JECH

ULTRAFILTERS ON ω

POSPÍŠIL THM

LET S BE INFINITE SET ST. $\omega \le \kappa = |S|$.

THUS, SINCE EVERY ULTRAFILTER U IS A SUBSET OF $\mathcal{P}(S)$, THEN THERE ARE AT MOST 2^{2^κ} ULTRAFILTERS ON S.

LET D BE AN ULTRAFILTER ON κ. D IS CALLED UNIFORM IF $|X| = \kappa$ $\forall X \in D$.

LET $\mathcal{A}$ BE A FAMILY OF SUBSETS OF κ. THEN $\mathcal{A}$ IS CALLED INDEPENDENT IF FOR ANY DISTINCT SETS $X_1, \dots, X_n, Y_1, \dots, Y_m$ IN $\mathcal{A}$, THE INTERSECTION

$$X_1 \cap \dots \cap X_n \cap (\kappa \setminus Y_1) \cap \dots \cap (\kappa \setminus Y_m) \quad (*)$$

HAS CARDINALITY κ.

LEMMA 7.7 THERE EXISTS AN INDEPENDENT FAMILY OF SUBSETS OF κ OF CARDINALITY 2^κ.

LET $F = \{X : X \in [\kappa]^{<\omega}\}$ & $\mathcal{F} = \{X : X \in [\kappa]^{<\omega}\}, |\mathcal{F}| < \omega$.

P IS THE SET OF PAIRS $(F, \mathcal{F})$.

$|P| = \kappa \Rightarrow$ SHOW $\exists \mathcal{A}$ (INDEPENDENT & $|\mathcal{A}| = 2^\kappa$), $\mathcal{A} \subseteq \mathcal{P}(P)$

[EXISTENCE] TAKE EACH $u \subseteq \kappa$. $(F, \mathcal{F}) \in P$

$$X_u = \{(F, \mathcal{F}) : F \cap u \in \mathcal{F}\}.$$

$$\mathcal{A} = \{X_u : u \subseteq \kappa\}.$$

U, V DISTINCT SUBSETS OF $\kappa \Rightarrow X_U \neq X_V$.

($X_U \subset X_V$, $X_V \subset X_U$ OR INCOMPARABLE)

$\alpha \in U$, $\alpha \notin V$

$\Rightarrow \{\alpha\} \in F$, $\{F\} \in \mathcal{F}$, $(F, \mathcal{F}) \in X_U$, BUT $(F, \mathcal{F}) \notin X_V$

HENCE, $|\mathcal{A}| = 2^\kappa$

BECAUSE $\mathcal{A} \subseteq \mathcal{P}(P)$.

[INDEPENDENCE] LET $U_1, U_2, \dots, U_n, V_1, \dots, V_m$ BE DISTINCT SUBSETS OF κ.

FOR EACH $i \leq n$, $j \leq m$ LET $\alpha_{i,j} \in \kappa$ S.T. $\alpha_{i,j} \in U_i \setminus V_j$ OR $\alpha_{i,j} \in V_j \setminus U_i$

LET $F \subseteq \kappa$, $|F| < \omega$.

$\{\alpha_{i,j} : i \leq n, j \leq m\} \subseteq F \Rightarrow F \cap U_i \neq F \cap V_j \ \forall i \leq n, j \leq m$ BY CONSTRUCTION.

LET $\mathcal{F} = \{F \cap U_i : i \leq n\}$ SO THAT $(F, \mathcal{F}) \in X_{U_i} \ \forall i \leq n$

$(F, \mathcal{F}) \notin X_{V_j} \ \forall j \leq m$

THUS $X_{U_1} \cap \dots \cap X_{U_n} \cap (P \setminus X_{V_1}) \cap \dots \cap (P \setminus X_{V_m})$ HAS CARDINALITY κ. ■

THM 7.6 (POSPÍŠIL) FOR EVERY INFINITE CARDINAL κ THERE ARE 2^{2^κ} ULTRAFILTERS ON κ.

