

Def: A filter on  $\mathcal{P}(S)$  is a collection  $F$  of subsets of  $S$  st.:  $F = \{F_n\}$

(i)  $\emptyset \notin F$ ,  $S \in F$

$$F_n \in \mathcal{P}(S)$$

(ii)  $X \in F, Y \in F \Rightarrow X \cap Y \in F$

$$F \subseteq \mathcal{P}(S).$$

(iii)  $X, Y \subseteq S, X \in F$  and  $X \subseteq Y \Rightarrow Y \in F$ .

$$\text{Ex: } S = \{a, b, c\}$$

$$F_1 = \{S\}$$

$$F_1 = \{S, \{a\}, \{a, b\}, \{a, c\}\}$$

$$F_2 = \{S, \{b\}, \{b, c\}, \{b, a\}\}$$

Def: An ideal on  $\mathcal{P}(S)$  is a collection  $I$  of subsets of  $S$  st:

(i)  $\emptyset \notin I$ ,  $S \notin I$

(ii)  $X, Y \in I \Rightarrow X \cup Y \in I$

(iii)  $X, Y \subseteq S, X \in I, Y \subseteq X \Rightarrow Y \in I$ .

$$S = \{a, b, c\}$$

$$I_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$



The set of all  $A \subseteq \mathbb{N}$  of density 0 is an ideal on  $\mathbb{N}$ .

DEFINITION: Density 0 if  $\lim_{n \rightarrow \infty} |\{A \cap n\}|/n = 0$

(k) Principal filter  $A \neq \emptyset A \subseteq S$

$F = \{X \subseteq S : A \subseteq X\} \rightarrow$  For  $S$  generated by  $A$

If  $A = \{a\}$ , then  $F = \{X \subseteq S : A \subseteq X\}$  is maximal, that is there is no filter  $F'$  on  $S$  st.  $F \subset F'$ .

Suppose  $X \in F' \setminus F$ . Then  $\{a\} \subseteq X$  by definition of  $A$ .

↳ this means  $a \notin X$ , otherwise  $X \in F$ .

But then  $X \cap \{a\} = \emptyset$ . Contradiction.  $\blacksquare$

Filter of all coinitial subsets of  $S$ .

$F = \{X \subseteq S : S \setminus X \text{ is finite}\}$

$X$  is coinitial subset of  $S$   
IFF  $S \setminus X$  is finite.

$F$  is not principal.

Let  $A \in F$ .  $\exists X \subseteq A$  st.  $X \in F$  (there is an  $X$  coinitial subset of  $S$  that is a proper subset of  $A$ ). Simply take  $X$  a coinitial subset of  $A$ .  $\blacksquare$

Def:  $S \neq \emptyset$ . An ideal on  $S$  is a collection  $I$  of subsets of  $S$  such that

(i)  $\emptyset \notin I, S \notin I$

(ii)  $X \in I, Y \in I \Rightarrow X \cup Y \in I$

(iii)  $X \in I \& Y \subseteq X \Rightarrow Y \in I$ .

Tivial ideal:  $\mathbb{I} = \{\emptyset\}$

Principal ideal:  $A \subseteq S \Rightarrow \mathbb{I} = \{x \in S : x \subseteq A\}$

Duality between filters and ideals:

Let  $F$  be a filter on  $S$ . The dual ideal to  $F$  is

$$\mathbb{I}^* = \{S \setminus X : X \in F\}$$

Analogously if  $I$  is an ideal on  $S$ , the dual filter to  $I$  is

$$F^* = \{S \setminus X : X \in I\}$$

Def: A nonempty collection  $G$  has the finite intersection property iff every finite nonempty collection  $H = \{X_1, \dots, X_n\} \subseteq G$  has nonempty intersection  $X_1 \cap \dots \cap X_n \neq \emptyset$ .

Lemma 7.2

(i) If  $F$  is a nonempty family of filters on  $S$ , then  $\cap F$  is a filter on  $S$ .

By definition,  $\cap F = \{x : \forall f \in F (x \in f) \& \forall f \in F (f \neq \emptyset)\}$ .  
Then  $x \neq \emptyset$ . Hence,  $\emptyset \notin \cap F$ . Moreover,  $S \in \cap F$ , since  
every member of  $F$  is a filter.

Let  $X \in f$  and  $Y \in g$  for all  $f, g \in F$ . Then  $X \cap Y \in f$ . Hence  
 $X \cap Y \in \cap F$ .

Let  $X \in f \in F$ . Then  $X \in \cap F$ . Suppose  $Y \supseteq X$ . Then  $Y \in f$ ,  $\forall f$ .  
Therefore,  $Y \in \cap F$ .

Consequently  $\cap F$  is a filter. ■

(ii) If  $C$  is a  $\subseteq$ -chain of filters on  $S$ , then  $UC$

is a filter on  $S$ .

Let  $C$  be a  $\subseteq$ -chain of filters on  $S$ .

Consider  $UC$ . Then  $\emptyset \notin UC$ , since  $\forall c \in C, \emptyset \notin c$ .

Moreover,  $S \in UC$ , since  $\exists c \in C (S \in c)$ .

Suppose  $X, Y \in UC$ . Then  $\exists c_1, c_2 \in C (X \in c_1)$  and  $\exists c_2, c_3 \in C (Y \in c_3)$ .

Since  $C$  is a  $\subseteq$ -chain, then  $c_1, c_2 \in C \Rightarrow c_1 \subseteq c_2 \vee c_2 \subseteq c_1$ .

If  $c_1 = c_2$ , then  $X \in c_1$  and  $Y \in c_1$ , and since  $c_1$  is a filter,  $X \cap Y \in c_1$ . Hence,  $X \cap Y \in UC$ . Otherwise, let  $c_1 \subset c_2$  and

let  $Y \in c_2 \setminus c_1$ . Since  $c_1 \subset c_2$ , then  $X \in c_1 \Rightarrow X \in c_2$ . Hence,

$X \cap Y \in c_2$ , therefore  $X \cap Y \in UC$ .

Suppose  $X \in UC$  and  $X \subseteq Y$ . Then  $\exists c \in C (X \in c)$ .

But then  $Y \in c$ . Therefore,  $Y \in UC$ .

Hence,  $UC$  is a filter.

(iii) If  $G \subseteq \mathcal{P}(S)$  has the finite intersection property,

then there is a filter  $f$  on  $S$  st.  $G \subseteq f$ .

$\nexists G \rightarrow \forall H = \{x_1, \dots, x_n\} \subseteq G \quad (x_1 \cap \dots \cap x_n \neq \emptyset)$ .

Let  $F = \{X \subseteq S : \exists H (\tilde{\bigcap} H \subseteq X)\}$ . Let  $X \in G$ . Then  $\tilde{\bigcap} H \subseteq X$ .  
Hence  $X \in F$ .

