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Equivalence Classes Under SLOCC for a tripartite System and Its Generalization

This is a short note on the mathematical tools used in the field of Quantum Computation and Quantum Information.Fundamental topics in Quantum Information and Computation are first introduced.There the various tools and methods required for studying entanglement of quantum systems that has been discussed. Further, the quantum mechanical properties of a three qubit entangled system are discussed with a final glimpse of generalization. Quantum entanglement is a quantum mechanical phenomenon that occurs when pairs or groups of particles are generated or they interact in ways such that the quantum state of each particle cannot be described independently. In other words, their wave functions become correlated. Using the concept of quantum entanglement, information can be transmitted in a speed faster than light. Equivalence class under SLOCC of an entangled system contains states which can be converted from one to another by ILO. Study of equivalence classes under SLOCC became a famous discipline under the field of quantum computation and information because each equivalence class of states under SLOCC contains states that are entangled in essentially different ways. Understanding equivalence classes under SLOCC for a system consisting two or more than two qubits is hence highly important to study the features of entanglement. When there are only two qubits, there is only one equivalence class, for three qubits there are two equivalence classes. Till this there was no problem. But for more than three qubits, one equivalence class contains infinite number of states, i.e., at least one non-local parameter correlating the entire Hilbert space or all possible quantum states. Hence generalizing the result found for three qubit system explores the properties of n-qubit system.

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I. INTRODUCTION

A. Quantum Entanglement and The Role of The Density Operator

Quantum entanglement is considered to be the most non-classical manifestation of quantum formalism. It was introduced by Einstein, Podolsky and Rosen in order to ascribe values to physical quantities prior to measurement. In short, quantum entanglement is a phenomena that links multiple number of particles together in such a way that the measurement done on one particle's quantum state readily determines the possible quantum states of other particles. Quantum entanglement represents co-relations between the parties those are entangled with each other in such a way that co-relations between them cannot be shared by any other party. This is the basic feature of entanglement. For an elaborate discussion, let us start with defining a bipartite system. [1, 2]

The properties of a multipartite system is much different than that of an one qubit system.The simplest possible example of a multipartite system is a bipartite system which is basically a system consisting two qubits. One of the qubits can be observed and another one is forbidden.

Let us consider that qubit A is allowed to observe or manipulate as desired. But qubit B is forbidden from any access. Let the orthonormal bases for qubit A and B are $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ respectively.A quantum state of the system can be represented as,

$$|\psi\rangle_{AB} = a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B \tag{1}$$

Qubits A and B are said to be correlated and measurement can be done on qubit A by projecting onto the basis $\{|0\rangle_A, |1\rangle_A\}$. With probability $|a|^2$, one can obtain the result $|0\rangle_A$ and the measurement prepares the state,

$$|0\rangle_A \otimes |0\rangle_B \tag{2}$$

and similarly with probability $|b|^2$, the result $|1\rangle_A$ is obtained and state

$$|1\rangle_A \otimes |1\rangle_B \tag{3}$$

can be prepared.In either case, a definite state of qubit B is picked out by the measurement.

Now let us consider any general observable which is acting on qubit A irrespective of whatever the outcome of qubit B is. The observable is expressed in the form,

$$\mathbf{M}_A \otimes \mathbf{1}_B \tag{4}$$

where $\mathbf{M}_A$ is a self-adjoint operator acting on qubit A and $\mathbf{1}_B$ is the identity operator acting on qubit B. The expectation value of the observable for the quantum state $|\psi\rangle_{AB}$ will be,

$$\begin{aligned} {}_{AB}\langle\psi|\mathbf{M}_A \otimes \mathbf{1}_B|\psi\rangle_{AB} &= (a^*_A\langle 0| \otimes_B \langle 0| + b^*_A\langle 1| \otimes_B \langle 1|)(\mathbf{M}_A \otimes \mathbf{1}_B)(a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B \\ &= |a|^2_A\langle 0|\mathbf{M}_A|0\rangle_A + |b|^2_A\langle 1|\mathbf{M}_A|1\rangle_A \end{aligned} \tag{5}$$

where the basis $\{|0\rangle_B, |1\rangle_B\}$ is considered to complete and orthogonal. The above expression can be rewritten in the form,

$$\langle \mathbf{M}_A \rangle = tr(\mathbf{M}_A \boldsymbol{\rho}_A) \tag{6}$$

$$\boldsymbol{\rho}_A = |a|^2|0\rangle_A\, {}_A\langle 0| + |b|^2|1\rangle_A\, {}_A\langle 1| \tag{7}$$

$\boldsymbol{\rho}_A$ is called the **density operator** [1]for qubit A which is self-adjoint, positive and has unit trace. It represents an ensemble of possible quantum states with a specified probability. Now an observable has a set of eigenstates in the Hilbert space which is known as the relevant space of the observable. When an operator $\boldsymbol{M}_A$ operates on any wave function $|\phi\rangle$ representing a Hilbert space, the outcome is obviously the projection of $|\phi\rangle$ onto the relevant eigenspace of that particular observable and may be denoted as $\boldsymbol{E}_A(a)$.Then the probability of outcome to be a summed over the ensemble.

$$\text{Prob}(a) = P_0\, {}_A\langle 0|\mathbf{E}_A(a)|0\rangle_A + p_1\, {}_A\langle 1|\mathbf{E}_A(a)|1\rangle_A \tag{8}$$

where $p_0 = |a|^2$ is the probability of the outcome to be $|0\rangle$ and $p_1 = |b|^2$ is the probability of the outcome to be $|1\rangle$. The difference between a coherent superposition of the states and the probabilistic ensemble is that the states occur with specified probabilities. For example, for a spin-$\frac{1}{2}$ object, if $\boldsymbol{\sigma}_1$ is measured over a state $\frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$, one can obtain the state $|\uparrow_x\rangle$ with probability 1 :

$$\begin{aligned} \boldsymbol{\sigma}_1 \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle) &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \\ &= \frac{1}{2}(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \\ &= |\uparrow_x\rangle \end{aligned} \tag{9}$$

But the ensemble in which $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$) occure with probability $\frac{1}{2}$ and is represented by the density operator

$$\begin{aligned}\boldsymbol{\rho} &= \frac{1}{2}(|\uparrow_z\rangle\langle\uparrow_z| + |\downarrow_z\rangle\langle\downarrow_z|) \\ &= \frac{1}{2}\mathbf{1}\end{aligned} \tag{10}$$

and the projection onto $|\uparrow_x\rangle$ then has the expectation value:

$$tr(|\uparrow_x\rangle\langle\uparrow_x|\rho) = \frac{1}{2} \tag{11}$$

As the identity is left unchanged by any unitary transform of basis, and the state $|\psi\rangle$ can be obtained by applying a suitable unitary transformation to $|\uparrow_z\rangle$, then for $\boldsymbol{\rho}$ given by equation(10) one can obtain,

$$\mathrm{tr}(|\psi\rangle\langle\psi|\boldsymbol{\rho}) = \frac{1}{2} \tag{12}$$

