SEQUENCES $\langle \alpha_\xi : \xi < \alpha \rangle$

DEFINITION BY RECURSION:

GIVEN A FUNCTION G (ON THE CLASS OF ALL TRANSFINITE SEQUENCES), THEN $\forall \Theta$ THERE EXISTS A UNIQUE Θ-SEQUENCE

$$\langle \alpha_\alpha : \alpha < \Theta \rangle$$

SUCH THAT

$$\alpha_\alpha = G(\langle \alpha_\xi : \xi < \alpha \rangle) \quad (1)$$

$\forall \alpha < \Theta$

WE ARE GOING TO DEFINE IT WITHOUT USING CLASSES.

NEED TO DEFINE $\boxed{A^\beta}$

[THM] LET α BE AN ORDINAL, A A SET AND $S = \boxed{\bigcup_{\beta<\alpha} A^\beta}$ THE SET OF ALL SEQUENCES IN A OF LENGTH $< \alpha$. LET $g: S \to A$ BE A FUNCTION.

THEN THERE IS A UNIQUE FUNCTION $f: \alpha \to A$ SUCH THAT $f(\beta) = g(f|_\beta)$, $\forall \beta < \alpha$.

$g(f|_\beta)$ CAN BE FOUND SINCE f IS WELL DEFINED ON α. AS WELL, S GUARANTEES THE EXISTENCE OF ALL β-SEQUENCES IN A.

THM 4.4 IS THE THM THAT SAYS THAT OUR (1) CAN BE DEFINED.

[THM] (TRANSFINITE RECURSION) LET G BE AN OPERATION. THEN THERE IS A PROPERTY P SUCH THAT $P(x)$ DEFINES A UNIQUE OPERATION F s.t. $F(\alpha) = G(F\restriction_\alpha)$ $\forall \alpha$ ORDINAL.

[LET $F(\alpha) = x$ iff $\exists \langle \alpha_\xi : \xi < \alpha \rangle$ st.
(i) $\forall \xi < \alpha$ $\alpha_\xi = G(\langle \alpha_\eta : \eta < \xi \rangle)$
(ii) $x = G(\langle \alpha_\xi : \xi < \alpha \rangle)$]

τ is A COMPUTATION OF LENGTH α BASED ON G IF:
(i) τ IS A FUNCTION
(ii) $\mathrm{Dom}(\tau) = \alpha + 1$
(iii) $\forall \beta \leq \alpha \, (\tau(\beta) = G(\tau \restriction \beta))$

DEFINE $P(x, y)$ TO BE AS FOLLOWS:
(i) $x \in \mathrm{ON}$ AND $y = \tau(x)$ FOR SOME COMPUTATION τ OF LENGTH x BASED ON G.
(ii) $x \notin \mathrm{ON}$ AN $y = \emptyset$.

MUST SHOW THAT P DEFINES AN OPERATION.

SHOW THAT FOR EACH x $\exists! y \, (P(x, y))$ HOLDS.

[$x \notin \mathrm{ON}$] OBVIOUS, SINCE $y = \emptyset$.

[$x \in \mathrm{ON}$] MUST BE SHOWN BY INDUCTION ON X.

$\forall x \, \exists! y \, (Y$ IS A COMPUTATION OF LENGTH $\alpha)$.

IH: $\forall \beta < \alpha$ THERE IS A UNIQUE COMPUTATION OF LENGTH β. PROVE EXISTENCE AND UNIQUENESS OF SUCH COMPUTATION FOR LENGTH α.

[EXISTENCE] BY AXIOM OF SEPARATION APPLIED ON THE PROPERTY

"y IS A COMPUTATION OF LENGTH x"

AND THE SET α. THE SET

$T = \{t : t$ IS A COMPUTATION OF LENGTH β FOR SOME $\beta < \alpha\}$.

$IH \Rightarrow \exists! t \in T$ S.T. LENGTH OF t IS β, $\forall \beta < \alpha$.

T IS SYSTEM OF FUNCTIONS.

