path divided by period: $v = \frac{2\pi r}{T}$
  - Rewriting: $$T = \frac{2\pi r}{v}$$



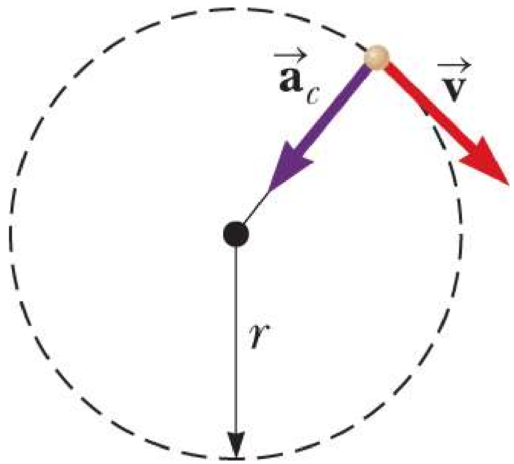
- Period of particle in uniform circular motion = measure of number of seconds for one revolution
- Inverse of period = *rotation rate*
  - Measured in revolutions per second
- One full revolution of particle around circle corresponds to angle 2π radians
- Product of 2π and rotation rate gives **angular speed** ω of particle: $$\omega = \frac{2\pi}{T}$$
  - Measured in radians/s or s^{-1}

4.4. Particle in Uniform Circular Motion

- Relationship between angular speed and translational speed with which particle travels in circular path:

$$\omega = 2\pi\left(\frac{v}{2\pi r}\right) = \frac{v}{r} \rightarrow v = r\omega$$

- For fixed angular speed: translational speed becomes larger as radial position becomes larger
  - Example: if merry-go-round rotates at fixed angular speed → rider at outer position at large r traveling through space faster than rider at inner position at smaller r



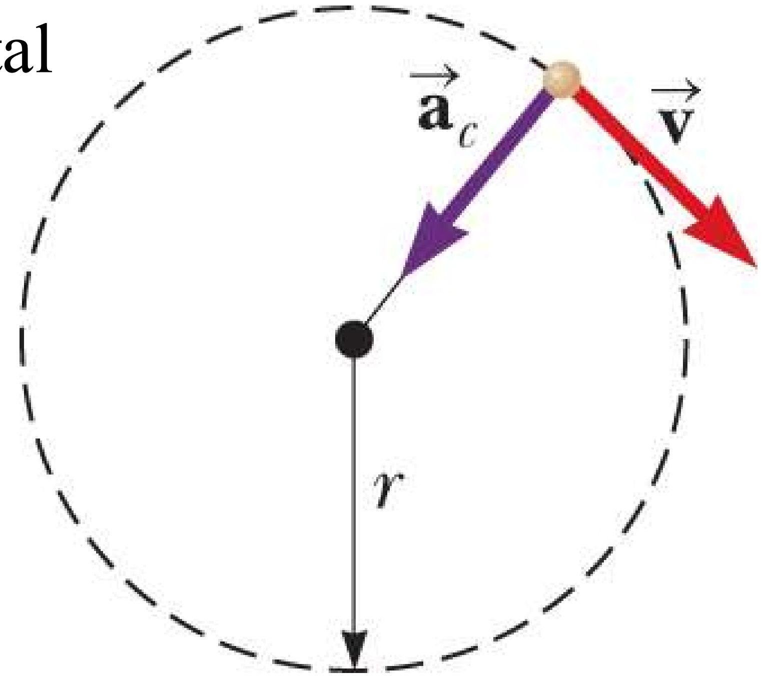
- Express centripetal acceleration of particle in uniform circular motion in terms of angular speed:

$$\boldsymbol{a_c} = \frac{(\boldsymbol{r\omega})^2}{\boldsymbol{r}} = \boldsymbol{r\omega^2}$$

Analysis Model:
Particle in Uniform Circular Motion

Imagine moving object that can be modeled as particle. If it moves in circular path of radius r at constant speed v, the magnitude of its centripetal acceleration is:

$$a_c = \frac{v^2}{r}$$



Period of the particle's motion is: $$T = \frac{2\pi r}{v}$$

Angular speed of particle is: $$\omega = \frac{2\pi}{T}$$

Examples:

- A rock twirled in a circle on a string of constant length
- A planet traveling around a perfectly circular orbit
- A charged particle moving in a uniform magnetic field
- An electron in orbit around a nucleus in the Bohr model of the hydrogen atom

Example 4.6: The Centripetal Acceleration of the Earth

(A) What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?

Conceptualize We will model the Earth as a particle and approximate the Earth's orbit as circular

Categorize The Conceptualize step allows us to categorize this problem as one of a *particle in uniform circular motion.*

Analyze We do not know the orbital speed of the Earth to substitute into the equation for the centripetal acceleration. But we can use the relationship between period and velocity to recast our equation:

$$a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

$$a_c = \frac{4\pi^2(1.496 \times 10^{11}\text{ m})}{(1\text{ yr})^2}\left(\frac{1\text{ yr}}{3.156 \times 10^7\text{ s}}\right)^2 = \boxed{5.93 \times 10^{-3}\text{ m/s}^2}$$

Finalize The acceleration is much smaller than the free-fall acceleration on the surface of the Earth. An important technique we learned here is replacing the speed v in the equation for centripetal acceleration in terms of the period T of the motion. In many problems, it is more likely that T is known rather than v.