Now, let us generalize our discussion of the two-qubit state by considering an arbitrary state. If one has a composite system with two sub-systems A and B and the Hilbert spaces for each sub-system is $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, then the Hilbert space corresponding to the bipartite system is given by $\mathcal{H}_A \otimes \mathcal{H}_B$.More specifically, if $|i\rangle_A$ is an orthonormal basis for $\mathcal{H}_A$ and $|\mu\rangle_B$ is an orthonormal basis for $\mathcal{H}_B$, then $|i\rangle_A \otimes |\mu\rangle_B$ can be an orthonormal basis for $\mathcal{H}_A \otimes \mathcal{H}_B$. Let us consider an arbitrary pure state of $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$|\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu}(|i\rangle_A \otimes |\mu\rangle_B) \tag{13}$$

where, $\sum_{i,\mu} |a_{i\mu}|^2 = 1$. Let us calculate the expectation value of the observable $\Xi M_A \otimes \Xi 1_B$ which, as described previously, acts only on system A.

$$\begin{aligned}\langle \boldsymbol{M}_A\rangle &= {}_{AB}\langle\psi|\boldsymbol{M}_A \otimes \mathbf{1}_B|\psi\rangle_{Ab} \\ &= \sum j,\nu a^*_{j\nu}({}_A\langle j| \otimes {}_B\langle\nu|)(\boldsymbol{M}_A \otimes \mathbf{1}_B)(\sum_{i,\mu} a_{i\mu}(|i\rangle_A \otimes |\mu\rangle_B) \\ &= \sum i,j,\nu,\mu a^*_{j\nu} a_{i\mu}\, {}_A\langle j|\boldsymbol{M}_A|i\rangle_A \delta_{\nu\mu} \\ &= \sum i,j,\mu a^*_{j\mu} a_{i\mu}\, {}_A\langle j|\boldsymbol{M}_A|i\rangle_A \\ &= \mathrm{tr}(\boldsymbol{M}_A\boldsymbol{\rho}_A)\end{aligned} \tag{14}$$

where,

$$\begin{aligned}\boldsymbol{\rho}_A &= \mathrm{tr_B}(|\psi\rangle_{AB}\, {}_{AB}\langle\psi|) \\ &= \sum i,j,\mu a^*_{j\mu} a_{i\mu}|i\rangle_A\, {}_A\langle j|\end{aligned} \tag{15}$$

Hence it can be stated that the density operator $\boldsymbol{\rho}_A$ for a subsystem A can be obtained by performing a partial trace over another subsystem B of the density matrix for the bipartite system.Some properties of this density matrix is:

(i) $\boldsymbol{\rho}_A$ is self-adjoint, i.e., $\boldsymbol{\rho}_A = \boldsymbol{\rho}_A^\dagger$

(ii) $\boldsymbol{\rho}_A$ is positive, i.e., for any $|\psi\rangle_A$

$$_A\langle\psi|\boldsymbol{\rho}_A|\psi\rangle_A = \sum \mu| \sum i a_{i\mu}\, {}_A\langle\psi|i\rangle_A|^2 \geq 0 \tag{16}$$

(iii) $\mathrm{tr}(\boldsymbol{\rho}_A) = 1$:

$$\mathrm{tr}(\boldsymbol{\rho}_A) = \sum i, \mu |a_{i,\mu}|^2 = 1 \tag{17}$$

Since $|\psi\rangle_{AB}$ is normalized.

For a larger quantum system, the state of the subsystem needs not to be, in general the state represented by the density operator. In the case where the state of the subsystem is a ray, the state is said to be pure and otherwise it is mixed.

For a pure state $|\psi\rangle_A$ the density matrix $\boldsymbol{\rho}_A = |\psi\rangle_A\, {}_A\langle\psi|$ is the projection onto the one-dimensional space spanned by $|\psi\rangle$ and it has the property $\boldsymbol{\rho}^2 = \boldsymbol{\rho}$. A general density matrix,expressed in the basis in which it is diagonal, has the form,

$$\boldsymbol{\rho}_A = \sum_a p_a |\psi_a\rangle\langle\psi_a| \tag{18}$$

where,$0 < p_a \leq 1$ and $\sum\limits_a p_a = 1$. If the state is not pure, there are two or more terms in this sum, and $\boldsymbol{\rho}^2 \neq \boldsymbol{\rho}$ and $\mathrm{tr}\boldsymbol{\rho}^2 = \sum p_a^2 < \sum p_a = 1$. The the density operator is said to be an incoherent superposition of the states $|\psi\rangle$,i.e., the relative phases of the $|\psi\rangle_a$ are experimentally inaccessible.Hence,

$$\langle\boldsymbol{M}\rangle = \mathrm{tr}\boldsymbol{M}\boldsymbol{\rho} = \sum_a p_a \langle\psi_a|\boldsymbol{M}|\psi_a\rangle \tag{19}$$

B. Bloch Sphere:

Bloch Sphere is also a very important concept that is required for understanding of entanglement.[2] A single qubit system can be expanded in the basis $\{\mathbf{1}, \boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \boldsymbol{\sigma_3}\}$ which is one of the most general set of self-adjoint 2X2 matrices. And as all the Pauli matrices are traceless then the the density matrix should be represented in the form,

$$\begin{aligned} \boldsymbol{\rho}(\overrightarrow{P}) &= \frac{1}{2}(\mathbf{1} + \overrightarrow{P}.\overrightarrow{\boldsymbol{\sigma}}) \\ &= \frac{1}{2}\begin{bmatrix} 1+P_3 & P_1 - iP_2 \\ P_1 + iP_2 & 1 - P_3 \end{bmatrix} \end{aligned} \tag{20}$$

where $\overrightarrow{P}$ is called the Bloch vector $(sin\theta cos\phi, sin\theta sin\phi, cos\theta)$ indicating a point on the surface of a unit three dimensional ball ($0 \leq |\overrightarrow{P}| \leq 1$) which is called the *Bloch Sphere*[1, 3]. Now while computing tr$\boldsymbol{\rho}$, the necessary condition is tr$\boldsymbol{\rho} \geq 0$ or $\overrightarrow{P}^2 \leq 0$. The boundary of the ball,i.e., the sphere contains the density matrices with vanishing determinant. Since tr$\boldsymbol{\rho} = 1$, then these density matrices must have eigenvalues 0 and 1. They are the one-dimensional projectors and hence pure states. And using the property of a pure state for a single-qubit system which can be envisioned as a spin pointing in the (θ, ϕ) direction, it can be easily verified that the states on the sphere are actually the pure states of the system.