To see that  $F$  is a filter,  $S \in F$ , since  $\exists H \subseteq G$  st.  $\tilde{\bigcap} H \subseteq S$ , naturally.

Let  $X, Y \in F$ . Thus,  $\exists H_1 (\tilde{\bigcap} H_1 \subseteq X)$  and  $\exists H_2 (\tilde{\bigcap} H_2 \subseteq Y)$ .

Thus,  $(\tilde{\bigcap} H_1) \cap (\tilde{\bigcap} H_2) \subseteq X \cap Y$ . Thus  $X \cap Y \in F$ .

If  $X \in F$  and  $X \subseteq Y$ , then by transitivity  $\exists H_1 (\tilde{\bigcap} H_1 \subseteq Y)$ ,  
Hence,  $Y \in F$ , as required.  $\blacksquare$

$F$  AS PROVED ABOVE IS THE SIMPLIEST FILTER THAT EXTENDS  $G$ . SINCE  $F$  CONTAINS ALL INTERSECTIONS OF  $\underbrace{\text{FINITE}}_{\text{SETS IN } G}$ .

$$F = \bigcap \{ D : D \text{ is a filter on } S \text{ AND } \underbrace{G \subseteq D}_{\text{EXTENDS } G} \}$$

F IS GENERATED BY G.

Def: A filter  $V$  on  $S$  is an ultrafilter if & only if for all  $X \subseteq S$  either  $X \in V$  or  $S \setminus X \in V$ .

Final notion: prime ideal

$$\forall X \subseteq S (X \in I \vee (S \setminus X) \in I)$$

$$I = \mathcal{P}(S) \setminus V$$

Lemma: A filter  $F$  on  $S$  is an ultrafilter iff it is maximal.

Proof: Let  $F$  be an ultrafilter on  $S$ . Then if  $X \subseteq S$ , then either  $X \in F$  or  $S \setminus X \in F$ . Let  $F \subset F'$ . Then  $\exists Y \in F' \setminus F$ .

Since  $F$  is ultrafilter, then  $S \setminus Y \in F$ . Thus  $S \setminus Y \in F'$ . Contradiction.

Let  $F$  be a maximal filter. Then  $\nexists F' (F \subset F')$ . Let  $X \subseteq S$ . Then, since  $F$  is maximal, only  $\emptyset$  is not a member of  $F$ . This means  $X \neq \emptyset$ ,  $X \in F$ . Otherwise,  $S \setminus X \in F$  as desired.

Suppose  $f$  is not an ultrafilter. Then there is some  $X \subseteq S$  such that neither  $X \in f$  nor  $S \setminus X \in f$ . Let  $F' = F \cup \{X\}$ .

[CLAIM]  $F'$  has the finite intersection property.

If  $Y \in F$ , then  $Y \cap X \neq \emptyset$ , otherwise  $S \setminus Y \supseteq X$ , and thus  $S \setminus Y \in F$ , since  $F$  is a filter. Thus, if  $Y_1, \dots, Y_n \in F$ , then  $(Y_1 \cap \dots \cap Y_n) \cap X \neq \emptyset$ . Hence  $F'$  has the finite intersection property.

By Lemma 7.2(iii) there is a filter  $F''$  that extends  $F'$ . Since  $X \notin F'' \setminus F$ , then  $F$  is not maximal.

Lemma 7.5 Every filter can be extended to an ultrafilter.

By Lemma 7.2(iii), if  $f$  is not maximal, then

(which happens)  
 $\exists F' \supseteq f$ . And then if  $F'$  is not maximal, one repeats this operation. Then we obtain a  $\subseteq$ -chain of filters  
 on  $S$ . By Lemma 7.2(iii),  $\cup F$  is also a filter, as well as upper bound of this chain. Since we have a chain, by Zorn's Lemma there is a maximal element to this chain. This element is an ultrafilter by Lemma 7.4.  
 And consider the partial order  $(\mathcal{P}, \subseteq)$

REMARK: There are principal filters which are maximal, whence ultrafilters.  $\Rightarrow$  finite  $\Rightarrow$  ultrafilter ( $f$  is principal).  
 Are there non-principal ultrafilters?

Let  $S$  be infinite. Then the filter that extends the Fréchet filter is non-principal and maximal.  
 (This follows from 7.5 and cannot be proved without AC).

Let  $K \geq \omega$   $|S| = K$

Since every ultrafilter is a subset of  $\mathcal{P}(S)$ ,  
then  $|U| \leq |\mathcal{P}(\mathcal{P}(S))| = 2^{2^K}$ .  
Next Thm:  $|U| = 2^{2^K}$ .

Let an ultrafilter  $D$  on  $K$  be uniform if  $\text{fd}(D) = K$   
and  $\text{fcd}(D) = \aleph_0$  (possibly) For every infinite cardinal there  
are  $2^{\aleph_0}$  uniform ultrafilters on  $K$ ,  
(We skip this one for now).

### $K$ -COMPLETE FILTERS AND IDEALS.

Filter on  $S$  is countably complete ( $\omega$ -complete)  
if whenever  $\{X_n : n \in \mathbb{N}\}$  is a family of subsets of  $S$   
and  $X_n \in F$  for every  $n$ , then  $\bigcap_{n=0}^{\infty} X_n \in F$ .

If  $I$  is an ideal and  $X_n \in I$  for every  $n$ , then  
 $\bigcup_{n=0}^{\infty} X_n$  and  $I$  is called COUNTABLY complete.

GENERALIZING

$K > \omega$ ,  $K$  regular cardinal. Filter on  $S$

$I$  ideal on  $S$

$F$  is  $K$ -complete iff  $F$  is closed under intersection  
of less than  $K$  sets, i.e.,

- $\mathcal{A} = \{X_\alpha : \alpha < \gamma\}$ ,  $X_\alpha \subseteq S$ , for each  $\alpha < \gamma\}$
  - $\gamma < K$
  - $X_\alpha \in F$ ,  $\forall \alpha < \gamma$
- $\Rightarrow \bigcap_{\alpha < \gamma} X_\alpha \in F$

The  $K$ -complete ideal is the DUAL notion.

A filter  $F$  is  $K$ -complete iff its dual ideal is  $K$ -complete.

## EXAMPLES

①  $\kappa$ -complete ideal  $|S| \geq \kappa$

$$\mathcal{D} = \{X \subseteq S : |X| < \kappa\}$$

②  $\sigma$ -complete filter  $\Rightarrow \kappa_1$ -complete filter

③ Nonprincipal  $\sigma$ -complete filter on  $S$  contains

$$|S| > \omega \Rightarrow \{X \subseteq S : X \leq \kappa_0\} \text{ is } \sigma\text{-ideal on } S$$

④  $\kappa > \omega$ ,  $\kappa$  regular,  $|S| \geq \kappa \Rightarrow \{X \subseteq S : |X| < \kappa\}$  is the  
smallest  $\kappa$ -complete ideal on  
 $S$  containing all singletons  
Last

## QUESTION

Can a nonprincipal ultrafilter on  $\omega$  be  $\sigma$ -complete?