Example 4.6: The Centripetal Acceleration of the Earth

(B) What is the angular speed of the Earth in its orbit around the Sun?

Analyze Substitute numerical values into the equation for angular speed:

$$\omega = \frac{2\pi}{1\ \text{yr}}\left(\frac{1\ \text{yr}}{3.156\times10^{7}\ \text{s}}\right) = \boxed{1.99\times10^{-7}\ \text{s}^{-1}}$$

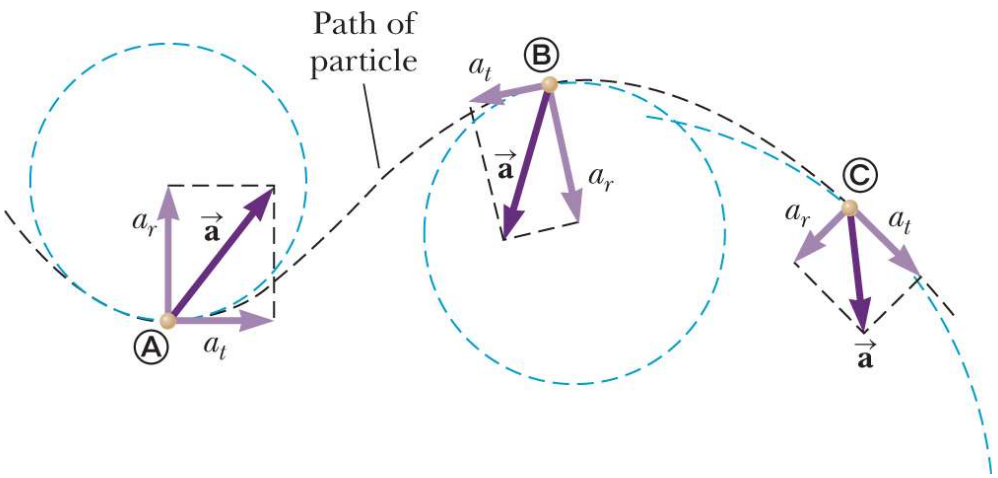
Finalize We see that the angular speed of the Earth is very small, which is to be expected because the Earth takes an entire year to go around the circular path once.

4.5. Tangential and Radial Acceleration

- Consider more general motion: Particle moves to right along curved path → velocity changes *both* in direction and in magnitude (figure)
    - Velocity vector always tangent to path
    - Acceleration vector **a** at some angle to path

- At each of three points A, B, and C: dashed blue circles represent curvature of actual path at each point
    - Radius of each circle = path's radius of curvature at each point

- As particle moves along curved path → direction of total acceleration vector **a** changes from point to point
- At any instant: this vector can be resolved into two components based on origin at center of dashed circle corresponding to that instant:
    - Radial component a_r along radius of circle
    - Tangential component a_t perpendicular to radius



4.5. Tangential and Radial Acceleration

- *Total* acceleration vector **a** written as vector sum of component vectors: $\vec{a} = \vec{a}_r + \vec{a}_t$
- Tangential acceleration component causes change in speed v of particle
  - Component parallel to instantaneous velocity
  - Magnitude given by:

$$a_t = \left|\frac{dv}{dt}\right|$$

- Radial acceleration component arises from change in directio velocity vector:

$$a_r = -a_c = -\frac{v^2}{r}$$

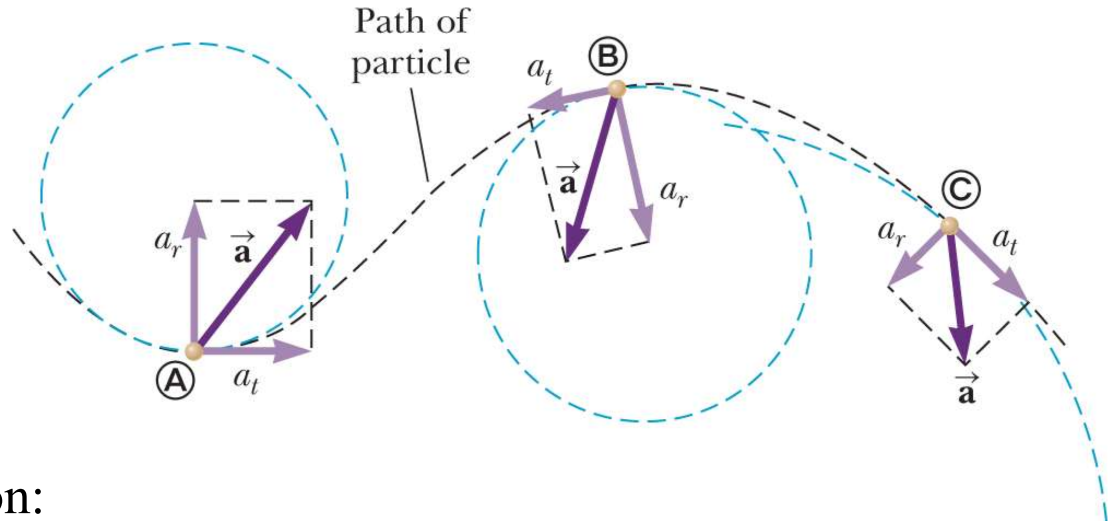
  - r = radius of curvature of path at point in question
- Magnitude of radial component of acceleration: centripetal acceleration
  - Even in situations in which particle moves along curved path with varying speed → can still use centripetal acceleration equation
    - Gives *instantaneous* centripetal acceleration at any time
- Negative sign in last equation indicates that direction of centripetal acceleration toward center of circle representing radius of curvature
  - Direction opposite that of radial unit vector **r** (always points away from origin at center of circle)
- Because $\mathbf{a}_r$ and $\mathbf{a}_t$ ⊥ component vectors of **a**: magnitude of **a** is:

