Assignment 2 Report

Problem 1: Weighted approximate common substring

Problem Overview

Given two strings $a = a_1a_2...a_m$ and $b = b_1b_2...b_n$, where each character a_i has an associated weight w_i, and a mismatch penalty δ, find the common substring with the maximum score. Matching characters contribute their weight, while mismatches incur a penalty.

Dynamic Programming Formulation

State Definition

$$OPT(i, j) = \text{Maximum score of any substring ending at position } a_i \text{ in string } a \text{ and } b_j \text{ in string } b$$

Key Insight: There is inherent similarity to Kadane's algorithm for maximum subarray sum.

Optimization Goal

$$\text{Answer} = \max_{1 \le i \le m,\, 1 \le j \le n} OPT(i, j)$$

We seek the maximum value across all entries in the DP table, which represents the best scoring common substring.

Bellman Equation

Assuming 1 based indexing of the two strings;

$$OPT(i, j) = \begin{cases} 0, & \text{if } i = 0 \text{ or } j = 0, \\ w_i + OPT(i-1, j-1), & \text{if } a_i = b_j, \\ \max\big(OPT(i-1, j-1) - \delta,\ 0\big), & \text{if } a_i \neq b_j \end{cases}$$

Correctness Justification

Base Case: When $i = 0$ or $j = 0$, we have exhausted one of the strings, so no substring can be formed. Thus $OPT(0, j) = OPT(i, 0) = 0$ for all i, j.

Case 1 - Match ($a_i = b_j$):

- When characters match, we extend the substring ending at $(i-1, j-1)$ by including the matching pair
- We gain the weight w_i associated with this character
- The score becomes $w_i + OPT(i-1, j-1)$
- **Optimal Substructure:** The best substring ending at (i, j) with a match must be built upon the best substring ending at $(i-1, j-1)$

Case 2 - Mismatch ($a_i \neq b_j$):

- We can extend the substring from $(i-1, j-1)$ and pay the penalty δ, giving score $OPT(i-1, j-1) - \delta$
- Alternatively, we can start a new substring at this position with score 0
- We take the maximum: $\max(OPT(i-1, j-1) - \delta, 0)$
- **Key Observation:** This allows negative-scoring segments to be discarded, ensuring we only track locally optimal alignments

Optimal Substructure Property:
The optimal solution to subproblem $OPT(i, j)$ depends only on the solution to subproblem $OPT(i-1, j-1)$. Each substring ending at (i, j) is constructed by either:

1. Extending the best substring ending at $(i-1, j-1)$ (if beneficial), or
2. Starting fresh at (i, j)

Overlapping Subproblems:
The value $OPT(i-1, j-1)$ may be computed multiple times in a naive recursive solution. The DP table eliminates redundant computation by storing each subproblem result exactly once.

Solution Extraction

Finding the Optimal Score

1. Compute all entries $OPT(i, j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$
2. Track the maximum value: $\text{max\_score} = \max_{i,j} OPT(i, j)$
3. Record the indices (i^*, j^*) where this maximum occurs

Reconstructing the Optimal Substring (Backtracking)

Starting from the position (i^*, j^*) where the maximum score was achieved:

1. **Initialize:** Set $i = i^*$, $j = j^*$, and create an empty result string
2. **Backtrack:** While $OPT(i, j) > 0$:
   - Append character a_i to the result string
   - Move to the previous position: $i \leftarrow i - 1$, $j \leftarrow j - 1$
3. **Termination:** Stop when $OPT(i, j) = 0$ (start of the optimal substring)
4. **Reverse:** Since we built the string backwards, reverse it to get the actual substring

Pseudo Code

```
string ans = "";
int i = i*, j = j*;
while(opt[i][j] > 0)
{
    ans.push_back(a[i]);
    i--;
    j--;
}
reverse(ans.begin(),ans.end());
```

Complexity Analysis

Time Complexity: $O(m \cdot n)$

Justification:

