Def A filter on $S \neq \emptyset$ is a collection $F$ of subsets of $S$ st.:

$F = \{F_\alpha\}$

$F_\alpha \in \mathcal{P}(S)$

$F \subseteq \mathcal{P}(S)$.

(i) $\emptyset \notin F$, $S \in F$

(ii) $X \in F, Y \in F \Rightarrow X \cap Y \in F$

(iii) $X, Y \subseteq S$, $X \in F$ and $X \subseteq Y \Rightarrow Y \in F$.

Ex: $S = \{a, b, c\}$

$F_T = \{S\}$.

$F_1 = \{S, \{a\}, \{a, b\}, \{a, c\}\}$

$F_2 = \{S, \{b\}, \{b, c\}, \{b, a\}\}$

Def: An ideal on $S \neq \emptyset$ is a collection $I$ of subsets of $S$ st:

(i) $\emptyset \in I$, $S \notin I$

(ii) $X, Y \in I \Rightarrow X \cup Y \in I$

(iii) $X, Y \subseteq S$, $X \in I$, $Y \subseteq X \Rightarrow Y \in I$.

$S = \{a, b, c\}$

$I_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$I_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}\}$

$I = \{S - X : X \in F\}$ $\quad F = \{S - X : X \in I\}$

$I = \{\emptyset\}$

Principal filter. $F = \{X \subseteq S : X_0 \subseteq X\}$

$X_0 \neq \emptyset$

$S = \{a, b, c, d, e\}$ $\quad X_0 = \{c, d, e\}$

$F = \{S, \{c,d,e\}, \{c,d,e,a\}, \{c,d,e,b\}\}$

Fréchet filter: $|S| \geq \omega$, $I = \{X \subseteq S : X \text{ finite}\}$

then $F = \{X \subseteq S : S \setminus X \text{ is finite}\}$

$S > \omega$ $\quad S \setminus X$ finite $\quad |X| \geq \omega$

$|A| \geq \omega$ $\quad S = [A]^{<\omega} = \{X \subseteq A : X \text{ finite}\}$

$P \in S$ $\quad \hat{P} = \{Q \in S : P \subseteq Q\}$

$F = \{X \subseteq S : \hat{P} \subseteq X\}$, for some $P$

The set of all $A \subseteq \mathbb{N}$ of density $0$ is an ideal on $\mathbb{N}$.

$A \subseteq \mathbb{N}$ has density $0$ iff $\lim_{n\to\infty} |A \cap n|/n = 0$

$A \subseteq \mathbb{R}$. $x$ is (limit point) accumulation point of $A$ iff

$$\forall x \in (a,b) \left( (a,b) \cap (A \setminus \{x\}) \neq \emptyset \right) \quad (*)$$

$A$ is open (in $\mathbb{R}$) iff $\forall a \in A\ \exists b < a\ \exists c > a\ \big((b,c) = \{x : b < x < c\}\big)$ & $(b,c) \subseteq A$. $A$ is closed iff $\mathbb{R} \setminus A$ is open.

A new notion of closed:

(i) $\alpha$ limit ordinal, $X$ set of ordinals.

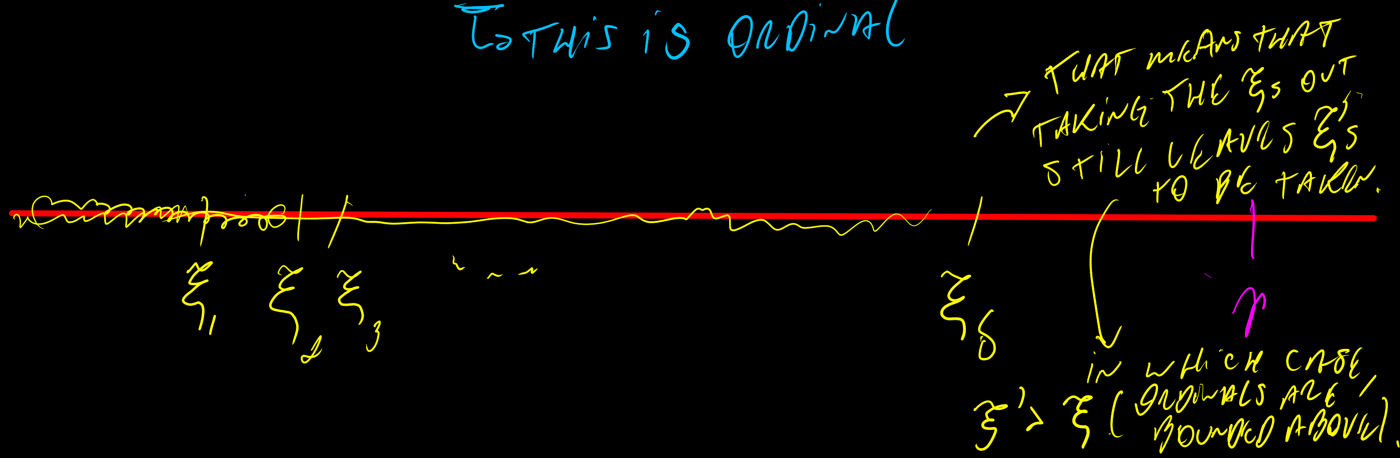
$\alpha$ is a limit point of $X$ iff $\sup(X \cap \alpha) = \alpha$. $(*)$