Answer: If yes, then there are large cardinals.

$A \subseteq \mathbb{R}$   $x$  is (limit point) accumulation point of  $A$  iff

$$\forall x \in (a, b) \left( (a, b) \cap (A \setminus \{x\}) \neq \emptyset \right) (*)$$

$A$  is open (in  $\mathbb{R}$ ) iff  $\forall a \in A \exists b < a \exists c > a ((b, c) = \{x : b < x < c\})$   
 $\& (b, c) \subseteq A$ .  $A$  is closed iff  $\mathbb{R} \setminus A$  is open.

Another notion of closed:

(i) Limit ordinal,  $X$  set of ordinals.

$\alpha$  is a limit point of  $X$  iff  $\sup(X \cap \alpha) = \alpha$ . (#)

(ii)  $X$  is closed iff  $\forall \alpha (\sup(X \cap \alpha) \in X)$ , that is,  
 $X$  is closed if every limit point of  $X$  is in  $X$ ,

(iii) If  $\gamma$  is an ordinal,  $X$  is unbounded in  $\gamma$  iff  
 $\forall \xi < \gamma (A \cap \gamma) \setminus \xi \neq \emptyset$

$\hookrightarrow$  This is ordinal

~~unbounded~~  
 $\xi_1, \xi_2, \xi_3, \dots$

$\nearrow$  THAT MEANS THAT  
TAKING THE  $\xi_s$  OUT  
STILL LEAVES  $\xi_s$   
TO BE TAKEN.  
 $\xi_6$   
 $\xi_7$   
 $\xi_8$   
 $\xi_9$   
 $\xi_{10}$   
 $\xi_{11}$   
 $\xi_{12}$   
 $\xi_{13}$   
 $\xi_{14}$   
 $\xi_{15}$   
 $\xi_{16}$   
 $\xi_{17}$   
 $\xi_{18}$   
 $\xi_{19}$   
 $\xi_{20}$   
 $\xi_{21}$   
 $\xi_{22}$   
 $\xi_{23}$   
 $\xi_{24}$   
 $\xi_{25}$   
 $\xi_{26}$   
 $\xi_{27}$   
 $\xi_{28}$   
 $\xi_{29}$   
 $\xi_{30}$   
 $\xi_{31}$   
 $\xi_{32}$   
 $\xi_{33}$   
 $\xi_{34}$   
 $\xi_{35}$   
 $\xi_{36}$   
 $\xi_{37}$   
 $\xi_{38}$   
 $\xi_{39}$   
 $\xi_{40}$   
 $\xi_{41}$   
 $\xi_{42}$   
 $\xi_{43}$   
 $\xi_{44}$   
 $\xi_{45}$   
 $\xi_{46}$   
 $\xi_{47}$   
 $\xi_{48}$   
 $\xi_{49}$   
 $\xi_{50}$   
 $\xi_{51}$   
 $\xi_{52}$   
 $\xi_{53}$   
 $\xi_{54}$   
 $\xi_{55}$   
 $\xi_{56}$   
 $\xi_{57}$   
 $\xi_{58}$   
 $\xi_{59}$   
 $\xi_{60}$   
 $\xi_{61}$   
 $\xi_{62}$   
 $\xi_{63}$   
 $\xi_{64}$   
 $\xi_{65}$   
 $\xi_{66}$   
 $\xi_{67}$   
 $\xi_{68}$   
 $\xi_{69}$   
 $\xi_{70}$   
 $\xi_{71}$   
 $\xi_{72}$   
 $\xi_{73}$   
 $\xi_{74}$   
 $\xi_{75}$   
 $\xi_{76}$   
 $\xi_{77}$   
 $\xi_{78}$   
 $\xi_{79}$   
 $\xi_{80}$   
 $\xi_{81}$   
 $\xi_{82}$   
 $\xi_{83}$   
 $\xi_{84}$   
 $\xi_{85}$   
 $\xi_{86}$   
 $\xi_{87}$   
 $\xi_{88}$   
 $\xi_{89}$   
 $\xi_{90}$   
 $\xi_{91}$   
 $\xi_{92}$   
 $\xi_{93}$   
 $\xi_{94}$   
 $\xi_{95}$   
 $\xi_{96}$   
 $\xi_{97}$   
 $\xi_{98}$   
 $\xi_{99}$   
 $\xi_{100}$   
 $\xi_{101}$   
 $\xi_{102}$   
 $\xi_{103}$   
 $\xi_{104}$   
 $\xi_{105}$   
 $\xi_{106}$   
 $\xi_{107}$   
 $\xi_{108}$   
 $\xi_{109}$   
 $\xi_{110}$   
 $\xi_{111}$   
 $\xi_{112}$   
 $\xi_{113}$   
 $\xi_{114}$   
 $\xi_{115}$   
 $\xi_{116}$   
 $\xi_{117}$   
 $\xi_{118}$   
 $\xi_{119}$   
 $\xi_{120}$   
 $\xi_{121}$   
 $\xi_{122}$   
 $\xi_{123}$   
 $\xi_{124}$   
 $\xi_{125}$   
 $\xi_{126}$   
 $\xi_{127}$   
 $\xi_{128}$   
 $\xi_{129}$   
 $\xi_{130}$   
 $\xi_{131}$   
 $\xi_{132}$   
 $\xi_{133}$   
 $\xi_{134}$   
 $\xi_{135}$   
 $\xi_{136}$   
 $\xi_{137}$   
 $\xi_{138}$   
 $\xi_{139}$   
 $\xi_{140}$   
 $\xi_{141}$   
 $\xi_{142}$   
 $\xi_{143}$   
 $\xi_{144}$   
 $\xi_{145}$   
 $\xi_{146}$   
 $\xi_{147}$   
 $\xi_{148}$   
 $\xi_{149}$   
 $\xi_{150}$   
 $\xi_{151}$   
 $\xi_{152}$   
 $\xi_{153}$   
 $\xi_{154}$   
 $\xi_{155}$   
 $\xi_{156}$   
 $\xi_{157}$   
 $\xi_{158}$   
 $\xi_{159}$   
 $\xi_{160}$   
 $\xi_{161}$   
 $\xi_{162}$   
 $\xi_{163}$   
 $\xi_{164}$   
 $\xi_{165}$   
 $\xi_{166}$   
 $\xi_{167}$   
 $\xi_{168}$   
 $\xi_{169}$   
 $\xi_{170}$   
 $\xi_{171}$   
 $\xi_{172}$   
 $\xi_{173}$   
 $\xi_{174}$   
 $\xi_{175}$   
 $\xi_{176}$   
 $\xi_{177}$   
 $\xi_{178}$   
 $\xi_{179}$   
 $\xi_{180}$   
 $\xi_{181}$   
 $\xi_{182}$   
 $\xi_{183}$   
 $\xi_{184}$   
 $\xi_{185}$   
 $\xi_{186}$   
 $\xi_{187}$   
 $\xi_{188}$   
 $\xi_{189}$   
 $\xi_{190}$   
 $\xi_{191}$   
 $\xi_{192}$   
 $\xi_{193}$   
 $\xi_{194}$   
 $\xi_{195}$   
 $\xi_{196}$   
 $\xi_{197}$   
 $\xi_{198}$   
 $\xi_{199}$   
 $\xi_{200}$   
 $\xi_{201}$   
 $\xi_{202}$   
 $\xi_{203}$   
 $\xi_{204}$   
 $\xi_{205}$   
 $\xi_{206}$   
 $\xi_{207}$   
 $\xi_{208}$   
 $\xi_{209}$   
 $\xi_{210}$   
 $\xi_{211}$   
 $\xi_{212}$   
 $\xi_{213}$   
 $\xi_{214}$   
 $\xi_{215}$   
 $\xi_{216}$   
 $\xi_{217}$   
 $\xi_{218}$   
 $\xi_{219}$   
 $\xi_{220}$   
 $\xi_{221}$   
 $\xi_{222}$   
 $\xi_{223}$   
 $\xi_{224}$   
 $\xi_{225}$   
 $\xi_{226}$   
 $\xi_{227}$   
 $\xi_{228}$   
 $\xi_{229}$   
 $\xi_{230}$   
 $\xi_{231}$   
 $\xi_{232}$   
 $\xi_{233}$   
 $\xi_{234}$   