- **Number of Subproblems:** There are $(m + 1) \times (n + 1)$ entries in the DP table
- **Time per Subproblem:** Each entry $OPT(i, j)$ requires $O(1)$ operations:
  - One character comparison: $a_i = b_j$
  - Constant arithmetic operations (addition, max)
- **Total Computation Time:** $O(m \cdot n) \times O(1) = O(m \cdot n)$
- **Solution Extraction:** Backtracking takes $O(\min(m, n))$ in the worst case (length of the substring), which is dominated by the table computation

Overall: $O(m \cdot n)$

Space Complexity: $O(m \cdot n)$

Justification:

- **DP Table Storage:** We maintain a 2D table of size $(m + 1) \times (n + 1)$ to store all $OPT(i, j)$ values
- **Auxiliary Space:** $O(1)$ for tracking maximum value and indices
- **Total Space:** $O(m \cdot n)$

Common Substring Analysis Results

The analysis compares the following two input strings:

- **String a**: `"ABCAABCAA"`
- **String b**: `"ABBCAACCBBBBBB"`

The scoring mechanism is based on the match weight (w_l) and the mismatch penalty (δ). The score is calculated as: $\text{Score} = (\sum w_l \times \text{Matches}) - (\delta \times \text{Mismatches})$.

Detailed Results Table

The match counts (A/B/C) refer to the number of times those letters appear in the **Best Common Substring**.

w_l **(Match Weight)**	δ **(Mismatch Penalty)**	**Best Common Substring**	**Score**	**Matches (A/B/C)**	**Mismatches**
$\mathbf{1}$	$\mathbf{10}$	`BCAA`	**4.0000**	2 / 1 / 1	0
Freq. of Letter	5.80398	`ABCAABCAA`	**22.7560**	4 / 1 / 1	3
Freq. of Letter	11.9731	`BCAA`	**20.4400**	2 / 1 / 1	0
Freq. of Letter	11.2369	`BCAA`	**20.4400**	2 / 1 / 1	0
Freq. of Letter	11.9383	`BCAA`	**20.4400**	2 / 1 / 1	0
Freq. of Letter	9.09142	`BCAA`	**20.4400**	2 / 1 / 1	0
Freq. of Letter	$\mathbf{0.245124}$	`AABCAA`	**28.3149**	3 / 1 / 1	1
Freq. of Letter	3.90712	`AABCAA`	**24.6529**	3 / 1 / 1	1
Freq. of Letter	9.83778	`BCAA`	**20.4400**	2 / 1 / 1	0
Freq. of Letter	5.3316	`ABCAABCAA`	**23.2284**	4 / 1 / 1	3
Freq. of Letter	3.02249	`AABCAA`	**25.5375**	3 / 1 / 1	1

Summary of Substrings

The analysis yielded three distinct "Best Common Substrings" depending on the penalty (δ):

1. **`BCAA`** : The highest-scoring result when the mismatch penalty (δ) is **very high**, as it has **0 mismatches**.
2. **`AABCAA`** : The highest-scoring result for an **intermediate** range of δ, offering a high match count for only **1 mismatch**.
3. **`ABCAABCAA`** : The highest-scoring result when the mismatch penalty is **very low**, as it maximizes the match score but incurs **3 mismatches**.

The analysis now compares the following two input strings:

- **String a**: `"ABCDEGHZIHJKLKKLKL"`
- **String b**: `"EFGHIJJKLKLLKKLLZ"`

The scoring mechanism is based on the match weight (w_l) and the mismatch penalty (δ). The score is calculated as: $\text{Score} = (\sum w_l \times \text{Matches}) - (\delta \times \text{Mismatches})$.

Detailed Results Table

The match counts refer to the number of times those letters appear in the **Best Common Substring**. For brevity, only letters that appear in a match (E, I, J, K, L) are listed.

