Chapter 11.7: Fourier Integral

❖ Introductions to Fourier integral

- Fourier series are powerful tools for problems involving functions that are periodic or are of interest on a finite interval only. Sections 11.1 and 11.2 first illustrated this.
- But many problems involve non-periodic functions that are of interest on the whole x-axis, we can extend the method of Fourier series to such functions. This idea will lead to "Fourier integrals."

❖ Representation of any function as Fourier Integral

- If $f(x)$ is piece-wise continuous on every finite interval, absolutely integrable, and has one-sided derivative at every point, then $f(x)$ can be represented by the Fourier integral:

$$\mathrm{f(x)} = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega, \tag{1}$$

where, $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt$ and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$. (2)

- At a point where $f(x)$ is discontinuous the value of the Fourier integral equals to the average of right-hand-side and the left-hand-side limits of $f(x)$ at that point, i.e., $f(x) = \frac{f(x+0)+f(x-0)}{2}$.

❖ Fourier Cosine Integral and Fourier Sine Integral

- Just as Fourier *series* simplify if a function is even or odd (see Sec. 11.2), so do Fourier *integrals*. Indeed, if $f(x)$ has a Fourier integral representation and is *even*, then $B(\omega) = 0$ in (2). This holds because the integrand of $B(\omega)$ is odd. Then (1) reduces to a **Fourier cosine integral**

$\mathrm{f(x)} = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega$, where $A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\,dt$. (3)

- Similarly, if f has a Fourier integral representation and is *odd*, then $A(\omega) = 0$ in (2). This is true because the integrand of $A(\omega)$ is odd. Then (5) becomes a ***Fourier sine integral***

$\mathrm{f(x)} = \int_0^\infty B(\omega)\sin(\omega x)\,d\omega$, where $B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t)\,dt$ (3)

❖ Some problems

1. Find the Fourier integral representation of the function $f(x) = \begin{cases} 1 & if\ |x| < 1 \\ 0 & if\ |x| > 1 \end{cases}$.

   Then show that $\int_0^\infty \frac{\sin(\omega)}{\omega}d\omega = \frac{\pi}{2}$.

Solution: We know that the Fourier integral of any function, $\mathrm{f(x)} = \int_0^\infty [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]d\omega$, and where $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt$ and $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt$. Now,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt = \frac{1}{\pi}\int_{-1}^{1}\cos(\omega t)\,dt = \left[\frac{2\sin(\omega t)}{w\pi}\right]_0^1 = \frac{2\sin(\omega)}{w\pi}$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt = \frac{1}{\pi}\int_{-1}^{1}\sin(\omega t)\,dt = 0$$

Therefore, we can represent as $f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)\cos(\omega x)}{\omega}d\omega$.

The average of the left- and right-hand limits of $f(x)$ at $x = 1$ is equal to $\frac{f(1+0)+f(1-0)}{2} = \frac{0+1}{2} = \frac{1}{2}$.

Furthermore, we can write the value of integration $\int_0^\infty \frac{\sin(\omega)\cos(\omega x)}{\omega}d\omega = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & if\ 0 \le x < 1, \\ \frac{\pi}{4} & if\ \quad x = 1, \\ 0 & if\ \quad x > 1. \end{cases}$

Now, if we put $x = 0$, this gives $\int_0^\infty \frac{\sin(\omega)}{\omega}d\omega = \frac{\pi}{2}$. *(Proved)*

2. Show that the integral represents the indicated function. Show your work in detail.

$$\int_0^\infty \frac{\cos(\omega x) + \omega \sin(\omega x)}{1+\omega^2} d\omega = \begin{cases} 0 & if\ x < 0 \\ \frac{\pi}{2} & if\ x = 0 \\ \pi e^{-x} & if\ x > 0 \end{cases}$$

Solution: If we compare the L.H.S with Fourier integral it gives $A(\omega) = \frac{1}{1+\omega^2}$ and $B(\omega) = \frac{\omega}{1+\omega^2}$.

Now, in Fourier integral representation, we have $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt = \frac{1}{\pi}\int_0^\infty \pi e^{-t}\cos(\omega t)\, dt$, from R.H.S

$$= \left[\frac{e^{-t}}{1+\omega^2}\left(-\cos(\omega t) + \omega \sin(\omega t)\right)\right]_0^\infty$$

$$= \frac{1}{1+\omega^2}.$$

and, $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\, dt = \frac{1}{\pi}\int_0^\infty \pi e^{-t}\sin(\omega t)\, dt = \left[\frac{e^{-t}}{1+\omega^2}\left(-\sin(\omega t) - \omega\cos(\omega t)\right)\right]_0^\infty = \frac{\omega}{1+\omega^2}$. *(Proved)*

3. Represent $f(x) = \begin{cases} x^2 & if\ \ 0 < x < 1 \\ 0 & if\ \ \ \ x > 1 \end{cases}$ as Fourier cosine integral.

Solution: From Fourier cosine integral representation, we obtain

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\cos(\omega t)\, dt = \frac{2}{\pi}\int_0^1 t^2 \cos(\omega t)\, dt = \frac{2}{\pi}\left[\frac{(\omega^2 - 2)\sin(\omega) + 2\omega\cos(\omega)}{\omega^3}\right]$$

and it gives the Fourier cosine integral representation as $f(x) = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2-2)\sin(\omega) + 2\omega\cos(\omega)}{\omega^3}\cos(\omega x)\, d\omega$.

4. Represent $f(x) = \begin{cases} e^{-x} & if\ \ 0 < x < 1 \\ 0 & if\ \ \ \ x > 1 \end{cases}$ as Fourier sine integral.