 $\xi_{235}$   
 $\xi_{236}$   
 $\xi_{237}$   
 $\xi_{238}$   
 $\xi_{239}$   
 $\xi_{240}$   
 $\xi_{241}$   
 $\xi_{242}$   
 $\xi_{243}$   
 $\xi_{244}$   
 $\xi_{245}$   
 $\xi_{246}$   
 $\xi_{247}$   
 $\xi_{248}$   
 $\xi_{249}$   
 $\xi_{250}$   
 $\xi_{251}$   
 $\xi_{252}$   
 $\xi_{253}$   
 $\xi_{254}$   
 $\xi_{255}$   
 $\xi_{256}$   
 $\xi_{257}$   
 $\xi_{258}$   
 $\xi_{259}$   
 $\xi_{260}$   
 $\xi_{261}$   
 $\xi_{262}$   
 $\xi_{263}$   
 $\xi_{264}$   
 $\xi_{265}$   
 $\xi_{266}$   
 $\xi_{267}$   
 $\xi_{268}$   
 $\xi_{269}$   
 $\xi_{270}$   
 $\xi_{271}$   
 $\xi_{272}$   
 $\xi_{273}$   
 $\xi_{274}$   
 $\xi_{275}$   
 $\xi_{276}$   
 $\xi_{277}$   
 $\xi_{278}$   
 $\xi_{279}$   
 $\xi_{280}$   
 $\xi_{281}$   
 $\xi_{282}$   
 $\xi_{283}$   
 $\xi_{284}$   
 $\xi_{285}$   
 $\xi_{286}$   
 $\xi_{287}$   
 $\xi_{288}$   
 $\xi_{289}$   
 $\xi_{290}$   
 $\xi_{291}$   
 $\xi_{292}$   
 $\xi_{293}$   
 $\xi_{294}$   
 $\xi_{295}$   
 $\xi_{296}$   
 $\xi_{297}$   
 $\xi_{298}$   
 $\xi_{299}$   
 $\xi_{300}$   
 $\xi_{301}$   
 $\xi_{302}$   
 $\xi_{303}$   
 $\xi_{304}$   
 $\xi_{305}$   
 $\xi_{306}$   
 $\xi_{307}$   
 $\xi_{308}$   
 $\xi_{309}$   
 $\xi_{310}$   
 $\xi_{311}$   
 $\xi_{312}$   
 $\xi_{313}$   
 $\xi_{314}$   
 $\xi_{315}$   
 $\xi_{316}$   
 $\xi_{317}$   
 $\xi_{318}$   
 $\xi_{319}$   
 $\xi_{320}$   
 $\xi_{321}$   
 $\xi_{322}$   
 $\xi_{323}$   
 $\xi_{324}$   
 $\xi_{325}$   
 $\xi_{326}$   
 $\xi_{327}$   
 $\xi_{328}$   
 $\xi_{329}$   
 $\xi_{330}$   
 $\xi_{331}$   
 $\xi_{332}$   
 $\xi_{333}$   
 $\xi_{334}$   
 $\xi_{335}$   
 $\xi_{336}$   
 $\xi_{337}$   
 $\xi_{338}$   
 $\xi_{339}$   
 $\xi_{340}$   
 $\xi_{341}$   
 $\xi_{342}$   
 $\xi_{343}$   
 $\xi_{344}$   
 $\xi_{345}$   
 $\xi_{346}$   
 $\xi_{347}$   
 $\xi_{348}$   
 $\xi_{349}$   
 $\xi_{350}$   
 $\xi_{351}$   
 $\xi_{352}$   
 $\xi_{353}$   
 $\xi_{354}$   
 $\xi_{355}$   
 $\xi_{356}$   
 $\xi_{357}$   
 $\xi_{358}$   
 $\xi_{359}$   
 $\xi_{360}$   
 $\xi_{361}$   
 $\xi_{362}$   
 $\xi_{363}$   
 $\xi_{364}$   
 $\xi_{365}$   
 $\xi_{366}$   
 $\xi_{367}$   
 $\xi_{368}$   
 $\xi_{369}$   
 $\xi_{370}$   
 $\xi_{371}$   
 $\xi_{372}$   
 $\xi_{373}$   
 $\xi_{374}$   
 $\xi_{375}$   
 $\xi_{376}$   
 $\xi_{377}$   
 $\xi_{378}$   
 $\xi_{379}$   
 $\xi_{380}$   
 $\xi_{381}$   
 $\xi_{382}$   
 $\xi_{383}$   
 $\xi_{384}$   
 $\xi_{385}$   
 $\xi_{386}$   
 $\xi_{387}$   
 $\xi_{388}$   
 $\xi_{389}$   
 $\xi_{390}$   
 $\xi_{391}$   
 $\xi_{392}$   
 $\xi_{393}$   
 $\xi_{394}$   
 $\xi_{395}$   
 $\xi_{396}$   
 $\xi_{397}$   
 $\xi_{398}$   
 $\xi_{399}$   
 $\xi_{400}$   
 $\xi_{401}$   
 $\xi_{402}$   
 $\xi_{403}$   
 $\xi_{404}$   
 $\xi_{405}$   
 $\xi_{406}$   
 $\xi_{407}$   
 $\xi_{408}$   
 $\xi_{409}$   
 $\xi_{410}$   
 $\xi_{411}$   
 $\xi_{412}$   
 $\xi_{413}$   
 $\xi_{414}$   
 $\xi_{415}$   
 $\xi_{416}$   
 $\xi_{417}$   
 $\xi_{418}$   
 $\xi_{419}$   
 $\xi_{420}$   
 $\xi_{421}$   
 $\xi_{422}$   
 $\xi_{423}$   
 $\xi_{424}$   
 $\xi_{425}$   
 $\xi_{426}$   
 $\xi_{427}$   
 $\xi_{428}$   
 $\xi_{429}$   
 $\xi_{430}$   
 $\xi_{431}$   
 $\xi_{432}$   
 $\xi_{433}$   
 $\xi_{434}$   
 $\xi_{435}$   
 $\xi_{436}$   
 $\xi_{437}$   
 $\xi_{438}$   
 $\xi_{439}$   
 $\xi_{440}$   
 $\xi_{441}$   
 $\xi_{442}$   
 $\xi_{443}$   
 $\xi_{444}$   
 $\xi_{445}$   
 $\xi_{446}$   
 $\xi_{447}$   
 $\xi_{448}$   
 $\xi_{449}$   
 $\xi_{450}$   
 $\xi_{451}$   
 $\xi_{452}$   
 $\xi_{453}$   
 $\xi_{454}$   
 $\xi_{455}$   
 $\xi_{456}$   
 $\xi_{457}$   
 $\xi_{458}$   
 $\xi_{459}$   
 $\xi_{460}$   
 $\xi_{461}$   
 $\xi_{462}$   
 $\xi_{463}$   
 $\xi_{464}$   
 $\xi_{465}$   
 $\xi_{466}$   
 $\xi_{467}$   
 $\xi_{468}$   
 $\xi_{469}$   
 $\xi_{470}$   
 $\xi_{471}$   
 $\xi_{472}$   
 $\xi_{473}$   
 $\xi_{474}$   
 $\xi_{475}$   
 $\xi_{476}$   
 $\xi_{477}$   
 $\xi_{478}$   
 $\xi_{479}$   
 $\xi_{480}$   
 $\xi_{481}$   
 $\xi_{482}$   
 $\xi_{483}$   
 $\xi_{484}$   
 $\xi_{485}$   
 $\xi_{486}$   
 $\xi_{487}$   
 $\xi_{488}$   
 $\xi_{489}$   
 $\xi_{490}$   
 $\xi_{491}$   
 $\xi_{492}$   
 $\xi_{493}$   
 $\xi_{494}$   
 $\xi_{495}$   
 $\xi_{496}$   
 $\xi_{497}$   
 $\xi_{498}$   
 $\xi_{499}$   
 $\xi_{500}$   
 $\xi_{501}$   
 $\xi_{502}$   
 $\xi_{503}$   
 $\xi_{504}$   
 $\xi_{505}$   
 $\xi_{506}$   
 $\xi_{507}$   
 $\xi_{508}$   
 $\xi_{509}$   
 $\xi_{510}$   
 $\xi_{511}$   
 $\xi_{512}$   
 $\xi_{513}$   
 $\xi_{514}$   
 $\xi_{515}$   
 $\xi_{516}$   
 $\xi_{517}$   
 $\xi_{518}$   
 $\xi_{519}$   
 $\xi_{520}$   
 $\xi_{521}$   
 $\xi_{522}$   
