Def: A filter on $S \neq \emptyset$ is a collection F of subsets of S st.:

$F = \{F_n\}$

$F_n \in \mathcal{P}(S)$

$F \subseteq \mathcal{P}(S)$

(i) $\emptyset \notin F$, $S \in F$

(ii) $X \in F, Y \in F \Rightarrow X \cap Y \in F$

(iii) $X, Y \subseteq S$, $X \in F$ and $X \subseteq Y \Rightarrow Y \in F$.

Ex: $S = \{a, b, c\}$

$F_T = \{S\}$.

$F_1 = \{S, \{a\}, \{a,b\}, \{a,c\}\}$

$F_2 = \{S, \{b\}, \{b,c\}, \{b,a\}\}$

Def: An ideal on $S \neq \emptyset$ is a collection I of subsets of S st:

(i) $\emptyset \in I$, $S \notin I$

(ii) $X, Y \in I \Rightarrow X \cup Y \in I$

(iii) $X, Y \subseteq S$, $X \in I$, $Y \subseteq X \Rightarrow Y \in I$.

$S = \{a, b, c\}$

$I_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$

$I_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}\}$

$I = \{S - X : X \in F\}$ $\quad F = \{S - X : X \in I\}$

$I = \{\emptyset\}$

(*) Principal filter. $F = \{X \subseteq S : X_0 \subseteq X\}$

$X_0 \neq \emptyset$

$S = \{a, b, c, d, e\}$ $\quad X_0 = \{c, d, e\}$

$F = \{S, \{c,d,e\}, \{c,d,e,a\}, \{c,d,e,b\}\}$

Fréchet filter: $|S| \geq \omega$, $I = \{X \subseteq S : X \text{ finite}\}$

then $F = \{X \subseteq S : S \setminus X \text{ is finite}\}$

$S > \omega$ $\quad S \setminus X$ finite $\quad |X| \geq \omega$

$|A| \geq \omega$ $\quad S = [A]^{<\omega} = \{X \subseteq A : X \text{ finite}\}$

$P \in S$ $\quad \hat{P} = \{Q \in S : P \subseteq Q\}$

$F = \{X \subseteq S : \hat{P} \subseteq X\}$, for some P

The set of all $A \subseteq \mathbb{N}$ of density 0 is an ideal on $\mathbb{N}$.

$A \subseteq \mathbb{N}$ has density 0 if $\lim_{n\to\infty} |A \cap n|/n = 0$

(*) Principal filter $A \neq \emptyset$, $A \subseteq S$

$F = \{X \subseteq S : A \subseteq X\} \to$ F on S generated by A

if $A = \{a\}$, then $F = \{X \subseteq S : A \subseteq X\}$ is maximal, that is there is no filter F' on S s.t. $F \subset F'$.

Suppose $X \in F' \setminus F$. Then $\{a\} \not\subseteq X$, by definition of A.

↳ this means $a \notin X$, otherwise $X \in F$.

But then $X \cap \{a\} = \emptyset$. Contradiction. ∎

Filter of all cofinite subsets of S.

$F = \{X \subseteq S : S \setminus X \text{ is finite}\}$

X is cofinite subset of S iff $S \setminus X$ is finite.

F is not principal.

Let $A \in F$. $\exists X \subset A$ st. $X \in F$ (there is an X cofinite subset of S that is a proper subset of A). Simply take X a cofinite subset of A. ∎

Def: $S \neq \emptyset$. An ideal on S is a collection I of subsets of S such that

(i) $\emptyset \in I$, $S \notin I$

(ii) $X \in I, Y \in I \Rightarrow X \cup Y \in I$

(iii) $X \in I$ & $Y \subseteq X \Rightarrow Y \in I$.

Trivial ideal: $I = \{\emptyset\}$

Principal ideal: $A \subseteq S \Rightarrow I = \{X \subseteq S : X \subseteq A\}$

Duality between filters and ideals:

Let F be a filter on S. The dual ideal to F is

$$I^* = \{S \setminus X : X \in F\}$$

Analogously if I is an ideal on S, the dual filter to I is

$$F^* = \{S \setminus X : X \in I\}.$$

Def: A nonempty collection G has the finite intersection property iff every finite nonempty collection $H = \{X_1, \dots, X_n\} \subseteq G$ has nonempty intersection $X_1 \cap \dots \cap X_n \neq \emptyset$.