w_l **(Match Weight)**	δ **(Mismatch Penalty)**	**Best Common Substring**	**Score**	**Matches (E/I/J/K/L)**	**Mismatches**
$\mathbf{1}$	$\mathbf{10}$	`JKLK`	**4.0000**	0 / 0 / 1 / 2 / 1	0
Freq. of Letter	7.31791	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	11.1879	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	4.38296	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	3.71391	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	4.95628	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	3.5106	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	2.45134	`EGHZIHJKLKKLK`	**17.2033**	1 / 1 / 1 / 3 / 2	5

w_l (Match Weight)	δ (Mismatch Penalty)	Best Common Substring	Score	Matches (E/I/J/K/L)	Mismatches
Freq. of Letter	5.26614	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0
Freq. of Letter	$\mathbf{0.377216}$	`EGHZIHJKLKKLK`	**27.5739**	1 / 1 / 1 / 3 / 2	5
Freq. of Letter	11.0693	`E`	**12.0200**	1 / 0 / 0 / 0 / 0	0

Summary of substrings

This dataset demonstrates a strong preference for the single-character match `E` across most δ values due to the high penalty of mismatches versus the gain from extra match weight.

1. **High Match Score (Low Penalty):** The longest match, **`EGHZIHJKLKKLK`**, is optimal when δ is very low (≤ 2.45134), accepting **5 mismatches** for the highest total score.
2. **Perfect Match Score (High Penalty):** When δ is high, the optimal match is the shortest perfect match, **`E`**, with **0 mismatches** and a score equal to the frequency weight of 'E' (12.02).

Problem 2: Largest zero sub-matrix

Problem Overview

Given an $m \times n$ boolean matrix B where entries are either 0 or 1, find the largest square submatrix consisting entirely of zeros.

Dynamic Programming Formulation

State Definition

$$OPT(i, j) = \text{Side length of the largest all-zeros square submatrix with bottom-right corner at position } (i, j)$$

Key Insight: A square can only grow if all three adjacent squares (top, left, and top-left diagonal) also contain all zeros.

Optimization Goal

$$\text{Answer} = \max_{1 \leq i \leq m,\, 1 \leq j \leq n} OPT(i, j)$$

The maximum value in the DP table represents the side length of the largest square. The position where this maximum occurs is the bottom-right corner of that square.

Bellman Equation

$$OPT(i,j) = \begin{cases} 0, & \text{if } B[i][j] = 1, \\ 1, & \text{if } B[i][j] = 0 \text{ and } (i = 0 \text{ or } j = 0), \\ 1 + \min\big(OPT(i-1, j-1),\ OPT(i-1, j),\ OPT(i, j-1)\big), & \text{if } B[i][j] = 0 \text{ and } i > 0 \text{ and } j > 0 \end{cases}$$

Correctness Justification

Base Case : If $B[i][j] = 0$ and we're at the first row ($i = 0$) or first column ($j = 0$), the largest square is just the single cell itself. Thus $OPT(i, j) = 1$.

Case 1: If $B[i][j] = 1$, the cell contains a 1, so no all-zeros square can have its bottom-right corner here. Thus $OPT(i, j) = 0$.

Case 2: If ($B[i][j] = 0$ and $i > 0, j > 0$):

To understand why we take the minimum of three neighbors plus 1, consider that a square of side length k with bottom-right corner at (i, j) requires:

1. **Top neighbor $(i-1, j)$:** Must support a square of at least side length $k-1$
2. **Left neighbor $(i, j-1)$:** Must support a square of at least side length $k-1$
3. **Diagonal neighbor $(i-1, j-1)$:** Must support a square of at least side length $k-1$

Visual Explanation:

```
Consider a 3×3 square with bottom-right at (i,j):
    j-2  j-1   j
i-2  0    0    0
i-1  0    0    0
i    0    0   (i,j)

For this to exist:
- Position (i-1,j) must support at least a 2×2 square
- Position (i,j-1) must support at least a 2×2 square
- Position (i-1,j-1) must support at least a 2×2 square
```

The **minimum** of the three values is the limiting factor—it represents the smallest square among the three neighbors. We can extend this by exactly 1 to include the current cell.

Optimal Substructure Property:
The largest square ending at (i, j) depends optimally on the largest squares ending at $(i-1, j)$, $(i, j-1)$, and $(i-1, j-1)$. We cannot form a larger square at (i, j) than what these three neighbors allow.

Overlapping Subproblems:
Computing $OPT(i, j)$ for different positions requires the same subproblems repeatedly (e.g., $OPT(i-1, j-1)$ is needed for computing $OPT(i, j)$, $OPT(i, j+1)$, and $OPT(i+1, j)$). The DP table stores each subproblem result once.

