# Pricing and Hedging American-Style Options with Deep Learning

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# Goal

- The goal of the article is to develop a deep learning method which learns the optimal exercise behavior, prices and hedging strategies from samples of underlying risk factors.
- The goal is achieved via learning the continuation values and designing a stopping rule based on comparing current payoff and the continuation value.
- The price lower and upper bound and hedging strategy is the computed using this candidate optimal stopping time.

# Paper Organization

- In section 2 neural network version of Longstaff-Schwartz algorithm is developed to estimate the continuation values. The candidate stopping time is then developed based on comparing the continuation value with the current payoff.
- In section 3 the constructed candidate optimal stopping time is used to derive a lower bound and upper bound and confidence interval for the option price.
- In section 4 two different dynamic hedging strategies are discussed.
- In the final section the theory is applied to a Bermudan call option on maximum of different underlying assets.

# Calculating a Candidate Optimal Stopping Strategy

- We consider an American-style option that can be exercised at any one of finitely many times $0 = t_0 < t_1 < \cdots < t_N = T$
- $\mathcal{T}_n$ is the set of all $\mathcal{F}$-stopping times satisfying $n \leq \tau \leq N$

$$\tau_n := \begin{cases} n & \text{if } G_n \geq E[G_{\tau_{n+1}} \mid X_n] \\ \tau_{n+1} & \text{otherwise} \end{cases}$$

# Computing a candidate optimal stopping policy

The algorithm for learning continuation values as explained in the paper the following steps are taken in order to fit the neural networks:

- Simulate paths $(x_n^k)$, $k = 1, 2, \ldots, K$ of the underlying process $(X_n)_{n=0}^N$.
- Set $s_N^k \equiv N$ for all $k$.
  $s_n$ will serve as the optimal stopping times when we stop after step $n$.
- For $1 \leq n \leq N-1$, approximate $\mathbb{E}[G_{\tau_{n+1}}|X_n]$ with $c^{\theta_n}(X_n)$ by minimizing the sum:

$$\sum_{k=1}^{K}(g(s_{n+1}^k, x_{s_{n+1}^k}^k) - c^{\theta_n}(x_n^k))^2$$

  where $c$ is the minimizer of the mean squared distance $\mathbb{E}[\{G_{\tau_{n+1}} - c(X_n)\}^2]$

# Calculating a Candidate Optimal Stopping Strategy

- Set $s_n^k = \begin{cases} n & \text{if } g(n, x_n^k) \geq c^{\theta_n}(x_n^k) \\ s_{n+1}^k & \text{otherwise} \end{cases}$
- Define $\theta_0 := \frac{1}{K} \sum_{k=1}^{K} g(s_1^k, x_{s_1^k}^k)$, and set $c^\theta$ constantly equal to $\theta_0$.

# Calculating a Candidate Optimal Stopping Strategy

- The function below calculates the stopping times based on continuation value models based on the recursion:

$$\tau_n = \begin{cases} n & \text{if } G_n \geq \mathbb{E}[G_{\tau_{n+1}} | X_n] \\ \tau_{n+1} & \text{otherwise} \end{cases}$$

we define stopping times as:

$$\tau^{\Theta} := min\{n \in \{0, 1, \dots, N-1\} : g(n, X_n) \geq c^{\theta_n}(X_n)\}$$

# Lower Bound

- In this function we generate independent sample paths

$$(x_n^k)_{n=0}^N, k = K, K+1, \dots, K+K_L$$

, of $(X_n)_{n=0}^N$ and approximate the lower bound $L$ with the Monte Carlo average:

$$\hat{L} = \frac{1}{K_L} \sum_{k=K+1}^{K+K_L} g^k$$

Where $g^k$ is the realization of $g(\tau^\Theta, X_{\tau^\Theta})$ along k'th sample.

# Lower Bound

- Also by computing the sample standard deviation:

$$\hat{\sigma}_L = \sqrt{\frac{1}{K_L - 1} \sum_{k=K+1}^{K+K_L} (g^k - \hat{L})^2}$$

and using central limit theorem if we denote by $z_{\alpha/2}$ the $1 - \alpha/2$ quantile of standard normal distribution we an asymptotically valid $1 - \alpha/2$ confidence interval:

$$\left[ \hat{L} - z_{\alpha/2} \frac{\hat{\sigma}_L}{\sqrt{K_L}}, \infty \right)$$