(ii) $X$ is closed iff $\forall \alpha \left( \sup(X \cap \alpha) \in X \right)$, that is, $X$ is closed if every limit point of $X$ is in $X$.

(iii) If $\gamma$ is an ordinal, $X$ is unbounded in $\gamma$ iff

$$\forall \xi < \gamma \left( (A \cap \gamma) \setminus \xi \neq \emptyset \right)$$

↳ this is ordinal



LET $\delta$ BE AN INFINITE REGULAR CARDINAL.

(iv) $X$ is $\delta$-CLOSED iff FOR ALL ORDINALS $\alpha$ WITH $cf(\alpha) = \delta$ AND s.t. $X \cap \alpha$ is UNBOUNDED in $\alpha$ $\alpha = \sup(X \cap \alpha) \in X$.

↳ THAT is, $X$ is UNBOUNDED FOR ANY LIMIT POINT OF SIZE $\leq \delta$.

↳ WE TOOK IT OUT PREVIOUSLY.

(v) $X$ IS CLUB in $\alpha$ iff $X \cup \{\alpha\}$ is CLOSED AND $X$ is UNBOUNDED in $\alpha$.

CONTAINS ALL LIMIT POINTS OF $X$

(vi) $X$ is $\delta$-club in $\alpha$ iff $X \cup \{\alpha\}$ is $\delta$-closed ∧ $X$ is UNBOUNDED in $\alpha$. (DEF 8.1 JECH).

(vii) $S \subseteq X$ is STATIONARY iff $S \cap X \neq \emptyset$ FOR ALL $X$ $\delta$-CLUB.

↳ $\delta$ INFINITE REGULAR CARDINAL.

If $\delta$ is INFINITE REGULAR CARDINAL WE CAN STATE (vi) AS FOLLOWS:

(vi*) $X \subseteq \delta$, $X$ UNBOUNDED is CLOSED iff FOR EVERY SEQUENCE $\alpha_0 < \alpha_1 < \dots < \alpha_\xi < \dots$, WITH $\xi < \gamma$, OF ELEMENTS OF $X$ WITH LENGTH $\gamma < \delta$ WE HAVE $\lim_{\xi \to \gamma} \alpha_\xi \in C$.

**Thm** If $C, D$ are club, then $C \cap D$ is club.

$C$ is closed $\Leftrightarrow$ every limit point of $C$ is in $C$.

$D$ is closed $\Leftrightarrow$ every limit point of $D$ is in $D$.

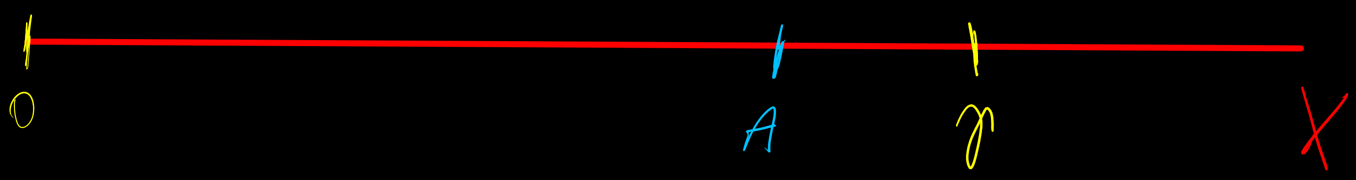
$C \cap D$ is $\underbrace{\text{closed}}$.

$$\forall \alpha \left(\sup\left((C\cap D)\cap \alpha\right) \in C \cap D\right)$$

Let $x$ be limit point of $C \cap D$. Then $x = \sup((C\cap D)\cap x)$. If $x \notin C \cap D$, then $x \notin C$ or $x \notin D$. But then, $C$ and $D$ are closed, which means that $x$ is not limit point. Hence $x \in C \cap D$.

