Pricing and Hedging American-Style Options with Deep Learning

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Goal

- The goal of the article is to develop a deep learning method which learns the optimal exercise behavior, prices and hedging strategies from samples of underlying risk factors.

- The goal is achieved via learning the continuation values and designing a stopping rule based on comparing current payoff and the continuation value.

- The price lower and upper bound and hedging strategy is the computed using this candidate optimal stopping time.

Paper Organization

- In section 2 neural network version of Longstaff-Schwartz algorithm is developed to estimate the continuation values. The candidate stopping time is then developed based on comparing the continuation value with the current payoff.
- In section 3 the constructed candidate optimal stopping time is used to derive a lower bound and upper bound and confidence interval for the option price.
- In section 4 two different dynamic hedging strategies are discussed.
- In the final section the theory is applied to a Bermudan call option on maximum of different underlying assets.

Calculating a Candidate Optimal Stopping Strategy

- We consider an American-style option that can be exercised at any one of finitely many times $0 = t_0 < t_1 < \cdots < t_N = T$
- $\mathcal{T}_n$ is the set of all $\mathcal{F}$-stopping times satisfying $n \leq \tau \leq N$

$$\tau_n := \begin{cases} n & \text{if } G_n \geq E[G_{\tau_{n+1}} \mid X_n] \\ \tau_{n+1} & \text{otherwise} \end{cases}$$

Computing a candidate optimal stopping policy

The algorithm for learning continuation values as explained in the paper the following steps are taken in order to fit the neural networks:

- Simulate paths (x_n^k), $k = 1, 2, \ldots, K$ of the underlying process $(X_n)_{n=0}^N$.
- Set $s_N^k \equiv N$ for all k.
  s_n will serve as the optimal stopping times when we stop after step n.
- For $1 \leq n \leq N-1$, approximate $\mathbb{E}[G_{\tau_{n+1}}|X_n]$ with $c^{\theta_n}(X_n)$ by minimizing the sum:

$$\sum_{k=1}^{K}(g(s_{n+1}^k, x_{s_{n+1}^k}^k) - c^{\theta_n}(x_n^k))^2$$

  where c is the minimizer of the mean squared distance $\mathbb{E}[\{G_{\tau_{n+1}} - c(X_n)\}^2]$

Calculating a Candidate Optimal Stopping Strategy

- Set $s_n^k = \begin{cases} n & \text{if } g(n, x_n^k) \geq c^{\theta_n}(x_n^k) \\ s_{n+1}^k & \text{otherwise} \end{cases}$
- Define $\theta_0 := \frac{1}{K}\sum_{k=1}^{K} g(s_1^k, x_{s_1^k}^k)$, and set c^θ constantly equal to θ_0.

Calculating a Candidate Optimal Stopping Strategy

- The function below calculates the stopping times based on continuation value models based on the recursion:

$$\tau_n = \begin{cases} n & \text{if } G_n \geq \mathbb{E}[G_{\tau_{n+1}} | X_n] \\ \tau_{n+1} & \text{otherwise} \end{cases}$$

we define stopping times as:

$$\tau^{\Theta} := min\{n \in \{0, 1, \dots, N-1\} : g(n, X_n) \geq c^{\theta_n}(X_n)\}$$

We take a mask in each step which indicates which samples we have not reached stopping time yet. This helps vectorizing the computation and each time we give the complete data of each time step as input instead of iterating over the samples.

Lower Bound

- In this function we generate independent sample paths

$$(x_n^k)_{n=0}^N, k = K, K+1, \dots, K+K_L$$

, of $(X_n)_{n=0}^N$ and approximate the lower bound L with the Monte Carlo average:

$$\hat{L} = \frac{1}{K_L} \sum_{k=K+1}^{K+K_L} g^k$$

Where g^k is the realization of $g(\tau^\Theta, X_{\tau^\Theta})$ along k'th sample.