 $\xi_{523}$   
 $\xi_{524}$   
 $\xi_{525}$   
 $\xi_{526}$   
 $\xi_{527}$   
 $\xi_{528}$   
 $\xi_{529}$   
 $\xi_{530}$   
 $\xi_{531}$   
 $\xi_{532}$   
 $\xi_{533}$   
 $\xi_{534}$   
 $\xi_{535}$   
 $\xi_{536}$   
 $\xi_{537}$   
 $\xi_{538}$   
 $\xi_{539}$   
 $\xi_{540}$   
 $\xi_{541}$   
 $\xi_{542}$   
 $\xi_{543}$   
 $\xi_{544}$   
 $\xi_{545}$   
 $\xi_{546}$   
 $\xi_{547}$   
 $\xi_{548}$   
 $\xi_{549}$   
 $\xi_{550}$   
 $\xi_{551}$   
 $\xi_{552}$   
 $\xi_{553}$   
 $\xi_{554}$   
 $\xi_{555}$   
 $\xi_{556}$   
 $\xi_{557}$   
 $\xi_{558}$   
 $\xi_{559}$   
 $\xi_{560}$   
 $\xi_{561}$   
 $\xi_{562}$   
 $\xi_{563}$   
 $\xi_{564}$   
 $\xi_{565}$   
 $\xi_{566}$   
 $\xi_{567}$   
 $\xi_{568}$   
 $\xi_{569}$   
 $\xi_{570}$   
 $\xi_{571}$   
 $\xi_{572}$   
 $\xi_{573}$   
 $\xi_{574}$   
 $\xi_{575}$   
 $\xi_{576}$   
 $\xi_{577}$   
 $\xi_{578}$   
 $\xi_{579}$   
 $\xi_{580}$   
 $\xi_{581}$   
 $\xi_{582}$   
 $\xi_{583}$   
 $\xi_{584}$   
 $\xi_{585}$   
 $\xi_{586}$   
 $\xi_{587}$   
 $\xi_{588}$   
 $\xi_{589}$   
 $\xi_{590}$   
 $\xi_{591}$   
 $\xi_{592}$   
 $\xi_{593}$   
 $\xi_{594}$   
 $\xi_{595}$   
 $\xi_{596}$   
 $\xi_{597}$   
 $\xi_{598}$   
 $\xi_{599}$   
 $\xi_{600}$   
 $\xi_{601}$   
 $\xi_{602}$   
 $\xi_{603}$   
 $\xi_{604}$   
 $\xi_{605}$   
 $\xi_{606}$   
 $\xi_{607}$   
 $\xi_{608}$   
 $\xi_{609}$   
 $\xi_{610}$   
 $\xi_{611}$   
 $\xi_{612}$   
 $\xi_{613}$   
 $\xi_{614}$   
 $\xi_{615}$   
 $\xi_{616}$   
 $\xi_{617}$   
 $\xi_{618}$   
 $\xi_{619}$   
 $\xi_{620}$   
 $\xi_{621}$   
 $\xi_{622}$   
 $\xi_{623}$   
 $\xi_{624}$   
 $\xi_{625}$   
 $\xi_{626}$   
 $\xi_{627}$   
 $\xi_{628}$   
 $\xi_{629}$   
 $\xi_{630}$   
 $\xi_{631}$   
 $\xi_{632}$   
 $\xi_{633}$   
 $\xi_{634}$   
 $\xi_{635}$   
 $\xi_{636}$   
 $\xi_{637}$   
 $\xi_{638}$   
 $\xi_{639}$   
 $\xi_{640}$   
 $\xi_{641}$   
 $\xi_{642}$   
 $\xi_{643}$   
 $\xi_{644}$   
 $\xi_{645}$   
 $\xi_{646}$   
 $\xi_{647}$   
 $\xi_{648}$   
 $\xi_{649}$   
 $\xi_{650}$   
 $\xi_{651}$   
 $\xi_{652}$   
 $\xi_{653}$   
 $\xi_{654}$   
 $\xi_{655}$   
 $\xi_{656}$   
 $\xi_{657}$   
 $\xi_{658}$   
 $\xi_{659}$   
 $\xi_{660}$   
 $\xi_{661}$   
 $\xi_{662}$   
 $\xi_{663}$   
 $\xi_{664}$   
 $\xi_{665}$   
 $\xi_{666}$   
 $\xi_{667}$   
 $\xi_{668}$   
 $\xi_{669}$   
 $\xi_{670}$   
 $\xi_{671}$   
 $\xi_{672}$   
 $\xi_{673}$   
 $\xi_{674}$   
 $\xi_{675}$   
 $\xi_{676}$   
 $\xi_{677}$   
 $\xi_{678}$   
 $\xi_{679}$   
 $\xi_{680}$   
 $\xi_{681}$   
 $\xi_{682}$   
 $\xi_{683}$   
 $\xi_{684}$   
 $\xi_{685}$   
 $\xi_{686}$   
 $\xi_{687}$   
 $\xi_{688}$   
 $\xi_{689}$   
 $\xi_{690}$   
 $\xi_{691}$   
 $\xi_{692}$   
 $\xi_{693}$   
 $\xi_{694}$   
 $\xi_{695}$   
 $\xi_{696}$   
 $\xi_{697}$   
 $\xi_{698}$   
 $\xi_{699}$   
 $\xi_{700}$   
 $\xi_{701}$   
 $\xi_{702}$   
 $\xi_{703}$   
 $\xi_{704}$   
 $\xi_{705}$   
 $\xi_{706}$   
 $\xi_{707}$   
 $\xi_{708}$   
 $\xi_{709}$   
 $\xi_{710}$   
 $\xi_{711}$   
 $\xi_{712}$   
 $\xi_{713}$   
 $\xi_{714}$   
 $\xi_{715}$   
 $\xi_{716}$   
 $\xi_{717}$   
 $\xi_{718}$   
 $\xi_{719}$   
 $\xi_{720}$   
 $\xi_{721}$   
 $\xi_{722}$   
 $\xi_{723}$   
 $\xi_{724}$   
 $\xi_{725}$   
 $\xi_{726}$   
 $\xi_{727}$   
 $\xi_{728}$   
 $\xi_{729}$   
 $\xi_{730}$   
 $\xi_{731}$   
 $\xi_{732}$   
 $\xi_{733}$   
 $\xi_{734}$   
 $\xi_{735}$   
 $\xi_{736}$   
 $\xi_{737}$   
 $\xi_{738}$   
 $\xi_{739}$   
 $\xi_{740}$   
 $\xi_{741}$   
 $\xi_{742}$   
 $\xi_{743}$   
 $\xi_{744}$   
 $\xi_{745}$   
 $\xi_{746}$   
 $\xi_{747}$   
 $\xi_{748}$   
 $\xi_{749}$   
 $\xi_{750}$   
 $\xi_{751}$   
 $\xi_{752}$   
 $\xi_{753}$   
 $\xi_{754}$   
 $\xi_{755}$   
 $\xi_{756}$   
 $\xi_{757}$   
 $\xi_{758}$   
 $\xi_{759}$   
 $\xi_{760}$   
 $\xi_{761}$   
 $\xi_{762}$   
 $\xi_{763}$   
 $\xi_{764}$   
 $\xi_{765}$   
 $\xi_{766}$   
 $\xi_{767}$   
 $\xi_{768}$   
 $\xi_{769}$   
 $\xi_{770}$   
 $\xi_{771}$   
 $\xi_{772}$   
 $\xi_{773}$   
 $\xi_{774}$   
 $\xi_{775}$   
 $\xi_{776}$   
 $\xi_{777}$   
 $\xi_{778}$   
 $\xi_{779}$   
 $\xi_{780}$   
 $\xi_{781}$   
 $\xi_{782}$   
 $\xi_{783}$   
 $\xi_{784}$   
 $\xi_{785}$   
 $\xi_{786}$   
 $\xi_{787}$   
 $\xi_{788}$   
 $\xi_{789}$   
 $\xi_{790}$   
 $\xi_{791}$   
 $\xi_{792}$   
 $\xi_{793}$   
 $\xi_{794}$   
 $\xi_{795}$   
 $\xi_{796}$   
 $\xi_{797}$   
 $\xi_{798}$   
 $\xi_{799}$   
 $\xi_{800}$   
 $\xi_{801}$   
 $\xi_{802}$   
 $\xi_{803}$   
 $\xi_{804}$   
 $\xi_{805}$   
 $\xi_{806}$   
 $\xi_{807}$   
 $\xi_{808}$   
 $\xi_{809}$   
 $\xi$

