

Def: A filter on $\mathcal{P}(S)$ is a collection F of subsets of S st.: $F = \{F_n\}$

(i) $\emptyset \notin F$, $S \in F$

$$F_n \in \mathcal{P}(S)$$

(ii) $X \in F, Y \in F \Rightarrow X \cap Y \in F$

$$F \subseteq \mathcal{P}(S).$$

(iii) $X, Y \subseteq S, X \in F$ and $X \subseteq Y \Rightarrow Y \in F$.

Ex: $S = \{a, b, c\}$ $F_T = \{S\}$.

$$F_1 = \{S, \{a\}, \{a, b\}, \{a, c\}\}$$

$$F_2 = \{S, \{b\}, \{b, c\}, \{b, a\}\}$$

Def: An ideal on $\mathcal{P}(S)$ is a collection I of subsets of S st:

(i) $\emptyset \notin I$, $S \notin I$

(ii) $X, Y \in I \Rightarrow X \cup Y \in I$

(iii) $X, Y \subseteq S, X \in I, Y \subseteq X \Rightarrow Y \in I$.

$$S = \{a, b, c\}$$

$$I_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

THE SET OR ALL $A^{\text{c}\mathbb{N}}$ OF DENSITY 0 IS AN IDEAL ON \mathbb{N} .

DEFINITION Density 0 IF $\lim_{n \rightarrow \infty} |A \cap \{n\}|/n = 0$

$A \subseteq \mathbb{R}$ x is (limit point) ACCUMULATION POINT OF A iff

$$\forall x \in (a, b) \left((a, b) \cap (A \setminus \{x\}) \neq \emptyset \right) (*)$$

A is OPEN (in \mathbb{R}) iff $\forall a \in A \exists b < a \exists c > a ((b, c) \subseteq A)$ & $(b, c) \subseteq A$. A is CLOSED iff $\mathbb{R} \setminus A$ is open.

Another notion of closed:

(i) LIMIT POINT, X SET OF ORDINALS.

α is a LIMIT POINT OF X IFF $\sup(X \cap \alpha) = \alpha$. (#)

(ii) X is CLOSED iff $\forall \alpha (\sup(X \cap \alpha) \in X)$, that is, X is CLOSED if every LIMIT POINT OF X IS IN X ,

(iii) If γ is an ORDINAL, X is UNBOUNDED IN γ IFF

$$\forall \xi < \gamma (A \cap \gamma) \setminus \xi \neq \emptyset$$

\hookrightarrow THIS IS ORDINAL

~~UNBOUNDED~~
 $\xi_1, \xi_2, \xi_3, \dots$

\nearrow THAT MEANS THAT
 TAKING THE ξ_s OUT
 STILL LEAVES ξ_j 'S
 TO BE TAKEN.
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LET δ BE AN INFINITE REGULAR CARDINAL.

(iv) X IS δ -CLOSED IFF FOR ALL ORDINALS α WITH $c_\delta(\alpha) = \delta$ AND ST. $X \cap \alpha$ IS UNBOUNDED IN α

$$\alpha = \sup(X \cap \alpha) \in X$$

↳ THAT IS, X IS UNBOUNDED FOR ANY LIMIT POINT OF SIZE $\leq \delta$.

(v) X IS CLUB IN α IFF $X \cup \{\alpha\}$ IS CLOSED AND
 X IS UNBOUNDED IN α .
→ WE TOOK IT OUT PREVIOUSLY.
CONTAINS ALL LIMIT POINTS OF X

{ (vi) X IS δ -CLUB IN α IFF $X \cup \{\alpha\}$ IS δ -CLOSED
AND X IS UNBOUNDED IN α . (DEF 2.1 TECH).

{ (vii) $S \subseteq X$ IS STATIONARY IF $S \cap X \neq \emptyset$ FOR
ALL X δ -CLUB.

↳ δ INFINITE REGULAR CARDINAL.

