Uniform Bose Gas Under Three Dimensional Harmonic Confinement

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This is a short note on the mathematical formulations revealing the basic concept of Bose Einstein condensation of dilute alkali metal vapour in a magnetic confinement. More precisely, it is a framework to understand the trapping of ultra-cold neutral atoms in a simple harmonic trap .The concept of laser cooling (to the Doppler limit) of the atoms utilizing "radiation pressure force" and the concept of "Zeeman effect" that has been exploited in construction of magnetic trap, discussed here in bellow. Also, the detailed calculation for finding total number of atoms confined in the trap and the total energy of the system has been done in this review. And finally the effect of interactions between the atoms has been discussed in a qualitative manner.

I. INTRODUCTION

Bose Einstein condensation is a new state of matter formed by bosons when they are cooled very near to the absolute zero temperature. The concept was first introduced by S.N.Bose in 1924 in his paper where he dealt with photons for which the total number of particle is not a fixed quantity. Then it was more elaborately discussed by Einstein in 1925 where he extended the concept to massive particles whose total number was fixed[1,2]. BEC was first observed experimentally in 1995 on dilute vapours of Rubidium and Sodium[3]. The atoms were confined in a trap which was nothing but a simple harmonic oscillator potential well and were cooled down to a very low temperature of the order of fractions of micro-kelvins. This is a path for understanding quantum mechanics and observe quantum mechanical phenomenon experimentally. Also atomic,condense matter,nuclear physics etc are very much involved in understanding the concept of Bose Einstein condensation.Hence it has become one of the most active research areas in contemporary physics.

Definition of Bose Einstein Condensation:

The phenomenon of dilute gas that a macroscopically large number of particles accumulating

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in a single state (ground state), is referred as Bose Einstein condensation. This phenomenon is closely similar to the process of vapour condensing to the liquid in certain sense. However, they are completely different phenomenon. And the most important difference is that BEC is purely of quantum origin.

Why Alkali metals? The Alkali metals [**Li,Na,K,Rb,Cs,Fr**] are favourable for BEC experiments.There are two important reasons behind this. Firstly, alkali metals have one electron, i.e., they have spin $\frac{1}{2}$ particle in their outermost orbit which is a very suitable condition for Zeeman effect and hence very helpful in trapping atoms in a magnetic trap.Secondly, the transition energy of alkali atoms are such that the near-resonant laser is easily available. That is why they perfectly fit for the experiments related to BEC and give best results.

II. LASER COOLING

Laser cooling is a much practised technique to use near-resonant radiation pressure forces to cool atoms to an extremely low temperature of the order 1 to 100 μK. For observing BEC experimentally laser Cooling plays a very important role.It is, in short, a method of control atoms by light which is basically based on quasi-resonant exchanges of energy, linear momentum and angular momentum between atoms and photons.

Basic Concept of laser Cooling: Light is an electromagnetic wave. Hence, when it incident on atoms, it exerts some forces on it. These forces[4] can be classified to two categories-

(i)Resonant radiation pressure or scattering force.

(ii)Non-resonant stimulated scattering force or dipole force.

The scattering force arises due to the momentum transfer from photons to atoms. But it does not play important role in laser cooling. The stimulated force arises from induced dipole moment of atom and it is related to the intensity gradient of incident E.M.wave. This force is heavily utilised in the process of laser cooling. More precisely, stimulated force is caused by photon absorption followed by stimulated emission and spontaneous force arises when photon absorption is followed by spontaneous emission.