Lemma 7.2

(i) If F is a nonempty family of filters on S, then $\bigcap F$ is a filter on S.

By definition, $\bigcap F = \{x : \forall f \in F (x \in f) \;\&\; \forall f \in F (f \neq \emptyset)\}$. Then $x \neq \emptyset$. Hence, $\emptyset \notin \bigcap F$. Moreover, $S \in \bigcap F$, since every member of F is a filter.

Let $X \in f$ and $Y \in f$ for all f. Then $X \cap Y \in f$ $\forall f$. Hence $X \cap Y \in \bigcap F$.

Let $X \in f$ $\forall f$. Then $X \in \bigcap F$. Suppose $Y \supseteq X$. Then $Y \in f, \forall f$. Therefore, $Y \in \bigcap F$.

Henceforth $\bigcap F$ is a filter. ∎

(ii) If C is a $\subseteq$-chain of filters on S, then $\bigcup C$ is a filter on S.

Let C be a $\subseteq$-chain of filters on S.

Consider $\bigcup C$. Then $\emptyset \notin \bigcup C$, since $\forall c \in C$, $\emptyset \notin c$.

Moreover, $S \in \bigcup C$, since $\exists c \in C (S \in c)$.

Suppose $X, Y \in \bigcup C$. Then $\exists c_1 \in C\ (X \in c_1)$ and $\exists c_2 \in C\ (Y \in c_2)$.

Since C is a $\subseteq$-chain, then $c_1, c_2 \in C \Rightarrow c_1 \subset c_2 \lor c_1 = c_2 \lor c_2 \subset c_1$.

If $c_1 = c_2$, then $X \in c_1$ and $Y \in c_1$, and since c_1 is a filter, $X \cap Y \in c_1$. Hence, $X \cap Y \in \bigcup C$. Otherwise, let $c_1 \subset c_2$ and let $Y \in c_2 \setminus c_1$. Since $c_1 \subset c_2$, then $X \in c_1 \Rightarrow X \in c_2$. Hence, $X \cap Y \in c_2$, therefore $X \cap Y \in \bigcup C$.

Suppose $X \in \bigcup C$ and $X \subseteq Y$. Then $\exists c_1 \in C (X \in c_1)$.

But then $Y \in c_1$. Therefore, $Y \in \bigcup C$.

Henceforth $\bigcup C$ is a filter.

(iii) If $G \subseteq \mathcal{P}(S)$ has the finite intersection property, then there is a filter F on S s.t. $G \subseteq F$.

$G \neq \emptyset$ $\Rightarrow \forall H = \{X_1, \ldots, X_n\} \subseteq G\ (X_1 \cap \ldots \cap X_n \neq \emptyset)$.

Let $F = \{X \subseteq S : \exists H (\tilde{\bigcap} H \subseteq X)\}$. Let $X \in G$. Then $\tilde{\bigcap} H \subseteq X$.

Hence $X \in F$.

To see that F is a filter, $S \in F$, since $\exists H \subseteq G$ s.t. $\tilde{\bigcap} H \subseteq S$, naturally.

Let $X, Y \in F$. Thus, $\exists H_1 (\tilde{\bigcap} H_1 \subseteq X)$ and $\exists H_2 (\tilde{\bigcap} H_2 \subseteq Y)$.

Thus, $(\tilde{\bigcap} H_1) \cap (\tilde{\bigcap} H_2) \subseteq X \cap Y$. Thus $X \cap Y \in F$.

If $X \in F$ and $X \subseteq Y$, then by transitivity $\exists H, (\tilde{\bigcap} H \subseteq Y)$, hence, $Y \in F$, as desired. ∎

F as proved above is the smallest F that extends G, since F contains all intersections of finite sets in G.

$$F = \bigcap \{ D : D \text{ is a filter on } S \text{ and } \underbrace{G \subseteq D}_{D \text{ extends } G} \}$$

F is generated by G.

Def: A filter U on S is an ultrafilter iff:
for all $X \subseteq S$ either $X \in U$ or $S \setminus X \in U$.

Dual notion: Prime ideal
$\forall X \subseteq S \, (X \in I \vee (S \setminus X) \in I)$

$I = \mathcal{P}(S) \setminus U$

Lemma: A filter F on S is an ultrafilter iff it is maximal.