LET  $\delta$  BE AN INFINITE REGULAR CARDINAL.

(iv)  $X$  IS  $\delta$ -CLOSED IFF FOR ALL ORDINALS  $\alpha$  WITH  $c_\delta(\alpha) = \delta$  AND ST.  $X \cap \alpha$  IS UNBOUNDED IN  $\alpha$

$$\alpha = \sup(X \cap \alpha) \in X$$

↳ THAT IS,  $X$  IS UNBOUNDED FOR ANY LIMIT POINT OF SIZE  $\leq \delta$ .

→ WE TOOK IT OUT PREVIOUSLY.

(v)  $X$  IS CLUB IN  $\alpha$  IFF  $X \cup \{\alpha\}$  IS CLOSED AND  
 $X$  IS UNBOUNDED IN  $\alpha$ .  
CONTAINS ALL LIMIT POINTS OF  $X$

{ (vi)  $X$  IS  $\delta$ -CLUB IN  $\alpha$  IFF  $X \cup \{\alpha\}$  IS  $\delta$ -CLOSED  
AND  $X$  IS UNBOUNDED IN  $\alpha$ . (DEF 2.1 TECH).

{ (vii)  $S \subseteq X$  IS STATIONARY IF  $S \cap X \neq \emptyset$  FOR  
ALL  $X$   $\delta$ -CLUB.

↳  $\delta$  INFINITE REGULAR CARDINAL.

If  $\delta$  IS INFINITE REGULAR CARDINAL WE CAN STATE

(vi) AS FOLLOWS:

(vi\*)  $X \subseteq \delta$ ,  $X$  UNBOUNDED IS CLOSED IFF  
FOR EVERY SEQUENTIAL  $\alpha_0 < \alpha_1 < \dots < \alpha_\xi < \dots$ ,  
WITH  $\xi < \tau$ , OR ELEMENTS OF  $X$  WITH  
LENGTH  $\tau < \delta$  WE MAY FIND  $\alpha_\xi \in C$ .

