Active Feedback Stabilization of a Continuous Z-Pinch (Draft Notes)

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1 Introduction

This document defines the physical and mathematical model used throughout the analysis. We first state the geometric assumptions and the magnetohydrodynamic (MHD) equilibrium conditions for an ideal, axisymmetric Z-pinch. We then introduce a small boundary perturbation and derive how this perturbation modifies the azimuthal magnetic field and magnetic pressure. Finally, we define the state variable used in the control-oriented instability model.

2 Geometric Assumptions

We consider a continuous, axisymmetric Z-pinch plasma column of:

- equilibrium radius a,
- chamber (or plasma column) length L.

The plasma is assumed to be in an axisymmetric equilibrium with no θ-dependence. The dominant self-generated magnetic field is the azimuthal field $B_\theta(r)$ produced by the pinch current. We assume quasi-static radial force balance, i.e., an ideal MHD equilibrium.

3 MHD Equilibrium Conditions

In ideal MHD, equilibrium is governed by the balance between the plasma pressure gradient and the magnetic body force:

$$\nabla P = \mathbf{J} \times \mathbf{B}. \tag{1}$$

For an axisymmetric Z-pinch with

$$\mathbf{B} = B_\theta(r)\,\hat{\theta}, \qquad \mathbf{J} = J_z(r)\,\hat{z},$$

the radial component of the equilibrium equation becomes

$$\frac{dP}{dr} = -J_z B_\theta. \tag{2}$$

Using Ampère's law, the azimuthal magnetic field at radius r is

$$B_\theta(r) = \frac{\mu_0 I_{\text{enc}}(r)}{2\pi r}. \tag{3}$$

At the plasma boundary $r = a$, we define the equilibrium boundary magnetic field as

$$B_0 \equiv B_\theta(a). \tag{4}$$

The associated magnetic pressure is

$$P_0 \equiv \frac{B_0^2}{2\mu_0}. \tag{5}$$

This expression follows from the diagonal components of the Maxwell stress tensor in magnetostatics. In the present case, the electric field is neglected ($\mathbf{E} = 0$), and the magnetic contribution dominates:

$$\mathbf{T} = \frac{1}{\mu_0}\left(\mathbf{BB} - \frac{1}{2}B^2\mathbf{I}\right). \tag{6}$$

The isotropic term $-\frac{1}{2}B^2\mathbf{I}$ corresponds to the magnetic pressure $B^2/(2\mu_0)$.

4 Boundary Perturbation and Magnetic Pressure Change

When the system becomes unstable, we model the plasma boundary as undergoing a small radial perturbation:

$$a \to a + \xi(t), \tag{7}$$

where initially

$$|\xi(t)| \ll a. \tag{8}$$

Assuming the pinch current I_p remains approximately constant over the short timescale of the instability growth, the azimuthal field is

$$B_\theta(r) = \frac{\mu_0 I_p}{2\pi r}. \tag{9}$$

Therefore, the perturbed boundary field is

$$B_\theta(a + \xi) = \frac{\mu_0 I_p}{2\pi(a + \xi)}. \tag{10}$$

The magnetic field perturbation is then

$$\delta B \equiv B_\theta(a+\xi) - B_\theta(a) \tag{11}$$

$$= \frac{\mu_0 I_p}{2\pi}\left(\frac{1}{a+\xi} - \frac{1}{a}\right) \tag{12}$$

$$= -\frac{\mu_0 I_p}{2\pi}\frac{\xi}{a(a+\xi)}. \tag{13}$$

Dividing by the equilibrium field $B_0 = B_\theta(a)$ yields the normalized magnetic field perturbation:

$$\frac{\delta B}{B_0} = -\frac{\xi}{a+\xi}. \tag{14}$$

In the linear regime $|\xi| \ll a$, this simplifies to

$$\frac{\delta B}{B_0} \approx -\frac{\xi}{a}. \tag{15}$$

Since magnetic pressure scales as $P \propto B^2$, the normalized magnetic pressure perturbation is

$$\frac{\delta P}{P_0} = \left(1 + \frac{\delta B}{B_0}\right)^2 - 1 = 2\frac{\delta B}{B_0} + \left(\frac{\delta B}{B_0}\right)^2. \tag{16}$$

In the strictly linear regime, the quadratic term may be neglected, giving

$$\frac{\delta P}{P_0} \approx 2\frac{\delta B}{B_0}. \tag{17}$$

5 Control-Oriented Instability Model

To analyze feasibility of active stabilization, we model the instability amplitude using the standard linear growth law

$$\dot{x}(t) = \lambda x(t), \tag{18}$$

where $\lambda > 0$ is the exponential growth rate of the dominant instability in the linear regime.

The remaining modeling decision is the definition of the dimensionless state variable $x(t)$.