To explain the concept of laser cooling, also referred as Doppler cooling, let us consider an atom cloud in which all atoms are constrained to move along x axis. Now let two near-resonant laser beams are incident from either side and they both have the same frequency which is very close the the atomic resonance frequency, slightly less than that. For an atom sitting at rest on the x-axis will absorb or scatter the photons coming from either side equally. However, if the atoms

have a finite velocity along +ve or -ve x-axis, then in their frame of references will be of different frequencies of opposite sign. This phenomena is known as Doppler Shift[5]. When velocity of atom $\overrightarrow{v}$ and wave vector of the incident quasi-monochromatic laser beam $\overrightarrow{k_L}$ are in opposite direction in the frame of reference where centre of mass of the atom is at rest, the light will appear as Doppler-shifted with a frequency:

$$\omega_L - \overrightarrow{k_L}.\overrightarrow{v} = \omega_L(1 + \frac{v}{c}) \tag{1}$$

When $\frac{\omega_L v}{c} \sim \delta$, the incident beam is said to be strongly resonant with the atomic transition. The beam with opposite direction will blue shifted and hence will get closer to the atomic resonance frequency and atoms will scatter more photons of the opposite direction beam. Whereas, the beam in the same direction will be red shifted and hence the frequency will become even more less than the resonant frequency of atoms. The net effect of this differential scattering rate is that the atoms experience a viscous drag of frictional force. The extension of this idea in three dimension is referred to the optical molasses which indicates the viscous effect on the atomic motion by incident of three sets of counter-propagating beams in three orthogonal directions. At steady state inside the molasses, the mean velocity of the atom cloud is zero in every direction. But still then the atoms are absorbing and emitting photons continuously. These spontaneously emitted photons provide a random recoil kick to the atoms the every time they emit photon with some finite velocity. This causes the atoms to slow down. By this process the atoms can be cooled down to an extremely low temperature. The lowest temperature is reached when the detuning is equal to half of the natural linewidth Γ of cooling transition which is called the Doppler cooling limit T_D given by,

$$T_D = \frac{\hbar\Gamma}{2\kappa_B} \tag{2}$$

III. MAGNETIC TRAPS [6]

Requirement of magnetic traps: To study the quantum mechanical properties of a system we need to solve the Schrödinger equation which will be exactly solvable when the centre of mass of the system is stationary. Thus, in order to have a precise comparison between theoretical calculations and experimental results, it is necessary to perform the experiment in such a frame of reference so that the centre of mass of the system is stationary with respect to that frame of reference. That condition can be achieved when atoms are brought to rest within a narrow region

of space before doing any measurement. Confining the atoms under the influence of some potential can provide this condition to be fulfilled. This technique is called tapping of atoms in a magnetic trap.

Concept of magnetic trap: Magnetic trapping of neutral atoms utilizes the concept of Zeeman effect. The energy of atomic states depends on the magnetic field applied on it. Therefore, when an atom experiences an inhomogeneous external magnetic field, the energy levels get shifted under its influence. If the applied magnetic moment is positive then the atoms tend to drive towards the higher field region and are referred as High-field seekers. On the other hand, if the magnetic moment is negative then the atoms will rush towards the lower field region, referred as Low-field seekers.

In three dimension the magnetic trap is constructed by applying inhomogeneous magnetic field which is zero at the origin and increases along all three orthogonal axes in both positive and negative direction. This is called Spherical quadrupole magnetic field. This is produced by passing current through a pair of coils in opposite directions(Anti-Helmholtz arrangement.

In case of interacting bosons, they give rise to unexpected features in presence of Harmonic field. The gas becomes inhomogeneous in presence of external field. Also, for three dimensional harmonic trap, it is a very useful feature that by changing the strength of field direction-wise as desired, the density of gas in different regions can be controlled. This is utilized in experiments for observing the properties of bose gas.

IV. CALCULATION OF T_c FOR BOSONS TRAPPED IN A HARMONIC TRAP: [6,7]

Let us consider N boson are confined in a Harmonic trap[1]. Hence the confining potential experienced by them can be written as,

$$V_e = \frac{m}{2}(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) \tag{3}$$

where m is mass of individual particle. And the Hamiltonian corresponding to it is given by,

$$\widehat{H} = \frac{\widehat{P^2}}{2m} + \frac{m}{2}\sum_i \omega_i^2 \widehat{x}^2 \tag{4}$$

[1] Here all subsequent calculations will follow non-relativistic Quantum Mechanics for identical indistinguishable particles

