THIS USES ULTRAFILTERS, CLUBS, STATIONARY SETS. BE WARNED.

<u>SILVER'S THM</u> (1) LET κ BE A SINGULAR CARDINAL s.t. $cf(\kappa) > \omega$. IF FOR EVERY $\alpha < \kappa$ $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, THEN $2^{\kappa} = \kappa^{+}$.

<u>SPECIAL CASE</u> $\lambda = \omega_{\omega_1}$ SATISFIES $cf(\lambda) > \omega$.

<u>SILVER'S THM (2)</u> IF SCH HOLDS FOR ALL SINGULAR CARDINALS OF COFINALITY ω, THEN IT HOLDS FOR ALL SINGULAR CARDINALS.

SINGULAR CARDINAL HYPOTHESIS (SCH)

FOR EVERY SINGULAR CARDINAL κ, IF $2^{cf(\kappa)} < \kappa$, THEN $\kappa^{cf(\kappa)} = \kappa^{+}$.

SCH $\Rightarrow$ GCH

- $2^{cf(\kappa)} \geq \kappa \Rightarrow \kappa^{cf(\kappa)} = 2^{cf(\kappa)}$
- $2^{cf(\kappa)} < \kappa \Rightarrow \kappa^{cf(\kappa)} = \kappa^{+}$

↳ THE LEAST POSSIBLE VALUE FOR $\kappa^{cf(\kappa)}$.

$\neg$SCH $\Rightarrow$ LARGE CARDINALS (LC)

Lemma 8.14 Let κ be a singular cardinal, let $cf(\kappa) > \omega$ and assume $\lambda^{cf(\kappa)} < \kappa$ for all $\lambda < \kappa$. If $\langle \kappa_\alpha : \alpha < cf(\kappa) \rangle$ is a normal sequence of cardinals s.t. $\lim \kappa_\alpha = \kappa$, and if the set $\{\alpha < cf(\kappa) : \kappa_\alpha^{cf(\kappa_\alpha)} = \kappa_{\alpha+1}\}$ is stationary in $cf(\kappa)$, then $\kappa^{cf(\kappa)} = \kappa^+$.

(i) $\langle \kappa_\alpha : \alpha < cf(\kappa) \rangle$ is normal sequence.
$\leftrightarrow$ increasing
continuous in α.
$\lim_{\xi \to \alpha}(\kappa_\xi) = \lim(\kappa_\alpha)$, for all $\xi < \alpha$ (limit).

(ii) $\{\alpha < cf(\kappa) : \kappa_\alpha^{cf(\kappa_\alpha)} = \kappa_{\alpha+1}\} = S$
$\to$ stationary set, $S \cap C \neq \emptyset$, $\forall C$ club.

(iii) GCH $\Rightarrow$
- conditions on Lemma 8.14 hold
- $2^\kappa = \kappa^{cf(\kappa)}$

Lemma 8.14 will be proved for special case: $\kappa = \aleph_{\omega_1}$ using theory of stationary subsets of $\aleph_1$.

Generalization can be proved in a similar way.

Def: Let f, g be functions on κ (infinite).

f, g are **almost disjoint** if there is an $\alpha_0 < \kappa$ s.t. $f(\alpha) \neq g(\alpha)$ for all $\alpha \geq \alpha_0$.

A family F of functions on κ is an **almost disjoint family** if for any two distinct elements f, g of F, f and g are A.D.

NOTE: EVERY ELEMENT OF F MUST HAVE CARDINALITY κ, AS PER DEF 1.1, CH. II ON KUNEN'S BOOK. THIS IS A MORE GENERAL DEFINITION.

LEMMA 8.14 IS A CONSEQUENCE OF THE FOLLOWING LEMMA:

LEMMA 8.15 ASSUME THAT $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ FOR ALL $\alpha < \omega_1$. LET F BE A A.D. FAMILY OF FUNCTIONS

$$F \subseteq \prod_{\alpha<\omega_1} A_\alpha$$

SUCH THAT THE SET $\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_{\alpha+1}\}$ IS STATIONARY. THEN $|F| \leq \aleph_{\alpha+1}$.

[IN THE GENERAL CASE, THE FUNCTIONS ARE DEFINED ON $cf(\kappa)$]

WE CAN, IN TURN, PROVE LEMMA 8.15 IN TWO STEPS, IN WHICH CASE THE FIRST STEP IS THE FOLLOWING LEMMA:

LEMMA 8.16 ASSUME THAT $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ FOR ALL $\alpha < \omega_1$. LET F BE AN A.D. FAMILY OF FUNCTIONS, $F \subseteq \prod_{\alpha<\omega_1} A_\alpha$, S.T. THE SET $\{\alpha < \omega_1 : |A_\alpha| \leq \aleph_\alpha\}$ IS STATIONARY. THEN $|F| \leq \aleph_{\omega_1}$.

(THIS IS CLOSE TO 8.15, THE DIFFERENCE BEING THAT THE STATIONARY SETS ARE THE $\alpha < \omega_1$ S.T. $|A_\alpha| \leq \aleph_\alpha$ AND THE BOUND FOR $|F|$ IS $\aleph_{\omega_1}$ INSTEAD OF $\aleph_{\omega_1+1}$.