Proof: Let F be an ultrafilter on S. Then if $X \subseteq S$, then either $X \in F$ or $S \setminus X \in F$. Let $F \subset F'$. Then $\exists Y \in F' (Y \notin F)$. Since F is ultrafilter, then $S \setminus Y \in F$. Thus $S \setminus Y \in F'$. Contradiction.

Let F be a maximal filter. Then $\nexists F' (F \subset F')$. Let $X \subseteq S$. Then, since F is maximal, only $\emptyset$ is not a member of F. Thus when $X \neq \emptyset$, $X \in F$. Otherwise, $S \setminus X \in F$ as desired.

Suppose F is not an ultrafilter. Then there is some $X \subseteq S$ such that neither $X \in F$ nor $S \setminus X \in F$. Let $F' = F \cup \{X\}$.

[Claim] F' has the finite intersection property.

If $Y \in F$, then $Y \cap X \neq \emptyset$, otherwise $S \setminus Y \supseteq X$, and thus $S \setminus Y \in F$, since F is a filter. Thus, if $Y_1, \dots, Y_n \in F$, then $(Y_1 \cap \dots \cap Y_n) \cap X \neq \emptyset$. Hence F' has the finite intersection property.

By Lemma 7.2 (iii) there is a filter F'' that extends F'. Since $X \in F' \setminus F$, then F is not maximal.

Lemma 7.5 Every filter can be extended to an ultrafilter.

By Lemma 7.2(iii), if F is not maximal, then $\exists F' \supseteq F$. And then if F' is not maximal, one repeat this operation. Then we obtain a $\subseteq$-chain of filters (which extend F) on S. By Lemma 7.2(ii), $\bigcup F$ is also a filter, as well an upper bound of the chain. Since we have a chain, by Zorn's Lemma there is a maximal element to this chain. This element is an ultrafilter by Lemma 7.4 ∎

And consider the partial order $(\mathcal{P}, \subseteq)$

Remark: There are principal filters which are maximal, whence ultrafilters. S finite $\Rightarrow \forall F$ ultrafilter (F is principal).

Are there non principal ultrafilters?

Let S be infinite. Then the filter that extends the Fréchet filter is non principal and maximal.

(This follows from 7.5 and cannot be proved without AC).

Let $\kappa \geq \omega$ $|S| = \kappa$

Since every ultrafilter is a subset of $\mathcal{P}(S)$,

then $|U| \leq |\mathcal{P}(\mathcal{P}(S))| = 2^{2^\kappa}$.

Next thm: $\underline{|U| = 2^{2^\kappa}}$.

Let an ultrafilter D on κ be uniform if $\forall X \in D (|X| = \kappa)$

Thm 7.6 (Pospíšil) For every infinite cardinal there are 2^{2^κ} uniform ultrafilters on κ.

(We skip this one for now).

$\underline{\kappa\text{-complete}}$ $\underline{\text{filters and ideals}}$.

F filter on S is countably complete (σ-complete) if whenever $\{X_n : n \in \mathbb{N}\}$ is a family of subsets of S and $X_n \in F$ for every n, then $\bigcap_{n=0}^{\infty} X_n \in F$.

If I is an ideal and $X_n \in I$ for every n, then $\bigcup_{n=0}^{\infty} X_n$ and I is called countably complete.

$\boxed{\text{Generalizing}}$

$\kappa > \omega$, κ regular cardinal. F filter on S

I ideal on S

F is κ-complete iff F is closed under intersection of less than κ sets, i.e.,

- $\mathcal{A} = \{X_\alpha : \alpha < \gamma\}$, $X_\alpha \subseteq S$, for each $\alpha < \gamma$
- $\gamma < \kappa$
- $X_\alpha \in F$, $\forall \alpha < \gamma$

$\Rightarrow \bigcap_{\alpha<\gamma} X_\alpha \in F$

The κ-complete ideal is the dual notion.

A filter F is κ-complete iff its dual ideal is κ-complete.

Examples

1) κ-complete ideal $|S| \geq \kappa$

$$I = \{X \subseteq S : |X| < \kappa\}$$

2) σ-complete filter $= \aleph_1$-complete filter

3) $\nexists$ nonprincipal σ-complete filter on S countable.