WE KNOW THAT THERE ARE AT MOST 2^{2^κ} ULTRAFILTERS ON κ.

WE MUST SHOW THAT THERE ARE AT LEAST 2^{2^κ} ULTRAFILTERS ON κ.

LET $\mathcal{A}$ BE AN INDEPENDENT FAMILY OF SUBSETS OF κ.

For each function $f : A \to \{0,1\}$, consider the following family of subsets of κ:

$G_f = \{X : |\kappa \setminus X| < \kappa\} \cup \{X : f(X) = 1\} \cup \{\kappa \setminus X : f(X) = 0\}$ (**)

By definition of independent family and (*) on p.1, G_f has the finite intersection property.

Thus, there is an ultrafilter D_f s.t. $D_f \supseteq G_f$.

By (**), D_f is uniform (?)

If $f \neq g$, $\Rightarrow X \in A$ $(f(X) \neq g(X))$

$f(X) = 1$ $\quad g(X) = 0$ $\quad X \in D_f$ & $\kappa \setminus X \in D_g$

Condition $\{X : |\kappa \setminus X| < \kappa\}$ guarantees that G_f contains all co-$<\kappa$, that is, the sets whose complement is less than κ.

Thus, there are 2^{2^κ} distinct uniform ultrafilters on κ. ∎

↳ Since we take the families generated by all $f : A \to \{0,1\}$.

Ultrafilters on ω

① P-point.

Let D be a nonprincipal ultrafilter on ω.

D is called a P-point if for every partition $\{A_n : n \in \omega\}$ of ω into $\aleph_0$ pieces st. $A_n \notin D$ for all $n \in \omega$, there exists $X \in D$ st. $X \cap A_n$ is finite for all $n \in \omega$.

$A=\{1,2,3\}$ $\quad P(A)=\{\{1\},\{2,3\}\}$

$\omega=\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\},\ldots\}$ $\quad P(\omega)=\{A_n : n\in\omega\}$ * $|A_n|=\omega_0$

$i\neq j \Rightarrow A_i\neq A_j$ $\quad A_n\subseteq\omega$ $\quad \bigcup_{n\in\omega}A_n=\omega$ $\quad A_n\neq\emptyset\ \forall n$

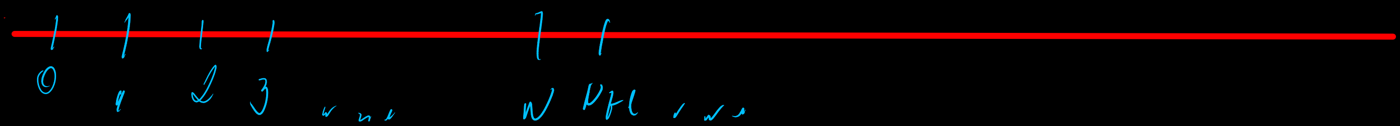
IMPORTANT

D is a P-point $\Leftrightarrow \forall n\in\omega\,(A_n\notin D)$

for all partitions $P(\omega)$

$\exists X\in D\,(|X\cap A_n|<\omega)$

D is ultrafilter on $\omega \Rightarrow X\subseteq\omega\ (X\in D$ or $\omega\setminus X\in D)$



NONPRINCIPAL ULTRAFILTER NOT A P-POINT

$\{A_n : n\in\omega\}$ partition of ω

$|A_n|=\aleph_0$.

F is the filter:

$X\in F$ iff except for finitely many n, $X\cap A_n$ contains all but finitely many elements of A_n.

$D\supseteq F$, then D is not $\boxed{\text{P-point}}$!

$\boxed{CH \Rightarrow \exists\text{ P-points}}$

$\Rightarrow$ P-point existence

STRONGER ASSERTION THAN P-POINT.

