FILTERS (HRBACEK-JECH)

EXAMPLES

① S EUCLIDEAN SPACE (A SPACE WITH EUCLIDEAN MEASURE).

LET a BE A POINT IN S.

$$G = \{U ; U \text{ is open in } S \text{ \& } a \in U\}.$$

$\Downarrow$

- G HAS THE FINITE INTERSECTION PROPERTY (PROPERTY OF OPEN SETS.)

$\Downarrow$

- G GENERATES A FILTER F ON S $\longrightarrow$ (LEMMA 7.2(iii))

F is THE NEIGHBORHOOD FILTER ON S.

② (DENSITY)

- LET A BE A SET OF NATURAL NUMBERS $(A \subseteq \mathbb{N})$.
- FOR EACH $n \in \mathbb{N}$, $A(n) = |A \cap [0, 1, \dots, n-1]|$ is THE NUMBER OF ELEMENTS OF A SMALLER THAN n.
- DEFINE $d(n) = \lim_{n \to \infty} \frac{A(n)}{n}$ TO BE THE DENSITY OF A, IF IT EXISTS.

PROPERTIES

(i) $A, B \subseteq \mathbb{N}$

$$A \subseteq B \Rightarrow A(n) \leq B(n), \forall n \in \mathbb{N}$$

IF A AND B HAVE DENSITY $\Rightarrow d(A) \leq d(B)$.

BY COMPARISON, $d(B) = 0 \Rightarrow d(A) = 0$ (BY CONVERGENCE)

(ii) $\forall n \in \mathbb{N}$ $(A \cup B)(n) \leq A(n) + B(n)$

IF $A \cap B = \emptyset \Rightarrow (A \cup B)(n) = A(n) + B(n)$.

HENCE, $d(A \cup B) \leq d(A) + d(B)$ AND $d(A \cup B) = d(A) + d(B)$ IF $A \cap B = \emptyset$.

IF $d(A), d(B) = 0 \Rightarrow d(A \cup B) = 0$.

• IDEAL OF SETS WITH DENSITY 0 (ON $\mathbb{N}$)

$$I_d = \{A : d(A) = 0\}$$

PROOF: $\emptyset \in I_d$, $\mathbb{N} \notin I_d$.

LET $A, B \in I_d \Rightarrow A \cup B \in I_d$ (BY PROPERTY (ii) ABOVE)

LET $B \in I_d$, $A \subseteq B \Rightarrow A \in I_d$ (BY (i) ABOVE) ∎

WICHTIG:

- I_d CONTAINS ALL FINITE SETS AND SOME INFINITE.
- NOT EVERY SET $A \subseteq \mathbb{N}$ HAS DENSITY (EX 1.9)
- SET OF ALL EVEN NUMBERS HAS DENSITY $1/2$.
- ALL FINITE SETS HAVE DENSITY 0 AND SOME INFINITE SETS HAVE DENSITY 0 (THAT'S WHY I_d HAS ALL THE FINITE SUBSETS OF $\mathbb{N}$).

DEF. A MEASURE ON A SET S IS A REAL-VALUED FUNCTION m DEFINED ON $\mathcal{P}(S)$ SATISFYING THE FOLLOWING CONDITIONS:

(i) $m(\emptyset) = 0$ AND $m(S) > 0$

(ii) $A \subseteq B \Rightarrow m(A) \leq m(B)$

(iii) $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$.

A <u>MEASURE</u> MEASURES SIZES OF SETS, WHILE A <u>METRIC</u> MEASURES THE DISTANCE BETWEEN TWO THINGS.

PROPERTIES

(i) For every set A, $m(A) \geq 0$

(ii) $m(S \setminus A) = m(S) - m(A)$

(iii) The third condition is called finite additivity:

If $\{A_1, \dots, A_n\}$ is a collection of pairwise disjoint, then

$$m(A_1 \cup \dots \cup A_n) = m(A_1) + \dots + m(A_n)$$

REMARK: Density satisfies condition (i)-(iii) on $\mathcal{P}(\mathbb{N})$, but is not defined for all subsets of $\mathbb{N}$.

EXAMPLE: $m(A) = |A|$, S finite, $A \subseteq S$.

Then m is called the counting measure on S

EXAMPLE: $S \neq \emptyset$, $a \in S$

Define $m(A) = \begin{cases} 1, & \text{if } a \in A \\ 0, & \text{if } a \notin A \end{cases}$

Trivial measure.

EXERCISES

JECH 7.1 If F is a filter and $X \in F$, then $\mathcal{P}(X) \cap F$ is a filter on X.

F a filter on S, let $X \in F$. We prove that $\mathcal{P}(X) \cap F$ is a filter on X.

(i) We know that $\emptyset \notin F$. Thus, $\emptyset \notin \mathcal{P}(X) \cap F$. At the same time, $X \in \mathcal{P}(X) \cap F$.