If δ IS INFINITE REGULAR CARDINAL WE CAN STATE

(vi) AS FOLLOWS:

(vi*) $X \subseteq \delta$, X UNBOUNDED IS CLOSED IFF
FOR EVERY SEQUENTIAL $\alpha_0 < \alpha_1 < \dots < \alpha_\xi < \dots$,
WITH $\xi < \tau$, OR ELEMENTS OF X WITH
LENGTH $\tau < \delta$ WE MAY FIND $\alpha_\xi \in C$.

Then C, D are club, taken $C \cap D$ is club.

C is closed \Leftrightarrow every limit point of C is in C .

D is closed \Leftrightarrow every limit point of D is in D .

$C \cap D$ is closed.

$$\forall \alpha (\sup(C \cap D) \cap \alpha) \in C \cap D$$

Let x be limit point of $C \cap D$. Then

$$x = \sup((C \cap D) \cap X)$$

If $x \notin C \cap D$, then $x \notin C$ or $x \notin D$. But then, C and D are closed, which means that x is not limit point. Hence, $x \in C \cap D$.

[Unboundedness] C, D unbounded $\Rightarrow C \cap D$ unbounded.

Suppose $C \cap D$ is bounded above. Then $\exists \xi' < \gamma$ such that $((C \cap D) \cap \gamma) \setminus \xi' = \emptyset$



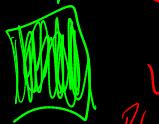
$$|A| = |\xi'| < \gamma$$

In another way, let K be a cardinal δ^+ .
 Both C, D are unbounded in K .

If $\alpha < K$, since C and D are both UNBOUNDED in K , then $\exists \alpha_j < K (\alpha_j > \alpha)$ and $\alpha_j \in C$. In the same way, $\exists \alpha_i < K (\alpha_i > \alpha)$ and $\alpha_i \in D$.

Hence, let $\alpha_0 = \alpha$, $j = 2K-1$ and $i = 2K$, for $K \in \mathbb{Z}^+$. Then \exists a sequence $\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$

And $\alpha_i \in D$, $\alpha_j \in C$, for all i, j .

Take $\beta = \lim_{\xi \rightarrow \gamma} \alpha_\xi$. Then $\beta \in C$ and $\beta \in D$,
 by (v, *). We are done. 

Remember that
 $\alpha = \sup(X \cap \alpha)$ is a limit point.

Remark (i) The collection of all CLUB SUBSETS OF K HAS THE FINITE INTERSECTION PROPERTY.

(ii) F_{CLUB} , THE FILTER GENERATED BY THE CLUB SETS, CONSISTS OF ALL $X \subseteq K$ THAT CONTAIN A CLUB SUBSET.
 We call it THE CLUB FILTER ON K .

Lemma α ordinal st. $cf(\alpha) > \omega$. Let F_{club} be the set of all $A \subseteq \alpha$ st. $\exists B \subseteq A$ (B is club in α).

Then F_{club} is a $\{cf(\alpha)\}$ -closed filter on α and F_{club} is non-trivial.

$cf(\alpha)$ is unbounded in α .

$cf(\alpha) = \alpha$ means that α is regular without a cardinal.

$cf(\alpha) > \omega$, α ordinal (limit).

$F_{club} = \{A \subseteq \alpha : \exists B(B \subseteq A \text{ & } B \text{ is club in } \alpha)\}$

Let \mathcal{A} be a family of subsets of α which are club in α . Let $A_i \in \mathcal{A}$, that is, $\mathcal{A} = \{A_i\}_{i \in \mathbb{P}}$, $\beta \leq cf(\alpha)$, each A_i club in α .

$\bigcap_{i \in \mathbb{P}} A_i$ is closed, since arbitrary intersections of closed sets are closed.

Unboundeness follows from a generalization of the previous argument.

Let $\delta^{\leq \alpha}$ define $f: \beta \cdot \omega \rightarrow \alpha$, where $\beta \cdot \omega$ is an ordinal multiplication. $\beta \cdot \omega \rightarrow \alpha$.