Nonprincipal filter D on ω is called Ramsey ultrafilter if for every partition $\{A_n : n\in\omega\}$ of ω into $\aleph_0$ pieces s.t. $A_n\notin D$ for all n, there exists $X\in D$ s.t. $X\cap A_n$ has one element for all $n\in\omega$.

Thm 7.8 CH $\Rightarrow \exists$ Ramsey ultrafilter

Let $\mathcal{A}_\alpha$, $\alpha < \omega_1$ enumerate all partitions of ω.

Construct ω_1-sequence of infinite subsets of ω:

Given X_α $\quad X_{\alpha+1} \subseteq X_\alpha$ $\begin{cases} X_{\alpha+1} \subseteq A, \text{ for some } A \in \mathcal{A}_\alpha \\ \text{or} \\ |X_{\alpha+1} \cap A| \leq 1, \forall A \in \mathcal{A}_\alpha \end{cases}$

If α is limit ordinal

Let X_α be s.t. $|X_\alpha - X_\beta| < \omega$, $\forall \beta < \alpha$

(X_α exists because α is countable)

Thus

$D = \{X : X \supseteq X_\alpha \text{ for some } \alpha < \omega_1\}$

is a Ramsey ultrafilter. ■

Exercises

Hrbacek-Jech

<u>2.1</u> If U is an ultrafilter on S, then $\mathcal{P}(S) \setminus U$ is a prime ideal.

Consider $\mathcal{P}(S)\setminus U$. We show it is an ideal.

(i) We know that $\emptyset \notin U$. Thus, $\emptyset \in \mathcal{P}(S)\setminus U$. Moreover, $S \in U$, whence $S \notin \mathcal{P}(S)\setminus U$.

(ii) Let $X, Y \in \mathcal{P}(S)\setminus U$. Thus, $S\setminus X \in U$ and $S\setminus Y \in U$. Suppose $X \cup Y \in U$. By exercise 7.2 on Jech, either $X \in U$ or $Y \in U$. Contradiction.

Henceforth $(X \cup Y) \notin U$. Thus, $X \cup Y \in \mathcal{P}(S) \setminus U$.

(iii) Let $X \in \mathcal{P}(S) \setminus U$ and $Y \subseteq X$. Since $X \notin U$, then $S \setminus X \in U$. But since $Y \subseteq X$, $S \setminus Y \supseteq S \setminus X$, hence, $S \setminus Y \in U$. Thus, $Y \notin U$. Therefore, $Y \in \mathcal{P}(S) \setminus U$.

To show that $\mathcal{P}(S) \setminus U$ is a prime ideal, let $X \subseteq S$. Thus either $X \in U$ or $S \setminus X \in U$. If $X \in U$, then $S \setminus X \notin U$. Since $S \setminus X \subseteq S$, then $S \setminus X \in \mathcal{P}(S) \setminus U$. Otherwise, if $S \setminus X \in U$, then $X \notin U$. Hence, $X \in \mathcal{P}(S) \setminus U$. Therefore, $\mathcal{P}(S) \setminus U$ is a prime ideal. ∎

2.2 If U is a nonprincipal ultrafilter, then every $X \in U$ is infinite.

Let U be a nonprincipal ultrafilter on S, $x \in S$. We know that $|S| \geq \omega$, otherwise U would be a principal ultrafilter (Jech, p. 75).

Since U is an ultrafilter, take $x \in S$ to be arbitrary. Thus, either $\{x\} \in U$ or $S \setminus \{x\} \in U$. Since U is nonprincipal, $\{x\}$ cannot be in U, otherwise we would have a filter generated by $\{x\}$, whence principal. Thus, for all $x \in S$, $S \setminus \{x\} \in U$. Now let F be a finite subset of S. By analogous reasoning, we have that for all finite $F \subseteq S$, $S \setminus F \in U$, given that

$S \setminus F = \bigcap_{x \in F} (S \setminus \{x\})$, and a filter is closed by finite intersections. Let $D = \{S \setminus F : F \subseteq S$ & F is finite$\}$. Then D is a filter. In fact, D is the Frechet filter on S. Since $D \subseteq U$, then by (Jech, p.75) U is a nonprincipal ultrafilter.