(ii) Suppose $Y \in \mathcal{P}(X) \cap F$ and $Z \in \mathcal{P}(X) \cap F$, where $Y, Z \subseteq X$. Thus $Y, Z \in \mathcal{P}(X)$ and $Y, Z \in F$. Since $Y, Z \in F$, $Y \cap Z \in F$. Y and Z cannot be disjoint, otherwise $\emptyset \in F$, which contradicts F being a filter. Thus $Y \cap Z \neq \emptyset$. Since $Y \cap Z \subseteq Y, Z$, then $Y \cap Z \in \mathcal{P}(X)$. Therefore $Y \cap Z \in \mathcal{P}(X) \cap F$.

$Y, Z \subseteq X$

(iii) Let $Y \in \mathcal{P}(X) \cap F$, $Y \subseteq Z$. Then $Y \subseteq X$, $Y \in F$. Since $Y \subseteq Z \subseteq X$, then $Z \in F$. Hence $Z \in \mathcal{P}(X) \cap F$

<u>Jech 7.2</u> The filter in Example 4 is generated by the sets $\{a\}^{\wedge}$, $a \in A$.

<u>Ex 4</u> $|A| \geq \omega$, $S = [A]^{<\omega}$

- For each $P \in S$, $\hat{P} = \{Q \in S : P \subseteq Q\}$.
- $F = \{X \subseteq S : \hat{P} \subseteq X\}$, for some $P \in S$.

Then F is a nonprincipal filter on S.

First, we prove F is a filter:

(i) Clearly, $\emptyset \notin F$, by definition of $\hat{P}$. Naturally, $S \in F$, since $S \supseteq \hat{P}$, for all $\hat{P}$.

(ii) Let $X, Y \in F$. Thus there is some $\hat{P} \subseteq S$, $\hat{P} \subseteq X$ and there is $\hat{Q} \subseteq S$ s.t. $\hat{Q} \subseteq Y$. Now consider $X \cap Y$. Clearly, $X \cap Y \supset \emptyset$. Take $\widehat{P \cup Q}$. Then $X \in \widehat{P \cup Q}$ iff $X \in S$ & $X \supseteq P \cup Q$. Thus, $X \cap Y \subseteq \hat{P} \cap \hat{Q} = \widehat{P \cup Q}$. Thus, $X \cap Y \in F$.

(iii) Let $X \in F$, $X \subseteq Y$, $X \subseteq S \Rightarrow \hat{P} \subseteq X \subseteq Y \Rightarrow \hat{P} \subseteq Y \subseteq S$. Thus $Y \in F$.

To show F is nonprincipal, let there be some $X_0 \subseteq S$ st. $F = \{X \subseteq S : X_0 \subseteq X\}$. Thus, there is some $p \in S$ st. $X_0 = \hat{p}$.

p is finite while A is infinite. Thus, there is $x \in A \setminus p$ st. $\widehat{\{a\}} \in F$, but $\hat{p} \notin \widehat{\{a\}}$, since $p \in \hat{p}$, but $p \notin \widehat{\{a\}}$. Thus, F is nonprincipal.

Now to show that F is generated by $\widehat{\{a\}}$, let $G = \{\widehat{\{a\}} : a \in A\}$. We need that G has the finite intersection property.

Let $a_1, \ldots, a_n \in A$. $\bigcap_{i<n} \widehat{\{a_i\}} = \{Q \in S : \forall_{i<n} (a_i \in Q)\}$ and $\{Q \in S : \forall_{i<n}(a_i \in Q)\} = \widehat{\{a_1, \ldots, a_n\}} \neq \emptyset$. Thus let H be a filter generated by G. We know that $G \subseteq F$, so that $H \subseteq F$, since F has all the extensions of $\hat{p}$, for any $p \in S$.

Moreover, if $X \in F$, then $\exists p \in S (X \supseteq \hat{p})$. If $p = \emptyset$, then $X = S \in H$. Otherwise $\exists n > 0$ and $a_1, \ldots, a_n \in A$ so that $p = \{a_1, \ldots, a_n\}$. By our previous argument $\widehat{\{a_1, \ldots, a_n\}} = \hat{p} \in H$. Thus $X \supseteq \hat{p}$ and $X \in H$. Therefore $F \subseteq H$.

$G \subseteq H$

Jech 7.3

If U is an ultrafilter and $X \cup Y \in U$, then $X \in U$ or $Y \in U$.

Let $X \cup Y \in U$. Suppose $X \notin U$ and $Y \notin U$. Thus, $S \setminus X \in U$ and $S \setminus Y \in U$; hence $(S \setminus X \cap S \setminus Y) \in U$.