By symmetry the N body Hamiltonian is the sum of the single particle Hamiltonians. The energy eigenvalue of a 3-dimensional one body Schrödinger equation is,

$$\varepsilon_{n_x n_y n_z} = (n_x + \frac{1}{2})\hbar\omega_x + (n_y + \frac{1}{2})\hbar\omega_y + (n_z + \frac{1}{2})\hbar\omega_z \tag{5}$$

where,$n_x = n_y = n_z = 0, 1, 2, 3...$ For Bose Einstein condensation every particle should be in the ground state.If $\phi(1, 2, 3..., N)$ be the wave function for N non-interacting bosons confined in the given potential then we need to find the ground state of ϕ only,i.e.,ϕ_0. Also the N body wave function can be written in terms of single body confined in potential [1],given by $\phi_0(1, 2, .., N) = \Pi_i \phi_0(i)$ and the energy eigenvalue corresponding to the wave function for each particle will be$\frac{1}{2}\hbar(\omega_x + \omega_y + \omega_z)$. The Schrödinger equation for one particle is given by,

$$-\frac{\hbar^2}{2m}\nabla^2\phi(x,y,z)+\frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)\phi(x,y,z) = [(n_x+\frac{1}{2})\omega_x+(n_y+\frac{1}{2})\omega_y+(n_z+\frac{1}{2})\omega_z]\hbar\phi(x,y,z) \tag{6}$$

And for a particle in ground state equation (6) reduces to,

$$-\frac{\hbar^2}{2m}\nabla^2\phi_0(x,y,z) + \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)\phi_0(x,y,z) = \frac{1}{2}\hbar(\omega_x + \omega_y + \omega_z)\phi_0(x,y,z) \tag{7}$$

The solution of equation(6)in one dimension is,

$$\phi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n x_{0!}}}(\sqrt{\frac{m\omega}{\hbar}})^{\frac{1}{4}} e^{-\frac{m}{2\hbar}\omega^2x^2} H_n(\frac{x}{x_0}) \tag{8}$$

where,$x_0 = (\frac{m\omega}{\hbar})^{\frac{1}{4}}$ and in ground state it reduces to,

$$\phi_0(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} e^{-\frac{m}{2\hbar}\omega^2x^2} \tag{9}$$

In three dimension, the solution will be,

$$\phi_0(\mathbf{r}) = (\frac{m\omega'}{\pi\hbar})^{\frac{3}{4}} e^{-\frac{m}{2\hbar}(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)} \tag{10}$$

where,$\omega' = (\omega_x\omega_y\omega_z)^{\frac{1}{3}}$. ω' is called the geometric average of oscillator frequencies. The density distribution will be $n(\mathbf{r}) = N|\phi_0(\mathbf{r})|^2$ and its value increases with N and the size of the cloud is fixed by the harmonic oscillator length a instead of N. a is given by,

$$a = \sqrt{\frac{\hbar}{m\omega}} \tag{11}$$

It is corresponding to the average width of the Gaussian in equation(10).

Trapped bosons in finite temperature:
(Thermodynamic Limit)

N number of particles, at temperature T, are in Grand Canonical Ensemble, Therefore,

$$N = \sum_{n_x,n_y,n_z} [e^{\beta(\varepsilon_{n_x n_y n_z} - \mu)} - 1]^{-1} \tag{12}$$

where μ is the Chemical potential and $\beta = \dfrac{1}{\kappa_B T}$
And the total energy of the system,

$$E = \sum_{n_x,n_y,n_z} \varepsilon_{n_x n_y n_z} [e^{\beta(\varepsilon_{n_x n_y n_z} - \mu)} - 1]^{-1} \tag{13}$$

For bosons, it is convenient to separate out the ε_{000} state from total number of particles N and it is denoted by N_0. This number achieves the order of N when μ becomes numerically equal to the value of lowest energy state,i.e.,

$$\mu = \frac{1}{2}\hbar\overline{\omega} \to \mu_c \tag{14}$$

where, $\overline{\omega} = (\omega_x + \omega_y + \omega_z)$
Substituting the value in equation (12), we can write,

$$N - N_0 = \sum_{n_x,n_y,n_z} [e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1]^{-1} \tag{15}$$

when $N \to \infty$, the sum can be replaced by an integral in order to evaluate its value explicitly.