Since S is infinite and each F is finite, $S \setminus F$ is infinite, as desired. ■

Jech

7.4 Let U be an ultrafilter on S. Then the set of all $X \subseteq S \times S$ st. $\{a \in S : \{b \in S : (a,b) \in X\} \in U\} \in U$ is an ultrafilter on $S \times S$.

First we check if V, the set of all $X \subseteq S \times S$ as defined above is a filter.

(i) $\emptyset \notin V$ since $\emptyset \notin U$. Moreover, since $S \in U$, then $S \times S \in V$.

(ii) Let $A, B \in V$. Let $(n,m) \in A$ and $(p,q) \in B$. Since $A, B \in U$, then $A' = \{m : (n,m) \in A\} \in U$, $B' = \{q : (p,q) \in B\} \in U$ and $A'' = \{n : A' \in U\} \in U$ and $B'' = \{p : B' \in U\} \in U$. Therefore $A' \cap B' \in U$ and $A'' \cap B'' \in U$, whence $A \cap B \in V$.

(iii) Let $X \in V$, and suppose $Y \supseteq X$. Thus, if $X = A \times B$, then either $Y = A' \times B$ or $Y = A \times B'$ (or both) where $A' \supseteq A$ and $B' \supseteq B$.

IN BOTH CASES, $A' \in U$ AND $B' \in U$. THUS $Y \in V$ AS DESIRED.

$X \subseteq S \times S$

$$V = \{ X \subseteq S \times S : \{a : \overbrace{\{b : (a,b) \in X\}}^{X'} \in U\} \in U \}$$

$X = \{ (a_1, b_1), (a_1, b_2), \dots, (a_2, b_1), \dots, (a_n, b_1), \dots \}$

$X' = \{ b : (a, b) \in X \} = \{ b_1, b_2, \dots, b_n, \dots \}$ $b_i \in S$ (given)

$X' \notin U \Rightarrow S \setminus X' \in U$. ($U$ ULTRAFILTER).

$X'' = \{ a : \{ b : (a,b) \in X \} \in U \}$

$(a,b) \in X$ $X' \notin U \Rightarrow S \setminus X' \in U$

$X'' = \{ a : \{ b : (a,b) \in X \} \in U \}$

$= \emptyset \notin U \Rightarrow S \setminus X'' \in U$

THUS $(S \setminus A) \times (S \setminus B) \in V$

$X \subseteq S \times S$

$= A \times B$

OTHERWISE, IF $(a,b) \in X$

$\{ b : (a,b) \in X \} \in U$

$(a,b) \in S \times S$ & $(a,b) \notin A \times B$

$(a \in S$ & $b \in S)$ & $(a \notin A \vee b \notin B)$

$(a,b) \in S$ & $(a \notin A \vee b \notin B)$

$((a \in S$ & $b \in S)$ & $a \notin A) \vee ((a \in S$ & $b \in S)$ & $b \notin B)$

$(a,b) \in S \times S \setminus X \Leftrightarrow (a,b) \in (S \setminus A) \times B \vee (a,b) \in A \times (S \setminus B)$

$X = A \times B$

$X'' = \{a : \{b : (a,b) \in X\} \in U\}$

$X'' \notin U \Rightarrow S \setminus X'' \in U$.

$\Rightarrow X \notin V$

$X' \in U$

$\Uparrow$

$(a,b) \in X$

THUS, $S \times S \setminus X \in V$ SINCE ONE OF THE COMPLEMENTS IS IN U.

