# Foundations of data science

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## 0. Exercise sheet
## Handin as announced on eCampus

fits to 01

**Exercise 0.1** (Recall: Random variables). ($\boxed{8}$ points)

(i) Consider the course' example: A player has a fair die $D$ resulting in one of $\mathcal{D} := \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$ with probability $\frac{1}{6}$ each, ie. $D \xleftarrow{⚄} \mathcal{D}$ uniformly, and a fair coin $C \xleftarrow{⚄} \{0, 1\}$ uniformly.

Actually, our player is a cheater and the coin merely tells whether he decides to cheat or not. In case the coin comes up heads, say that's encoded $1$, he changes the die's outcome to ⚅. Denote the variable describing the faked die by $F$.

(a) Describe $F$ as a function of $C$ and $D$. $\boxed{2}$

**Solution.** We have (if we identify the die's results with the numbers 1 to 6)

$$F = \begin{cases} D & \text{if } C = 0 \\ 6 & \text{else.} \end{cases}$$

(b) Note that the pair $(C, F)$ is again a random variable, namely with outputs in $\{0, 1\} \times \mathcal{D}$. Write down a $2 \times 6$-table with its distribution. $\boxed{2}$

**Solution.** We obtain for all combinations of $C \in \{0, 1\}$ and $F \in \{1, 2, 3, 4, 5, 6\}$ that the probability of $\operatorname{prob}((C, F) = (c, f))$ takes the values of:

| $(c, f)$ | $f = 1$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $c = 0$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |
| 1 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ |

(c) Prove that $(C, F)$ is not independent. $\boxed{2}$

**Solution.** If the variables were independent, it should hold that for all $c \in \{0,1\}$, $f \in \{1,2,3,4,5,6\}$ we would have

$$\operatorname{prob}(C = c \wedge F = f) = \operatorname{prob}(C = c) \cdot \operatorname{prob}(F = f)\,.$$

Now we can choose one of the many counterexamples:

$$\operatorname{prob}(C = 1 \wedge F = 1) = 0 \neq \frac{1}{24} = \operatorname{prob}(C = 1) \cdot \operatorname{prob}(F = 1)\,,$$

or just as well

$$\operatorname{prob}(C = 1 \wedge F = 6) = \frac{1}{2} \neq \frac{7}{24} = \operatorname{prob}(C = 1) \cdot \operatorname{prob}(F = 6)\,.$$

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(ii) Consider a continuous random variable $U$ with outcomes in $[0,1] \subset \mathbb{R}$ with uniform density, ie. $U \longleftarrow [0,1]$ with density $p(x) = 1$ for $x \in [0,1]$ and $p(x) = 0$ otherwise.

$\boxed{2}$ Determine the density of $U^2$, ie. a function $q$ such that

$$\operatorname{prob}\left(a < U^2 < b\right) = \int_a^b q(x)\ \mathrm{d}x\,.$$

**Solution.** A nice way to see this is to consider the function $Q(x) = \operatorname{prob}\left(U^2 < x\right)$. As it holds that

$$\int_a^b q(x)\ \mathrm{d}x = Q(b) - Q(a)$$

we know that $Q$ must be the anti derivative of $q$. We compute

$$Q(x) = \operatorname{prob}\left(U^2 < x\right) = \begin{cases} 0 & x \leq 0 \\ \operatorname{prob}\left(U < \sqrt{x}\right) = \sqrt{x} & 0 < x \leq 1 \\ \operatorname{prob}\left(U < \sqrt{x}\right) = 1 & \text{else.} \end{cases}$$

Now we can compute

$$q(x) = \operatorname{prob}\left(U^2 < x\right) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{2}x^{-\frac{1}{2}} & 0 < x \leq 1 \\ 0 & \text{else.} \end{cases}$$

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**Exercise 0.2** (Maximum of two dice). (8 points)

Take two independent fair dice $D_i \xleftarrow{} \mathcal{D} := \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$ and consider the larger outcome

$$M := \max(D_0, D_1).$$

2 (i) Compute $\mathrm{prob}\left(M \leq a\right)$ for $a \in \mathbb{N}_{\leq 6}$.

**Solution.** We have by definition

$$\mathrm{prob}\left(M \leq a\right) = \mathrm{prob}\left(D_0 \leq a \wedge D_1 \leq a\right),$$

and as the dice are independent

$$\mathrm{prob}\left(D_0 \leq a \wedge D_1 \leq a\right) = \mathrm{prob}\left(D_0 \leq a\right) \cdot \mathrm{prob}\left(D_1 \leq a\right) = \frac{a}{6} \cdot \frac{a}{6} = \frac{a^2}{36}.$$

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(ii) Compute $\mathrm{prob}\left(M = a\right)$ for $a \in \mathbb{N}_{\leq 6}$. 2

**Solution.** As the distribution is discrete (meaning: there are just finitely many outputs for $M$) we have

$$\mathrm{prob}\left(M = a\right) = \mathrm{prob}\left(M \leq a\right) - \mathrm{prob}\left(M \leq (a-1)\right),$$

such that we can insert from (i), that

$$\mathrm{prob}\left(M = a\right) = \frac{a^2}{36} - \frac{(a-1)^2}{36} = \frac{2a-1}{36}.$$

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(iii) Compute $\mathrm{E}(M)$. 2

**Solution.** We can now sum up

$$\begin{aligned}
\mathrm{E}(X) &= 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} \\
&= \frac{1 + 6 + 15 + 28 + 45 + 66}{36} \\
&= \frac{161}{36} = 4.471.
\end{aligned}$$

[1]

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(iv) Compute $\mathrm{E}(M^2)$ and derive $\mathrm{var}(M)$. $\boxed{2}$

**Solution.** We can similarly sum up

$$\begin{aligned}\mathrm{E}(X^2) &= 1\cdot\frac{1}{36}+4\cdot\frac{3}{36}+9\cdot\frac{5}{36}+16\cdot\frac{7}{36}+25\cdot\frac{9}{36}+36\cdot\frac{11}{36}\\ &= \frac{1+12+45+112+225+396}{36}\\ &= \frac{791}{36} = 21.97⫰\end{aligned}$$

and derive the variance

$$\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{764\cdot 36-(161)^2}{(36)^2} = \frac{1583}{1296} = 1.22⫰.$$

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[1]Side remark: to indicate how a real number was rounded we append a special symbol. Examples: $\pi = 3.14⫰ = 3.142⫯ = 3.1416⫱ = 3.14159⫰$. The height of the platform shows the size of the left-out part and the direction of the antenna indicates whether actual value is larger or smaller than displayed. We write, say, $e = 2.72⫱ = 2.71⫯$ as if the shorthand were exact.