Lower Bound

- Also by computing the sample standard deviation:

$$\hat{\sigma_L} = \sqrt{\frac{1}{K_L - 1} \sum_{k=K+1}^{K+K_L} (g^k - \hat{L})^2}$$

and using central limit theorem if we denote by $z_{\alpha/2}$ the $1 - \alpha/2$ quantile of standard normal distribution we an asymptotically valid $1 - \alpha/2$ confidence interval:

$$\left[\hat{L} - z_{\alpha/2} \frac{\hat{\sigma_L}}{\sqrt{K_L}}, \infty\right)$$

Snell's envelope

- Snell's envelope is the smallest supermartingale with respect to $(F_n)_{n=0}^N$ that dominates $(g(n, X_n))_{n=0}^N$. It is given by:

$$H_n = esssup_{\tau \in \mathrm{T}_n} \mathbb{E}[g(\tau)|F_n]$$

- It's Doob-Meyer decomposition is:

$$H_n = H_0 + M_n^H - A_n^H$$

Where M^H is the (F_n)-martingale given by:

$$M_0^H = 0, M_n^H - M_{n-1}^H = H_n - \mathbb{E}[H_n|F_{n-1}]$$

and A^H is the nondecreasing (F_n)-predictable process given by:

$$A_0^H = 0, A_n^H - A_{n-1}^H = H_{n-1} - \mathbb{E}[H_n|F_{n-1}$$

Upper bound

- The derivation of upper bound is based on following two propositions.
- Let $(\epsilon_n)_{n=0}^N$ be a sequence of integrable random variables on $(\Omega, F, \mathbb{P})$. Then:

$$V_0 \geq \mathbb{E}[max_{0\leq n\leq N}(g(n, X_n) - M_n^H - \epsilon_n)] + \mathbb{E}[min_{0\leq n\leq N}(A_n^H + \epsilon_n)]$$

- Moreover, if $\mathbb{E}[\epsilon_n | F_n] = 0$ for all $n \in \{0, 1, \ldots, N\}$ we have:

$$V_0 \leq \mathbb{E}[max_{0\leq n\leq N}(g(n, X_n) - M_n - \epsilon_n)]$$

for every F_n-martingale $(M_n)_{n=0}^N$ starting from 0.

- The resulting upper bound will be tight if M_n is the martingale part of snell's envelope and $\epsilon_n = 0$ for all n.

Upper bound algorithm

- To compute an upper bound we need to get a good enough martingale(one that is close enough to martingale part of snell's envelope. Using the candidate optimal stopping time we can get out approximation of snell's envelope by:

$$H_n^{\Theta} = \mathbb{E}[g(\tau_n^{\Theta}, X_{\tau_n^{\Theta}})], \quad n = 0, 1, \ldots, N$$

- Then the approximation of martingale part of snell's envolope is given by:

$$M_n^{\Theta} - M_{n-1}^{\Theta} = H_n^{\Theta} - \mathbb{E}[H_N^{\Theta} | F_{n-1}]$$

$$= \mathbb{1}_{g(n,X_n) \geq c^{\theta_n}(X_n)} g(n, X_n) + \mathbb{1}_{g(n,X_n) < c^{\theta_n}(X_n)} C_n^{\Theta} - C_{n-1}^{\Theta}$$

Estimating continuation values

- If we use $c^{\theta_n}(X)$ for estimating the continuation values the resulting continuation value won't be unbiased.
- In order to get unbiased estimates we generate a third set of independent realizations $(z_n^k)_{n=0}^N, k = 1, 2, \dots, K_U$, of $(X_n)_{n=0}^N$. In addition, for every z_n^k, we simulate J continuation paths $z_{n+1}^{\widetilde{k,j}}, \dots, z_N^{\widetilde{k,j}}, \quad j = 0, \dots, J$ that are conditionally independent of each other and $z_{n+1}^k, \dots, z_N^k$.
- We denote $\tau_{n+1}^{k,j}$ the value of τ_{n+1}^{Θ} along $\tilde{z}_{n+1}^{k,j}, \dots, \tilde{z}_N^{k,j}$. Then we can get an unbiased estimate of continuation value by:

$$C_n^k = \frac{1}{J} \sum_{j=1}^{J} g(\tau_{n+1}^{k,j}, \tilde{z}_{\tau_{n+1}^{k,j}}^{k,j})$$