$$N - N_0 = \int_0^\infty \frac{dn_x dn_y dn_z}{e^{\beta\hbar(\omega_x n_x + \omega_y n_y + \omega_z n_z)} - 1} \tag{16}$$

Let, $n'_x = \omega_x n_x \beta\hbar, n'_y = \omega_y n_y \beta\hbar, n'_z = \omega_z n_z \beta\hbar$ Then substituting the values in equation (16)

$$\begin{aligned} N - N_0 &= (\frac{\kappa_B T}{\hbar\omega'})^3 \int_0^\infty \frac{dn'_x dn'_y dn'_z}{e^{(n'_x + n'_y + n'_z)} - 1} \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \int_0^\infty \frac{e^{-(n'_x + n'_y + n'_z)} dn'_x dn'_y dn'_z}{1 - e^{-(n'_x + n'_y + n'_z)}} \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \int_0^\infty \frac{e^{-(n'_y + n'_z)} e^{-n'_x} dn'_x dn'_y dn'_z}{1 - e^{-(n'_y + n'_z)} e^{-n'_x}} \end{aligned} \tag{17}$$

Now,

$$\begin{aligned}\frac{e^{-x}}{1-ze^{-x}} &= e^{-x}[1+ze^{-x}+z^2e^{-x^2}+....] \\ &= \sum_{p=1}^{\infty} z^{p-1}e^{-px} \end{aligned} \tag{18}$$

Therefore,euation (17) can be written as,

$$\begin{aligned}N-N_0 &= (\frac{\kappa_B T}{\hbar\omega'})^3 \sum_{p=1}^{\infty}\int_0^{\infty}(e^{-(n'_y+n'_z)})^p e^{-pn'_x}dn'_x dn'_y dn'_z \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \sum_{p=1}^{\infty}\int_0^{\infty}\int_0^{\infty}(e^{-(n'_y+n'_z)})^p dn'_y dn'_z \int_0^{\infty} e^{-pn'_x}dn'_x \end{aligned} \tag{19}$$

Using the formula of Gamma function and putting $pn'_x = u_x$ we get,

$$\begin{aligned}N-N_0 &= (\frac{\kappa_B T}{\hbar\omega'})^3 \sum_{p=1}^{\infty}\frac{1}{p}\int_0^{\infty}\int_0^{\infty}(e^{-(n'_y+n'_z)})^p dn'_y dn'_z \int_0^{\infty} e^{-u_x}du_x \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \sum_{p=1}^{\infty}\frac{1}{p}\int_0^{\infty}\int_0^{\infty} e^{-p(n'_y+n'_z)} dn'_y dn'_z \Gamma(1) \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \sum_{p=1}^{\infty}\frac{1}{p}\int_0^{\infty}\int_0^{\infty} e^{-p(n'_y+n'_z)} dn'_y dn'_z \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \sum_{p=1}^{\infty}\frac{1}{p^3} \\ &= (\frac{\kappa_B T}{\hbar\omega'})^3 \zeta(3) \end{aligned} \tag{20}$$

where $\zeta(n)$ is Riemann ζ function.

To obtain the Transition temperature T_c of Bose Einstein condensation we must impose the condition $N_0 \to 0$ at transition point. Then,

$$N = (\frac{\kappa_B T_c}{\hbar\omega'})^3 \zeta(3) \tag{21}$$

T_c is the transition temperature. And therefore,

$$\kappa_B T_c = \hbar\omega'(\frac{\zeta(3)}{N})\frac{1}{3} \tag{22}$$

$$= 0.94\hbar\omega' N\frac{1}{3} \tag{23}$$

And hence,

$$N = (\frac{\kappa_B T_c}{\hbar\omega'})^3 \zeta(3) \tag{24}$$

For temperatures higher than T_c, the chemical potential is less than μ_c and becomes N dependent while the population at ground state is of the order 1 instead of N. The thermodynamic limit of such a system can be defined as $N \to \infty$ and $\omega' \to 0$ while $N\omega'$ is constant. Hence the transition temperature is well defined at thermodynamic limit. Substituting the value of T_c from equation (22) to equation (20) we get,