Thus $\delta I(X \cup Y) \in U$. Contradiction. $\blacksquare$

Hrbacek - Jech

Filters

1.1 If S is a finite nonempty set, then every filter on S is a principal filter.

$|S| < \omega$, $S \neq \emptyset$

Let F be a filter on S, suppose F is nonprincipal. Thus, for all $A \in F$ there is a proper subset of A s.t. $X \in F$. Hence, whenever $A \in F$, there is $X \subset A$ st. $X \in F$. Now suppose $|S| = n$, for some $n \in \mathbb{N}$. Suppose, moreover, that $X_0 = \{a\}$, for $a \in A$ and that $X_0 \in F$. Thus, there is an $X \subset X_0$ st. $X \in F$. But then $X = \emptyset \in F$, which contradicts F being a filter. $\blacksquare$

1.2 Let S be an uncountable set, and let I be the collection of all $X \subseteq S$ st. $|X| \leq \aleph_0$. I is a nonprincipal ideal on S.

Principal ideal: $I = \{X : X \subseteq A\}$, $A \subseteq S$.

$|S| > \omega$ $I = \{X \subseteq S : |X| \leq \omega\}$

The argument goes: let $X \subseteq S$ st. $|X| \leq \omega$. If $X_0 = \emptyset$, then clearly $\exists X \supset X_0$ st. $X \in I$. If $|X_0| < \omega$ and $X_0 \neq \emptyset$, then, again, is easy to see that $\exists X \supset X_0$ and $X \in I$, just take $X_0 \cup \{a\}$ st. $a \in S$ & $a \notin X_0$.

When $|X_0| = \omega$, then let $f : \omega \to X_0$ be a 1-1 onto function. Since $|S| > \omega$, $\exists a \in S$ st. $a \notin X_0$. Let $X = X_0 \cup \{a\}$. Then $\exists g$ $g : \omega \to X$, which is 1-1 and onto, hence, well-defined. Moreover,

$|X| = \omega$ AND $X \supset X_0$ AND $X \in I$.

THEREFORE, I IS NON PRINCIPAL.

1.3 LET S BE AN INFINITE SET AND $Z \subseteq S$ BE SUCH THAT Z AND $S \setminus Z$ ARE INFINITE. THE COLLECTION

$$I = \{ X \subseteq S : X \setminus Z \text{ is finite} \}$$

IS A NONPRINCIPAL IDEAL.

$|Z| \geq \omega \quad |S| \geq \omega \quad |S \setminus Z| \geq \omega$

$$I = \{ X \subseteq S : |X \setminus Z| < \omega \}$$

LET S AND $S \setminus Z$ BE INFINITE SET AND $Z \subseteq S$ BE SUCH THAT Z IS INFINITE. LET I BE THE COLLECTION OF ALL SETS $X \in S$ ST. $X \setminus Z$ IS FINITE.

FIRST WE PROVE I IS AN IDEAL.

(i) $\emptyset$ IS CLEARLY IN I, AND SINCE $|S \setminus Z| \geq \omega$, $S \notin I$ AS WELL

(ii) LET $X, Y \in I$. THUS, $X \setminus Z$ AND $Y \setminus Z$ ARE FINITE. NOW CONSIDER $(X \cup Y) \setminus Z$. THUS, $(X \cup Y) \setminus Z = (X \setminus Z) \cup (Y \setminus Z)$. SINCE $X \setminus Z$ AND $Y \setminus Z$ ARE FINITE AND UNION OF FINITE SETS ARE FINITE, THEN $(X \cup Y) \setminus Z$ IS FINITE, HENCE, $(X \cup Y) \setminus Z \in I$,

(iii) LET $X \in I$ AND $Y \subseteq X$. THUS $Y \setminus Z \subseteq X \setminus Z$, HENCE, $Y \setminus Z$ IS FINITE AND $Y \setminus Z \in I$.

LET $|Z| \geq \omega$ AND $|S \setminus Z| \geq \omega$ ST. Z IS FIXED. THUS $\forall X \in I, \exists x \in S (x \notin X \ \& \ X \cup \{x\} \in I)$ HENCE, I IS NONPRINCIPAL, SINCE $x \notin X$ AND $(X \cup \{x\}) \setminus Z$ IS FINITE.

This is guaranteed by $|S\setminus Z|$ being infinite.

1.4 If a set $A \subseteq S$ has more than one element, then the filter generated by A is not maximal.

Let $A \subseteq S$ be such that $|A| \geq 2$. Thus, $F=\{X\subseteq S : X \supseteq A\}$ is a filter. Let $A' = \{a\}$ for $a \in A$. Thus, $F' = \{X \subseteq S : X \supseteq A'\}$ is a filter. Since $a \in A$ and $|A| > 2$, then $A \supset A'$. Thus, $F \subseteq F'$ and $F \notin F'$. ■

1.5 If $\mathcal{F}$ is a nonempty set of filters on S, then $\bigcap\{F : F \in \mathcal{F}\}$ is a filter on S.

Since $\mathcal{F}$ is a set of filters, then

(i) $\emptyset \notin \bigcap\mathcal{F}$ and $S \in \bigcap\mathcal{F}$ trivially.