THE ABOVE CRATINGS SHOWS THAT WHEN $X \notin V$, $(S \times S) \setminus X \in V$. OTHERWISE, WHEN $(S \times S) \setminus X \notin V$, $(S \times S) \setminus (A \times B) \notin V$ iff EITHER

$S \setminus X' = \{b : (a,b) \in (S \times S) \setminus (A \times B)\} \notin U$

OR

$S \setminus X'' = \{a : S \setminus X' \in U\} \notin U$

$\Downarrow$

$X'' \in U$

$\Rightarrow X' \in U$, BECAUSE OTHERWISE, $X' \notin U \Rightarrow X'' = \emptyset \in U$. CONTRADICTION.

$\Downarrow$

$X \in V$.

$\Downarrow$ $X' \in U$

$X \in V \left\{ \begin{array}{l} X' \in U \\ \Downarrow \\ S \setminus X'' \notin U \\ \Downarrow \\ X'' \in U \end{array} \right.$

THEREFORE, V IS AN ULTRAFILTER $\blacksquare$

7.5 Let U be an ultrafilter on S and let $f: S \to T$. Then the set $f_*(U) = \{X \subseteq T : f^{-1}(X) \in U\}$ is an ultrafilter on T.

[$f_*(U)$ is a filter]

$(a, b) \in f$. $a \in S$, $b \in T$

$f(X) = \{b : (a,b) \in f \ \& \ a \in X\}$

$f^{-1}(X) = \{a : (a,b) \in f \ \& \ b \in X\}$.

(i) Clearly $\emptyset \notin f_*(U)$ since $\forall b \in T (b \notin \emptyset)$, which means $f^{-1}(\emptyset) = \emptyset$, thus, $f^{-1}(\emptyset) \notin U$.

Otherwise $f^{-1}(T) \in U$, since $f^{-1}(T) = \{a : (a,b) \in f \ \& \ b \in T\}$.

Since every $b \in T$, $f^{-1}(T) = S$, whence is in U.

Thus, $\emptyset \notin f_*(U)$ and $T \in f_*(U)$

(ii) Let $X, Y \in f_*(U)$. Thus, $f^{-1}(X) \in U$ and $f^{-1}(Y) \in U$. Hence, $(f^{-1}(X) \cap f^{-1}(Y)) \in U$.

Thus $\{a : (a,b) \in f \ \& \ b \in X\} \cap \{c : (c,d) \in f \ \& \ d \in Y\} \in U$.

Thus, there is some $x \in S$ s.t. $(x, b) \in f$ & $(x, d) \in f$.

Therefore $b = d$. Hence $\{x : (x,y) \in f \ \& \ y \in X \cap Y\} \in U$.

Thus, $X \cap Y \in f_*(U)$.

(iii) Let $X \in f_*(U)$ and $Y \supseteq X$. Thus, $f^{-1}(Y) \supseteq f^{-1}(X)$, for $f^{-1}(X) = \{a : (a,b) \in f \ \& \ b \in X\}$ $f^{-1}(Y) = \{c : (c,d) \in f \ \& \ d \in Y\}$. Since $X \subseteq X$, then every $a \in f^{-1}(X)$ is also a member of $f^{-1}(Y)$. Thus, $f^{-1}(Y) \in U$. Hence $Y \in f_*(U)$.

To show that $f_*(U)$ is an ultrafilter, let $X \subseteq T$. Suppose $X \notin f_*(U)$. Thus, $f^{-1}(X) \notin U$. That means that $\{a ; (a,b) \in f \ \& \ b \in X\} \notin U$. Thus $S \setminus f^{-1}(X) \in U$. Thus, $\{a : (a,b) \in f \ \& \ b \notin X\} \in U$. Thus $f^{-1}(S \setminus X) \in U$. Hence $S \setminus X \in f_*(U)$.

$\{a : (a,b) \in f \ \& \ b \in S \setminus X\} \in U$.