[Unboundedness] $C, D$ unbounded $\Rightarrow$ $C \cap D$ unbounded.

Suppose $C \cap D$ is bounded above. Then $\exists \xi' < \gamma$ such that $((C\cap D)\cap \gamma) \setminus \xi' = \emptyset$




$$|A| = |\xi| < \gamma$$

In another way, let $\kappa$ be a cardinal s.t. both $C, D$ are unbounded in $\kappa$.

If $\alpha < \kappa$, since $C$ and $D$ are both unbounded in $\kappa$, then $\exists\, \alpha_j < \kappa\ (\alpha_j > \alpha)$ and $\alpha_j \in C$. In the same way, $\exists\, \alpha_i < \kappa\ (\alpha_i > \alpha)$ and $\alpha_i \in D$.

Hence, let $\alpha_0 = \alpha$, $j = 2k-1$ and $i = 2k$, for $k \in \mathbb{Z}^+$. Then $\exists$ a sequence

$$\alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \dots$$

and $\alpha_i \in D$, $\alpha_j \in C$, for all $i, j$.

Take $\beta = \lim_{\xi \to \gamma} \alpha_\xi$. Then $\beta \in C$ and $\beta \in D$, by $(v_i^*)$. We are done. ∎

Remember that $\alpha = \sup(X \cap \alpha)$ is a limit point.

**Remark** (i) The collection of all club subsets of $\kappa$ has the finite intersection property.

(ii) $\mathcal{F}_{\text{club}}$, the filter generated by the club sets, consists of all $X \subseteq \kappa$ that contain a club subset. We call it the club filter on $\kappa$.

LEMMA $\alpha$ ORDINAL st. $cf(\alpha) > \omega$. LET $\mathcal{F}_{CLUB}$ BE THE SET OF ALL $A \subseteq \alpha$ st. $\exists B \subseteq A$ ($B$ is CLUB in $\alpha$).

THEN $\mathcal{F}_{CLUB}$ IS A $< cf(\alpha)$-CLOSED FILTER ON $\alpha$ AND $\mathcal{F}_{CLUB}$ is NON-TRIVIAL.

$cf(\alpha)$ is UNBOUNDED IN $\alpha$.

$cf(\alpha) = \alpha$ MEANS THAT $\alpha$ is REGULAR, WHENCE A CARDINAL.

$cf(\alpha) > \omega$, $\alpha$ ORDINAL (Limit).

$$\mathcal{F}_{CLUB} = \{A \subseteq \alpha : \exists B (B \subseteq A \;\&\; B \text{ is CLUB in } \alpha)\}$$

LET $\mathcal{A}$ BE A FAMILY OF SUBSETS OF $\alpha$ WHICH ARE CLUB IN $\alpha$. LET $A_i \in \mathcal{A}$, THAT IS, $\mathcal{A} = \{A_i\}_{i \in \beta}$, $\beta < cf(\alpha)$, EACH $A_i$ CLUB in $\alpha$.

$\bigcap_{i \in \beta} A_i$ IS CLOSED, SINCE ARBITRARY INTERSECTIONS OF CLOSED SETS ARE CLOSED.

UNBOUNDEDNESS FOLLOWS FROM A GENERALIZATION OF THE PREVIOUS ARGUMENT.

LET $\xi < \alpha$. DEFINE $f : \beta \cdot \omega \to \alpha$, WHERE $\beta \cdot \omega$ IS AN ORDINAL MULTIPLICATION. $\beta \cdot \omega \to \alpha$.

LET $f^{\text{NEW}}(\beta\cdot n+i)$ BE THE LEAST $\delta \in A_i$

$\delta > \sup(\{f(\eta) : \eta < (\beta\cdot n+i)\} \cup \{\xi\}), i \in \beta$

[$\beta\cdot\omega < cf(\alpha)$ AND EACH $A_i$ IS UNBOUNDED, HENCE $f$ IS WELL DEFINED.]

$\ell = \sup(\{f(\eta) : \eta < \beta\cdot\omega\})$ $\Rightarrow \ell \in \bigcap_{i\in\beta} A_i$

$\ell < \alpha$

TAKING $\beta\cdot n+i$ AS THE MEMBERS OF EACH $A_i$, WE GUARANTEE THAT $A_i$ HAS INFINITE MEMBERS, WHENCE, IS UNBOUNDED.