$$\begin{aligned} N - N_0 &= (\frac{\kappa_B T}{\hbar\omega'})^3 \zeta(3) \\ \Rightarrow 1 - \frac{N_0}{N} &= \frac{(\frac{\kappa_B T}{\hbar\omega'})^3 \zeta(3)}{(\frac{\kappa_B T_c}{\hbar\omega'})^3 \zeta(3)} \end{aligned} \tag{25}$$

$$\Rightarrow \frac{N_0}{N} = 1 - \frac{T^3}{{T_c}^3} \tag{26}$$

The $\frac{N_0}{N}$ vs. $\frac{T}{T_c}$ plot is given bellow.

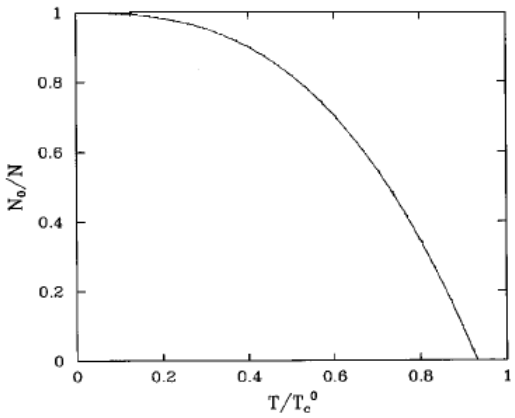

FIG. 1: $\frac{N_0}{N}$ vs. $\frac{T}{T_c}$ plot

This result can also be calculated from the integral,

$$N - N_0 = \int_0^\infty \frac{g(\varepsilon)d\varepsilon}{e^{\beta\varepsilon} - 1} \tag{27}$$

where, $g(\varepsilon)$ is called the density of states. It can be calculated from spectrum (5) in the following way, Defining $\varepsilon_i = \hbar\omega_i n_i$ and then $G(\varepsilon)$, number of states can be calculated from the integral,

$$\begin{aligned} G(\varepsilon) &= \frac{1}{\hbar^3\omega_x\omega_y\omega_z} \int_0^{\varepsilon_{n_x n_y n_z}} d\varepsilon_{n_x} \int_0^{\varepsilon_{n_x n_y n_z} - \varepsilon_{n_x}} d\varepsilon_{n_y} \int_0^{\varepsilon_{n_x n_y n_z} - \varepsilon_{n_x} - \varepsilon_{n_y}} \varepsilon_{n_z} \\ &= \frac{\varepsilon^3_{n_x n_y n_z}}{6\hbar^3\omega_x\omega_y\omega_z} \end{aligned} \tag{28}$$

Therefore, $g(\varepsilon)$ will be,

$$\begin{aligned} g(\varepsilon) &= \frac{dG(\varepsilon)}{d\varepsilon_{n_x n_y n_z}} \\ &= \frac{\varepsilon^2_{n_x n_y n_z}}{2\hbar^3 \omega_x \omega_y \omega_z} \\ &= \frac{\varepsilon^2_{n_x n_y n_z}}{2(\hbar\omega')^3} \end{aligned} \tag{29}$$

Using the value of $g(\varepsilon)$ one can easily obtain the total energy E from the integral,

$$\begin{aligned} E &= \int_0^\infty \frac{\varepsilon g(\varepsilon) d\varepsilon}{e^{\beta\varepsilon} - 1} \\ &= \frac{1}{2(\hbar\omega')^3} \int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta\varepsilon} - 1} \\ &= \frac{1}{2(\hbar\omega')^3} \int_0^\infty \frac{\varepsilon^3 e^{-\beta\varepsilon} d\varepsilon}{1 - e^{-\beta\varepsilon}} \end{aligned} \tag{30}$$

Using the formula of Riemann ζ function from equation (18),

$$\begin{aligned} \frac{\varepsilon^3 e^{-\beta\varepsilon}}{1 - e^{-\beta\varepsilon}} &= \varepsilon^3 e^{-\beta\varepsilon}[1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + ...] \\ &= \varepsilon^3 [e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + e^{-3\beta\varepsilon} + ...] \\ &= \varepsilon^3 \sum_{p=1}^{\infty} e^{-p\beta\varepsilon} \end{aligned} \tag{31}$$