Otherwise, suppose $S \setminus X \notin f_*(U)$. Thus $f^{-1}(S \setminus X) \notin U$. Hence $\{a : (a,b) \in f \ \& \ b \in S \setminus X\} \notin U$. Henceforth, $\{a : (a,b) \in f \ \& \ b \in X\} \in U$. Therefore $f^{-1}(X) \in U$ and $X \in f_*(U)$, as desired. ∎

[Need to show f is well-defined!]

Proof by reference:

Rather than V⊆U, you want to write V≤_{RK} U, "V is Rudin-Keisler reducible to U".
The reduction is the function f. Here, g[C]=f_{−1}[C]={t∈S|f(t)∈C} is the preimage of C under f.
Note that this is well-defined, and makes sense, regardless of whether f is injective.
This is a subset of S, so it may or may not be in U, and that f is a reduction says that a subset of T is in V iff its preimage is in U. Note in particular that g[T]={a∈S|f(a)∈T}=S is certainly in U.
– Andrés E. Caicedo Aug 4 '13 at 5:55

As Andres comments, this is not the inverse function, but rather the preimage.
Recall that f is a function then it is also a relation, then f^{-1} is just the inverse relation.
When we write $f^{-1}[A]$ (and in many places using parentheses rather than brackets)
we write the set $\{x \mid f(x)\in A\}$. This is the dual notion of the direct image, $f[B]=\{f(b)\mid b\in B\}$.

If $f:X\to Y$ then both the direct image and preimage are functions between the power sets of X and Y. That is, the preimage, which we shall denote by $f\hat{}(A)=f^{-1}(A)$ is a function from P(Y) to P(X). The direct image is in the other way.

Now we say that $V\leq_{RK}U$ if there is a function f such that $V=f\hat{}^{-1}[U]$. That is to say, V is exactly the preimages of the sets which are in U.

Asaf Karagila (https://math.stackexchange.com/users/622/asaf-karagila),
Definition of Rudin-Keisler ordering, URL (version: 2013-08-04):
https://math.stackexchange.com/q/459630

7.6 Let U be an ultrafilter on $\mathbb{N}$ and let $\langle a_n\rangle_{n=0}^{\infty}$ be a bounded sequence of real numbers. Prove that there exists a unique U-limit $a = \lim_U a_n$ s.t. $\forall \varepsilon>0$, $\{n : |a_n - a| < \varepsilon\}\in U$.

Definition of U-limit.

[Uniqueness] Let a, b be two U-limits of $\langle a_n\rangle_{n=0}^{\infty}$. Assume WLOG $a<b$. Thus, we have that $\lim_U a_n = a$ and $\lim_U a_n = b$. Let $\varepsilon = |b-a|/2$. Thus $\{n : |a_n - a|<\varepsilon\}$ and $\{n : |a_n - b| < \varepsilon\}$ are disjoint sets. Since U is a filter, it has the f.i.p, hence, disjoint sets cannot be both in U. Contradiction.

[Existence]

Let $\langle a_n \rangle_{n=0}^{\infty}$ be a bounded sequence of real numbers. Since $\langle a_n \rangle$ is bounded, there are numbers a, b s.t. $a < a_n < b$, for all n.

Let for every $x \in [a, b]$, $A_x = \{n : a_n < x\}$.

Thus, $A_a = \emptyset$ and $A_b = \mathbb{N}$.

Moreover, if $x \le y$, $A_x \subseteq A_y$.

Hence, $A_a \notin U$, $A_b \in U$ and if $A_x \in U$ and $x \le y$, then $A_y \in U$.

Now let $c = \sup\{x : A_x \notin U\}$.

(Claim) c is $\lim_U a_n$.

Let $\varepsilon > 0$. Thus $A_{c-\varepsilon} \notin U$, since $(c-\varepsilon) < c$.

