ODE for Damped Oscillation -

$$Mx'' + bx' + kx = 0 \quad \text{--- (i)}$$

let $\frac{b}{M} = 2\nu$, $\frac{k}{M} = \omega_0^2$

From (i),

$$y'' + 2\nu x' + \omega_0^2 x = 0$$

$$A.E \Rightarrow \lambda^2 + 2\nu\lambda + \omega_0^2 = 0$$

Using quadratic formula,

$$\lambda = \frac{-2\nu \pm \sqrt{4\nu^2 - 4\omega_0^2}}{2}$$

$$= \frac{2(-\nu \pm \sqrt{\nu^2 - \omega_0^2})}{2}$$

$$\therefore \lambda_1 = -\nu + \sqrt{\nu^2 - \omega_0^2}$$

$$\lambda_2 = -\nu - \sqrt{\nu^2 - \omega_0^2}$$

$$\therefore x(t) = A e^{(-\nu + \sqrt{\nu^2 - \omega_0^2})t} + B e^{(-\nu - \sqrt{\nu^2 - \omega_0^2})t}$$

$$= e^{-\nu t}\left(A e^{t\sqrt{\nu^2 - \omega_0^2}} + B e^{t\sqrt{\nu^2 - \omega_0^2}}\right)$$

Underdamped oscillation ($\omega_0 > \nu$) —

we have general ODE,

$$x(t) = e^{-\nu t}\left(A e^{t\sqrt{\nu^2 - \omega_0^2}} + B e^{t\sqrt{\nu^2 - \omega_0^2}}\right)$$

let $\sqrt{\omega_0^2 - \nu^2} = q$

$$x(t) = e^{-\nu t}(A e^{iqt} + B e^{-iqt})$$

$$= A e^{-\nu t}(e^{iqt} + e^{-iqt})$$

$$= 2A e^{-\nu t}\, i \sin(qt) \quad \text{--- (ii)} \qquad \left[\tfrac{1}{2}(e^{ix} + e^{-ix}) = i\sin(x)\right]$$

$$x'(t) = 2A\,(q e^{\cancel{-\nu t}}\, i \underbrace{\cos(qt)}_{1} - \cancel{\nu e^{-\nu t} \sin(qt)})$$

$$x'(0) = 2Aqi = V_0$$

$$\text{or, } A = \frac{V_0}{2qi}$$

From (ii),

$$x(t) = 2A e^{-\nu t}\, i \sin(qt)$$

$$\text{or, } x(t) = \cancel{2}\,\frac{V_0}{\cancel{2}q\cancel{i}}\, e^{-\nu t}\, \cancel{i} \sin(qt)$$

$$\boxed{x(t) = \frac{V_0}{q} e^{-\nu t} \sin(qt)}$$

term $\frac{V_0}{q}$ physically signifies that amplitude is directly proportional to how powerful the impulse is and inversely proportional to q which is $\sqrt{\omega_0^2 - \nu^2}$.

Term $e^{-\gamma t}$ physically signifies the exponential decay of amplitude. $\sin(qt)$ is the nature of the oscillation.

Overdamped oscillation $(\gamma > \omega_0)$ —

$$x(t) = e^{-\gamma t}\left(A e^{t\sqrt{\gamma^2-\omega_0^2}} + B e^{-t\sqrt{\gamma^2-\omega_0^2}}\right)$$

Let $p = \sqrt{\gamma^2 - \omega_0^2}$

$$\therefore x(t) = e^{-\gamma t}\left(A e^{pt} + B e^{-pt}\right)$$

$$x(0) = A + B = 0$$

or, $A = -B$

$$x'(t) = \frac{d}{dt}\left(e^{-\gamma t}\left(A e^{pt} + B e^{-pt}\right)\right)$$

$$= -\gamma e^{-\gamma t}\left(Ape^{pt} - Bpe^{-pt}\right)$$

$$= -\gamma e^{-\gamma t}\left(A e^{pt} + Ape^{-pt}\right)$$

$$x'(0) = -\gamma\left(Ap + Ap\right) = V_0$$

$$= -2A\gamma p = V_0$$

or, $A = \dfrac{-V_0}{2\gamma p}$

$$\therefore x(t) = e^{-\gamma t}\left(A e^{pt} + B e^{-pt}\right)$$

$$= e^{-\gamma t}\left(\frac{-V_0}{2\gamma p} e^{pt} + \frac{V_0}{2\gamma p} e^{-pt}\right)$$

$$= \frac{V_0}{\gamma p} e^{-\gamma t} \, \frac{1}{2}\left(e^{-pt} - e^{pt}\right)$$

$$\boxed{x(t) = \frac{-V_0}{\gamma p} e^{-\gamma t} \sinh(pt)}$$

Critically damped oscillation $(\gamma = \omega_0)$ —

$$x(t) = e^{-\gamma t}\left(A e^{\sqrt{\gamma^2-\omega_0^2}\,t} + Bt e^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) \quad [t * B \text{ for LI root}]$$

$$= e^{-\gamma t}\left(A + Bt\right)$$

$$x(0) = 0 = A$$

$$x'(t) = -\gamma A e^{-\gamma t} - B\gamma t e^{-\gamma t} + B e^{-\gamma t}$$

$$x'(0) = -\gamma A + B = V_0$$

or, $B = V_0$

$$\boxed{x(t) = V_0 t \, e^{-\gamma t}}$$

Total energy of an oscillator = Kinetic Energy + Potential Energy

$\therefore \quad E(t) = \frac{1}{2} M \left(\frac{d^2x}{dt^2}\right)^2 + \frac{1}{2} K(x^2)$

Put different values of $x(t)$ for different types of oscillations.

$$\langle E \rangle = \int_0^T E(t)\,dt$$

$\boxed{E_{avg} = E_0 e^{-2\nu t}}$ ← Expression for average energy

$\therefore$ Energy decays twice at the rate of decay of amplitude.

Relaxation Time →

Time required for an oscillator's energy to decay from E_0 to E_0/e.

$\boxed{\tau = 1/2\nu}$ ← Expression for relaxation time

Time period →

$\boxed{T^* = \dfrac{2\pi}{\sqrt{\omega_0^2 - \nu^2}}}$ ← Expression for time period.

Amplitude of n^{th} oscillation -

$$\boxed{A_n(t) = A_0 e^{-\nu(2n+1)T^*/4}}$$

Logarithmic decrement → The natural logarithm of the ratio of two consecutive amplitudes.

$$\boxed{\ln\left(A_n / A_{n+1}\right) = \nu T^*/2}$$

Quality factor → The ratio between energy stored in oscillator to the energy dissipated in 1 radian of oscillation.

Energy of oscillator $= E_0 e^{-2\nu t} = E(t)$

Energy after 1 oscillation $= E(T^*) = E_0 e^{-2\nu T^*}$

$\boxed{Q = \dfrac{\omega_0}{2\nu}}$

if $Q > 1/2$, underdamped
$Q = 1/2$, critically damped
$Q < 1/2$, overdamped.