(ii) Let $X \in \bigcap\mathcal{F}$ and $Y \in \bigcap\mathcal{F}$. Thus, both X and Y are members of every $F \in \mathcal{F}$. Thus, $X \cap Y \in F$, for every $F \in \mathcal{F}$, by definition. Hence, $X \cap Y \in \bigcap\mathcal{F}$.

(iii) Let $X \in \bigcap\mathcal{F}$ and $Y \supseteq X$. Then $X \in \bigcap\mathcal{F}$ implies $X \in F$, $\forall F \in \mathcal{F}$. Thus, $Y \in F$, $\forall F \in \mathcal{F}$. Therefore, $Y \in \bigcap\mathcal{F}$.

By (i), (ii) and (iii) $\bigcap\mathcal{F}$ is a filter on S. ■

1.6 The filter constructed in the proof of Lemma 1.7 is the smallest filter on S that includes the collection G.

$F = \{X \subseteq S : \exists H \subseteq G (\bigcap H \subseteq X)\}$

$G \subseteq F$. ||

$F' = \bigcap \{D : D \text{ is filters } \& \ G \subseteq D\}$

WE MUST SHOW THAT $F = F'$. WE KNOW THAT F IS A FILTER AND $G \subseteq F$. HENCE, $F' \subseteq F$. WE MUST SHOW $F \subseteq F'$.

SUPPOSE $F' \subset F$. THEN $\exists X \in F (X \notin F')$. IF $X \notin F'$, THEN $\forall D$, D FILTER, $D \supseteq G$, $X \notin D$. BUT THEN, $X \notin F$. CONTRADICTION. THUS, $F \subseteq F'$.

SINCE $F \subseteq F'$ AND $F' \subseteq F$, $F = F'$. ∎

1.7 LET A BE THE SET OF ALL NATURAL NUMBERS DIVISIBLE BY A NUMBER $p > 0$. SHOW THAT $d(A) = 1/p$.

$$A(n) = |A \cap \{0, 1, \dots, n-1\}|$$

$$d(A) = \lim_{n \to \infty} \frac{A(n)}{n} = |A \cap \{0, 1, \dots, n-1\}|$$

LET $A_n = \frac{A(n)}{n}$. THUS $d(A) = \lim_{n \to \infty} A_n$. NOW LET p BE A NUMBER SUCH THAT $x \in A$ IFF x IS DIVISIBLE BY p. THUS WHENEVER $n = k \cdot p$, WHERE k IS AN INTEGER, $|A(n)| = |A((k-1)p)| + 1$

THUS, LET $p = 3$. THUS $A(1) = 0$, $A(2) = 0$, $A(3) = 1$

$A_1 = 0$, $A_2 = 0$, $A_3 = 1/3$, $A_4 = 1/4$, $A_5 = 1/5$, $A_6 = 1/3$

$A_7 = 2/7$, $A_8 = 1/4$, $A_9 = 1/3$, $A_{10} = 3/10$, $A_{11} = 3/11$, $A_{12} = 1/3$
$A_{13} = 4/13$, $A_{14} = 2/7$, $A_{15} = 1/3$

Thus, given $n \to \infty$ $$\frac{A(n)}{n} = \frac{|\{3, 6, 9, 12, 15, \ldots\}|}{n} = \frac{1}{3}$$

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To argument for $p > 0$, let $A = \{x : x \in \mathbb{N} \text{ \& } x \text{ is divisible by } p\}$. Thus, $A_{k \cdot p} = \frac{k}{k \cdot p} = \frac{1}{p}$.

Taking $d(A) = \lim_{n \to \infty} A_n$, then $\lim_{n \to \infty} A_n = \lim_{k \to \infty} A_{k \cdot p} = \frac{1}{p}$. ■

1.8 Show that the set $\{2^n : n \in \mathbb{N}\}$ has density 0.

This can be done by thinking abou how frequently A has a member intersecting $\{0, 1, 2, \ldots, n-1\}$.

Thus $A_2 = 1/2$, $A_4 = 1/2$, $A_8 = 3/8$, $A_{16} = 1/4$, $A_{32} = 5/32$, $A_{64} = 3/32$, $A_{128} = 7/128$ and so forth.

Thus, when n gets larger, we have:

$$\lim_{n \to \infty} A_{2^n} = \lim_{n \to \infty} \frac{n}{2^n} = \lim_{n \to \infty} \frac{1}{\infty} = 0. \blacksquare$$