Substituting the value in equation (31)

$$E = \frac{1}{2(\hbar\omega')^3} \sum_{p=1}^{\infty} \int_0^\infty \varepsilon^3 e^{-p\beta\varepsilon} d\varepsilon \tag{32}$$

Let, $p\beta\varepsilon = \xi$ Therefore,

$$\begin{aligned} \varepsilon &= \frac{\xi}{p\beta} \\ \Rightarrow d\varepsilon &= \frac{d\xi}{p\beta} \end{aligned} \tag{33}$$

Substituting the values in equation(33)

$$\begin{aligned} E &= \frac{1}{2(\hbar\omega')^3} \sum_{p=1}^{\infty} \frac{1}{(p\beta)^4} \int_0^\infty \xi^3 e^{-\xi} d\xi \\ &= \frac{1}{2\beta^4(\hbar\omega')^3} \sum_{p=1}^{\infty} \frac{1}{(p)^4} \Gamma(4) \\ &= \frac{3}{\beta^4(\hbar\omega')^3} \sum_{p=1}^{\infty} \frac{1}{(p)^4} \\ &= \frac{3}{\beta^4(\hbar\omega')^3} \zeta(4) \end{aligned} \tag{34}$$

And from equation (24) and equation (35) we get,

$$\frac{E}{N} = \frac{\frac{3}{\beta^4(\hbar\omega')^3}\zeta(4)}{(\frac{\kappa_B T_c}{\hbar\omega'})^3\zeta(3)}$$

$$\Rightarrow \frac{E}{N} = \frac{3\kappa_B T^4}{{T_c}^3}\frac{\zeta(4)}{\zeta(3)}$$

$$\Rightarrow \frac{E}{N\kappa_B T_c} = \frac{3\zeta(4)}{\zeta(3)}(\frac{T}{T_c})^4 \tag{35}$$

Hence, starting from the total energy of the system, all other thermodynamic quantities like entropy, specific heat can be calculated. The non-interacting gas in Harmonic potential allows us to calculate the transition temperature T_c and the value that has been found experimentally is very near to the calculated value.

V. THE EFFECT OF INTERACTION

When a gas is non-interacting, the width of condensate depends on a'. Also the Hamiltonian can be written as the summation of Hamiltonian of individual atoms.But things become very different when the particles in harmonic trap are considered to be interacting. The Hamiltonian that defined earlier becomes invalid. Then an extra potential term, referred as two-body interaction potential $V(\mathbf{r}-\mathbf{r}')$, must be added to the previous $\widehat{H}$. The Hamiltonian then takes the form for a many-body system,

$$\widehat{H} = \int d\mathbf{r}\Psi^\dagger(\mathbf{r})[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r})]\Psi(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}d\mathbf{r}'\Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\Psi(\mathbf{r})\Psi(\mathbf{r}') \tag{36}$$

When the gas under consideration is very dilute then the inter atomic separation is very large so that we can neglect the three particle interaction part and can assume that only two body interaction is significant,i.e., only short distance $(\mathbf{r}-\mathbf{r}')$ interactions are involved. Therefore, a proper interaction term where only binary collisions at very low energy are involved , can be written as,

$$V(\mathbf{r}-\mathbf{r}') = g\delta^3(\mathbf{r}-\mathbf{r}') \tag{37}$$

where,

$$g = \frac{4\pi\hbar^2 a}{m} \tag{38}$$

g is the strength of the potential and it is connected to "a" which is defined as s-wave scattering length. From equation (37) it is clear that interaction potential is significant only when the particles are in contact or extremely close to each other.

Again if we consider the bose gas to be very dilute then the strength of the potential $gn^3 << 1$,where n is the number of particles per unit volume.This is valid for all condensate systems. The interaction energy, estimated to be $\frac{N^2 g}{a'^3}$ and kinetic energy to be $N\hbar\omega$.