WE PROVE SILVER'S THMS IN THE FOLLOWING ORDER:
FIRST WE PROVE 8.16, WHICH HELPS US IN PROVING 8.15. HAVING PROVED 8.15, WE SHOW HOW 8.14 FOLLOWS FROM 8.15. THEN WE PROVE SILVER'S THMS.

PROOF OF THM 8.16

Assume A_α is a set of ordinals s.t. for all α in some stationary subset of ω_1, $A_\alpha \subseteq \omega_\alpha$. Let $S_0 = \{\alpha < \omega_1 : \alpha$ is a limit ordinal & $A_\alpha \subseteq \omega_\alpha\}$.

If $f \in F$, then $f(\alpha) < \omega_\alpha$ for all $\alpha \in S_0$, since each α is in some stationary subset of ω_1. Thus let $g(\alpha)$ denote the least $\beta < \alpha$ s.t. for $\alpha > 0$ ($\in S_0$), $f(\alpha) < \omega_\beta$.
Since $g(\alpha) < \alpha$ for each $\alpha \in S_0$, then Fodor's thm (see my notes on filters and clubs) implies that there is $S \subseteq S_0$ stationary s.t. g is constant on S.

This means that for every $\alpha \in S$, $f(\alpha) < \omega_\beta = g(\alpha)$. Which, in turn, guarantees that $f \restriction S$ is a function from S into ω_β, for some $\beta < \omega_1$. In other words $f \restriction S$ is bounded in S by some $\omega_\beta < \omega_{\omega_1}$.

We assign for each f a pair $(S, f \upharpoonright S)$, where S is a stationary set $S \subseteq S_0$ and $f \upharpoonright S$ is a bounded function. If $f \neq g$ $(f, g \in F)$, then so are $f \upharpoonright S$ and $g \upharpoonright S$, even when $\mathrm{Dom}(f)$ and $\mathrm{Dom}(g)$ are equal, since f and g are almost disjoint. Hence we call the aforementioned assignment $\varphi(f)$ for each $f \in F$, so that φ is a 1-1 correspondence with domain F and $\varphi(f) = (S, f \upharpoonright S)$, which is a function with domain S and value $f \upharpoonright S = \omega_\gamma < \omega_{\omega_1}$ for some $\gamma < \omega_1$.

Thus, for each S there are at most $\sum_{\gamma<\omega_1} \aleph_\gamma^{|S|}$ bounded functions. That is

$$\sum_{\gamma<\omega_1} \aleph_\gamma^{|S|} \leq \sup_{\gamma<\omega_1} \aleph_\gamma^{\aleph_1} \leq \aleph_{\omega_1}$$

Since $|F| = |\mathcal{P}(\omega_1)| = 2^{\aleph_1} \leq \aleph_{\omega_1}$, then we have that the number of pairs $(S, f \upharpoonright S)$ is

$$|\varphi| = \underset{\text{number of } S_s}{2^{\aleph_1}} \cdot \underset{\text{number of bounded functions for each } S.}{\sum_{\gamma<\omega_1} \aleph_\gamma^{|S|}} \leq \underbrace{2^{\aleph_1} \cdot \aleph_{\omega_1}}_{\text{"} \max\{2^{\aleph_1}, \aleph_{\omega_1}\}.} = \aleph_{\omega_1}.$$

Given that $|F| \leq |\varphi|$, then $|F| \leq \aleph_{\omega_1}$. $\blacksquare$

We can prove the following before moving on to Lemma 8.15:

Lemma Let f be a function on ω_1 s.t. $f(\alpha) < \aleph_{\alpha+1}$ for all $\alpha < \omega_1$. Let F be a family of A.D. functions on ω_1, and let

$F_f = \{g \in F : \text{for some stationary set } T \subseteq \omega_1, g(\alpha) < f(\alpha) \text{ for all } \alpha \in T\}$

Then $|F_f| \leq \aleph_{\omega_1}$.

Proof: Let T be a fixed stationary set. Then the set $\{g \in F : \forall \alpha \in T (g(\alpha) < f(\alpha))\}$ has cardinality at most $\aleph_{\omega_1}$, as we have seen in the proof of Lemma 8.16. Thus, $|F_f| \leq \underbrace{2^{\aleph_1}}_{} \cdot \aleph_{\omega_1} = \aleph_{\omega_1}$.

↳ number of subsets of ω_1.

Proof of 8.15 from 8.16 and the previous lemma.

Let U be an ultrafilter on ω_1 which extends the club filter,

club filter: $F_{club} = \{X \subseteq K : \exists C (C \subseteq X \,\&\, C \text{ is club on } K)\}$

Here, $K = \omega_1$

Thus every $S \in U$ is stationary, since it intersects every $X \in F_{club}$, given filter definition.