Two natural choices are:

$$x(t) \equiv \frac{\xi(t)}{a}, \qquad \text{or} \qquad x(t) \equiv \frac{\delta B(t)}{B_0}. \tag{19}$$

In the linear regime $|\xi| \ll a$, these definitions are equivalent up to a sign:

$$\frac{\delta B}{B_0} \approx -\frac{\xi}{a}. \tag{20}$$

For the remainder of this work, we adopt the state variable

$$x(t) \equiv \frac{\xi(t)}{a}, \tag{21}$$

since it directly represents the normalized radial deformation of the pinch boundary and provides a clear criterion for nonlinear onset (e.g., $|x| \sim 0.1$).

5.1 Definition and Scaling of the Growth Rate λ

Having defined the normalized boundary displacement

$$x(t) \equiv \frac{\xi(t)}{a},$$

we now specify the meaning and expected scaling of the linear growth rate λ in the instability model

$$\dot{x}(t) = \lambda x(t). \tag{22}$$

Since $x(t)$ is dimensionless and $\dot{x}(t)$ has units of s^{-1}, the growth rate λ must also have units of s^{-1}.

In this work we restrict attention to the ideal-MHD $m = 0$ (sausage) instability of a continuous, axisymmetric Z-pinch. In the ideal limit, the characteristic timescale of low-order MHD motion is set by Alfvénic propagation, i.e., by the ratio of a characteristic length scale to the Alfvén speed. The Alfvén speed is defined in terms of a characteristic magnetic field magnitude B_0 and mass density ρ as

$$v_A \equiv \frac{B_0}{\sqrt{\mu_0 \rho}}. \tag{23}$$

For the present Z-pinch equilibrium, the dominant self-field is azimuthal. Evaluated at the plasma boundary $r = a$, this field is

$$B_0 \equiv B_\theta(a) = \frac{\mu_0 I_p}{2\pi a}. \tag{24}$$

The ideal $m = 0$ instability is not characterized by a single universal growth rate; rather, λ depends on the equilibrium current and pressure profiles, as well as on the axial wavenumber k of the perturbation. However, the growth rate must be constructed from the available characteristic quantities

$$\{v_A,\ a,\ k\},$$

and therefore admits the general scaling form

$$\lambda = \Gamma(ka)\, \frac{v_A}{a}, \tag{25}$$

where $\Gamma(ka)$ is a dimensionless function of the dimensionless parameter ka.

Equation (25) may equivalently be written as

$$\lambda = \Gamma(ka)\, \frac{1}{\tau_A}, \qquad \tau_A \equiv \frac{a}{v_A}, \tag{26}$$

where τ_A is the Alfvén transit time across the plasma radius.

The factor $\Gamma(ka)$ encodes the dependence of the sausage-mode growth rate on the axial wavelength

$$\lambda_z = \frac{2\pi}{k}.$$

In particular:

- For long-wavelength perturbations ($ka \ll 1$), the axial variation is weak, and the dominant timescale is set primarily by radial Alfvénic motion.
- For shorter-wavelength perturbations ($ka \gtrsim 1$), axial structure becomes important, and the growth rate is generally expected to increase toward an Alfvénic rate set by the axial scale, i.e. $\lambda \sim kv_A$ up to profile-dependent factors.

Thus, rather than assuming a fixed numerical value for λ, we parameterize it in terms of the physically measurable equilibrium quantities I_p, a, and ρ, together with an explicit dependence on the mode wavenumber through $\Gamma(ka)$. This representation is sufficient for the delay-limited feasibility analysis developed in later sections.

6 Energy-Based Control Model with Delay

6.1 Mode Energy Definition

To analyze the feasibility of active stabilization, we introduce an energy-like scalar functional associated with the instability amplitude.

Let the normalized boundary displacement be defined as

$$x(t) \equiv \frac{\xi(t)}{a}, \tag{27}$$

where $|x(t)| \ll 1$ in the linear regime.

We define the quadratic Lyapunov-style energy proxy

$$W(t) \equiv \frac{x^2(t)}{2}. \tag{28}$$

This quantity is positive definite and vanishes only when $x(t) = 0$. While $W(t)$ is not the full MHD energy functional, it serves as a consistent scalar measure of instability amplitude in the linear regime.

6.2 Lemma 1: Open-Loop Energy Growth

Lemma 1 (Open-loop energy growth rate). If the instability amplitude satisfies

$$\dot{x}(t) = \lambda x(t), \tag{29}$$

with $\lambda > 0$, then the energy proxy satisfies

$$\dot{W}(t) = 2\lambda W(t). \tag{30}$$

Proof.

$$\begin{aligned}
\dot{W}(t) &= \frac{d}{dt}\left(\frac{x^2(t)}{2}\right) && (31)\\
&= x(t)\dot{x}(t) && (32)\\
&= x(t)\left(\lambda x(t)\right) && (33)\\
&= \lambda x^2(t) && (34)\\
&= 2\lambda W(t). && (35)
\end{aligned}$$

□

Interpretation. The parameter λ represents the exponential amplitude growth rate of the instability, while 2λ represents the corresponding energy growth rate.