$$(\widehat{n}.\overrightarrow{\boldsymbol{\sigma}})^2 = \mathbf{1} \tag{21}$$

Then the density matrix can be written in the form,

$$\begin{aligned} \boldsymbol{\rho}(\widehat{n}) &= \frac{1}{2}(\mathbf{1} + \widehat{n}.\boldsymbol{\sigma}) \\ &= |\psi(\widehat{n})\rangle\langle\psi(\widehat{n})| \\ &= |\psi(\theta,\phi)\rangle\langle\psi(\theta,\phi)| \\ &= \begin{bmatrix} e^{-i\phi/2} & cos\frac{\theta}{2} \\ e^{i\phi/2} & sin\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} e^{i\phi/2} & cos\frac{\theta}{2} \\ e^{-i\phi/2} & sin\frac{\theta}{2} \end{bmatrix} \\ &= \begin{bmatrix} cos^2\frac{\theta}{2} & cos\frac{\theta}{2}sin\frac{\theta}{2}e^{-i\phi} \\ cos\frac{\theta}{2}sin\frac{\theta}{2}e^{i\phi} & sin^2\frac{\theta}{2} \end{bmatrix} \\ &= \frac{1}{2}\mathbf{1} + \frac{1}{2}\begin{bmatrix} cos\theta & sin\theta e^{-i\phi} \\ sin\theta e^{i\phi} & -cos\theta \end{bmatrix} \\ &= \frac{1}{2}(\mathbf{1} + \widehat{n}.\overrightarrow{\boldsymbol{\sigma}}) \end{aligned} \tag{22}$$

where $\widehat{n} = (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$ The Bloch parametrization of pure states is such that $|\psi(\theta, \phi)\rangle$ has an arbitrary overall phase that has no significance and hence there is no phase ambiguity in the density matrix $\boldsymbol{\rho}(\theta, \phi) = |\psi(\theta, \phi)\rangle\langle\psi(\theta, \phi)|$. And as,

$$\frac{1}{2}\mathrm{tr}\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \delta_{ij} \tag{23}$$

Therefore,

$$\begin{aligned}
\langle \widehat{n}.\overrightarrow{\boldsymbol{\sigma}}\rangle_{\overrightarrow{P}} &= \mathrm{tr}(\widehat{n}.\overrightarrow{\boldsymbol{\sigma}}\,\boldsymbol{\rho}(\overrightarrow{P})) \\
&= \mathrm{tr}((\widehat{n}.\overrightarrow{\boldsymbol{\sigma}})\frac{1}{2}(\mathbf{1} + \overrightarrow{P}.\overrightarrow{\boldsymbol{\sigma}})) \\
&= \frac{1}{2}\mathrm{tr}((\widehat{n}.\overrightarrow{\boldsymbol{\sigma}})\mathbf{1} + (\overrightarrow{P}.\overrightarrow{\boldsymbol{\sigma}})(\widehat{n}.\overrightarrow{\boldsymbol{\sigma}})) \\
&= \frac{1}{2}[\mathrm{tr}[\widehat{n}.\overrightarrow{\boldsymbol{\sigma}}] + \mathrm{tr}[\widehat{n}.(\overrightarrow{\boldsymbol{\sigma}}.(\overrightarrow{P}.\overrightarrow{\boldsymbol{\sigma}}))]] \\
&= \frac{1}{2}\mathrm{tr}[\widehat{n}.(\overrightarrow{\boldsymbol{\sigma}}(\overrightarrow{P}.\overrightarrow{\boldsymbol{\sigma}}))] \\
&= \frac{1}{2}\mathrm{tr}[\widehat{n}.(\overrightarrow{P}(\overrightarrow{\boldsymbol{\sigma}}.\overrightarrow{\boldsymbol{\sigma}}))] \\
&= \widehat{n}.\overrightarrow{P}
\end{aligned} \tag{24}$$

Thus $\overrightarrow{P}$ parametrizes the polarization of the spin. Because it is known to us that the eigenvectors of the matrix $\widehat{n}.\overrightarrow{\boldsymbol{\sigma}}$ are the states describing the pure states of a single-qubit system and is now proved to be dependent upon $\overrightarrow{P}$.

C. Schmidt Decomposition

Density operators and the partial trace are just the beginning of a wide array of tools useful for the study of composite quantum systems, which are at the heart of quantum computation and quantum information. There are additional tools of great value and among them Schmidt decomposition is very famous. [2, 3]

*Theorem:*Suppose $|\psi\rangle$ is a pure state of a composite system AB then there exist orthonormal states $|i_A\rangle$ for system A and orthonormal states $|i_B\rangle$ of system B such that,

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle|i_B\rangle, \tag{25}$$

whereλ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt coefficients.

This result is very useful. As a taste of its power, consider the following consequence:let $|\psi\rangle$ be a pure state of a composite system AB.Then by the Schmidt decomposition $\rho_A = \sum\limits_i \lambda_i|i_A\rangle\langle i_A|$

and $\rho_B = \sum_i \lambda_i |i_B\rangle\langle i_B|$, so the eigenvalues of ρ_A and ρ_B are identical, namely λ_i^2 for both density operators. Many important properties of quantum systems are completely determined by the eigenvalues of the reduced density operator of the system, so for a pure state of a composite system such properties will be the same for both systems.

As an example, considering the state of two qubits,$(|00\rangle + |01\rangle + |11\rangle)/\sqrt{3}$. This has no obvious symmetry property, yet if one can calculate $\mathrm{tr}(\rho_A^2)$ and $\mathrm{tr}(\rho_B^2)$ it will be discovered that they have the same value, 7/9 in each case. This is but one small consequence of the Schmidt decomposition.

Proof: Let us consider the case where systems A and B have state spaces of the same dimension.Let $|j\rangle$ and $|k\rangle$ be any fixed orthonormal bases for systems A and B, respectively. Then $|\psi\rangle$ can be written as,

$$|\psi\rangle = \sum_{jk} a_{jk} |j\rangle |k\rangle \tag{26}$$

,for some matrix a of complex numbers a_{jk}.By the singular value decomposition, $a = udv$, where d is a diagonal matrix with non-negative elements, and u and v are unitary matrices.Thus,

$$|\psi\rangle = \sum_{ijk} u_{ji} d_{ii} v_{ik} |j\rangle |k\rangle \tag{27}$$

Defining,

$$|i_A\rangle = \sum_j u_{ji} |j\rangle \tag{28}$$

$$|i_B\rangle = \sum_k v_{ik} |k\rangle \tag{29}$$

$$\lambda_i = d_{ii} \tag{30}$$

It is easy to check that $|i_A\rangle$ forms an orthonormal set of bases,from the unitarity of u and the orthonormality of $|j\rangle$ and similarly that the $|i_B\rangle$ also form an orthonormal set of bases.

The bases $|i_A\rangle$ and $|i_B\rangle$ are called the Schmidt bases for A and B respectively and the number of non-zero values λ_i is called the Schmidt number for the state $|\psi\rangle$.The Schmidt number is an important property of a composite quantum system, which in some sense quantifies the amount of entanglement between systems A and B. The Schmidt number is preserved under unitary transformations on system A or system B alone.

II. UNDERSTANDING OF CONDITIONS FOR A CLASS OF TRANSFORMATION

In this section it will be established that for a bipartite system, any pure state $|\psi\rangle_{AB}$ can be transformed to another state $|\phi\rangle_{AB}$ by only using local operations.[5]
The necessary and sufficient condition for entanglement transformation exhibit an unexpected connection between entanglement and the linear-algebraic theory of majorization. But the problem is that in case of a pure state into EPR pairs, the limits are asymptotic. The investigations for the finite case can serve some benefits for recovering the constrain of limits. For that, the concept of majorization is important.

A. What is majorization?

Majorization[6] is a very active area of research in linear algebra. It states that, **if $x = (x_1, x_2, ..., x_d)$ and $y = (y_1, y_2, ..., y_d)$ are real d-dimensional vectors, then x is majorized by y if for each k in the range $1 \leq k \leq d$,**

$$\sum_{j=1}^{k} x_j^{\downarrow} \leq \sum_{j=1}^{k} y_j^{\downarrow} \tag{31}$$

and is denoted as,

$$x \prec y \tag{32}$$

The equality holds when $k = d$ and this condition is valid iff $x_j^{\downarrow}$'s and $y_j^{\downarrow}$'s are taken in a decreasing order denoted by the $\downarrow$ mark.

