THEOREM (CSB) IF $|X| \le |Y|$ AND $|Y| \le |X|$, THEN $|X| = |Y|$.

LET $|X| \le |Y|$,

$\exists f: X \to Y$ s.t. f is INJECTIVE

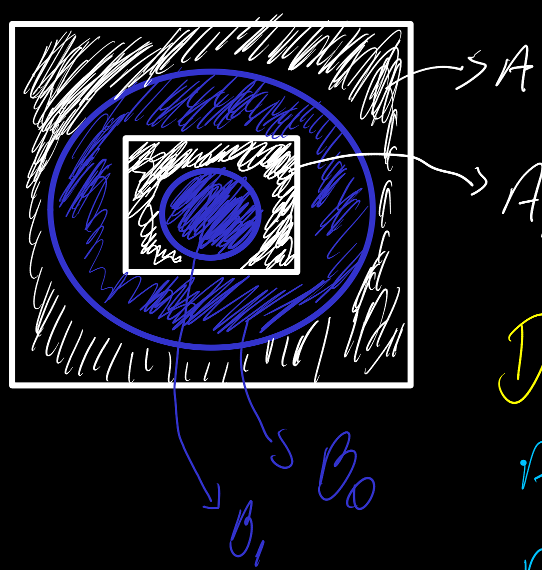
ANALOGOUSLY, $|Y| \le |X| \Rightarrow \exists g: Y \to X$, INJECTIVE.

SHOW THAT $|X| = |Y|$, THAT IS, THAT THERE IS $f: X \to Y$, BIJECTIVE!

WE NEED A LEMMA TO DO THAT!

LEMMA: $A_1 \subseteq B \subseteq A$ AND $|A_1| = |A| \Rightarrow |B| = |A|$

THIS IS KEY



f is 1-1, SURJECTIVE

$f: A \to A_1$

DEFINE RECURSIVELY:

$A_0, A_1, A_2, \dots, A_n, \dots$

$B_0, B_1, B_2, \dots, B_n, \dots$

So that: $A_0 = A$ $\quad B_0 = B$

$A_{n+1} = f[A_n]$ $\quad B_{n+1} = f[B_n]$ $\quad (*)$

Since $A_1 \subseteq B \subseteq A_0$, then:

$\forall n,\ A_{n+1} \subseteq A_n$ by induction

Let $n=0$. Then $A_1 \subseteq A_0$.

Let for all $m < n+1$, $A_{m-1} \subseteq A_m$.

Suppose $A_{n+1} \not\subseteq A_n$. Thus, $f[A_n] \not\subseteq A_n$

But $A_n \subseteq A_{n-1}$, I.H. Then $f[A_n] \not\subseteq f[A_{n-1}]$

Contradiction.

Take $C_n = A_n \setminus B_n$, $\forall n \in \omega$

$C = \bigcup_{n \in \omega} C_n$ $\quad D = A \setminus C$

Using $(*)$ again, $f[C_n] = C_{n+1}$.

which implies $f[C] = \bigcup_{n \in \omega} C_n \setminus C_0$

Let $g : A \to B$

$$g(x) = \begin{cases} f(x), & x \in C \\ x, & x \in D. \end{cases}$$

SINCE

$g|C$ AND $g|D$ ARE INJECTIVE

AND $RAN(g|D) \cap RAN(g|C) = \emptyset$, THEN

$RAN(g) = f[C] \cup D = B$

g IS INJECTIVE AND SURJECTIVE.

GOING BACK TO CSB

$|X| \leq |Y| \Rightarrow f: X \to Y$, INJ.

$|Y| \leq |X| \Rightarrow g: Y \to X$, INJ.

$g \circ f : X \to X$, INJ.

$g[f[X]] \subseteq g[Y] \subseteq X$ $\to$ ~~$g \circ f : X \to X$ 1-1~~

f, g 1-1 $\Rightarrow \begin{cases} |X| = |g[f[X]]| & (?) \\ |Y| = |g[Y]| \end{cases}$ $\to$ SAME ARGUMENT!

SET $A = X$ $\quad A_1 = g[f(X)]$ $\Rightarrow |X| = |Y|$

$B = g[Y]$

BY LEMMA $\square$