The ground state energy of the system and other thermodynamic properties can be calculated from the Hamiltonian in equation (36). But the calculation is impracticable when the number of atoms N is considerably large. This problem is overcome by taking the approach of Mean Field Theory. The basic idea of Mean field theory is to separate out the condensate contribution to the bosonic field operator. If ψ is the wave function representing the behaviour of the trapped particles then it is split out in two parts likewise,

$$\psi = \phi + \psi' \tag{39}$$

Then the equation of motion,

$$i\hbar\frac{\partial\phi}{\partial t} = [\psi, H] \tag{40}$$

to the lowest order yields the Gross-Pitaeski equation[8,9],

$$i\hbar\frac{\partial\phi}{\partial t} = [-\frac{\hbar^2}{2m}\nabla^2 + V + g|\phi|^2]\phi \tag{41}$$

The solution of the above equation is given by,

$$\phi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-\frac{i\mu t}{\hbar}} \tag{42}$$

The constrain that has been applied is,

$$\int |\phi(r)|^2 d^3r = N \tag{43}$$

Form the above equation and putting $V(r) = \frac{1}{2}m\omega'^2 r^2$ rescaling $\phi(r)$ and $\mathbf{r}$, we find,

$$[-\nabla^2 + r^2 + 8\pi\frac{Na}{a'}\phi^2]\phi = 2\mu\phi \tag{44}$$

The effect of interaction[7] that has been discussed above can be revealed more easily by observing the deviation of the Gaussian picture as defined in equation(10)for non-interacting model. If the forces of interaction are repulsive then "a" is positive. Then the atoms always try to make the

inter-atomic separation to be larger and hence they are pushed outwards with a flat density in the central region.Therefore the ground state Gaussian wave function gradually spreads out. In this case there is no limitation of total number of atom and the wave function can spread out upto the length of the trap.

To the contrary, when the forces of interaction are attractive, the scattering length "a" becomes negative. Then the atoms pull each other towards themselves and hence the distribution starts to squeeze. Now if one starts increasing the number of particles, the interaction increases simultaneously and the ground state wave function becomes thinner and at a certain limitation it becomes a delta function. This implies that all atoms are confined in a single state. At this limit the zero point energy is no longer capable of balancing the interaction energy. Hence a break down occurs for further increment of N. This limit is called the critical limit, denoted by N_c. It is the largest possible number of atoms that can be confined within the trap when the interaction is negative. The change in the nature of wave function with "a" has been shown bellow.

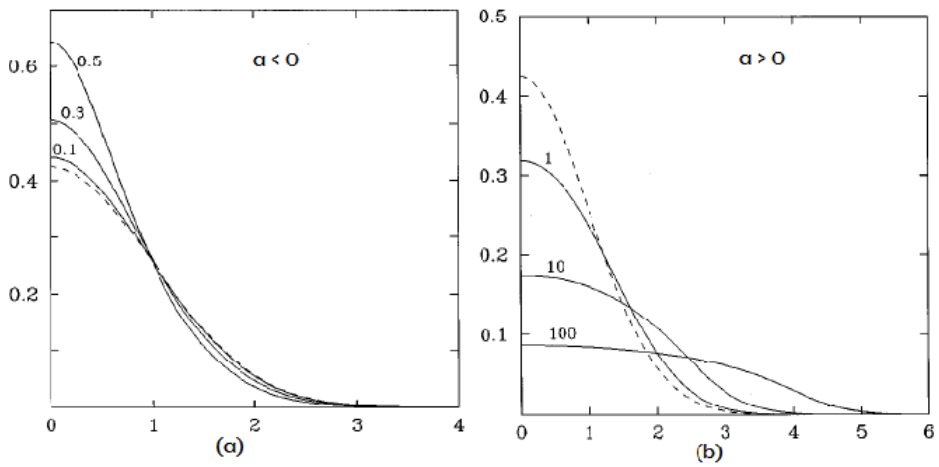

FIG. 2: (a) Condensate of wave function at T = 0 due to attractive interaction between the atoms; (b) Wave function due to repulsive interaction between the atoms[7]

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