KUNEN - CH. 2 § 3

THM 3.4: $\kappa \geq \omega$ IMPLIES EQUIVALENCE OF THE FOLLOWING:

a) $MA(\kappa)$

b) $MA(\kappa)$ RESTRICTED TO PO OF CARDINALITY $\leq \kappa$

c) $MA(\kappa)$ RESTRICTED TO COMPLETE BOOLEAN ALGEBRAS

d) X IS ANY COMPACT c.c.c. HAUSDORFF SPACE AND U_α ARE DENSE OPEN SETS FOR $\alpha < \kappa$, THEN $\bigcap_\alpha U_\alpha \neq \emptyset$.

↳ IMPORTANT RESULT!

SUSLIN PROBLEM §4

(a) EVERY $\underbrace{\text{SEPARABLE SPACE}}$ IS $\underbrace{\text{C.C.C.}}$

X IS SEPARABLE IFF CONTAINS A COUNTABLE DENSE SUBSET.

X HAS C.C.C. IFF ∄ UNCOUNTABLE FAMILY OF PAIRWISE DISJOINT NON-EMPTY OPEN SUBSETS OF X

(b) NOT EVERY C.C.C. IS SEPARABLE.

CONSIDERING THESE PROPERTIES FOR ORDERED SPACES, THE QUESTION: C.C.C. $\Longleftrightarrow$ SEPARABLE? IS INDEPENDENT OF ZFC!

DEF: SUSLIN LINE IS A TOTAL ORDER $\langle X, < \rangle$ S.T. IN THE ORDER TOPOLOGY, X IS C.C.C. BUT NOT SEPARABLE.

(SH) THERE ARE NO SUSLIN LINES.

I.E. $\forall X$, $\langle X; < \rangle$ IS TOTAL, IN THE ORDER TOPOLOGY, IF X IS C.C.C. $\Rightarrow$ X IS SEPARABLE.

ORDER TOPOLOGY

X simple ordered set with $|X|>1$.

$\mathcal{B}$ is the collection of all sets such that:

(1) All open intervals $(a,b) \in X$.

(2) All intervals $[a_0,b) \in X$

(3) All intervals $(a,b_0] \in X$

} if a_0, b_0 exist. (if there are minimal or maximal elements).

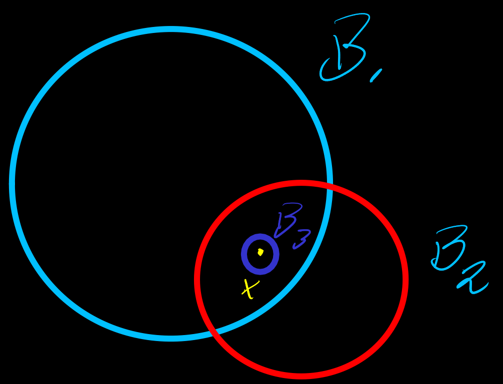
$\mathcal{B}$ is a basis for a $\underbrace{\text{topology}}_{\text{ORDER TOPOLOGY}}$ in X.

(a) For each $x \in X$, $\exists B \in \mathcal{B}$ st. $x \in B$.

(b) If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$

$$\Downarrow$$

$$\exists B_3 \subseteq B_1 \cap B_2 \text{ st. } x \in B_3.$$



The topology generated by $\mathcal{B}$ is such that $U \subseteq X$ is open in X ($U \in \tau$) if for each $x \in U$, $\exists B \in \mathcal{B}$ st. $x \in B$ and $B \subseteq U$.

(SH) Suslin Hypothesis

- Attempt to characterize the order type of the reals, i.e. $\langle \mathbb{R}, < \rangle$.

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If $\langle X, < \rangle$ is total and satisfies:

ⓐ X has no first element

ⓑ X is connected in the order topology.

ⓒ X is separable in the order topology.

Then $\langle X, < \rangle$ is isomorphic to $\langle \mathbb{R}, < \rangle$.

Can c be replaced by

(c') X has the c.c.c. in the order topology?

$$SH \Rightarrow (c) \Leftrightarrow (c')$$

SH is equivalent to saying that a, b, c entails the isomorphism.

4.2 $MA(\omega_1) \to SH$

By Thm 2.24, $MA(\omega_1) \Rightarrow$ any product of c.c.c. spaces is c.c.c.

The next lemma proves the implication:

Lemma 4.3. If X is a Suslin line, X^2 is not c.c.c.