$|S| > \omega \Rightarrow \{X \subseteq S : X \leq \aleph_0\}$ is σ-ideal on S.

4) $\kappa > \omega$, κ regular, $|S| \geq \kappa \Rightarrow \{X \subseteq S : |X| < \kappa\}$ is the smallest κ-complete ideal on S containin all singletons $\{a\}$.

Question

Can a nonprincipal ultrafilter on a set be σ-complete?

Answer: If yes, then there are large cardinals.

$A \subseteq \mathbb{R}$. x is accumulation point (limit point) of A iff

$$\forall x \in (a,b) \left((a,b) \cap (A \setminus \{x\}) \neq \emptyset \right) \quad (*)$$

A is open (in $\mathbb{R}$) iff $\forall a \in A\ \exists b < a\ \exists c > a$ ($(b,c) = \{x : b < x < c\}$) & $(b,c) \subseteq A$. A is closed iff $\mathbb{R} \setminus A$ is open.

A new notion of closed:

(i) α limit ordinal, X set of ordinals.

α is a limit point of X iff $\sup(X \cap \alpha) = \alpha$. (*)

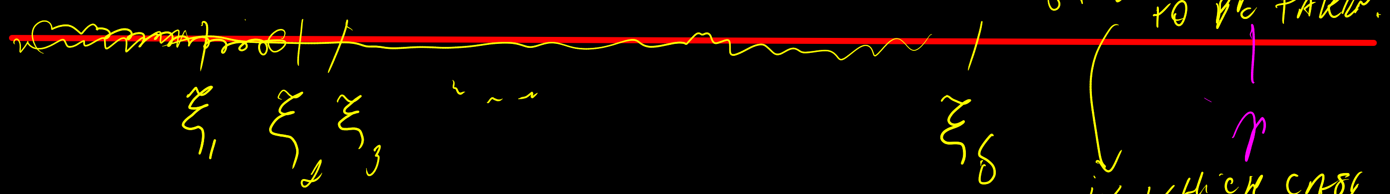
(ii) X is closed iff $\forall \alpha \left(\sup(X \cap \alpha) \in X \right)$, that is, X is closed if every limit point of X is in X.

(iii) If γ is an ordinal, X is unbounded in γ iff

$$\forall \xi < \gamma \left((A \cap \gamma) \setminus \xi \neq \emptyset \right)$$

this is ordinal

THAT MEANS THAT TAKING THE ξs OUT STILL LEAVES ξ's TO BE TAKEN.



in which case $\xi' > \xi$ (ordinals are bounded above).

LET δ BE AN INFINITE REGULAR CARDINAL.

(iv) X is δ-CLOSED iff FOR ALL ORDINALS α WITH $cf(\alpha) = \delta$ AND s.t. $X \cap \alpha$ is UNBOUNDED in α $\alpha = \sup(X \cap \alpha) \in X$.

↳ THAT is, X is UNBOUNDED FOR ANY LIMIT POINT OF SIZE $\leq \delta$.

WE TOOK IT OUT PREVIOUSLY.

(v) X IS CLUB in α iff $X \cup \{\alpha\}$ is CLOSED AND X is UNBOUNDED in α.

CONTAINS ALL LIMIT POINTS OF X

(vi) X is δ-club in α iff $X \cup \{\alpha\}$ is δ-closed $\wedge$ X is UNBOUNDED in α. (DEF 8.1 JECH).

(vii) $S \subseteq X$ is STATIONARY iff $S \cap X \neq \emptyset$ FOR ALL X δ-CLUB.

↳ δ INFINITE REGULAR CARDINAL.

IF δ is INFINITE REGULAR CARDINAL WE CAN STATE (Vi) AS FOLLOWS:

(vi*) $X \subseteq \delta$, X UNBOUNDED is CLOSED iff FOR EVERY SEQUENCE $\alpha_0 < \alpha_1 < \ldots < \alpha_\xi < \ldots$, WITH $\xi < \gamma$, OF ELEMENTS OF X WITH LENGTH $\gamma < \delta$ WE HAVE $\lim_{\xi \to \gamma} \alpha_\xi \in C$.

Thm If C, D are club, then $C \cap D$ is club.

C is closed $\Leftrightarrow$ every limit point of C is in C.