Then  $C, D$  are club, taken  $C \cap D$  is club.

$C$  is closed  $\Leftrightarrow$  every limit point of  $C$  is in  $C$ .

$D$  is closed  $\Leftrightarrow$  every limit point of  $D$  is in  $D$ .

$C \cap D$  is closed.

$$\forall \alpha (\sup(C \cap D) \cap \alpha) \in C \cap D$$

Let  $x$  be limit point of  $C \cap D$ . Then

$$x = \sup((C \cap D) \cap X)$$

If  $x \notin C \cap D$ , then  $x \notin C$  or  $x \notin D$ . But then,  $C$  and  $D$  are closed, which means that  $x$  is not limit point. Hence,  $x \in C \cap D$ .

[Unboundedness]  $C, D$  unbounded  $\Rightarrow C \cap D$  unbounded.

Suppose  $C \cap D$  is bounded above. Then  $\exists \xi' < \gamma$  such that  $((C \cap D) \cap \gamma) \setminus \xi' = \emptyset$



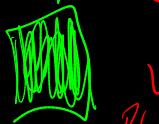
$$|A| = |\xi'| < \gamma$$

In another way, let  $K$  be a cardinal  $\delta^+$ .  
 Both  $C, D$  are unbounded in  $K$ .

If  $\alpha < K$ , since  $C$  and  $D$  are both UNBOUNDED in  $K$ , then  $\exists \alpha_j < K (\alpha_j > \alpha)$  and  $\alpha_j \in C$ . In the same way,  $\exists \alpha_i < K (\alpha_i > \alpha)$  and  $\alpha_i \in D$ .

Hence, let  $\alpha_0 = \alpha$ ,  $j = 2K-1$  and  $i = 2K$ , for  $K \in \mathbb{Z}^+$ . Then  $\exists$  a sequence  $\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$

And  $\alpha_i \in D$ ,  $\alpha_j \in C$ , for all  $i, j$ .

Take  $\beta = \lim_{\xi \rightarrow \gamma} \alpha_\xi$ . Then  $\beta \in C$  and  $\beta \in D$ ,  
 by (v, \*). We are done. 

Remember that  
 $\alpha = \sup(X \cap \alpha)$  is a limit point.

Remark (i) The collection of all CLUB SUBSETS OF  $K$  HAS THE FINITE INTERSECTION PROPERTY.

(ii)  $F_{\text{CLUB}}$ , THE FILTER GENERATED BY THE CLUB SETS, CONSISTS OF ALL  $X \subseteq K$  THAT CONTAIN A CLUB SUBSET.  
 We call it THE CLUB FILTER ON  $K$ .

Lemma  $\alpha$  ordinal st.  $cf(\alpha) > \omega$ . Let  $F_{club}$  be the set of all  $A \subseteq \alpha$  st.  $\exists B \subseteq A$  ( $B$  is club in  $\alpha$ ).

Then  $F_{club}$  is a  $\{cf(\alpha)\}$ -closed filter on  $\alpha$  and

$F_{club}$  is non-trivial.

$cf(\alpha)$  is unbounded in  $\alpha$ .

$cf(\alpha) = \alpha$  means that  $\alpha$  is regular without a cardinal.

$cf(\alpha) > \omega$ ,  $\alpha$  ordinal (limit).

$F_{club} = \{A \subseteq \alpha : \exists B(B \subseteq A \text{ & } B \text{ is club in } \alpha)\}$

Let  $\mathcal{A}$  be a family of subsets of  $\alpha$  which are club in  $\alpha$ . Let  $A_i \in \mathcal{A}$ , that is,  $\mathcal{A} = \{A_i\}_{i \in \mathbb{P}}$ ,  $\beta \leq cf(\alpha)$ , each  $A_i$  club in  $\alpha$ .

$\bigcap_{i \in \mathbb{P}} A_i$  is closed, since arbitrary intersections of closed sets are closed.

Unboundeness follows from a generalization of the previous argument.

Let  $\delta^{\leq \alpha}$  define  $f: \beta \cdot \omega \rightarrow \alpha$ , where  $\beta \cdot \omega$  is an ordinal multiplication.  $\beta \cdot \omega \rightarrow \alpha$ .

LET  $\{f(\beta \cdot n + i)\}$  BE THE LIMIT OF  $\{e_i\}_{i \in A_i}$

$$\beta > \sup(\{f(\eta) : \eta < (\beta \cdot n + i)\} \cup \{\gamma\}), i \in \beta$$

$\beta \cdot \omega < cf(\alpha)$  AND EACH  $A_i$  IS UNBOUNDED,  
HENCE  $f$  IS WELL DEFINED.

$$\ell = \sup(\{f(\eta) : \eta < \beta \cdot n\}) \Rightarrow \ell \in \bigcap_{i \in \beta} A_i$$

$$\ell < \alpha$$



$$\ell \in \bigcap_{i \in \beta} A_i$$

TAKING  $\beta \cdot n + i$  AS THE  
MEMBERS OF EACH  $A_i$ , WE  
GUARANTEED THAT  $A_i$  WAS INFINITE  
MEMBER, whence, IT IS UNBOUNDED.

SINCE EACH  $A_i$  IS CLOSED.

THAT IS,  $\ell$  IS LIMIT POINT OF EVERY  $A_i$ .

$$\text{IN SYMBOLS, } \ell = \sup(\{f(\beta \cdot n + i) : n \in \omega\}) = \sup(A_i \cap \alpha).$$

AND ALSO STRICTLY GREATER THAN  $\square$

EVERY POINT THAT OCCURS BEFORE IT, BY DEF.

REMARK (i) (more on last lemma)

THE SET OF ALL LIMIT ORDINALS  $\kappa$   
IS CLUB IN  $\kappa$ . IF  $A$  IS UNBOUNDED IN  $\kappa \Rightarrow$  THE SET OF  
ALL LIMIT POINTS OF  $A$  IS CLUB IN  $\kappa$ . (WHAT WAS  
USED ON LAST LEMMA).

Def:  $f$  is a function,  $f: K \rightarrow K$ , then  $f$  is normal iff  $f$  is increasing and continuous.

That is: (i)  $\alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)$

(ii)  $f(\alpha) = \lim_{\xi \rightarrow \alpha} f(\xi)$ , for every limit  $\xi < K$ .

Def: If  $f$  is normal, then Ran(f) is club in  $K$ .

Remark  $C$  is club  $\Rightarrow \exists f$  ( $f$  is uncountable  
 $C$ )

Thm The intersection of fewer than  $\kappa$  club subsets of  $K$  is club.

$\mathcal{A} = \{A_i\}_{i \in \beta} \quad \beta < \kappa \quad A_i$  is club.

That  $\bigcap_{i \in \beta} A_i$  is closed is immediate.