$\Downarrow$

$\ell \in \bigcap_{i\in\beta} A_i$

SINCE EACH $A_i$ IS CLOSED.

THAT IS, $\ell$ IS LIMIT POINT OF EVERY $A_{i\in\beta}$.
IN SYMBOLS, $\ell = \sup(\{f(\beta\cdot n+i) : n \in \omega\} = \sup(A_i \cap \alpha)$.
AND ALSO STRICTLY GREATER THAN EVERY POINT THAT OCCURS BEFORE IT, BY DEF. $\blacksquare$

(MORE OR LESS LAST LEMMA)

REMARK (i) THE SET OF ALL LIMIT ORDINALS $< \kappa$ IS CLUB IN $\kappa$. IF $A$ IS UNBOUNDED IN $\kappa \Rightarrow$ THE SET OF ALL LIMIT POINTS OF $A$ IS CLUB IN $\kappa$. (WHAT WAS USED ON LAST LEMMA).

**Def:** $f$ is a function, $f: \kappa \to \kappa$, then $f$ is <u>normal</u> iff $f$ is increasing and continuous.

That is: (i) $\alpha < \beta \Rightarrow f(\alpha) \leq f(\beta)$

(ii) $f(\alpha) = \lim_{\xi \to \alpha} f(\xi)$, for every limit $\alpha < \kappa$.

**Def:** If $f$ is normal, then <u>$\mathrm{Ran}(f)$ is club in $\kappa$.</u>

<u>Remark</u> $C$ is club $\Rightarrow \exists! f$ ($f$ enumerates $C$).

**Thm** The intersection of fewer than $\kappa$ club subsets of $\kappa$ is club.

$\mathcal{A} = \{A_i\}_{i \in \beta}$ $\beta < \kappa$ $A_i$ is club.

That $\bigcap_{i \in \beta} A_i$ is closed is immediate.

Now let $\beta < \kappa$. If $\beta$ is successor, we can iterate the pairwise intersection $\beta - 1$ times, and a previous lemma guarantees that it is club.

Now suppose $\gamma$ is limit. Assume that, for all $\alpha < \gamma$, the assertion is true. Thus we can replace each $A_\alpha$ with $\bigcap_{\xi < \alpha} C_\xi$.

This gives us a descending sequence of club sets:

$$C_0 \supseteq C_1 \supseteq \ldots \supseteq C_\alpha \supseteq \ldots \quad (\alpha < \gamma)$$

Thus, $C = \bigcap_{\alpha < \gamma} C_\alpha$. $C$ is closed, since the arbitrary intersection of closed sets is closed.

To prove unboundedness, let $\alpha < \kappa$, and let the following be a $\gamma$-sequence:

$$\beta_0 < \beta_1 < \ldots < \beta_\xi < \ldots (\xi < \gamma)$$

Let $\beta_0 \in C_0$ and $\beta_0 > \alpha$. Let for each $\xi < \gamma$, $\beta_\xi \in C_\xi$ and $\beta_\xi = \sup(\{\beta_\nu : \nu < \xi\})$. This sequence exists because $\kappa$ is regular and $\gamma < \kappa$. Moreover, $\lim_{\xi \to \gamma} \beta_\xi = \beta$ and $\beta < \kappa$.

Which means that for each $\eta < \kappa$ $\beta = \sup(\langle \beta_\xi : \eta \leq \xi < \kappa \rangle)$

Thus, $\beta \in C_\eta$. Since that is true for every $\eta$, then $\beta \in C$, by our construction of $C$. ∎

DEF: LET $\langle X_\alpha : \alpha < \kappa \rangle$ BE A SEQUENCE OF SUBSETS OF $\kappa$. THE DIAGONAL INTERSECTION OF $X_\alpha$, $\alpha < \kappa$ IS DEFINED AS FOLLOWS:

$$\mathop{\Delta}_{\alpha<\kappa} X_\alpha = \{\xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha\}$$

REMARK (i) $\Delta X_\alpha = \Delta Y_\alpha$, $Y_\alpha = \{\xi \in X_\alpha : \xi > \alpha\}$

(ii) $\Delta X_\alpha = \bigcap_\alpha (X_\alpha \cup \{\xi : \xi \leq \alpha\})$

LEMMA THE DIAGONAL INTERSECTION OF A $\kappa$-SEQUENCE OF CLUB SETS IS CLUB.