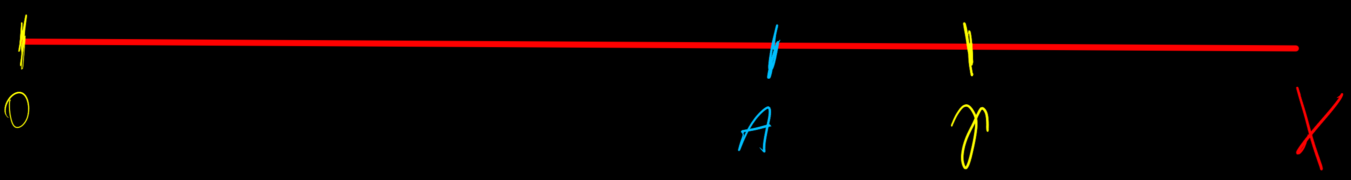
D is closed $\Leftrightarrow$ every limit point of D is in D.

$C \cap D$ is $\underbrace{\text{closed}}_{\forall \alpha\, (\sup(C\cap D)\cap\alpha) \in C\cap D}$.

Let x be limit point of $C \cap D$. Then $x = \sup((C\cap D)\cap x)$. If $x \notin C\cap D$, then $x \notin C$ or $x \notin D$. But then, C and D are closed, which means that x is not limit point. Hence, $x \in C \cap D$.

[Unboundedness] C, D unbounded $\Rightarrow$ $C\cap D$ unbounded.

Suppose $C \cap D$ is bounded above. Then $\exists \xi' < \gamma$ such that $((C\cap D)\cap\gamma)\setminus\xi' = \emptyset$



$|A| = |\xi| < \gamma$

In another way, let κ be a cardinal s.t. both C, D are unbounded in κ.

If $\alpha < \kappa$, since C and D are both unbounded in κ, then $\exists \alpha_j < \kappa (\alpha_j > \alpha)$ and $\alpha_j \in C$. In the same way, $\exists \alpha_i < \kappa (\alpha_i > \alpha)$ and $\alpha_i \in D$.

Hence, let $\alpha_0 = \alpha$, $j = 2k-1$ and $i = 2k$, for $k \in \mathbb{Z}^+$. Then $\exists$ a sequence

$$\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$$

and $\alpha_i \in D$, $\alpha_j \in C$, for all i, j.

Take $\beta = \lim_{\xi \to \gamma} \alpha_\xi$. Then $\beta \in C$ and $\beta \in D$, by (v;*) we are done. ∎

Remember that $\alpha = \sup(X \cap \alpha)$ is a limit point.

Remark (i) The collection of all club subsets of κ has the finite intersection property.

(ii) $\mathcal{F}_{club}$, the filter generated by the club sets, consists of all $X \subseteq \kappa$ that contain a club subset. We call it the club filter on κ.

Lemma α ordinal s.t. $cf(\alpha) > \omega$. Let $\mathcal{F}_{club}$ be the set of all $A \subseteq \alpha$ s.t. $\exists B \subseteq A$ (B is club in α).

Then $\mathcal{F}_{club}$ is a $\underbrace{< cf(\alpha)}$-closed filter on α and $\mathcal{F}_{club}$ is non-trivial.

$cf(\alpha)$ is unbounded in α.

$cf(\alpha) = \alpha$ means that α is regular, whence a cardinal.

$cf(\alpha) > \omega$, α ordinal (limit).

$$\mathcal{F}_{club} = \{A \subseteq \alpha : \exists B (B \subseteq A \ \& \ B \text{ is club in } \alpha)\}$$

Let $\mathcal{A}$ be a family of subsets of α which are club in α. Let $A_i \in \mathcal{A}$, that is, $\mathcal{A} = \{A_i\}_{i \in \beta}$, $\underline{\beta < cf(\alpha)}$, each A_i $\underline{\text{club in } \alpha}$.

$\bigcap_{i \in \beta} A_i$ is closed, since arbitrary intersections of closed sets are closed.

Unboundedness follows from a generalization of the previous argument.