LET $\{f(\beta \cdot n + i)\}$ BE THE LIMIT OF $\{e_i\}_{i \in A_i}$

$$\beta > \sup(\{f(\eta) : \eta < (\beta \cdot n + i)\} \cup \{\gamma\}), i \in \beta$$

$\beta \cdot \omega < cf(\alpha)$ AND EACH A_i IS UNBOUNDED,
HENCE f IS WELL DEFINED.

$$\ell = \sup(\{f(\eta) : \eta < \beta \cdot n\}) \Rightarrow \ell \in \bigcap_{i \in \beta} A_i$$

$$\ell < \alpha$$



$$\ell \in \bigcap_{i \in \beta} A_i$$

TAKING $\beta \cdot n + i$ AS THE
MEMBERS OF EACH A_i , WE
GUARANTEED THAT A_i WAS INFINITE
MEMBER, whence, IT IS UNBOUNDED.

SINCE EACH A_i IS CLOSED.

THAT IS, ℓ IS LIMIT POINT OF EVERY A_i .

$$\text{IN SYMBOLS, } \ell = \sup(\{f(\beta \cdot n + i) : n \in \omega\}) = \sup(A_i \cap \alpha).$$

AND ALSO STRICTLY GREATER THAN \square

EVERY POINT THAT OCCURS BEFORE IT, BY DEF.

REMARK (i) (more on last lemma)

THE SET OF ALL LIMIT ORDINALS κ
IS CLUB IN κ . IF A IS UNBOUNDED IN $\kappa \Rightarrow$ THE SET OF
ALL LIMIT POINTS OF A IS CLUB IN κ . (WHAT WAS
USED ON LAST LEMMA).

Def: f is a function, $f: K \rightarrow K$, then f is normal iff f is increasing and continuous.

That is: (i) $\alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)$

(ii) $f(\alpha) = \lim_{\xi \rightarrow \alpha} f(\xi)$, for every limit $\xi < K$.

Def: If f is normal, then Ran(f) is club in K .

Remark C is club $\Rightarrow \exists f$ (f is uncountable) such that $C = \text{Ran}(f)$.

Thm The intersection of fewer than κ club subsets of K is club.

$\mathcal{A} = \{A_i\}_{i \in \beta}$, $\beta < \kappa$: A_i is club.

That $\bigcap_{i \in \beta} A_i$ is closed is immediate.

Now let $\beta < \kappa$. If β is successor, we can iterate the pairwise intersection $\beta - 1$ times, and a previous lemma guarantees that it is club.

Now suppose \mathcal{P} is limit. Assume that, for all $\alpha < \mathcal{P}$, the assertion is true. Thus we can replace each A_α with $\bigcap_{\xi > \alpha} C_\xi$.

This gives us a descending sequence of club sets:

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_\alpha \supseteq \dots \quad (\alpha < \mathcal{P})$$

Thus, $C = \bigcap_{\alpha < \mathcal{P}} C_\alpha$. C is closed, since the arbitrary intersection of closed sets is closed.

To prove unboundedness, let $\alpha < K$, and let the following be a \mathcal{P} -sequence:

$$\beta_0 < \beta_1 < \dots < \beta_\xi < \dots \quad (\xi < \mathcal{P})$$

Let $\beta_0 \in C_0$ and $\beta_0 > \alpha$. Let for each $\xi < \mathcal{P}$, $\beta_\xi \in C_\xi$ and $\beta_\xi = \sup(\{\beta_\eta : \eta < \xi\})$. This sequence exists because K is regular and $\mathcal{P} < K$. Moreover, $\lim_{\xi \rightarrow \mathcal{P}} \beta_\xi = \beta$ and $\beta < K$.

which means that for $\eta < K$ $\beta = \sup(\{\beta_\xi : \eta \leq \xi < K\})$. Thus, $\beta \in C_\eta$. Since that is true for every η , then $\beta \in C$, by our construction of C . ■

Def: Let $\langle X_\alpha : \alpha < K \rangle$ be a sequence of subsets of K . The diagonal intersection of X_α , $\alpha < K$ is defined as follows:

$$\Delta_{\alpha < K} X_\alpha = \{ \xi < K : \xi \in \bigcap_{\alpha < \xi} X_\alpha \}$$

Remark (i) $\Delta X_\alpha = \Delta Y_\alpha$, $Y_\alpha = \{ \xi \in X_\alpha : \xi > \alpha \}$

(ii) $\Delta X_\alpha = \bigcap_\alpha (X_\alpha \setminus \{\xi : \xi \leq \alpha\})$

Lemma The diagonal intersection of a K -sequence of club sets is club.

