Cantor's Normal Form

Remark: A sequence of ordinals $\langle \gamma_\alpha : \alpha \in ON \rangle$ is normal iff it is increasing and continuous

$\gamma_\delta < \gamma_\beta$ when $\delta < \beta$

$\forall \alpha$ limit, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$

Remark: $\alpha + \beta$, $\alpha \cdot \beta$ and α^β are continuous on the second variable

If γ is limit and $\beta = \lim_{\delta < \gamma} \beta_\delta$ then

$\alpha + \beta = \sup_{\delta < \gamma} (\alpha + \beta_\delta)$

$\alpha^\beta = \sup_{\delta < \gamma} (\alpha^{\delta})$

$\alpha \cdot \beta = \sup_{\delta < \gamma} (\alpha \cdot \beta_\delta)$

Lemma: If $0 < \alpha \leq \gamma$, then there is a greatest β s.t. $\alpha \cdot \beta \leq \gamma$.

If $1 < \alpha \leq \gamma$ then there is a greatest β s.t. $\alpha^\beta \leq \gamma$

Proof: Taking $\beta > \gamma$ let $\beta = (\gamma + 1)$.

Then $\alpha \cdot (\gamma + 1) \geq \gamma + 1 > \gamma$. Similarly, $\alpha^{(\gamma+1)} = \alpha^\gamma \cdot \alpha \geq \gamma + 1 > \gamma$. Hence, there is a δ s.t. $\alpha\delta > \gamma$ and $\alpha^\delta > \gamma$.

The least such δ must be a successor, since, otherwise, it contradicts continuity. Take $\delta = \beta + 1$. Then β is the greatest ordinal that satisfies (i) and (ii).

Lemma: γ ordinal, $\alpha \neq 0 \Rightarrow \exists! \beta \, \exists! \rho < \alpha \, (\gamma = \alpha \cdot \beta + \rho)$

Proof: By previous lemma, let β be the greatest ordinal st $\alpha \cdot \beta \leq \gamma$ ($\alpha > \gamma \Rightarrow \beta = 0$) and ρ be the unique ρ st. $\alpha \cdot \beta + \rho = \gamma$

ρ must be less than α

otherwise $\gamma = \alpha \cdot \beta + \rho \geq \alpha \cdot \beta + \alpha = \alpha(\beta + 1)$

But β is the greatest ordinal st. $\alpha \cdot \beta \leq \gamma$.

Contradiction. $\perp$

[Uniqueness] $\gamma = \alpha \beta_1 + \rho_1 = \alpha \cdot \beta_2 + \rho_2$

$\beta_1 < \beta_2 \Rightarrow \beta_1 + 1 \leq \beta_2 \Rightarrow \alpha(\beta_1 + 1) + \rho_2 \leq \alpha \beta_2 + \rho_2$

$\alpha(\beta_1+1)+\rho_2 = \alpha \beta_1 + (\alpha + \rho_2)$, $\quad \alpha\beta_2+\rho_2 = \alpha \beta_1 + \rho_1$

$\alpha \beta_1 + \rho_1 \geq \alpha \beta_2 + \rho_1$

But $\rho_1 \geq \alpha + \rho_2 \geq \alpha$. $\perp$

Uniqueness follows. ■

Thm Every ordinal $\alpha > 0$ can be uniquely expressed as:

$$\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \dots + \omega^{\beta_n} \cdot k_n$$

where $\beta_1 > \beta_2 > \dots > \beta_n$

and $k_i > 0$, for $0 < i \leq n$ are finite.

Proof: [Existence]

induction on α.

$\alpha = 1 \Rightarrow \alpha = \omega^0 \cdot 1 = 1$.

Suppose α is arbitrary.

By Lemma 6.2, there is a greatest β s.t. $\omega^{\beta} \leq \alpha$.

$\hookrightarrow$ If $\alpha < \omega \Rightarrow \beta = 0$.

By Lemma 6.3 there is a unique ρ and a unique δ s.t.

$$\omega^{\beta} \cdot \delta + \rho = \alpha.$$

Since $\omega^{\beta} \leq \alpha \Rightarrow \delta > 0$ and $\rho < \alpha$

Claim δ is finite, since,

if δ were infinite, then

$$\alpha \geq \omega^{\beta} \cdot \delta \geq \omega^{\beta} \cdot \omega = \omega^{\beta+1}$$

$\Rightarrow \alpha \geq \omega^{\beta+1}$ which contradicts the maximality of β.

Thus, take $\beta_1 = \beta$ and $k_1 = \delta$.

Now if $\rho = 0 \Rightarrow \alpha = \omega^{\beta} \cdot \delta$ is in normal form.

Otherwise, by IH, $\rho > 0$ implies

$$\rho = \omega^{\beta_2} \cdot k_2 + \dots + \omega^{\beta_n} \cdot k_n$$

for $\beta_2 > \dots > \beta_n$ and $k_i > 0$ $2 \leq i \leq n$.

Since $\rho < \omega^{\beta_1} \Rightarrow \omega^{\beta_2} \leq \rho < \omega^{\beta_1} \Rightarrow \beta_1 > \beta_2$.

Hence, $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \dots + \omega^{\beta_n} \cdot k_n$.

As desired $\Rightarrow$

[UNIQUENESS]

$\beta < \gamma \Rightarrow \omega^{\beta} \cdot k < \omega^{\gamma} \quad \forall k$ FINITE.

$\beta + 1 \leq \gamma \Rightarrow \omega^{\beta} \cdot k < \omega^{\beta} \cdot \omega = \omega^{\beta+1} \leq \omega^{\gamma}$.

THEN IF $\alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \ldots + \omega^{\beta_n} \cdot k_n$ IS IN NF, AND $\beta_1 < \gamma \Rightarrow \alpha < \gamma$. SINCE β_1 IS GREATEST OF ALL β's.

BY INDUCTION:

$\alpha = 1 \Rightarrow \omega^{0} \cdot 1 = 1$ IS UNIQUE.

$\alpha > 1 \Rightarrow \alpha = \omega^{\beta_1} \cdot k_1 + \omega^{\beta_2} \cdot k_2 + \ldots + \omega^{\beta_n} \cdot k_n$

$\alpha = \omega^{\gamma_1} \cdot l_1 + \omega^{\gamma_2} \cdot l_2 + \ldots + \omega^{\gamma_m} \cdot l_m$

THE OBSERVATION IN GREEN YIELDS $\beta_1 = \gamma_1$

SINCE $\gamma_1 < \beta_1 \Rightarrow \omega^{\gamma_1} \cdot k < \omega^{\beta_1}$ AND THUS, A $\perp$.

LET $\delta = \omega^{\beta_1} = \omega^{\gamma_1}$, $\rho = \omega^{\beta_2} \cdot k_2 + \ldots + \omega^{\beta_n} \cdot k_n$ AND

$\sigma = \omega^{\gamma_2} \cdot l_2 + \ldots + \omega^{\gamma_m} \cdot l_m$

THUS, $\alpha = \delta \cdot k_1 + \rho = \delta \cdot l_1 + \sigma$

BY LEMMA 6.3, $k_1 = l_1$ AND $\rho = \sigma$

THUS, THE NF FOR ρ IS UNIQUE AND, CONSEQUENTLY, THE NF FOR α IS UNIQUE.