Submodularity of Variance

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1 Proof of Submodularity of Variance function

The variance function can be used as a candidate symmetric potential function for multilabel higher order cliques, below is a proof for submodularity of variance function:

1.1 Proof

Let $L^a = (l_i^a)_1^k$, $L^b = (l_i^b)_1^k$ be two labelings on a k size clique where each label l_i^x belongs to the set of labels S, We use μ^x, σ^x to represent the mean and standard deviations for given labeling x, Now to prove submodularity we need:

$$(\sigma^a)^2 + (\sigma^b)^2 \geq (\sigma^\vee)^2 + (\sigma^\wedge)^2 \tag{1}$$

Where $\vee, \wedge$ are the component wise maximum and minimum operators defined on the set of all labelings. Also we have:

$$k\mu^a = \sum_{j=1}^{k} l_i^a \tag{2}$$

$$k\mu^b = \sum_{j=1}^{k} l_i^b \tag{3}$$

$$k\mu^\vee = \sum_{i=1}^{k} 1(l_i^a > l_i^b)[l_i^a] + 1(l_i^a \leq l_i^b)[l_i^b] \tag{4}$$

$$k\mu^\wedge = \sum_{i=1}^{k} 1(l_i^a > l_i^b)[l_i^b] + 1(l_i^a \leq l_i^b)[l_i^a] \tag{5}$$

Here 1() is the indicator function, From 2- 5 we can see that

$$\mu^\vee \geq \mu^a, \mu^b \geq \mu^\wedge \tag{6}$$

$$\mu^a + \mu^b = \mu^\vee + \mu^\wedge \tag{7}$$

Now for submodularity enough to prove that:

$$k((\sigma^a)^2 + (\sigma^b)^2 - (\sigma^\vee)^2 - (\sigma^\wedge)^2) \geq 0 \tag{8}$$

The LHS in 8 can be simplified as:

$$\begin{aligned}&\sum_{j=1}^{k} (l_j^a - \mu^a)^2 + (l_j^a - \mu^b)^2 - \{1(l_i^a > l_i^b)(l_j^a - \mu^\vee)^2 + 1(l_i^a \leq l_i^b)(l_j^b - \mu^\vee)^2 \\ &+ 1(l_i^a > l_i^b)(l_j^b - \mu^\wedge)^2 + 1(l_i^a \leq l_i^b)(l_j^a - \mu^\wedge)^2\}\end{aligned} \tag{9}$$

On opening the squares in 9 we get:

$$\begin{aligned}&\sum_{j=1}^{k} (\mu^a)^2 + (\mu^b)^2 - ((\mu^\vee)^2 + (\mu^\wedge)^2) + \{1(l_i^a > l_i^b)[2l_j^a\mu^\vee + 2l_j^b\mu^\wedge] \\ &+ 1(l_i^a \leq l_i^b)[2l_j^b\mu^\vee + 2l_j^a\mu^\wedge]\}\end{aligned} \tag{10}$$

$$= 2\sum_{j=1}^{k} \mu^\vee\mu^\wedge - \mu^a\mu^b + \{1(l_i^a > l_i^b)[2l_j^a\mu^\vee + 2l_j^b\mu^\wedge] + 1(l_i^a \leq l_i^b)[2l_j^b\mu^\vee + 2l_j^a\mu^\wedge]\} \tag{11}$$

11 is obtained by using 7 which gives

$$(\mu^a)^2 + (\mu^b)^2 - (\mu^\vee)^2 + (\mu^\wedge)^2 = 2(\mu^\vee\mu^\wedge - \mu^a\mu^b) \tag{12}$$

Also using univariate calculus along with property 6 it can be seen that:

$$(\mu^a)^2 + (\mu^b)^2 - (\mu^\vee)^2 + (\mu^\wedge)^2 = 2(\mu^\vee\mu^\wedge - \mu^a\mu^b) \leq 0 \tag{13}$$

11 can be rewritten as:

$$2\sum_{j=1}^{k} \mu^\vee\mu^\wedge - \mu^a\mu^b + \{\mu^\vee[1(l_i^a > l_i^b)l_j^a + 1(l_i^a \leq l_i^b)l_j^b] + \mu^\wedge[1(l_i^a > l_i^b)l_j^b + 1(l_i^a \leq l_i^b)l_j^a]\} \tag{14}$$

$$= 2[k\mu^\vee\mu^\wedge - k\mu^a\mu^b + k(\mu^\vee)^2 + k(\mu^\wedge)^2] \tag{15}$$

$$= 2k[(\mu^\vee)^2 + (\mu^\wedge)^2 + \mu^\vee\mu^\wedge - \mu^a\mu^b] \tag{16}$$

$$= 2k[(\mu^a)^2 + (\mu^b)^2 - \mu^\vee\mu^\wedge + \mu^a\mu^b] \tag{17}$$

Where 15 can be obtained by using the definitions 4, 5 and equation 17 can be obtained by using property 7, Finally we have:

$$(\mu^a)^2, (\mu^b)^2 \geq 0 \tag{18}$$

Which along with 13 proves that 17 is always non negative, hence proving the submodularity of variance.