Now let  $\beta < \kappa$ . If  $\beta$  is successor, we can iterate the pairwise intersection  $\beta - 1$  times, and a previous lemma guarantees that it is club.

Now suppose  $\mathcal{P}$  is limit. Assume that, for all  $\alpha < \mathcal{P}$ , the assertion is true. Thus we can replace each  $A_\alpha$  with  $\bigcap_{\xi > \alpha} C_\xi$ .

This gives us a descending sequence of club sets:

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_\alpha \supseteq \dots \quad (\alpha < \mathcal{P})$$

Thus,  $C = \bigcap_{\alpha < \mathcal{P}} C_\alpha$ .  $C$  is closed, since the arbitrary intersection of closed sets is closed.

To prove unboundedness, let  $\alpha < K$ , and let the following be a  $\mathcal{P}$ -sequence:

$$\beta_0 < \beta_1 < \dots < \beta_\xi < \dots \quad (\xi < \mathcal{P})$$

Let  $\beta_0 \in C_0$  and  $\beta_0 > \alpha$ . Let for each  $\xi < \mathcal{P}$ ,  $\beta_\xi \in C_\xi$  and  $\beta_\xi = \sup(\{\beta_\eta : \eta < \xi\})$ . This sequence exists because  $K$  is regular and  $\mathcal{P} < K$ . Moreover,  $\lim_{\xi \rightarrow \mathcal{P}} \beta_\xi = \beta$  and  $\beta < K$ .

which means that for  $\eta < K$   $\beta = \sup(\{\beta_\xi : \eta \leq \xi < K\})$ . Thus,  $\beta \in C_\eta$ . Since that is true for every  $\eta$ , then  $\beta \in C$ , by our construction of  $C$ . ■

Def: Let  $\langle X_\alpha : \alpha < K \rangle$  be a sequence of subsets of  $K$ . The diagonal intersection of  $X_\alpha$ ,  $\alpha < K$  is defined as follows:

$$\Delta_{\alpha < K} X_\alpha = \{ \xi < K : \xi \in \bigcap_{\alpha < \xi} X_\alpha \}$$

Remark (i)  $\Delta X_\alpha = \Delta Y_\alpha$ ,  $Y_\alpha = \{ \xi \in X_\alpha : \xi > \alpha \}$

(ii)  $\Delta X_\alpha = \bigcap_\alpha (X_\alpha \setminus \{\xi : \xi \leq \alpha\})$

Lemma The diagonal intersection of a  $K$ -sequence of club sets is club.

Proof: Let  $A = \langle A_\xi : \xi \leq K \rangle$  be a  $K$ -sequence of club sets.

$$\Delta_{\xi < K} A_\xi = \{ \eta < K : \eta \in \bigcap_{\xi < \eta} A_\eta \}$$

Let  $A_\xi = \bigcap_{\zeta \leq \xi} A_\zeta$ . Thus  $\{A_\xi\}_{\xi \leq K}$  is a decreasing sequence of club sets:

$A_0 \supseteq A_1 \supseteq \dots \supseteq A_\xi \supseteq \dots$  ( $\xi < \kappa$ ).

We take  $A = \bigcap_{\xi < \kappa} A_\xi$ . By our definition of diagonal intersection  $A = \bigtriangleup_{\xi < \kappa} A_\xi$ .

[closed] To show that  $A$  is closed, let  $\xi$  be a limit point of  $A$ . Since  $A = \bigtriangleup_{\xi < \kappa} A_\xi$ , write  $A = \{\eta < \kappa : \eta \in \bigcap_{\xi < \eta} A_\eta\}$ .

Let for each  $\nu < \xi$ ,  $\xi_\nu = \sup \{\nu < \xi : \nu \in A_\nu\}$ .

Then  $\xi_\nu \in \bigcap_{\alpha < \xi} A_\alpha$ . Thus,  $\xi_\nu \in A$  and  $A$  is closed.

[unbounded] Let  $v_0 \in A_0$  and  $v_0 > \xi_1$  for  $\xi_1$ .

Now take an increasing sequence

$v_0 < v_1 < \dots < v_\eta < \dots$  ( $\eta < \kappa$ )

such that for each  $\eta < \kappa$ ,  $v_\eta \in A_{v_\eta}$ . We must show that  $\nu = \lim_{\eta \rightarrow \xi} v_\eta$  is in  $A$ .

$$A = \bigcap_{\xi < \kappa} A_\xi$$

We know that  $\delta < \eta$ ,  $v_\delta \in A_{v_\delta}$ ,

Hence each  $v_\eta \in A_{v_\delta}$ . Thus,  $v_\eta \in A_{v_\eta}$  for all  $\eta < \xi$ . Hence,  $\nu \in A_\xi$ . Therefore  $\nu \in A$ .  
For each  $\xi < \kappa$ .



Def: An ORDINAL function  $f$  On a SET  $S$  is REGRESSIVE if  $f(\alpha) < \alpha$ ,  $\forall \alpha \in S$ ,

$\alpha \neq 0$ .

Thm (Fodor) If  $f$  is A REGRESSIVE function on A <sup>STATIONARY</sup> SET  $S \subseteq \kappa$ ,  $\kappa$  REGULAR, THEN THERE IS A STATIONARY SET  $T \subseteq S$  AND SOME  $\gamma < \kappa$  SUCH THAT  $f(\alpha) = \gamma$ ,  $\forall \alpha \in T$ .

LET  $f$  BE A REGRESSIVE FUNCTION ON  $S \subseteq \kappa$ ,  $S$  STATIONARY, i.e.,  $S \cap C \neq \emptyset$ , FOR ALL  $C$  CLUB IN  $\kappa$ .  
WHEN  $f(\alpha) < \alpha$ ,  $\forall \alpha \in S$ ,  $\alpha \neq 0$ . USING THE CONSTRUCTION OF  $C$  IN THE PREVIOUS THMS, LET  $C = \bigtriangleup_{\eta < \kappa} C_\eta$ , WHEN  $\eta < \kappa$ .  
SUCH THAT EACH  $C_\eta$  IS CLUB IN  $\kappa$ . WHEN  $C$  IS CLUB IN  $\kappa$ . MONOCHROM,  $C \cap S \neq \emptyset$ . NOW LET  $D = \{\alpha \in S : f(\alpha) = \eta\}$  BE NON-STATIONARY. CHOOSE  $C_\eta$  ST.  $f(\alpha) \neq \eta$  FOR  $\alpha \in S \cap C_\eta$ . SINCE  $S$  IS STATIONARY,  $S \cap C$  IS STATIONARY AS WELL. THUS, AS  $C = \bigtriangleup_{\eta < \kappa} C_\eta$ , WHEN IF  $\alpha \in S \cap C$ ,  $f(\alpha) \neq \eta$  FOR EVERY  $\eta < \alpha$ . THEREFORE,  $f(\alpha) \geq \alpha$ , WHICH CONTRADICTS REGRESSIVENESS. ■