Section 1: The model

To model $m=0$ instability growth rates in ideal MHD cylindrical Z-pinches, we have used a toy model

$$\dot{x}(t) = \lambda x(t) + m x(t - T_d)$$

$x(t) = \frac{\xi}{a}$ where a is your z pinch radius, ξ is the magnitude of your perturbation.

λ is the perturbation growth rate.
$\lambda = c_1 V_A a^{-1}$ where c_1 is a dimensionless constant and V_A is our alven speed of sound in the plasma.

Our control law m has units of $[s^{-1}]$.

Z pinch equilibrium is set by the radial force balance between inward magnetic Lorentz force $(j \times B)$ and outward plasma pressure gradient ∇p. This balance can be equivalently expressed using the Maxwell stress tensor; in order-of-magnitude terms the relevant magnetic stress scale is $B_\theta^2/2\mu_0$ often written as

$$P_B \sim \frac{B_\theta^2}{2\mu_0}$$

Low order MHD instabilities such as sausage or kink instabilities ($m=0$ and $m=1$ respectively) destabalize the equilibrium by perturbing the plasma boundary and the current channel, thereby producing perturbed $j \times B$ forces that drive rapid radial or helical deformation. Active stabalization with external coils is therefore modelled as the application of fast mode coupled magnetic perturbation δB that produces a restoring force (via perturbed Maxwell stress or Lorentz force) phased to oppose the unstable displacement. The goal is not to increase the equilibrium confinement globally, but to introduce a targetted restoring perturbation that reduces effective linear growth rate.

In this work, the objective of active stabalization is not to maintain indefinite equilibrium, but to delay the onset of non-linear distortion by keeping the unstable perturbations in the linear regime for as long as possible. The distinction is essential. In linear regime perturbations remain small relative to pinch radius ($\xi/a \ll 1$) the dynamics of low order modes can be approximated by a small number of dominant eigenmodes with well defined growth rates and spatial structure. Under these conditions, the instability amplitude evolves approximately exponentially $x(t) \sim x_0 e^{\lambda t}$, enabling meaningful definitions of a growth rate λ, a feedback delay T_d and an effective actuator authority.

Once perturbations grow to an order $\xi/a \sim 0.1$, non-linear effects become significant. Mode coupling, waveform steepening, geometric distortion of current channel and changes in equilibrium invalidate the assumption of a fixed growth rate and a single mode description. In this regime, feedback becomes fundamentally less well posed: the actuator no longer acts on a stable linearization about a fixed equilibrium, and control actions can easily couple into unintended modes or amplify existing distortions. Consequently, any feasible active stabalization architecture must act early during this linear phase, where the instability dynamics remain predictable and the feedback gains can be meaningfully related to a reduction in the effective growth rate.

What is our control law 'm'? (1.2)

Let the instability amplitude be the boundary poturbation

$$x(t) = \xi_a$$

assume linear regime exponential growth in the absence of control

$$\dot{x}(t) = \lambda x(t), \quad \lambda > 0$$

Define the mode "energy" proxy

$$W(t) = \frac{1}{2} x^2(t) \quad \text{[Quadratic Lyapunov-style energy function]}$$

Lemma 1: (open loop energy growth rate)

If $\dot{x} = \lambda x$, then

$$\dot{W}(t) = 2\lambda W(t)$$

Proof: $\dot{W} = \frac{d}{dt}\left(\frac{1}{2}x^2(t)\right) = x\dot{x} = x(\lambda x) = \lambda x^2 = 2\lambda\left(\frac{1}{2}x^2\right)$

$= 2\lambda W$

Interpretation: λ is amplitude growth rate

2λ is energy growth rate

Assumption A (Actuator modifies the power driving the mode)

Let actuator reduce the instantaneous power driving the mode by a dimensionless fraction α evaluated on delayed information

$$\dot{W}(t) = 2\lambda(1-\alpha g(t))\, W(t)$$

where $g(t)$ is a normalized feedback signal chosen as

$$g(t) = \frac{x(t-T_d)}{x(t)}$$

This ensures that feedback is proportional to mode amplitude while keeping α dimensionless.

equiv: $\dot{W}(t) = 2\lambda W(t) - 2\alpha\lambda x(t)\, x(t-T_d)\frac{1}{2}$

Lemma 2: This implies a delayed amplitude equation.

$$\dot{x}(t) = \lambda x(t) - (\alpha\lambda)x(t-T_d)$$

Proof :

Start from $\dot{W} = x\dot{x}$, using the assured form

$$x(t)\dot{x}(t) = 2\lambda\left(1-\alpha\frac{x(t-T_d)}{x(t)}\right)W(t)$$

But $W = \frac{1}{2}x^2$, so RHS becomes $2\lambda\left(1-\alpha\frac{x(t-T_d)}{x(t)}\right)\left(\frac{1}{2}x^2(t)\right)$

$$= \lambda x^2(t) - \alpha \lambda x(t) x(t - T_d)$$

Thus,

$$x(t)\dot{x}(t) = \lambda x^2(t) - \alpha x(t) \lambda x(t - T_d)$$

divide by $x(t) \neq 0$ in the linear regime.

$$\dot{x}(t) = \lambda x(t) - \alpha \lambda x(t - T_d)$$

which is equivalent to our toy model

$$\dot{x}(t) = \lambda x(t) - m x(t - T_d)$$

Thus under assumption A ; $\boxed{m = \alpha \lambda}$

So m is exactly the actuator-induced reduction in growth rate : a fraction α of the natural rate scale λ.

$$\alpha = k \frac{\delta p_B}{p_{B_0}} \Rightarrow m = \alpha \lambda = k \left(\frac{\delta p_B}{p_{B_0}} \right) \lambda \quad \text{and} \quad \frac{\delta p_B}{p_{B_0}} = 2 \frac{\delta B}{B_0} + \left(\frac{\delta B}{B_0} \right)^2$$

$\dot{x}(t) = \lambda x(t) - m\,x(t-T_d)$ (laplace transform)

$x(t) = \phi(t)$ on $[-T_d, 0]$

$\mathcal{L}\{x(t-T_d)\} = e^{-sT_d}X(s) + \int_0^{T_d} e^{-sT_d}\phi(t-T_d)\,dt$ [delay differential equation]

$sX(s) - x(0) = \lambda X(s) - m\left[e^{-sT_d}X(s) + \int_0^{T_d} e^{-sT_d}\phi(t-T_d)\,dt\right]$

$[s - \lambda + m e^{-sT_d}]X(s) = \xi_0 - m\int_0^{T_d} e^{-sT_d}\phi(t-T_d)\,dt$