Assume WLOG $A_\alpha \subseteq \omega_{\alpha+1}$ for each $\alpha < \omega_1$. Let $<^*$ be a relation defined on F as follows:

$f <^* g \iff \{\alpha < \omega_1 : f(\alpha) < g(\alpha)\} \in U$.

CLAIM: $<^*$ IS A TOTAL ORDERING ON F

[TRANSITIVITY]

LET $\{\alpha : f(\alpha) < g(\alpha)\} \in U$ AND $\{\alpha : g(\alpha) < h(\alpha)\} \in U$.

THUS $\{\alpha : f(\alpha) < g(\alpha)\} \cap \{\alpha : g(\alpha) < h(\alpha)\} \in U$, SINCE U IS A FILTER.

GIVEN THAT

$$\{\alpha : f(\alpha) < h(\alpha)\} \supseteq \{\alpha : f(\alpha) < g(\alpha)\} \cap \{\alpha : g(\alpha) < h(\alpha)\}$$

THEN

$$\{\alpha : f(\alpha) < h(\alpha)\} \in U.$$

HENCE $f <^* g$ AND $g <^* h$ IMPLY $f <^* h$.

[TRICHOTOMY]

$f, g \in F$, $f \neq g \Rightarrow \{\alpha : f(\alpha) = g(\alpha)\} \notin U$.

SINCE $\{\alpha : f(\alpha) = g(\alpha)\}$ IS AT MOST COUNTABLE.

THIS COMES FROM THE FACT THAT EVERY TWO DISTINCT ELEMENTS OF F ARE ALMOST DISJOINT, WHICH MEANS THAT FOR $f, g \in F$ $|f \cap g| < \omega_1$. HENCE, $|\{\alpha : f(\alpha) = g(\alpha)\}| < \omega_1$.

SINCE U EXTENDS THE CLUB FILTER, EVERY $X \in F_{club}$ HAS CARDINALITY AT LEAST ω_1. THAT EXPLAINS WHY $\{\alpha : f(\alpha) = g(\alpha)\} \notin U$.

THUS, SINCE U IS AN ULTRAFILTER, EITHER $\{\alpha : f(\alpha) < g(\alpha)\} \in U$ OR $\{\alpha : g(\alpha) < f(\alpha)\} \in U$. HENCE, EITHER $f <^* g$ OR $g <^* f$, AS DESIRED. ∎

Since $<^*$ is total on F:

Recall our set in the previous lemma, F_f, and let for every $f \in F$,

$$F_f = \{g \in F : \text{for some stationary } T,\ g(\alpha) < f(\alpha) \text{ for all } \alpha \in T\}.$$

By the previous lemma, $|F_f| \leq \aleph_{\omega_1}$. Since $<^*$ is total, then $g <^* f$ implies $g \in F_f$, whence $|\{g \in F : g <^* f\}| \leq \aleph_{\omega_1}$.

Since this is the case for all $f \in F$, then

$$|F| \leq \underbrace{2^{\aleph_1}}_{\text{number of } f \in F} \cdot \underbrace{\aleph_{\omega_1}}_{\text{number of (at most) } g \in F \text{ s.t. } g <^* f \text{ for each } f.} = \aleph_{\omega_1}. \ \blacksquare$$

Proof of 8.14 from 8.15:

Assume $\aleph_\alpha^{\aleph_1} < \aleph_{\omega_1}$ and that $\aleph_\alpha^{cf(\aleph_\alpha)} = \aleph_{\alpha+1}$ for a stationary set of α's. We want to show that $\aleph_{\omega_1}^{cf(\aleph_1)} = \aleph_{\omega_1+1}$.

For every $h : \omega_1 \to \aleph_{\omega_1}$ let $f_h = \underbrace{\langle h_\alpha : \alpha < \omega_1 \rangle}_{\text{increasing continuous sequence}}$

where $Dom(h_\alpha) = \omega_1$ and

$$h_\alpha(\xi) = \begin{cases} h(\xi), & \text{if } h(\xi) < \aleph_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

And let $F = \{f_h : h \in \underbrace{\aleph_{\omega_1}^{\omega_1}}_{\text{set of functions from } \omega_1 \text{ into } \aleph_{\omega_1}.}\}$

If $h \neq g$, then f_h and f_g are A.D, since for some $\alpha < \omega_1$, for all $\beta \geq \alpha$, $h_\beta(\xi) \neq g_\beta(\xi)$.

Moreover, $F \subseteq \prod_{\alpha<\omega_1} \aleph_\alpha^{\omega_1}$, since for a stationary set of α's, $\underbrace{\aleph_\alpha^{\aleph_1} = \aleph_{\alpha+1}}_{\forall \alpha \text{ s.t. } \aleph_\alpha > 2^{\aleph_1} \;\&\; \aleph_\alpha^{\aleph_0} = \aleph_{\alpha+1}}$.

Thus we have that $|F| \leq \aleph_{\omega_1+1}$, whence $|\aleph_{\omega_1+1}^{\omega_1}| = \aleph_{\omega_1+1}$.

Silver's Thm (1) follows directly from 8.14, since $2^\kappa = \kappa^{cf(\kappa)}$.