LET $\bar{t} = \bigcup T$ AND $\tau = \bar{t} \cup \{(\alpha, G(\bar{t}))\}$.

WE PROVE THAT τ IS A COMPUTATION OF LENGTH α.

[CLAIM] τ IS A FUNCTION & $\mathrm{Dom}(\tau) = \alpha + 1$.

(i) $\mathrm{Dom}(\bar{t}) = \bigcup_{t \in T} \mathrm{Dom}\, t = \bigcup_{\beta \in \alpha} (\beta + 1) = \alpha$. HENCE,

$\mathrm{Dom}\, \tau = \alpha + 1 = \mathrm{Dom}\, \bar{t} \cup \{\alpha\}$.

(ii) SINCE $\alpha \notin \mathrm{Dom}\, \bar{t}$, THEN IT SUFFICES TO PROVE THAT $\bar{t}$ IS A FUNCTION WHICH FOLLOWS FROM THE FACT THAT T IS A COMPATIBLE SYSTEM OF FUNCTIONS.

$\forall t_1, t_2 \in T\ \exists r \in T (r \leq t_1\ \&\ r \leq t_2)$

Let $t_1, t_2 \in T$. Dom $t_1 = \beta_1$ Dom $t_2 = \beta_2$.

Assume $\beta_1 \le \beta_2$ then $\beta_1 \subseteq \beta_2$ and it suffices to show that $t_1(\gamma) = t_2(\gamma)$ $\forall \gamma < \beta_1$. or $\gamma \in \beta_1 \cap \beta_2$

Proceed by transfinite induction:

Assume $\delta < \beta_1$ and $t_1(\delta) = t_2(\delta)$ for all $\delta < \gamma$

Then $t_1 \restriction \gamma = t_2 \restriction \gamma \Rightarrow t_1(\gamma) = G(t_1 \restriction \gamma) = G(t_2 \restriction \gamma) = t_2(\gamma)$

We conclude $t_1(\gamma) = t_2(\gamma)$ for all $\gamma < \beta_1$.

Claim $\tau(\beta) = G(\tau \restriction \beta)$ $\forall \beta \le \alpha$.

[$\beta = \alpha$] $\tau(\alpha) = G(\tau \restriction \alpha)$ by our definition of τ.

$$\tau(\alpha) = G(\bar{t}) = G(\tau \restriction \alpha)$$

[$\beta < \alpha$] Pick $t \in T$ s.t. $\beta \in \operatorname{dom} t$. Then

$$\tau(\beta) = G(\tau \restriction \beta) = G(t \restriction \beta) = t(\beta) \quad \text{IH}$$

Since t is a computation of length β.

From Claims 1 and 2, existence of the computation follows.

[Uniqueness] Let σ be another computation of length α. Prove $\tau = \sigma$.

τ, σ functions and $\operatorname{dom} \tau = \operatorname{dom} \sigma = \alpha + 1$.

It suffices to show that $\tau(\gamma) = \sigma(\gamma)$ $\forall \gamma \le \alpha$.

Assume $\forall \delta < \gamma$ that $\tau(\delta) = \sigma(\delta)$. Then

$$\tau(\gamma) = G(\tau \restriction \gamma) \underbrace{=}_{\text{by IH}} G(\sigma \restriction \gamma) = \sigma(\gamma).$$

Uniqueness follows.

Hence, we have proved that φ defines a unique operation F.

Notes: For any computation t

$F \restriction \mathrm{dom}\, t = t$.

↳ $\beta \in \mathrm{dom}\, t \Rightarrow t_\beta = t(\beta+1)$.

which is a computation of length β. Thus by definition of F, $F(\beta) = G(F\restriction\beta) = t(\beta) = t_\beta(\beta)$.

To prove $\forall\alpha\ F(\alpha) = G(F\restriction\alpha)$ let t be a unique computation of length α. Thus $F(\alpha) = t(\alpha) = G(t\restriction\alpha) = G(F\restriction\alpha)$ ■