Estimating upper bound

- We can get the noisy estimate for approximation of snell's envelope by:

$$\Delta M_n^k = \mathbb{1}_{g(n,X_n)\geq c^{\theta_n}(X_n)} g(n, X_n) + \mathbb{1}_{g(n,X_n)<c^{\theta_n}(X_n)} C_n^{\Theta} - C_{n-1}^{\Theta}$$

The corresponding martingales are:

$$M_n^k = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{m=1}^{n} \Delta M_m^k & \text{otherwise} \end{cases}$$

- After constructing the martingales we estimate the upper bound by:

$$\hat{U} = \frac{1}{K_U} \sum_{k=1}^{K_U} max_{0\leq n\leq N}(g(n, z_n^k) - M_n^k)$$

Which is an unbiased estimate for:

$$U = \mathbb{E}[max_{0\leq n\leq N}(g(n, X_n) - M_n^{\Theta} - \epsilon_n)]$$

Lower confidence interval

- In order to compute lower confidence interval we also compute the sample standard deviation of components used in upper bound estimation:

$$\hat{\sigma}_U = \sqrt{\frac{1}{K_u - 1}\sum_{k=1}^{K_U}(max_{0\leq n\leq N}(g(n, z_n^k) - M_n^k) - \hat{U})^2}$$

- The lower confidence interval is then computed as:

$$(-\infty, \hat{U} + z_{\alpha/2}\frac{\hat{\sigma}_U}{\sqrt{K_U}}]$$

Point estimate and confidence interval

- Computing the lower bound and upper bound and the corresponding sample variances out point estimate will be equal to:

$$\frac{\hat{L} + \hat{U}}{2}$$

- The corresponding $1 - \alpha$ confidencer interval will be:

$$[\hat{L} - z_{\alpha/2} \frac{\hat{\sigma_L}}{\sqrt{K_L}}, \hat{U} + z_{\alpha/2} \frac{\hat{\sigma}_U}{\sqrt{K_U}}]$$

Assumptions

- We consider a saving account together with $e \in \mathbb{N}$ financial securities as Hedging instruments.
- We fix a positive integer M and introduce a time grid $0 = u_0 < u_1 < \dots u_{NM}$ such that $u_{nM} = t_n$ for all $n = 0, 1, \dots N$.
- We suppose that the information available at time u_m is described by H_m where $\mathbb{H} = (H_m)_{m=0}^{MN}$ is a filteration satisfying $H_{nM} = F_n$ for all n.
- If any of the financial securities pay dividends, they are immediately reinvested. We assume the resulting discounted value processes are of form $P_{u_m} = p_m(Y_m)$ for measurable functions $p_m : \mathbb{R}^d \to \mathbb{R}^e$ and an H-markov process $(Y_m)_{m=0}^{MN}$ such that $Y_{nM} = X_n$ for all $n = 0, \dots N$.

Assumptions

- A hedging strategy consists of a sequence $h = (h_m)_{m=0}^{NM-1}$ of functions $h_m : \mathbb{R}^d \to \mathbb{R}^e$ specifying the time u_m holdings in $P^1_{u_m}, \dots, P^e_{u_m}$.
- As usual, money is dynamically deposited or borrowed from the saving account to make the strategy self-financing.
- The resulting discounted gain at time u_m are given by:

$$(h.P)_{u_m} := \sum_{j=0}^{m-1} h_j(Y_j).(p_{j+1}(Y_{j+1}) - p_j(Y_j))$$

$$:= \sum_{j=0}^{m-1} \sum_{i=0}^{e} h^i_j(Y_j)(p^i_{j+1}(Y_{j+1}) - p^i_j(Y_j))$$