Solution: From Fourier sine integral representation, we obtain

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(t)\sin(\omega t) dt = \frac{2}{\pi}\int_0^1 e^{-t}\sin(\omega t) dt = \frac{2}{\pi}\left[\frac{-\sin(\omega) - \omega\cos(\omega) + \omega e}{e\omega^2 + e}\right]$$

and it gives the Fourier sine integral representation as $f(x) = \frac{2}{\pi}\int_0^\infty \frac{-\sin(\omega) - \omega\cos(\omega) + \omega e}{e\omega^2 + e}\ \sin(\omega x)\, d\omega$.

Chapter 11.8: Fourier Cosine and Sine Transforms

The Fourier transform is the extension of this idea to non-periodic functions by taking the limiting form of Fourier series when the fundamental period is made very large (infinite). Fourier transform finds its applications in astronomy, signal processing, linear time invariant systems etc.

Fourier Cosine Transform

Fourier Cosine transform of a function $f(x), 0 < x < \infty$, denoted by $\overline{f_c}(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx dx$

Also inverse Fourier Cosine Transform of $\overline{f_c}(w)$ gives $f(x)$ as:

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \overline{f_c}(w)\cos wx dw$$

Fourier sine Transform

Fourier Sine transform of $f(x), 0 < x < \infty$, given by $\overline{f_s}(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx dx$

Also inverse Fourier Sine Transform of $\overline{f_c}(w)$ gives $f(x)$ as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{f_s}(w) \sin wx dw$$

Question: *Find the Fourier cosine and Fourier sine transform of the function*

$f(x) = k$ *if 0<x<a, and 0 if x>a.*

Solution: Fourier cosine transform and sine transform of *f(x)* is given by

$$\overline{f_c}(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos wx dx = \sqrt{\frac{2}{\pi}} k (\frac{\sin aw}{w})$$

and $\overline{f_s}(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin wx dx = \sqrt{\frac{2}{\pi}} k (\frac{1 - \cos aw}{w})$.

Chapter 11.9: Linearity property, Fourier transform of derivatives, Convolution

Linearity of the Fourier Transform:

The Fourier transform is a *linear operation*; that is, for any functions $f(x)$ and $g(x)$ whose Fourier transforms exist and any constants **a** and **b**, the Fourier transform of $af + bg$ exists, and

$$\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g).$$

Fourier Transform of the Derivative of f(x):

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = i\omega\,\mathcal{F}\{f(x)\}.$$

$$\mathcal{F}\{f''(x)\} = i\omega\,\mathcal{F}\{f'^{(x)}\} = (i\omega)^2 \mathcal{F}\{f(x)\} = -\,\omega^2 \mathcal{F}\{f(x)\}.$$

Ex

Find the Fourier transform of $\mathbf{xe^{-x^2}}$.

Solution: We use above theorem, i.e.,

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}$$

$$\mathcal{F}(\mathrm{xe}^{-\mathrm{x}^2}) = \mathcal{F}\{-\frac{1}{2}(\mathrm{e}^{-\mathrm{x}^2})'\}$$

$$= -\frac{1}{2}\mathcal{F}\{(\mathrm{e}^{-\mathrm{x}^2})'\}$$

$$= -\frac{1}{2}\mathrm{i}\omega\mathcal{F}(\mathrm{e}^{-\mathrm{x}^2})$$

$$= -\frac{1}{2}\mathrm{i}\omega\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\omega^2}{4}}$$

$$= -\frac{\mathrm{i}\omega}{2\sqrt{2}}\mathrm{e}^{-\frac{\omega^2}{4}}$$

Convolution

- The convolution $f * g$ of two functions f and g is defined by

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

- Convolution theorem for Fourier transforms

Suppose that $f(t)$ and $g(x)$ are piece-wise continuous, bounded, and absolutely integrable on the x-axis. Then,

$$\mathcal{F}[f * g] = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$$

From this we can write from inverse Fourier transformation

$$f * g = \mathcal{F}^{-1}\left(\sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)\right) = \int_{-\infty}^{\infty} \hat{f}(\omega)\hat{g}(\omega)e^{i\omega x}d\omega$$

- **Ex.** Using convolution theorem for Fourier transforms, find $\mathcal{F}^{-1}\left[\frac{1}{6+5i\omega-\omega^2}\right]$.

Solution: Let, $F(\omega) = \frac{1}{6+5i\omega-\omega^2} = \frac{1}{(i\omega+2)(i\omega+3)} = \frac{1}{(i\omega+2)}\frac{1}{(i\omega+3)} = G(\omega)H(\omega).$

Also, we know that, $\mathcal{F}^{-1}\big(G(\omega)\big) = e^{-2x}U(x-0)$ and $\mathcal{F}^{-1}\big(H(\omega)\big) = e^{-3x}U(x-0)$,

Where $U(t)$ is the unit step function.

Therefore $\mathcal{F}^{-1}\big(F(\omega)\big) = \mathcal{F}^{-1}\big(G(\omega)H(\omega)\big) = \mathcal{F}^{-1}\big(G(\omega)\big) * \mathcal{F}^{-1}\big(H(\omega)\big) = e^{-2x}U(x-0) * e^{-3x}U(x-0)$

$$\boxed{U(\tau-0)U(t-\tau-0) = \begin{cases} 1 & if\ 0 < \tau < t \\ 0 & if\ \quad \tau > t \end{cases}}$$

$$= \int_{-\infty}^{\infty} e^{-2\tau}U(\tau-0)e^{-3t+3\tau}U(t-\tau-0)d\tau$$

$$= \int_0^t e^{-3t+\tau}\ d\tau$$

$$= e^{-3t}(e^t - 1).$$