Solution Extraction

Finding the Largest Square

1. Compute all entries $OPT(i, j)$ for $0 \leq i < m$ and $0 \leq j < n$
2. Track the maximum side length: $k = \max_{i,j} OPT(i, j)$
3. Record the bottom-right corner indices (i^*, j^*) where this maximum occurs

Reconstructing the Square Boundaries

Given:

- Side length: $k = OPT(i^*, j^*)$
- Bottom-right corner: (i^*, j^*)

The top-left corner is computed as:

$$i_{\text{top}} = i^* - k + 1 \tag{1}$$
$$j_{\text{left}} = j^* - k + 1 \tag{2}$$

The square occupies:

- Rows: $[i_{\text{top}}, i^*]$
- Columns: $[j_{\text{left}}, j^*]$

Pseudo Code

```
int k = opt[i*][j*];
int i_top = i* - k + 1;
int j_top = j* - k + 1;
for(int i = i_top ; i <= i* ; i++)
{
    for(int j = j_top; j <= j* ; j++)
    {
        cout << B[i][j] << " ";
    }
    cout << "\n";
}
```

Complexity Analysis

Time Complexity: $O(m \cdot n)$

Justification:

- **Number of Subproblems:** There are $m \times n$ entries in the DP table (one for each cell)
- **Time per Subproblem:** Each entry $OPT(i, j)$ requires $O(1)$ operations:
  - One condition check: $B[i][j] = 0$ or $B[i][j] = 1$
  - Computing minimum of three values: $O(1)$
  - One addition: $O(1)$
- **Total Computation Time:** $O(m \cdot n) \times O(1) = O(m \cdot n)$
- **Solution Extraction:** Finding the maximum and its position requires one pass through the table: $O(m \cdot n)$, which doesn't increase the overall complexity

Overall: $O(m \cdot n)$

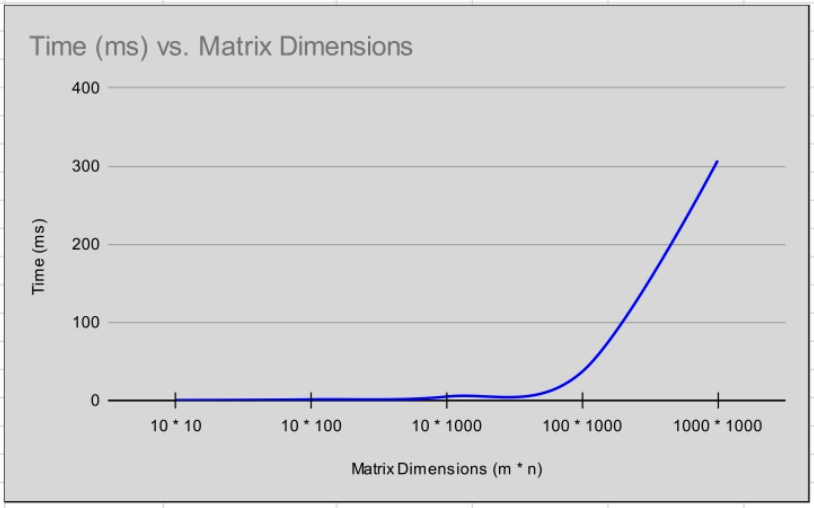
Space Complexity: $O(m \cdot n)$

Justification:

- **DP Table Storage:** We maintain a 2D table of size $m \times n$ to store all $OPT(i, j)$ values
- **Auxiliary Space:** $O(1)$ for tracking the maximum value and its position
- **Total Space:** $O(m \cdot n)$

Synthetic Data Memory and Running Time

Matrix Dimensions	Time (ms)
10 * 10	0.241208
10 * 100	1.15679
10 * 1000	4.83879
100 * 1000	36.7714
1000 * 1000	307.061



Matrix Dimensions	Memory Usage (KB)
10 * 10	1264
10 * 100	1280
10 * 1000	1344
100 * 1000	1760
1000 * 1000	5568