Let $\xi < \alpha$. Define $f : \beta \cdot \omega \to \alpha$, where $\beta \cdot \omega$ is an ordinal multiplication. $\beta \cdot \omega \to \alpha$

LET NEW $f(\beta \cdot n + i)$ BE THE LEAST $\delta \in A_i$

$\delta > \sup(\{f(\eta) : \eta < (\beta \cdot n + i)\} \cup \{\xi\}), i \in \beta$

[$\beta \cdot \omega < cf(\alpha)$ AND EACH A_i IS UNBOUNDED, HENCE f IS WELL DEFINED.]

$\ell = \sup(\{f(\eta) : \eta < \beta \cdot \omega\})$

$\ell < \alpha$ $\Rightarrow \ell \in \bigcap_{i \in \beta} A_i$

TAKING $\beta \cdot n + i$ AS THE MEMBERS OF EACH A_i, WE GUARANTEE THAT A_i HAS INFINITE MEMBERS, WHENCE, IS UNBOUNDED.

$\Downarrow$

$\ell \in \bigcap_{i \in \beta} A_i$

SINCE EACH A_i IS CLOSED.

THAT IS, ℓ IS LIMIT POINT OF EVERY $A_{i \in \beta}$.
IN SYMBOLS, $\ell = \sup(\{f(\beta n + i) : n \in \omega\} = \sup(A_i \cap \alpha)$.
AND ALSO STRICTLY GREATER THAN EVERY POINT THAT OCCURS BEFORE IT, BY DEF. $\blacksquare$

(MORE OR LESS LAST LEMMA)

REMARK (i) THE SET OF ALL LIMIT ORDINALS $< \kappa$ IS CLUB IN κ. IF A IS UNBOUNDED IN $\kappa \Rightarrow$ THE SET OF ALL LIMIT POINTS OF A IS CLUB IN κ. (WHAT WAS USED ON LAST LEMMA).

Def: f is a function, $f: \kappa \to \kappa$, then f is normal iff f is increasing and continuous.

That is: (i) $\alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)$

(ii) $f(\alpha) = \lim_{\xi \to \alpha} f(\xi)$, for every limit $\alpha < \kappa$.

Def: If f is normal, then $Ran(f)$ is club in κ.

Remark C is club $\Rightarrow \exists! f$ (f enumerates C).

Thm The intersection of fewer than κ club subsets of κ is club.

$\mathcal{A} = \{A_i\}_{i \in \beta}$ $\beta < \kappa$: A_i is club.

That $\bigcap_{i \in \beta} A_i$ is closed is immediate.

Now let $\beta < \kappa$. If β is successor, we can iterate the pairwise intersection $\beta - 1$ times, and a previous lemma guarantees that it is club.

Now suppose γ is limit. Assume that, for all $\alpha < \gamma$, the assertion is true. Thus we can replace each A_α with $\bigcap_{\xi < \alpha} C_\xi$.

This gives us a descending sequence of club sets:

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_\alpha \supseteq \dots \quad (\alpha < \gamma)$$

Thus, $C = \bigcap_{\alpha < \gamma} C_\alpha$. C is closed, since the arbitrary intersection of closed sets is closed.

To prove unboundedness, let $\alpha < \kappa$, and let the following be a γ-sequence:

$$\beta_0 < \beta_1 < \dots < \beta_\xi < \dots (\xi < \gamma)$$

Let $\beta_0 \in C_0$ and $\beta_0 > \alpha$. Let for each $\xi < \gamma$, $\beta_\xi \in C_\xi$ and $\beta_\xi = \sup(\{\beta_\nu : \nu < \xi\})$. This sequence exists because κ is regular and $\gamma < \kappa$. Moreover, $\lim_{\xi \to \gamma} \beta_\xi = \beta$ and $\beta < \kappa$.

Which means that for each $\eta < \kappa$ $\beta = \sup(\langle \beta_\xi : \eta \le \xi < \kappa \rangle)$

Thus, $\beta \in C_\eta$. Since that is true for every η, then $\beta \in C$, by our construction of C. $\blacksquare$

Def: Let $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of subsets of κ. The diagonal intersection of X_α, $\alpha < \kappa$ is defined as follows:

$$\mathop{\Delta}_{\alpha<\kappa} X_\alpha = \{ \xi < \kappa : \xi \in \bigcap_{\alpha < \xi} X_\alpha \}$$

Remark (i) $\Delta X_\alpha = \Delta Y_\alpha$, $Y_\alpha = \{ \xi \in X_\alpha : \xi > \alpha \}$

(ii) $\Delta X_\alpha = \bigcap_\alpha (X_\alpha \cup \{ \xi : \xi \leq \alpha \})$

Lemma The diagonal intersection of a κ-sequence of club sets is club.