Hedging Until the First Possible Exercise Time

- We assume τ^{Θ} does not stop at time zero. Otherwise there is nothing to hedge.
- In a first step we compute the hedge until time t_1. If the option is still alive at time t_1 the hedge can then be computed until time t_2.
- We approximate the time t_1 value of the option with $V_{t_1}^{\Theta} = v^{\theta_1}(X_1)$ for the function $v^{\theta_1}(X_1) = g(1, x) \vee c^{\theta_1}(x)$, where $c^{\theta_1} : \mathbb{R}^d \to \mathbb{R}$ is the time t_1 continuation value function estimated in section 2.
- We search for hedging positions $h_m, m = 0, 1, \ldots, M-1$, that minimize the mean squared error:

$$\mathbb{E}[(\hat{V} + (h.P)_{t_1} - V_{t_1}^{\Theta})]$$

Approximating using neural network

- Instead of searching the entire space for functions h_m we approximate h_m with neural networks $h^{\lambda} : \mathbb{R}^d \to \mathbb{R}^e$ and try to find parameters $\lambda_0, \dots, \lambda_{M-1}$ that minimize:

$$\sum_{k=1}^{K_H} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - v^{\theta_1}(y_M^k))^2$$

for Independent realizations of $(y_m^k)_{m=0}^M, k = 1, \dots, K_H$ of $(Y_m)_{m=0}^M$.

- We train the networks $h_{\lambda_0}, \dots, h_{\lambda_{M-1}}$ together using the Adam optimization method.

Assessing the hedge quality

- We assess the quality of hedge by simulating new independent realizations $(y_m^k)_{m=0}^M$, $k = K_H + 1, \dots, K_H + K_E$ of $(Y_m)_{m=0}^M$ and calculating the average hedging error.

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - v^{\theta_1}(y_M^k))$$

and the empirical hedging shortfall:

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - v^{\theta_1}(y_M^k))^-$$

over the time interval $[0, t_1]$

Hedging Until the Exercise Time

- Alternatively we can precompute the whole hedging strategy from time 0 to T and use it until the option is exercised.
- We introduce the functions:

$$v^{\theta_n}(x) = g(n, x) \vee c^{\theta_n}(x), C^{\theta_n}(x) := 0 \vee c^{\theta_n}(x), x \in \mathbb{R}^d$$

- We aim to hedge the difference $v^{\theta_n}(Y_{nM}) - C^{\theta_{n-1}}(Y_{(n-1)M})$ on each of the time intervals $[t_{n-1}, t_n]$ for $n = 1, \ldots, N$ separately.

Approximating using neural network

- Again we seek for parameters $\lambda_{(n-1)M}, \ldots, \lambda_{nM-1}$ for $n = 1, \ldots, N$ for approximating the hedging functions.

- The loss function is defined as:

$$\sum_{k=1}^{K_H} \left(C^{\theta_{n-1}}(y^k_{(n-1)M}) + \sum_{m=0}^{M-1} h^{\lambda_m}(y^k_m) \cdot (p_{m+1}(y^k_{m+1}) - p_m(y^k_m)) - v^{\theta_n}(y^k_{nM})\right)^2$$

where $(y^k_m)_{m=0}^{NM}, k = 1, \ldots, K_H$ are independent samples of $(Y_m)_{m=0}^{MN}$.

Assessing the hedge quality

- In order to assess the hedge quality we generate independent samples $(y_m^k)m = 0^{MN}$, $k = K_H + 1, \ldots, K_H + K_E$, of $(Y_m)_{m=0}^{MN}$ and denote the realization of τ^{Θ} along each sample path $(y_m^k)_{m=0}^{MN}$ by τ^k.

- The corresponding hedging error is given by:

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - g(\tau^k, X_{\tau^k}))$$

and the empirical hedging shortfall by:

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - g(\tau^k, X_{\tau^k}))^-$$

Thank You!

Any Question?