Theorem: $|\psi\rangle$ *transforms to* $|\phi\rangle$ *using local operations and classical communication*(LOCC)*if and only if* λ_ψ *is majorized by* λ_ϕ*, i.e.,*

$$|\psi\rangle \longrightarrow |\phi\rangle \qquad \Longleftrightarrow \lambda_\psi \prec \lambda_\phi \tag{33}$$

From the above theorem, it can be visualize that neither $|\psi\rangle \longrightarrow |\phi\rangle$ nor $|\phi\rangle \longrightarrow |\psi\rangle$ is providing an example of essentially different type of entanglement, from the point of view of LOCC and hence $|\phi\rangle$ and $|\psi\rangle$ are incomparable. To prove the theorem, some useful facts can be taken into account, (i) Alice performs a generalized measurement in her system and sends her outcome to Bob. Bob performs an operation on his system, conditional on the Alice's result.

(ii) For any matrix A, the polar decomposition states that $A = \sqrt{AA^{\dagger}} = U\sqrt{A^{\dagger}A}$.

(iii) Suppose $\rho' = \sum_i p_i U \rho U_i^{\dagger}$, where $p_i \geq 0$, $\sum_i p_i = 1$ and U_i is unitary. Then the vector of eigenvalues of ρ' is majorized by the vector of eigenvalues of ρ, $\lambda_{\rho' \prec \lambda_\rho}$.

(iv) If $x \prec y$, then $x = Dy$ where D is a matrix that may be written as a product of at most $(d-1)$ numbers of T transform, where d is the dimension of x and y. And T transform is defined as,

$$T = \begin{bmatrix} t & 1-t \\ 1-t & t \end{bmatrix} \tag{34}$$

proof: Assuming that we have the implication for dimension upto $(n-1)$, let $x, y \in R^n$. Since $\mathbf{x}^{\downarrow}$ and $\mathbf{y}^{\downarrow}$ can be obtained from x and y by permutations and each permutation is a product of transpositions which are surely T-transforms. It can be assumed without loosing generality that $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$. Then,

$$x_1 = ty_1 + (1-t)y_2 \tag{35}$$

for some $0 \leq t \leq 1$.

By choosing k such that $y_k \leq x_1 \leq y_{k-1}$, iff $x \prec y$.

Let,

$$T_1 z = (tz_1 + (1-t)z_k, z_2, z3, ..., z_{k-1}, (1-t)z_1 + tz_k, z_{k+1}, ..., y_n) \qquad \forall z \in R^n \tag{36}$$

Then it can be noted that the first coordinate of $T_1 y$ is x_1. Let,

$$x' = (x_2, ..., x_n) \tag{37}$$

and

$$y' = (y_2, ..., y_{k-1}, (1-t)y_1 + ty_k, y_{k+1}, ..., y_n) \tag{38}$$

It can be shown that, $x' \prec y'$. Since $y_1 \geq ... \geq y_{k-1} \geq x_1 \geq ... \geq\geq x_n$, one can have for $2 \leq m \leq k-1$,

$$\sum_{j=2}^{m} x_j \leq \sum_{j=2}^{m} y_j \tag{39}$$

And for $k \leq m \leq n$,

$$\begin{aligned} \sum_{j=2}^{m} y'_j &= \sum_{j=2}^{k-1} y_j + [(1-t)y_1 + ty_k] + \sum_{j=k+1}^{m} y_j \\ &= \sum_{j=1}^{m} y_j - ty_1 + (1-t)y_k \\ &= \sum_{j=1}^{m} y_j - x_1 \geq \sum_{j=1}^{m} x_j - x_1 \end{aligned} \tag{40}$$

The last inequality is an equality when $m = n$ since $x \prec y$. Thus $x' \prec y'$. So, by the induction hypothesis there exist a finite number of T-transforms $T_2, ..., T_r$ on R^{n-1} such that $x' = (T_r, ..., T_2)y'$. Each of them can be regarded as one T-transform on R^n if they are prohibited from touching the first co-ordinate of any vector. Then one can have,

$$(T_r, ..., T_1)y = (T_r, ..., T_2)(x_1, y') = (x_1, x') = x \tag{41}$$

Now a T-transform is a convex combination of the identity map and some permutation. So a product of such maps is a convex combination of permutation.
Now the another fact that is required to be proven that two wave functions related by a stochastic transform then the later wave function must be majorized by the former one.

Theorem: *A matrix A is doubly stochastic if and only if $Ax \prec x$ for all vectors.*

Let A be a doubly stochastic matrix and $y = Ax$. Lets assume that the co-ordinates of both x and y are in decreasing order. Let, for any k, $1 \le k \le n$, where n is the dimension of x and y. Then,

$$\sum_{j=1}^{k} y_j = \sum_{j=1}^{k} \sum_{i=1}^{n} a_{ji} x_i \tag{42}$$

If $t_i = \sum_{j=1}^{k} a_{ji}$, then $0 \le t_i \le 1$ and

$$\sum_{i=1}^{k} t_i = k \tag{43}$$

therefore,

$$\begin{aligned}
\sum_{j=1}^{k} y_i - \sum_{j=1}^{k} x_j &= \sum_{i=1}^{n} t_i x_i - \sum_{i=1}^{k} x_i \\
&= \sum_{i=1}^{k} t_i x_i - \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} t_i x_i + (k - \sum_{i=1}^{n} t_i) x_k \\
&= \sum_{i=1}^{k} (t_i - 1) x_i + \sum_{i=k+1}^{n} x_i t_i + k x_k - \sum_{i=1}^{k} t_i x_k - \sum_{i=k+1}^{n} t_i x_k \\
&= \sum_{i=1}^{k} (t_i - 1)(x_i - x_k) + \sum_{i=k+1}^{n} t_i (x_i - x_k) \\
&\le 0
\end{aligned} \tag{44}$$

Therefore, if $y \prec x$, then y can be represented as Ax where A is a doubly stochastic matrix. Hence defining a T-transform as,

$$T_x = (x_1, x_2, ..., x_j, tx_j + (1-t)y_k, y_{j+1}, ..., (1-t)y_j + ty_k, x_{k+1}, ..., x_n) \tag{45}$$

And hence $Tx \prec x, \forall x$.

Now, back to our previous job of proving the former theorem,

Suppose, $|\psi\rangle \longrightarrow |\phi\rangle$. It is assumed that Alice performs a generalized measurement described by the operator M_m on her system, satisfying the completeness relation $\sum_m M_m^{\dagger} M_m = 1$. Then she sends the outcomes to Bob who performs an operation E_m^B, possibly nonunitary, on his system, conditional on the result m. Thus,

$$|\phi\rangle\rangle\phi| = \sum_m E_m^b (M_m |\psi\rangle\rangle\psi| M_m^{\dagger}) \tag{46}$$

Since $|\phi\rangle$ is a pure state,

$$E_m^b (M_m |\psi\rangle\rangle\psi| M_m^{\dagger}) \propto |\phi\rangle\rangle\phi| \tag{47}$$

Hence, tracing out system B gives,

$$tr_B E_m^b (M_m \rho_\psi M_m^{\dagger}) \propto |\phi\rangle\rangle\phi| \tag{48}$$

with non-negative proportionality constant p_m satisfying $\sum_m p_m = 1$.