6.3 Control Authority Modeling Assumption

Assumption A (Control authority representation).

The instability is driven by magnetic pressure gradients at the plasma boundary. A correction actuator modifies the local azimuthal magnetic field B_θ, thereby modifying the magnetic pressure

$$P_B = \frac{B^2}{2\mu_0}. \tag{36}$$

Since linear instability growth arises from imbalance in radial force equilibrium, a fractional modification of the magnetic pressure produces a proportional modification of the instantaneous energy injection rate into the mode.

We therefore model the actuator as reducing the instantaneous energy growth rate by a dimensionless fraction α, evaluated on delayed information due to finite sensor and actuator latency T_d.

6.4 Delayed Energy Growth Model

Let the feedback factor be defined as

$$g(t) \equiv \frac{x(t-T_d)}{x(t)}. \tag{37}$$

This form ensures that the feedback is proportional to measured amplitude while keeping α dimensionless.

The delayed energy evolution model is then written as

$$\dot{W}(t) = 2\lambda\left(1 - \alpha g(t)\right) W(t). \tag{38}$$

Substituting $W(t) = x^2(t)/2$ gives

$$x(t)\dot{x}(t) = 2\lambda \left(1 - \alpha \frac{x(t-T_d)}{x(t)}\right) \frac{x^2(t)}{2}. \tag{39}$$

Simplifying,

$$x(t)\dot{x}(t) = \lambda x^2(t) - \alpha\lambda x(t)x(t-T_d). \tag{40}$$

Assuming $x(t) \neq 0$ in the linear regime, division by $x(t)$ yields the delayed amplitude equation.

6.5 Lemma 2: Delayed Amplitude Equation

Lemma 2. Under Assumption A, the energy-based delayed model implies the amplitude equation

$$\dot{x}(t) = \lambda x(t) - \alpha\lambda x(t-T_d). \tag{41}$$

Proof.
From the previous derivation,

$$x(t)\dot{x}(t) = \lambda x^2(t) - \alpha\lambda x(t)x(t-T_d). \tag{42}$$

Dividing both sides by $x(t)$ gives

$$\dot{x}(t) = \lambda x(t) - \alpha\lambda x(t-T_d). \tag{43}$$

□

6.6 Model Validity Conditions

The derivation assumes:

- $|x(t)| \ll 1$, ensuring linear MHD behavior,
- the actuator produces a correction magnetic field proportional to the measured instability amplitude,
- the plasma response remains within the superposition regime,
- $x(t) \neq 0$ during division in the derivation.

The parameter α is a dimensionless control authority coefficient determined by the achievable fractional modification of the boundary magnetic field $\delta B/B_0$ via the correction actuator.

6.7 Characteristic Equation

Seeking exponential solutions of the form

$$x(t) = e^{st}, \tag{44}$$

substitution into the delayed amplitude equation yields

$$se^{st} = \lambda e^{st} - \alpha\lambda e^{s(t-T_d)}. \tag{45}$$

Dividing by e^{st} gives the characteristic equation

$$s = \lambda - \alpha\lambda e^{-sT_d}. \tag{46}$$

Equation (46) governs the stability of the delayed feedback system. Its roots determine whether the instability can be attenuated, stabilized, or further destabilized depending on the parameters (λ, α, T_d).

6.8 Real and Imaginary Decomposition of the Characteristic Equation

The closed-loop dynamics are governed by the characteristic equation

$$s = \lambda - \alpha\lambda e^{-sT_d}. \tag{47}$$

To analyze stability, let the complex root be written as

$$s = c_1 + j\omega, \tag{48}$$

where c_1 is the real part (growth/decay rate) and ω is the oscillation frequency.

The delayed exponential term becomes

$$\begin{aligned}
e^{-sT_d} &= e^{-(c_1+j\omega)T_d} && (49)\\
&= e^{-c_1T_d}e^{-j\omega T_d} && (50)\\
&= e^{-c_1T_d}\left(\cos(\omega T_d) - j\sin(\omega T_d)\right). && (51)
\end{aligned}$$

Substituting into (47) gives

$$c_1 + j\omega = \lambda - \alpha\lambda e^{-c_1T_d}\left(\cos(\omega T_d) - j\sin(\omega T_d)\right). \tag{52}$$

Separating real and imaginary parts yields the coupled nonlinear equations

$$c_1 = \lambda - \alpha\lambda e^{-c_1T_d}\cos(\omega T_d), \tag{53}$$

$$\omega = \alpha\lambda e^{-c_1T_d}\sin(\omega T_d). \tag{54}$$

Equations (53) and (54) determine the closed-loop pole locations as functions of (λ, α, T_d).