$$a = \sqrt{a_r{}^2 + a_t{}^2}$$

4.5. Tangential and Radial Acceleration

- For given speed:
  - a_r large when radius of curvature small (points A and B)
  - a_r small when r large (point C).

- Direction of $\mathbf{a}_t$ either in same direction as $\mathbf{v}$ (if v increasing) or opposite $\mathbf{v}$ (if v decreasing) (e.g., at point B)

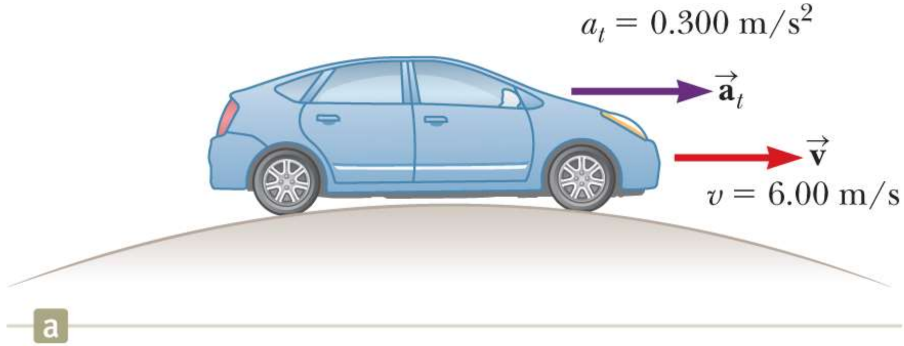


- Uniform circular motion:
  - v constant
  - $a_t = 0$
  - Acceleration always completely radial
- Uniform circular motion: special case of motion along general curved path
- If direction of $\mathbf{v}$ does not change $\rightarrow$ no radial acceleration:
  - motion one dimensional (in this case, $a_r = 0$, but a_t may not be zero)

Example 4.7: Over the Rise

A car leaves a stop sign and exhibits a constant acceleration of 0.300 m/s^2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like an arc of a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s. What are the magnitude and direction of the total acceleration vector for the car at this instant?

Conceptualize Conceptualize the situation using the figure and any experiences you have had in driving over rises on a roadway.



Categorize Because the accelerating car is moving along a curved path, we *categorize this problem as one involving a particle experiencing both tangential and radial acceleration*. We recognize that it is a relatively simple substitution problem. The tangential acceleration vector has magnitude 0.300 m/s^2 and is horizontal. The radial acceleration is given by $-v^2/R$, with v = 6.00 m/s and r = 500 m. The radial acceleration vector is directed straight downward.

Example 4.7: Over the Rise

Evaluate the radial acceleration:

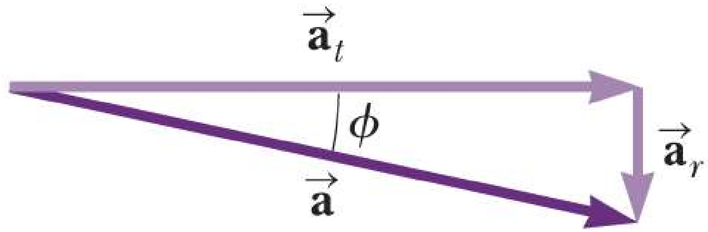
$$a_r = -\frac{v^2}{r} = -\frac{(6.00\text{ m/s})^2}{500\text{ m}} = -0.0720\text{ m/s}$$

Find the magnitude of **a**:

$$a = \sqrt{a_r^{\ 2} + a_t^{\ 2}}$$

$$= \sqrt{(-0.0720\text{ m/s}^2)^2 + (0.300\text{ m/s}^2)^2} = \boxed{0.309\text{ m/s}^2}$$

For the angle, use a sketch of the vectors. Find the angle ϕ between **a** and the horizontal:



$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.0720\text{ m/s}^2}{0.300\text{ m/s}^2}\right) = \boxed{-13.5°}$$