Now,

$$\begin{aligned} M_m \sqrt{\rho_\psi} &= \sqrt{M_m \rho_\psi M_m^{\dagger}} U_m \\ &= \sqrt{p_m} \sqrt{\rho_\psi} U_m \end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned} \rho_\psi &= \sum_m \sqrt{\rho_\psi} M_m^{\dagger} M_m \sqrt{\rho_\psi} \\ &= \sum_m \sqrt{\rho_\psi} M_m^{\dagger} \sqrt{p_m} \sqrt{\rho_\psi} U_m \\ &= \sum_m \sqrt{p_m} \sqrt{\rho_\phi} U_m^{\dagger} \sqrt{p_m} \sqrt{\rho_\phi} U_m \\ &= \sum_m p_m U_m^{\dagger} \rho_\phi U_m \end{aligned} \tag{50}$$

And hence from the fact provided in assumption (iii), it can be concluded that $\lambda_\psi \prec \lambda_\phi$ condition is required for a transformation $|\phi\rangle \longrightarrow |\psi\rangle$ using LOCC.

Conclusion: If Alice and Bob jointly posses a pure state $|\psi\rangle$, then using local operations on their respective systems and classical communication, it may possible for Alice and Bob to transform$|\psi\rangle$ into another pure entangled state $|\phi\rangle$ and this connection or relation is thhe necessary and sufficient condition for the process of entanglement transform to be possible for a bipartite system. These condition reveal a partial ordering on the entangled states and connect quantum entanglement to algebraic theory of majorization.[4, 6, 7]

III. EQUIVALENCE CLASSES OF A THREE QUBIT SYSTEM UNDER SLOCC

Two sates are said to be under any equivalence class if both of them can be obtained from the another by the mean of LOCC with non-zero probability. And they are said to have the same kind of entanglement.Before analysing a more generalized case, we will identify and characterize all possible kinds of pure-state entanglement of three qubit under SLOCC. It will be established that there are two kinds of genuine tripartite entanglement by means of SLOCC and hence any pure state can be categorized into either of the two kind of entanglement[7], namely **GHZ state**,

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{51}$$

and **W state,**

$$|\mathrm{W}\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \tag{52}$$

These two classes split the set of genuinely trifold entangled states. If $|\psi\rangle$ and $|\phi\rangle$, two arbitrary pure trifold entangled states, are categorized to GHZ and W states respectively, then it is neither possible to transform $|\psi\rangle$ into $|\phi\rangle$ nor the other way round. The basic difference between the two kind of states rise from the fact that not all entangled states can be expressed as a linear combination of only two product states as GHZ states.

The GHZ state can be regarded as the maximally entangled state of three qubits. But, if one of the three qubits is traced out, the remaining state is completely unentangled, i.e., GHZ states are collapsible under particle losses. But for W states, after tracing out one of a particle, the co-relation between the other two remains intact.

A. What kind of entanglement is possible under SLOCC?(Requirement of operators to be invertible)

Before going for classification one should find out which states are related by SLOCC. The local protocols can be visualized as a series of rounds of operations, where in each round a certain party manipulates its subsystem of entangled locally and then communicates classically the result of its operation to rest of the parties. Subsequent operations can be made dependent on the previous results and the protocol splits into several branches. If one particular state $|\psi\rangle$ can be converted into another state $|\phi\rangle$ with non-zero probability then at least one branch of the protocol participate in the process of transformation. Mathematically in a three qubit case two states can be converted iff an operator $A \otimes B \otimes C$ exists such that,

$$|\psi\rangle = A \otimes B \otimes C|\phi\rangle \tag{53}$$

where, A, B, C respectively contain contributions coming from any round in which party A, B , C acted on their respective subsystems. Then the reduced density matrices can be represented as,

$$\rho_A \;=\; \mathrm{tr}_{BC}(|\psi\rangle\langle\psi|) \tag{54}$$

$$\rho_B \;=\; \mathrm{tr}_{AC}(|\psi\rangle\langle\psi|) \tag{55}$$

$$\rho_C \;=\; \mathrm{tr}_{AB}(|\psi\rangle\langle\psi|) \tag{56}$$

Each of the operation participating in the process of conversion is necessarily required to be invertible if $|\psi\rangle$ and $|\phi\rangle$ belong to the same class under SLOCC and are related by means of those operator. In other words local ranks of a pure state, $r(\rho_\kappa), \kappa = A, B, C, ..$ are invariant under SLOCC.

Lemma. If the bipartite vectors $|\psi\rangle$ and $|\phi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ fulfil,

$$|\phi\rangle = A \otimes \mathrm{I}_B|\psi\rangle \tag{57}$$

then the ranks of the corresponding reduced density matrices satisfy $r(\rho_A^\psi) \geqslant r(\rho_A^\phi)$ and $r(\rho_B^\psi) \geqslant r(\rho_B^\phi)$.

Proof Lets consider the Schmidt decomposition of $|\psi\rangle$,

$$|\psi\rangle = \sum_{i=1}^{n_\psi} \sqrt{\lambda_i^\psi}|i\rangle_A|i\rangle_B, \qquad \lambda_i^\psi > 0, \qquad n_\psi \leqslant \min(n, m) \tag{58}$$

and writing the operator A as,

$$A = \sum_{i=1}^{n} |\mu_i\rangle\langle i| \tag{59}$$

where,$|\mu_i\rangle$ is not necessarily required to be normalized or linearly independent.Then one can have,

$$\rho_A^{\psi} = \sum_{i=1}^{n^{\psi}} \sqrt{\lambda_i^{\psi}} |i\rangle\langle i| \tag{60}$$

and

$$\rho_A^{\phi} = A\rho_A^{\psi}A^{\dagger} = \sum_{i=1}^{n^{\phi}} \sqrt{\lambda_i^{\psi}} |\mu_i\rangle\langle\mu_i| \tag{61}$$

so, that $r(\rho_A^{\phi}) \leqslant n_{\psi}$. The second inequality of the lemma follows from the fact that for any bipartite vector $r(\rho_A) \geqslant r(\rho_B)$.

Corollary: If the vectors $|\psi\rangle$, $|\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes .... \otimes \mathcal{H}_N$ are connected by a local operator as $|\phi\rangle = A \otimes B \otimes ..... \otimes N|\psi\rangle$, then the local ranks satisfy $r(\rho_{\kappa}^{\psi}) \geqslant r(\rho_{\kappa}^{\phi})$, $\kappa = A, B, C, ..., N$.

Proof: For each of the parties, the operators takes the form $A \otimes I_{B....N}$ and so on and the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{B....N}$. According to the previous discussion it can be concluded that rank of the density matrix corresponding to each local party cannot change.

Theorem: The pure states of a multipartite system are equivalent under SLOCC if they are related by a local operator.

