# Tutorial 6

## Question 1: Invertibility of Composite Linear Maps

Suppose $T \in L(U, V)$ and $S \in L(V, W)$ are both invertible linear maps.
Prove that $ST \in L(V, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

### Solution:

Given $T$ and $S$ are invertible, they satisfy:

$$TT^{-1} = I_U, \quad T^{-1}T = I_V$$

$$SS^{-1} = I_W, \quad S^{-1}S = I_V$$

To prove $ST$ is invertible:

$$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SI_VS^{-1} = SS^{-1} = I_W$$

$$(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}I_VT = T^{-1}T = I_U$$

Thus, $(ST)^{-1} = T^{-1}S^{-1}$.

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## Question 2: Noninvertible Operators Not Forming a Subspace

Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible operators on $V$ is not a subspace of $L(V)$.

### Solution:

To disprove that noninvertible operators form a subspace, we need to show that they do not satisfy closure under addition or scalar multiplication. Consider $T$ and $S$ in $L(V)$ where:

$$T(v) = 0 \quad \text{for all } v \in V$$

and $S$ is the identity map $I_V$. Clearly, $T$ is noninvertible (null operator), and $S$ is invertible.

Now, consider their sum $T + S = S + T$:

$$(T + S)(v) = T(v) + S(v) = 0 + v = v$$

$(T + S)$ is invertible (being the identity map), which contradicts the subspace requirement that the sum of two noninvertible operators should also be noninvertible.

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## Question 3: Extension of an Injective Operator

Suppose $V$ is finite-dimensional, $U$ is a subspace of $V$, and $S \in L(U, V)$.
Prove there exists an invertible operator $T \in L(V)$ such that $Tu = Su$ for every $u \in U$ if and only if $S$ is injective.

## Solution:

**If Part:** Assume $S$ is injective. Extend the basis of $U$ to a basis of $V$, and define $T$ on this basis by setting $T(u) = S(u)$ for $u \in U$ and arbitrarily for the basis elements not in $U$ such that $T$ remains invertible. By construction, $T$ is invertible and agrees with $S$ on $U$.

**Only If Part:** Assume such a $T$ exists. Since $T$ is invertible, it must be injective. Hence, $S$ is injective because for $u \in U$, $Su = 0$ implies $Tu = 0$, which implies $u = 0$ because $T$ is injective.

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# Question 4: Equality of Null Spaces

Suppose $W$ is finite-dimensional and $T_1, T_2 \in L(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{null} T_2$ if and only if there exists an invertible operator $S \in L(W)$ such that $T_1 = ST_2$.

## Solution:

**If Part:** Assume $T_1 = ST_2$ where $S$ is invertible. For $v \in V$, if $v \in \operatorname{null} T_2$, then $T_2(v) = 0$, and thus $T_1(v) = S(T_2(v)) = 0$, which means $v \in \operatorname{null} T_1$. The argument reverses due to $S$ being invertible, proving $\operatorname{null} T_1 = \operatorname{null} T_2$.

**Only If Part:** Assume $\operatorname{null} T_1 = \operatorname{null} T_2$. By the Rank-Nullity Theorem, $\operatorname{range} T_1$ and $\operatorname{range} T_2$ have the same dimension. Extend bases from these ranges to bases of $W$, and define $S$ on these bases to map $T_2$ onto $T_1$. By construction, $S$ is invertible, and $T_1 = ST_2$.

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# Question 5: Null and Range Equality Condition

Suppose $V$ is finite-dimensional and $T_1, T_2 \in L(V, W)$. Prove that $\operatorname{null} T_1 = \operatorname{range} T_2$ if and only if there exists an invertible operator $S \in L(V)$ such that $T_1 = T_2 S$.

## Solution:

**If Part:** Assume $T_1 = T_2 S$ where $S$ is invertible. For $v \in V$, if $v \in \operatorname{null} T_1$, then $T_1(v) = T_2(S(v)) = 0$. Thus $S(v) \in \operatorname{null} T_2 = \operatorname{range} T_1$. Similarly, for $w \in \operatorname{range} T_2$, $w = T_2(v)$ for some $v$, hence $T_1(S^{-1}(v)) = T_2(v) = w$, proving the equality.

**Only If Part:** Assume $\operatorname{null} T_1 = \operatorname{range} T_2$. Define $S$ to map $T_2$ onto $T_1$. This involves choosing $S$ so that its nullity complements the rank of $T_2$, aligning with the nullity of $T_1$. With careful construction, $S$ can be made invertible, ensuring $T_1 = T_2S$.