Proof: Let $A = \langle A_\xi : \xi \leq K \rangle$ be a K -sequence of club sets.

$$\Delta_{\xi < K} A_\xi = \{ \eta < K : \eta \in \bigcap_{\xi < \eta} A_\eta \}$$

Let $A_\xi = \bigcap_{\zeta \leq \xi} A_\zeta$. Thus $\{A_\xi\}_{\xi \leq K}$ is a decreasing sequence of club sets:

$A_0 \supseteq A_1 \supseteq \dots \supseteq A_\xi \supseteq \dots$ ($\xi < \kappa$).

We take $A = \bigcap_{\xi < \kappa} A_\xi$. By our definition of diagonal intersection $A = \bigtriangleup_{\xi < \kappa} A_\xi$.

[closed] To show that A is closed, let ξ be a limit point of A . Since $A = \bigtriangleup_{\xi < \kappa} A_\xi$, write $A = \{\eta < \kappa : \eta \in \bigcap_{\xi < \eta} A_\eta\}$.

Let for each $\nu < \xi$, $\xi_\nu = \sup \{\nu < \xi : \nu \in A_\nu\}$.

Then $\xi_\nu \in \bigcap_{\alpha < \xi} A_\alpha$. Thus, $\xi_\nu \in A$ and A is closed.

[unbounded] Let $v_0 \in A_0$ and $v_0 > \xi_1$ for ξ_1 .

Now take an increasing sequence

$v_0 < v_1 < \dots < v_\eta < \dots$ ($\eta < \kappa$)

such that for each $\eta < \kappa$, $v_\eta \in A_{v_\eta}$. We must show that $\nu = \lim_{\eta \rightarrow \xi} v_\eta$ is in A .

$$A = \bigcap_{\xi < \kappa} A_\xi$$

We know that $\delta < \eta$, $v_\delta \in A_{v_\delta}$,

Hence each $v_\eta \in A_{v_\delta}$. Thus, $v_\eta \in A_{v_\eta}$ for all $\eta < \xi$. Hence, $\nu \in A_\xi$. Therefore $\nu \in A$.
For each $\xi < \kappa$.



Def: An ORDINAL function f On a SET S is REGRESSIVE if $f(\alpha) < \alpha$, $\forall \alpha \in S$,

$\alpha \neq 0$.

Thm (Fodor) If f is A REGRESSIVE function on A ^{STATIONARY} SET $S \subseteq \kappa$, κ REGULAR, THEN THERE IS A STATIONARY SET $T \subseteq S$ AND SOME $\gamma < \kappa$ SUCH THAT $f(\alpha) = \gamma$, $\forall \alpha \in T$.

LET f BE A REGRESSIVE FUNCTION ON $S \subseteq \kappa$, S STATIONARY, i.e., $S \cap C \neq \emptyset$, FOR ALL C CLUB IN κ .
 WHEN $f(\alpha) < \alpha$, $\forall \alpha \in S$, $\alpha \neq 0$. USING THE CONSTRUCTION OF C IN THE PREVIOUS THMS, LET $C = \bigtriangleup_{\eta < \kappa} C_\eta$, WHERE $\eta < \kappa$.
 SUCH THAT EACH C_η IS CLUB IN κ . WHEN C IS CLUB IN κ . MONOCHROM, $C \cap S \neq \emptyset$. NOW LET $D = \{\alpha \in S : f(\alpha) = \eta\}$ BE NON-STATIONARY. CHOOSE C_η ST. $f(\alpha) \neq \eta$ FOR $\alpha \in S \cap C_\eta$. SINCE S IS STATIONARY, $S \cap C$ IS STATIONARY AS WELL. THUS, AS $C = \bigtriangleup_{\eta < \kappa} C_\eta$, WHEN IF $\alpha \in S \cap C$, $f(\alpha) \neq \eta$ FOR EVERY $\eta < \alpha$. THEREFORE, $f(\alpha) \geq \alpha$, WHICH CONTRADICTS REGRESSIVENESS. ■