Proof: If,

$$|\phi\rangle = A \otimes B \otimes .... \otimes N|\psi\rangle \tag{62}$$

then a local protocol exists for the parties to transform $|\psi\rangle$ into $|\phi\rangle$ with a finite probability of success. Indeed, each party needs to perform a local POVM defined by the operators $\sqrt{p_A}A$ and $\sqrt{1_A - p_A A^{\dagger}A}$, where $p_A \leqslant 1$ is a positive weight such that $p_A A^{\dagger}A \leqslant i_A$ and in the similar way for the rest of the parties. Then such local protocol converts the wave functions successfully with probability $p_A p_B ... p_N$. If in addition $A, B, ..., N$ are invertible operators then,

$$|\psi\rangle = A^{-1} \otimes B^{-1} \otimes .... \otimes N^{-1}|\phi\rangle \tag{63}$$

and the conversion can be reserved totally. To prove the converse,i.e., the equivalence of $|\psi\rangle$ and $|\phi\rangle$ under SLOCC implies that this operator can always be chosen to be invertible. Considering Schmidt decomposition for both the wave functions,

$$|\psi\rangle = \sum_{i=1}^{n_{\psi}} \sqrt{\lambda_i^{\psi}} |i\rangle_A |\tau_i\rangle_{B,...,N} \qquad \lambda_i^{\psi} > 0 \tag{64}$$

$$|\phi\rangle = \sum_{i=1}^{n_{\phi}} \sqrt{\lambda_i^{\phi}} (U_A|i\rangle) |\tau_i\rangle_{B,...,N} \qquad \lambda_i^{\phi} > 0 \tag{65}$$

,where the local unitary U_A relates the two local Schmidt bases in Alice's part, $|i\rangle_{i=1}^{n} \in \mathcal{H}_A = \mathbb{C}^n$, $|\tau_i\rangle \in \mathcal{H}_B \otimes .... \otimes \mathcal{H}_N$ and $n_\psi = n_\phi$ because of the previous corollary. Now, the operator A in defined above must be of the form,

$$A = U_A(A_1 + A_2) \tag{66}$$

where,

$$A_1 \equiv \sum_{i=1}^{n_\psi} \sqrt{\frac{\lambda_i^\phi}{\lambda_i^\psi}} |i\rangle\langle i| \tag{67}$$

$$A_2 \equiv \sum_{i=n_\psi+1}^{n} |\mu_i\rangle\langle i| \tag{68}$$

where, $|\mu_i\rangle$ are arbitrary unnormalized vectors and it plays no role in the process of conversion. Therefore A_2 can be redefined as,

$$A_2 \equiv \sum_{i=n_\psi+1}^{n} |i\rangle\langle i| \tag{69}$$

which implies that A is an invertible operator.

Therefore, it is proven that,

$$|\psi\rangle = A^{-1} \otimes B^{-1} \otimes C^{-1} |\phi\rangle \tag{70}$$

Hence states $|\psi\rangle$and $|\phi\rangle$ are equivalent under SLOCC if an ILO relating them exits.

B. Entanglement of Pure states of three qubits

For a system of three parties, each consists of single qubit, SLOCC split the set of pure states into six inequivalent classes which further structure themselves into a three-grade hierarchy when non-invertible local operation local operations are used to relate them.

1. *Non-entangled states:*

If atleast one of the local ranks $r(\rho_A)$,$r(\rho_B)$ or $r(\rho_C)$ is 1, then the pure states of the three qubits factors out as the tensor product of two pure states and hence atleast one qubit is uncorrelated with the others.

a. Class: Product States: In this class rank of each local density matrix is of rank 1. One of the possible form by which an completely un-entangled form can be expressed is given by,

$$|\psi_{A-B-C}\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C = |0\rangle_A |0\rangle_B |0\rangle_C \tag{71}$$

Similarly by taking the various combinations of $|0\rangle$ and $|1\rangle$ states for each party, in a product form as shown above, states for the composite system can be developed where the parties are not correlated.

b. Class: Bipartite entangled states: In this class one of the reduced density matrices has rank 1 and others have rank 2. Foe example, the states in class $A - BC$ possesss entanglement between the systems B and C $[r(\rho_B) = r(\rho_C) = 2]$ and are product with respect to system A $[r(\rho_A) = 1]$. Then,

$$|\psi_{A-BC}\rangle = |0\rangle_A (c_\delta |0\rangle_B |0\rangle_C - s_\delta |1\rangle_B |1\rangle_C) \tag{72}$$

Any state belonging to this class can be obtained from the above equation with certainty by means of LOCC.

2. True three-qubit entangled states:

Convertibility of states under SLOCC and the way entangled states can be expressed minimally as a linear combination of product states are closely connected. GHZ states and W states have different number of terms in their minimal product decompositions. Hence it is obvious that **there is no way to convert one GHZ state to a W state or the converse by means of an ILO $\boldsymbol{A} \otimes \boldsymbol{B} \otimes \boldsymbol{C}$.**

Let us consider the most general pure state that can be obtained reversibly from a $|\mathrm{GHZ}\rangle$ state.

$$A \otimes B \otimes C|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|A0\rangle|B0\rangle|C0\rangle + |A1\rangle|B1\rangle + |C1\rangle) \tag{73}$$

$|A0\rangle$ and $|A1\rangle$ are linearly independent vectors and this is true for other two parties also. Hence the minimal number of terms in a product decomposition for the above state is also 2. hence conclusion can be drawn that **the minimal number of product terms for any given state remains unchanged under SLOCC.**

Therefore, **in three qubit system there are at least two inequivalent classes of genuine tripartite entanglement under SLOCC.**

Whereas $|W\rangle$ cannot be expressed as a linear combination of two product vectors.

Lemma: Let $\sum_{i=1}^{I} |e_i\rangle|f_i\rangle$ be a product decomposition for the state $|\eta\rangle \in \mathcal{H}_E \otimes \mathcal{H}_F$. Then the set of states $\{|e_i\}_{i=1}^{I}$ spans the range of $\rho_E = \mathrm{Tr}_F|\eta\rangle\langle\eta|$.

Proof:

$$rho_E = \sum_{i,j=1}^{I} \langle f_i|f_j\rangle|e_j\rangle\langle e_i| \tag{74}$$

On the other hand, $|\nu\rangle$ is in the range of ρ_E if a state $|\mu\rangle$ exists such that $|\nu\rangle = \rho_E|\mu\rangle$,i.e.,$|\mu\rangle = \sum_{i,j=1}^{I} \langle f_i|f_j\langle e_i|\mu\rangle|e_j\rangle$.

Basically, $r(\rho_A) = 2$ implies that at least two product terms are needed to expand $|\psi\rangle \in \mathbb{D}^2 \otimes \mathbb{D}^2 \otimes \mathbb{D}^2$. It is supposed that a product decomposition with only two terms is possible,

$$|\psi\rangle = |a_1\rangle|b_1\rangle|c_1\rangle + |a_2\rangle|b_2\rangle|c_2\rangle \tag{75}$$

Then also according to the previous lemma $|b_i\rangle|c_i\rangle$ have to span the range of $\rho_{BC}, R(\rho_{BC})$ which is a two-dimensional subspace of $\mathbb{D}^2 \otimes \mathbb{D}^2$. Hence it always contains either only one or only two product states. Only one vector in $R(\rho_{BC})$ and thus the impossibility of the above decomposition is going to be precisely the trait of the states in the W class.