Proof: Let $\mathcal{A} = \langle A_\xi : \xi \leq \kappa \rangle$ be a κ-sequence of club sets.

$$\mathop{\Delta}_{\xi<\kappa} A_\xi = \{ \eta < \kappa : \eta \in \bigcap_{\xi < \eta} A_\eta \}$$

Let $A_\xi = \bigcap_{\xi \leq \kappa} A_\xi$. Thus there is a decreasing sequence of club sets:

$A_0 \supseteq A_1 \supseteq \ldots \supseteq A_\xi \supseteq \ldots \quad (\xi < \kappa)$.

WE TAKE $A = \bigcap_{\xi<\kappa} A_\xi$. BY OUR DEFINITION OF DIAGONAL INTERSECTION $A = \underset{\xi<\kappa}{\Delta} A_\xi$.

[CLOSED] TO SHOW THAT A IS CLOSED, LET ξ BE A LIMIT POINT OF A. SINCE $A = \underset{\xi<\kappa}{\Delta} A_\xi$, WRITE

$A = \{\eta < \kappa : \eta \in \bigcap_{\xi<\eta} A_\eta\}$.

LET FOR EACH $\nu < \xi$, $\xi = \sup(\{\nu < \xi : \nu \in \bigcap_{\alpha<\nu} A_\alpha\})$

THEN $\xi \in \bigcap_{\alpha<\xi} A_\alpha$. THUS, $\xi \in A$ AND A IS CLOSED.

[UNBOUNDED] LET $\nu_0 \in A_0$ AND $\nu_0 > \xi$, FOR $\xi < \kappa$.

NOW TAKE AN INCREASING SEQUENCE

$\nu_0 < \nu_1 < \ldots < \nu_\eta < \ldots \quad (\eta < \kappa)$

SUCH THAT FOR EACH $\eta < \kappa$, $\nu_\eta \in A_{\nu_\eta}$. WE MUST SHOW THAT $\nu = \lim_{\eta \to \xi} \nu_\eta$ IS IN A.

$\boxed{A = \bigcap_{\xi<\kappa} A_\xi}$

WE KNOW THAT $\delta < \eta$, $\nu_\delta \in A_{\nu_\delta}$,

HENCE EACH $\nu_\eta \in A_{\nu_\delta}$. THUS, $\nu_\eta \in A_{\nu_\eta}$ FOR ALL $\eta < \xi$. HENCE, $\nu \in A_\xi$ FOR EACH $\xi < \kappa$. HENCEFORTH $\nu \in A$. ∎

Def: An ordinal function f on a set S is *regressive* if $f(\alpha) < \alpha$, $\forall \alpha \in S$, $\alpha \neq 0$.

Thm (Fodor) If f is a regressive function on a stationary set $S \subseteq \kappa$, κ regular, then there is a stationary set $T \subseteq S$ and some $\gamma < \kappa$ such that $f(\alpha) = \gamma$, $\forall \alpha \in T$.

Let f be a regressive function on $S \subseteq \kappa$, S stationary, i.e., $S \cap C \neq \emptyset$, for all C club in κ. Then $f(\alpha) < \alpha$, $\forall \alpha \in S$ $\alpha \neq 0$. Using the construction of C in the previous thms, let $C = \underset{\gamma<\kappa}{\Delta} C_\gamma$, where $\gamma < \kappa$. Such that each C_γ is club in κ. Then C is club in κ. Moreover, $C \cap S \neq \emptyset$. Now let $D = \{\alpha \in S : f(\alpha) = \gamma\}$ be non stationary. Choose C_γ s.t. $f(\alpha) \neq \gamma$ for each $\alpha \in S \cap C_\gamma$. Since S is stationary, $S \cap C$ is stationary as well. Thus, as $C = \underset{\gamma<\kappa}{\Delta} C_\gamma$, then if $\alpha \in S \cap C$, $f(\alpha) \neq \gamma$ for every $\gamma < \alpha$. Therefore, $f(\alpha) \geq \alpha$, which contradicts regressiveness. ∎