PROOF: LET $\mathcal{A} = \langle A_\xi : \xi \leq \kappa \rangle$ BE A $\kappa$-SEQUENCE OF CLUB SETS.

$$\mathop{\Delta}_{\xi<\kappa} A_\xi = \{\eta < \kappa : \eta \in \bigcap_{\xi<\eta} A_\eta\}$$

LET $A_\xi = \bigcap_{\xi \leq \kappa} A_\xi$. THUS THERE IS A DECREASING SEQUENCE OF CLUB SETS:

$A_0 \supseteq A_1 \supseteq \ldots \supseteq A_\xi \supseteq \ldots \quad (\xi < \kappa)$.

We take $A = \bigcap_{\xi<\kappa} A_\xi$. By our definition of diagonal intersection $A = \Delta_{\xi<\kappa} A_\xi$.

[closed] To show that $A$ is closed, let $\xi$ be a limit point of $A$. Since $A = \Delta_{\xi<\kappa} A_\xi$, write

$A = \{\eta < \kappa : \eta \in \bigcap_{\xi<\eta} A_\eta\}$.

Let for each $\nu < \xi$, $\xi = \sup(\{\nu < \xi : \nu \in \bigcap_{\alpha<\nu} A_\alpha\})$

Then $\xi \in \bigcap_{\alpha<\xi} A_\alpha$. Thus, $\xi \in A$ and $A$ is closed.

[unbounded] Let $\nu_0 \in A_0$ and $\nu_0 > \xi$, for $\xi < \kappa$.

Now take an increasing sequence

$\nu_0 < \nu_1 < \ldots < \nu_\eta < \ldots \quad (\eta < \kappa)$

such that for each $\eta < \kappa$, $\nu_\eta \in A_{\nu_\eta}$. We must show that $\nu = \lim_{\eta \to \xi} \nu_\eta$ is in $A$.

$\boxed{A = \bigcap_{\xi<\kappa} A_\xi}$

We know that $\delta < \eta$, $\nu_\delta \in A_{\nu_\delta}$,

Hence each $\nu_\eta \in A_{\nu_\delta}$. Thus, $\nu_\eta \in A_{\nu_\eta}$ for all $\eta < \xi$. Hence, $\nu \in A_\xi$ for each $\xi < \kappa$. Henceforth $\nu \in A$.

**DEF**: AN ORDINAL FUNCTION $f$ ON A SET $S$ IS *REGRESSIVE* IF $f(\alpha) < \alpha$, $\forall \alpha \in S$, $\alpha \neq 0$.

**THM (FODOR)** IF $f$ IS A REGRESSIVE FUNCTION ON A STATIONARY SET $S \subseteq \kappa$, $\kappa$ REGULAR, THEN THERE IS A STATIONARY SET $T \subseteq S$ AND SOME $\gamma < \kappa$ SUCH THAT $f(\alpha) = \gamma$, $\forall \alpha \in T$.

LET $f$ BE A REGRESSIVE FUNCTION ON $S \subseteq \kappa$, $S$ STATIONARY, i.e., $S \cap C \neq \emptyset$, FOR ALL $C$ CLUB IN $\kappa$. THEN $f(\alpha) < \alpha$, $\forall \alpha \in S$ $\alpha \neq 0$. USING THE CONSTRUCTION OF $C$ IN THE PREVIOUS THMS, LET $C = \underset{\gamma<\kappa}{\Delta} C_\gamma$, WHERE $\gamma < \kappa$. SUCH THAT EACH $C_\gamma$ IS CLUB IN $\kappa$. THEN $C$ IS CLUB IN $\kappa$. MOREOVER, $C \cap S \neq \emptyset$. NOW LET $\gamma = \{\alpha \in S : f(\alpha) = \gamma\}$ BE NON STATIONARY. CHOOSE $C_\gamma$ ST. $f(\alpha) \neq \gamma$ FOR EACH $\alpha \in S \cap C_\gamma$. SINCE $S$ IS STATIONARY, $S \cap C$ IS STATIONARY AS WELL. THUS, AS $C = \underset{\gamma<\kappa}{\Delta} C_\gamma$, THEN IF $\alpha \in S \cap C$, $f(\alpha) \neq \gamma$ FOR EVERY $\gamma < \alpha$. THEREFORE, $f(\alpha) \geq \alpha$, WHICH CONTRADICTS REGRESSIVENESS. $\blacksquare$