On the other hand $A_{c+\varepsilon} \in U$, since

$A_{c+\varepsilon} = A_{c-\frac{\varepsilon}{2}} \cup \{n : \underbrace{c-\varepsilon < a_n < c+\varepsilon}\}$ then

$\{n : |a_n - c| < \varepsilon\} \in U$, which

satisfies our definition. ■

Properties of U-limits:

- If U is a principal ultrafilter, $U = \{A : n_0 \in A\}$, for some $n_0 \Rightarrow \lim_U a_n = a_{n_0}$ for any sequence $\langle a_n \rangle_{n=0}^{\infty}$, since $\{n : |a_n - a_{n_0}| < \varepsilon\} \supseteq \{n_0\} \in U$ for all $\varepsilon > 0$.

- If $\langle a_n \rangle_{n=0}^{\infty}$ is convergent and $\lim a_n = a$, then for all ultrafilter U nonprincipal, $\lim_U a_n = a$.
This is because for every $\varepsilon > 0$ there is a K st.
$\{n : |a_n - a| < \varepsilon\} \supseteq \{n : n \geq K\} \in U$.

Thm 2.9 (Hrbacek-Jech)

There exists a measure m on $\mathbb{N}$ st $m(A) = d(A)$ for every set A that has a density.

Proof: Since A has a density, take $m(A) := \lim_U \frac{A(n)}{n}$ (where $A(n)$ = $|A\cap n|$ as earlier).
Clearly $m(A) = d(A)$ if A has density. Also
$m(\emptyset) = \lim_U \frac{|\emptyset \cap n|}{n} = 0$, $m(\mathbb{N}) = \lim_U \frac{|\mathbb{N} \cap n|}{n} = 1$.
Also, $A \subseteq B \Rightarrow m(A) \leq m(B)$.

If A, B are disjoint, then $(A \cup B)(n) = A(n) + B(n)$ and additivity is defined as follows:

$$\lim_U (a_n + b_n) = \lim_U a_n + \lim_U b_n = m(A) + m(B)$$

[Proof of $\lim_U (a_n + b_n)$]

Given $\varepsilon > 0$, let a, b be such that

$$\{n : |a_n - a| < \tfrac{\varepsilon}{2}\} \in U$$

$$\{n : |b_n - b| < \tfrac{\varepsilon}{2}\} \in U$$

Let $a' = \max\{a, b\}$ so that

$\lim_{U} (a_n + b_n) = c$

$\Rightarrow \forall \varepsilon > 0, \{n : |(a_n+b_n) - c| < \varepsilon\} \in U$

SUPPOSE

$\forall \varepsilon > 0 \ \{n : |(a_n - a| < \varepsilon\} \in U$

$\forall \varepsilon > 0 \ \{n : |b_n - b| < \varepsilon\} \in U$

THUS $\{n : |a_n - a| < \cancel{\varepsilon}\tfrac{\varepsilon}{2}\} \cap \{n : |b_n - b| < \cancel{\varepsilon}\tfrac{\varepsilon}{2}\} \in U$

SINCE $\langle a_n \rangle$ AND $\langle b_n \rangle$ ARE BOUNDED,

LET $a' < a_n < a''$ AND $b' < a_n < b''$.

$\forall x \in [a', a'']$, $A_x = \{n : a_n < x\}$

$\forall y \in [b', b'']$, $B_y = \{n : b_n < y\}$

LET $c = \sup\{x : A_x \notin U\}$

$d = \sup\{y : B_y \notin U\}$

BY OUR PREVIOUS PROOF $c = \lim_U a_n$

$d = \lim_U b_n$

THEN $\lim_U a_n + \lim_U b_n = c + d$

$= \sup\{x : A_x \notin U\} + \sup\{y : B_y \notin U\}$

$= \sup\{x + y : A_x \notin U \ \& \ B_y \notin U\}$

$= \lim_U (a_n + b_n)$.

$\{1, 2, 3\}$ $\quad 3 + 6 = 9$

$\{4, 5, 6\}$ $\quad \sup\{x+y\} \overset{?}{=} 9$