ASSIGNMENT # 02

LINEAR ALGEBRA

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Matrix Determinant :-

For every square matrix, $A = [a_{ij}]$ of order $n \times n$, a determinant can be defined as scalar value that is real or a complex number, where C_{ij} is the $(i, j)^{th}$ element of matrix A. The determinant can be denoted as $\det(A)$ or $|A|$.

It is sum of of n terms of form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots a_{1n}$ are from first row of A. In symbols

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \ldots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Properties Of Determinants :-

① The determinant of identity matrix is always 1.

Example:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore |I| = (1)(1) - (0)(0)$$

$$= 1$$

(2) If the Matrix A^T is the transpose of Matrix A, then $\det(A^T) = \det(A)$

Example:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

$$|A| = (4)(1) - (3)(2) \qquad |A^T| = (4)(1) - (2)(3)$$

$$= 4 - 6 \qquad = 4 - 6$$

$$= -2 \qquad = -2$$

Hence $|A| = |A^T|$

(3) If Matrix A^{-1} is the inverse of Matrix A, then $\det(A^{-1}) = \dfrac{1}{\det(A)} = \det(A)^{-1}$

Example:

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(4)(1)-(3)(2)}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -1/2 & 1 \\ 3/2 & -2 \end{bmatrix}$$

$$\det(A^{-1}) = \left(\frac{-1}{2}\right)(-2) - \left(\frac{3}{2}\right)(1)$$

$$= 1 - \frac{3}{2} = -0.5$$

$$\det(A)^{-1} = \frac{1}{\det(A)} = \frac{1}{-2} = -0.5$$

Hence $\det(A^{-1}) = \det(A)^{-1}$

④ If two square matrices A and B are of equal size, then det(AB) = det(A) det(B)

Example:

$$A = \begin{bmatrix} 1 & 6 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 \\ 3 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} (1\times4)+(6\times3) & (1\times2)+(6\times4) \\ (3\times4)+(5\times3) & (3\times2)+(5\times4) \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 26 \\ 27 & 26 \end{bmatrix}$$

$$|A| = (1)(5) - (3)(6)$$
$$= 5 - 18$$
$$= -13$$

$$|B| = (4)(4) - (3)(2)$$
$$= 16 - 6$$
$$= 10$$

$$|A||B| = (-13)(10)$$
$$= -130$$

$$|AB| = (22)(26) - (27)(26)$$
$$= 572 - 702$$
$$= -130$$

Hence $|AB| = |A||B|$

⑤ If a square matrix A has a zero row or a zero column, then det(A) = 0

Example:-

① $A = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$

$$|A| = (0)(4) - (0)(2)$$
$$= 0 - 0$$
$$= 0$$

② $A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$

$$|A| = (2)(0) - (0)(4)$$
$$= 0 - 0$$
$$= 0$$

⑥ If A is an upper or lower triangular matrix then det(A) is the product of all its diagonal entries.

Example:

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$|A| = (4)(5)(6)$$
$$= 120$$

$$|A| = 4(30-0) - 1(0) + 2(0)$$
$$= 4(30)$$
$$= 120$$

Hence proved.

⑦ If A is a square matrix, then if its row is multiplied by a constant K, then the constant can be taken out of determinant.

Example:

$$A = \begin{bmatrix} Ka & Kb \\ c & d \end{bmatrix}, \quad A = K \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 8 \\ 3 & 4 \end{bmatrix} \quad \text{or} \quad A = 4 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$|A| = (4)(4) - (3)(8)$$
$$= 16 - 24$$
$$= -8$$

$$|A| = 4 \times \left((1)(4) - (3)(2)\right)$$
$$= 4 \times (4 - 6)$$
$$= 4 \times (-2)$$
$$= -8$$

Hence proved.

(8) If a is A square matrix with same rows or columns then its determinant is equal to 0.

Example:

$$A = \begin{bmatrix} 5 & 5 \\ 6 & 6 \end{bmatrix}$$

$$|A| = (5)(6) - (6)(5)$$
$$= 30 - 30$$
$$= 0$$

Hence proved.

(9) If A is a square matrix then interchanging any two rows (or column), the sign of determinant changes.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} , \quad A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$|A| = (1)(4) - (3)(2) , \quad |A| = (2)(3) - (4)(1)$$
$$= 4 - 6 \qquad = 6 - 2$$
$$= -2 \qquad = 2$$

Hence Proved

(10) If each element of any row (or column) consists of two or more terms then the determinant can be expressed as sum of two (or more) determinants.

Example:

$$\begin{vmatrix} a_1 + b_1 & c_1 \\ a_2 + b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Suppose

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 5 \end{bmatrix} , \quad B = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 2+3 & 4 \\ 1+4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 5 & 5 \end{bmatrix}$$

$$|C| = \begin{vmatrix} 5 & 4 \\ 5 & 5 \end{vmatrix} = (5)(5) - (5)(4)$$
$$= 25 - 20$$
$$= 5$$

$$|A| = \begin{vmatrix} 2 & 4 \\ 1 & 5 \end{vmatrix} = (2)(5) - (1)(4)$$
$$= 10 - 4$$
$$= 6$$

$$|B| = \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = (3)(5) - (4)(4)$$
$$= 15 - 16$$
$$= -1$$

$$|A| + |B| = 6 + (-1)$$
$$= 5$$

Hence proved

(11) If a determinant becomes 0 on putting $x = \alpha$, then $(x-\alpha)$ is factor of the determinant

Example:

$$A = \begin{bmatrix} x & 1 & 4 \\ x+1 & 2 & 5 \\ x+2 & 3 & 6 \end{bmatrix}, \quad \text{then at } x = 1$$

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix}$$

$$|A| = 1(12-15) - 1(12-15) + 4(6-6)$$
$$= 1(-3) - 1(-3) + 4(0)$$
$$= -\cancel{3} + \cancel{3} + 0$$
$$= 0$$

Hence $(x-1)$ is a factor of determinant