3. GHZ Class

When it is assumed that $R(\rho_{BC})$ contains two product vectors, $|b_1\rangle|c_1\rangle$ and $|b_2\rangle|c_2\rangle$, then the previous decomposition is possible uniquely with $|a_i\rangle = \langle\xi_i|\psi\rangle, i = 1, 2,$ where $|\xi_i\rangle$ are the two vectors supported in $R(\rho_{BC})$ that are biorthonormal to the $|b_i\rangle|c_i\rangle$. In this case one can use LU in order to take the wave function into a useful standard form,

$$|\psi_{\mathrm{GHZ}}\rangle = \sqrt{K}(c_\delta|0\rangle|0\rangle|0\rangle + s_\delta e^{i\varphi}|\varphi_A\rangle|\varphi_B\rangle|\varphi_C\rangle) \tag{76}$$

where,

$$|\varphi_A\rangle = c_\alpha|0\rangle + s_\alpha|1\rangle \tag{77}$$

$$|\varphi_B\rangle = c_\beta|0\rangle + s_\beta|1\rangle \tag{78}$$

$$|\varphi_C\rangle = c_\gamma|0\rangle + s_\gamma|1\rangle \tag{79}$$

where, $K = (1 + 2c_\delta s_\delta c_\alpha c_\beta c_\gamma)^{-1} \in (\frac{1}{2}, \infty)$ is a normalizing factor.

All these states are in the same equivalence class as the $|\mathrm{GHZ}\rangle$ under SLOCC and the ILO is given by,

$$\sqrt{2K}\begin{bmatrix} c_\delta & s_\delta c_\alpha e^{i\varphi} \\ 0 & s_\delta s_\alpha e^{i\varphi} \end{bmatrix} \otimes \begin{bmatrix} 1 & c_\beta \\ 0 & s_\beta \end{bmatrix} \otimes \begin{bmatrix} 1 & c_\gamma \\ 0 & s_\gamma \end{bmatrix} \tag{80}$$

The GHZ state is a remarkable representative of this class. It is maximally entangled in several senses.For instance, it maximally violates Bell-type inequalities, the mutual information of measurement outcomes is maximal, it is maximally stable against(white) noise and one can locally obtain from a GHZ state with unit probability an EPR state shared between any two of the three parties. Another relevant feature is that when any one of the three qubits is traced out, the remaining two are in a separable — and therefore unentangled — state.

4. W Class

When $R(\rho_{BC})$ contains only one product vector. Then,

$$|\psi\rangle = |a_1\rangle|b_1\rangle|c_1\rangle + |a_2\rangle|\phi_{BC}\rangle \tag{81}$$

where $|\phi_{BC}\rangle$ is the vector of $R(\rho_{BC})$ which is orthogonal to $|b_1\rangle|c_1\rangle$, and $|a_1\rangle$ and $|a_2\rangle$ are given by $\langle b_1|\langle c_1|\psi\rangle$ and $\langle\phi_{BC}|\psi\rangle$. Therefore, by means of LU,

$$|\psi\rangle = (\sqrt{c}|1\rangle + \sqrt{d}|0\rangle)|00\rangle + |0\rangle(\sqrt{a}|01\rangle + \sqrt{b}|10\rangle) \tag{82}$$

Where, $|b_1\rangle|c_1\rangle$ are first considered to be $|0\rangle|0\rangle$ and since $|\phi_{BC}\rangle$ is chosen to be orthogonal to $|b_1\rangle|c_1\rangle$, it will take the form $x|01\rangle + y|10\rangle + z|11\rangle$. By requiring that a linear combination of these two vectors has no second product vector, one can obtain $z = 0$. Hence three product terms always do the job for instance, $(\sqrt{c}|1\rangle + \sqrt{d}|0\rangle)|00\rangle$, $\sqrt{a}|0\rangle|01\rangle$ and $\sqrt{b}|0\rangle|10\rangle$ once the original state into the standard, unique form,

$$|\psi_{\mathrm{W}}\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle \tag{83}$$

where, $a, b, c > 0$ and $d \equiv 1 - (a + b + c) \geqslant 0$

The parties can locally obtain the state mentioned above from the state $|\mathrm{W}\rangle$ mentioned earlier in equation (52) by application of the form,

$$\begin{bmatrix} \sqrt{a} & \sqrt{d} \\ 0 & \sqrt{c} \end{bmatrix} \otimes \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \frac{\sqrt{3b}}{\sqrt{a}} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{84}$$

The states within the GHZ-class and the W-class depend,respectively,on 5 and 3 parameters that cannot be changed by means of LU. As a generic state of three qubits depends, up to LU, on

5 parameters. This means that states typically belong to the GHZ-class or equivalently that a generic pure state of three qubits can be locally transformed into a GHZ with finite probability of success.The W-class is of zero measure compared to the GHZ-class. This does not mean, however, that it is irrelevant. In a similar way as separable mixed states are not of zero measure with respect to entangled states, even though product states are,it is in principle conceivable that mixed states having only W-class entanglement are also not of zero measure in the set of mixed states.

IV. LOCAL INACCESSIBILITY OF STATES IN GENERAL MULTIPARTITE SYSTEM

In this section we will consider a multipartite system to analyze the entanglement. It will be shown that the set of entangled states is a inaccessible for a loacal operator and two pure states are then typically not connected by means of LOCC. And as a result, the parties are usually unable to convert locally.[7, 8]

A. Stochastic local operation assisted with classical communication(SLOCC) for a multipartite system

The parties sharing a composite system in an entangled state are typically allowed to communicate through a classical channel- to only act locally on their subsystems. This communication is even restricted to local operations assisted with classical communication as shown for bipartite system in previous section and hence they can modify their entanglement properties and can convert themselves to another state.Entanglement is never increasing on average under LOCC ,i.e., LOCC will either preserve or destroy entanglement. The LOCC equivalence classes are actually equivalence class under local unitary operation(LU) or they are classes of states with same co-efficients in Schmidt decomposition.[8, 9]

However, for the studies of multipartite systems ($n > 2$), it is difficult to identify the equivalence classes under LOCC. Those cases demand the conversion of states through Stochastic local operations and classical communication (SLOCC), also known as local filter operations. *“An advantage of the SLOCC classification lies in the fact that we can grasp essence, by successfully characterizing the equivalence class of entanglement and monotone(irreversible)properties of entanglement under LOCC.”*

Let us consider a single copy of a multipartite state $|\psi\rangle$, each party consists of single qubit, on the l-partite Hilbert space,

$$\mathcal{H} = \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes .... \otimes \mathbb{C}^2}_{l} \tag{85}$$

The equivalence classes are defined as rays and each of them are the projective Hilbert space determined by $\mathcal{H}$.These sets are often denoted by,

$$\mathcal{P}(\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes .... \otimes \mathbb{C}^2}_{l}) \tag{86}$$

And the state can be represented in the form,

$$|\psi\rangle = \sum_{0,1} \psi_{i_1,i_2,..,i_l} |i\rangle_1 \otimes |i\rangle_2 \otimes ... \otimes |i\rangle_l \tag{87}$$

where, a set of $|i\rangle_1 \otimes |i\rangle_2 \otimes ... \otimes |i\rangle_l$ constitutes the standerd basis. Our aim is to identify two states that inverts from one to another under SLOCC with non-vanishing probability. Let two states $|\psi\rangle$ and $|\psi^{'}\rangle$ are under SLOCC equivalent class and are connected by invertible local operations,

$$|\psi^{'}\rangle = M_1 \otimes ... \otimes M_l |\psi\rangle \tag{88}$$

and

$$|\psi\rangle = M_1^{-1} \otimes ... \otimes M_l^{-1} |\psi^{'}\rangle \tag{89}$$

