Tutorial 5

Question 1: Matrix Nonzero Entries

Suppose V and W are finite-dimensional and $T \in L(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Solution:

Given $T : V \to W$, let $\{v_1, \ldots, v_n\}$ be a basis for V, and let $\{w_1, \ldots, w_m\}$ be a basis for W. The matrix representation of T under these bases is formed by the coefficients that express $T(v_i)$ as linear combinations of the w_j's.

Since $\dim \operatorname{range} T$ (rank of T) represents the maximum number of linearly independent images of the basis vectors of V under T, there are at least $\dim \operatorname{range} T$ vectors in $\{T(v_1), \ldots, T(v_n)\}$ that contribute to this rank. Each such vector must have at least one nonzero coordinate in the basis of W, thereby ensuring at least $\dim \operatorname{range} T$ nonzero entries in the matrix of T.

Question 2: Matrix of Differentiation Map

Suppose $D \in L(P_3(\mathbb{R}), P_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $P_3(\mathbb{R})$ and a basis of $P_2(\mathbb{R})$ such that the matrix of D with respect to these bases is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution:

Choose the basis $\{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$ and $\{1, x, x^2\}$ for $P_2(\mathbb{R})$. Applying the differentiation map D:

$$D(1) = 0, \quad D(x) = 1, \quad D(x^2) = 2x, \quad D(x^3) = 3x^2$$

Express these in terms of the basis of $P_2(\mathbb{R})$:

$$D(1) = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$D(x) = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

$$D(x^2) = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

$$D(x^3) = 0 \cdot 1 + 0 \cdot x + 3 \cdot x^2$$

Hence, the matrix of D with respect to these bases is as required.

Question 3: Basis for Matrix Representation

Suppose V and W are finite-dimensional and $T \in L(V, W)$. $Provethatthereexistsabasisof$V$ and a basis of W such that with respect to these bases, all entries of $M(T)$ are 0 except that the entries in row j, column j, equal 1 for $1 \leq j \leq$ dim range T.

Solution:

Let $\dim \text{range } T = r$. Select a basis $\{u_1, \ldots, u_r\}$ for $\text{range } T$ and extend it to a basis $\{w_1, \ldots, w_m\}$ for W. Choose vectors $\{v_1, \ldots, v_n\}$ in V such that $T(v_j) = u_j$ for $1 \leq j \leq r$ and extend $\{v_1, \ldots, v_r\}$ to a basis for V. In this basis, $T(v_j) = w_j$ for $1 \leq j \leq r$, and $T(v_j) = 0$ for $j > r$. Therefore, the matrix of T will have the desired form.

Question 6: Rank-1 Linear Map

Suppose V and W are finite-dimensional and $T \in L(V, W)$. $Provethat\$ \dim \text{range } T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $M(T)$ equal 1.

Solution:

If $\dim \text{range } T = 1$, then T maps all of V to a single line in W. Choose a basis for V and let the basis for W include a vector from this line and extend it arbitrarily. Then, each vector in V under T maps to a scalar multiple of a single vector in W, and we can scale the bases so that these are all 1. Conversely, if the matrix entries are all 1, T is rank-1 because all column vectors are scalar multiples of any nonzero column.

Question 10: Matrix Multiplication Row Operation

Suppose A is an $m-$by$-n$ matrix and C is an $n-$by$-p$ matrix. Prove that:

$$(AC)_{j}, . = A_{j}. C$$

for $1 \leq j \leq m$. In other words, show that row j of AC equals (row j of A) times C.

Solution:

Consider the element in row j, column k of AC. This is given by:

$$(AC)_{jk} = \sum_{i=1}^{n} A_{ji} C_{ik}$$

This expression means that row j of AC is formed by multiplying row j of A by matrix C. Each element of row j of AC is a linear combination of the elements of row j of A with columns of C, as required.