M_i is any inverting ILO only on the i^{th} party. In other words M_i is an element of the general linear group $GL_2(\mathbb{C})$. The overall normalization constant and phase is unimportant here and hence the determinant of the operation on the i^{th} party can be taken as unity,i.e., $M_i \in SL_2(\mathbb{C})$. SLOCC consists of successive rounds of measurements and communication of the outcomes which, as a whole, is given by the local operation of equation (44). Moreover, SLOCC are a set of trace decreasing, completely positive maps by the postselection of a sequence of successful outcomes. In other words,*the SLOCC classification of multipartite entanglement means clarifying the moduli space under the SLOCC equivalence,i.e., the structure of orbits generated by a direct product of special linear groups $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times ... \times SL_2(\mathbb{C})$.*

In case of non-invertible local operators, the equivalence classes under SLOCC can be converted only in one direction with finite probability.

In case of the bipartite system, the SLOCC classification means classifying the whole states by the Schmidt local rank. But this rank is the SLOCC invariant under an invertible local operation as the wave function transforms as,

$$\Psi^{'} = M_1 \Psi M_2^T \tag{90}$$

where the non-inverting ILO $M_1 \otimes M_2 \in SL_2 \times SL_2$.

B. Equivalence classes of a multipartite system under SLOCC

Discussing about the equivalence classes of a multipartite system under SLOCC, concept of space and subspace is required. As it is known that each equivalence class is a subspace of the entire Hilbert space of interest.

Theorem : *An equivalence class is defined as* $U = \frac{V}{S}$*, where,* $V \subseteq U$ *and* S *is called the stabilizer. And also,*

$$\mathrm{dim}U = \mathrm{dim}V - \mathrm{dim}S \tag{91}$$

For the Hilbert space $\mathcal{H}$ of our system of l parties each possesing single qubit, any non-zero element of it can be viewed as states of the quantum system. Now an equivalence class of those quantum states must be formed by the mean of the difference of some non-zero complex factor.Hence the SLOCC equivalence classes can be related to the Hilbert space by the above relation where the ILOs will perform the job of the stabilizer.A generic vector of $\mathcal{H}$ depends on $2(2^l - 1)$ real parameters and a general one-party invertible operator depends on six real parameters.Therefore, the set of equivalence classess under SLOCC,

$$\frac{\mathcal{H}}{SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times ... \times SL_2(\mathbb{C})} \tag{92}$$

depends at least on $2(2^l - 1) - 6l$ parameters.[7] The dimension of the Hilbert space expands exponentially as the number of parties increases, while the dimension of the ILOs increases linearly. If the value $2(2^l - 1) - 6l$ is positive then there are infinitely many orbits in the generic class, otherwise there are just finitely many orbits.

Equivalence classes can be found under local unitary operations as the states are also equivalent by means of their non-local properties.There is a similar kind of concept in group theory.For a group G, the transformation acts transitively on an orbit O which is given by,

$$O = G/S \tag{93}$$

where S is the stabilizer of the orbit and is a subgroup of G.The stabilizer S consists of all the permutations of G that can produce all the elements of the orbit from a single element belonging to O. This requirement is equivalent to invariance under the choice of local Hilbert space basis.Hence an equivalence class has characteristics similar to that of the orbit.[3, 8]

The total number of parameters describing the Hilbert space $\mathcal{H}$ minus the number of parameters describing a generic vector on the orbit gives the number of parameters describing the location of the orbit in the space of orbits, i.e., the number of parameters describing the non-local properties of the states.It is called the dimension of the orbit.The dimension of the orbit can be determined from the knowledge of the dimension of group G and that of stabilizer S,

$$\dim O = \dim G - \dim S \tag{94}$$

Not all orbits have the same dimension.And the number of parameters describing a generic orbit is nothing but the number of parameters characterizing the stabilizer.

For the Hilbert space $\mathcal{H}$ of our system of l parties each possessing single qubit, any non-zero element of the space can be viewed as a state of the quantum system. Now an equivalence class of those quantum states must be formed by the mean of the difference of some non-zero complex factor.Hence the SLOCC equivalence classes can be related to the Hilbert space by the above relation where the ILOs will perform the job of the stabilizer.

Any pure state of the entangled system is therefore described by $2(2^l - 1)$ real parameters. And a general one-party invertible operator depends on six real parameters.Thus there cannot be more than $6l$ real parameters describing local properties of the states of any equivalence class and therefore, the set of equivalence classes under SLOCC,

$$\frac{\mathcal{H}}{SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times ... \times SL_2(\mathbb{C})}$$

$$= \frac{\mathbb{C}^{2^l}}{SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times ... \times SL_2(\mathbb{C})} \tag{95}$$

depends at least on $2(2^l - 1) - 6l$ non-local parameters.

For $l \leq 3$ the lower bound of non-local is negative. But for $l > 3$ this lower bound becomes positive. The positive number of parameters implies that there are infinitely many classes which are labeled by at least that many number of continuous parameters. Hence, for the positive value of the term '$2(2^l - 1) - 6l$' there are infinitely many generic states in a single equivalence class, otherwise there are just finitely many states in each of them.

The dimension of the Hilbert space expands exponentially as the number of parties increases, while the dimension of the ILOs increases linearly.Then it can be immediately seen that for large l almost all parameters have non-local significance.[10]

C. Generalization

If the multipartite system under consideration consists of parties possessing different number of qubits then the Hilbert space for the l-partite system is given by,

$$\mathcal{H} = \underbrace{\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes .... \otimes \mathbb{C}^{n_l}}_{l} \tag{96}$$

And then the set of equivalence classes under SLOCC,

$$\begin{aligned} & \frac{\mathcal{H}}{SL_{n_1}(\mathbb{C}) \times SL_{n_2}(\mathbb{C}) \times ... \times SL_{n_l}(\mathbb{C})} \\ = & \frac{\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} .... \otimes \mathbb{C}^{n_l}}{SL_{n_1}(\mathbb{C}) \times SL_{n_2}(\mathbb{C}) \times ... \times SL_{n_l}(\mathbb{C})} \end{aligned} \tag{97}$$

depends on the number of non-local parameters to describe its position given by,

$$\begin{aligned} & \dim \ \frac{\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} .... \otimes \mathbb{C}^{n_l}}{SL_{n_1}(\mathbb{C}) \times SL_{n_2}(\mathbb{C}) \times ... \times SL_{n_l}(\mathbb{C})} \\ = \ & 2(n_1 n_2 ... n_l) - 2\sum_{i=1}^{l}(n_i^2 - 1) \end{aligned} \tag{98}$$

Hence from the above equation, one can find the lower bound of non-local parameters for any multipartite system.[10] In this way the entire Hilbert space, containing all the possible states for a given system is divided into several parts according to the features of the system. Each part contains an equivalence class under SLOCC and accessing from one equivalence class to another is not possible by means of ILO.

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