# Snell's envelope

- Snell's envelope is the smallest supermartingale with respect to $(F_n)_{n=0}^N$ that dominates $(g(n, X_n))_{n=0}^N$. It is given by:

$$H_n = esssup_{\tau \in \mathrm{T}_n} \mathbb{E}[g(\tau)|F_n]$$

- It's Doob-Meyer decomposition is:

$$H_n = H_0 + M_n^H - A_n^H$$

Where $M^H$ is the $(F_n)$-martingale given by:

$$M_0^H = 0, M_n^H - M_{n-1}^H = H_n - \mathbb{E}[H_n|F_{n-1}]$$

and $A^H$ is the nondecreasing $(F_n)$-predictable process given by:

$$A_0^H = 0, A_n^H - A_{n-1}^H = H_{n-1} - \mathbb{E}[H_n|F_{n-1}$$

# Upper bound

- The derivation of upper bound is based on following two propositions.
- Let $(\epsilon_n)_{n=0}^{N}$ be a sequence of integrable random variables on $(\Omega, F, \mathbb{P})$. Then:

$$V_0 \geq \mathbb{E}[max_{0\leq n\leq N}(g(n, X_n) - M_n^H - \epsilon_n)] + \mathbb{E}[min_{0\leq n\leq N}(A_n^H + \epsilon_n)]$$

- Moreover, if $\mathbb{E}[\epsilon_n | F_n] = 0$ for all $n \in \{0, 1, \dots, N\}$ we have:

$$V_0 \leq \mathbb{E}[max_{0\leq n\leq N}(g(n, X_n) - M_n - \epsilon_n)]$$

  for every $F_n$-martingale $(M_n)_{n=0}^{N}$ starting from 0.
- The resulting upper bound will be tight if $M_n$ is the martingale part of snell's envelope and $\epsilon_n = 0$ for all $n$.

# Upper bound algorithm

- To compute an upper bound we need to get a good enough martingale(one that is close enough to martingale part of snell's envelope. Using the candidate optimal stopping time we can get out approximation of snell's envelope by:

$$H_n^{\Theta} = \mathbb{E}[g(\tau_n^{\Theta}, X_{\tau_n^{\Theta}})], \quad n = 0, 1, \dots, N$$

- Then the approximation of martingale part of snell's envolope is given by:

$$M_n^{\Theta} - M_{n-1}^{\Theta} = H_n^{\Theta} - \mathbb{E}[H_N^{\Theta} | F_{n-1}]$$

$$= \mathbb{1}_{g(n, X_n) \geq c^{\theta_n}(X_n)} g(n, X_n) + \mathbb{1}_{g(n, X_n) < c^{\theta_n}(X_n)} C_n^{\Theta} - C_{n-1}^{\Theta}$$

# Estimating continuation values

- If we use $c^{\theta_n}(X)$ for estimating the continuation values the resulting continuation value won't be unbiased.
- In order to get unbiased estimates we generate a third set of independent realizations $(z_n^k)_{n=0}^N, k = 1, 2, \dots, K_U$, of $(X_n)_{n=0}^N$. In addition, for every $z_n^k$, we simulate $J$ continuation paths $z_{n+1}^{\tilde{k,j}}, \dots, z_N^{\tilde{k,j}}, \quad j = 0, \dots, J$ that are conditionally independent of each other and $z_{n+1}^k, \dots, z_N^k$.
- We denote $\tau_{n+1}^{k,j}$ the value of $\tau_{n+1}^{\Theta}$ along $\tilde{z}_{n+1}^{k,j}, \dots, \tilde{z}_N^{k,j}$. Then we can get an unbiased estimate of continuation value by:

$$C_n^k = \frac{1}{J}\sum_{j=1}^{J} g(\tau_{n+1}^{k,j}, \tilde{z}_{\tau_{n+1}^{k,j}}^{k,j})$$

# Estimating upper bound

- We can get the noisy estimate for approximation of snell's envelope by:

$$\Delta M_n^k = \mathbb{1}_{g(n,X_n) \geq c^{\theta_n}(X_n)} g(n, X_n) + \mathbb{1}_{g(n,X_n) < c^{\theta_n}(X_n)} C_n^{\Theta} - C_{n-1}^{\Theta}$$

The corresponding martingales are:

$$M_n^k = \begin{cases} 0 & \text{if } n = 0 \\ \sum_{m=1}^{n} \Delta M_m^k & \text{otherwise} \end{cases}$$

- After constructing the martingales we estimate the upper bound by:

$$\hat{U} = \frac{1}{K_U} \sum_{k=1}^{K_U} max_{0 \leq n \leq N}(g(n, z_n^k) - M_n^k)$$

Which is an unbiased estimate for:

$$U = \mathbb{E}[max_{0 \leq n \leq N}(g(n, X_n) - M_n^{\Theta} - \epsilon_n)]$$

# Lower confidence interval

- In order to compute lower confidence interval we also compute the sample standard deviation of components used in upper bound estimation:

$$\hat{\sigma}_U = \sqrt{\frac{1}{K_u - 1}\sum_{k=1}^{K_U}(max_{0\leq n\leq N}(g(n, z_n^k) - M_n^k) - \hat{U})^2}$$

- The lower confidence interval is then computed as:

$$(-\infty, \hat{U} + z_{\alpha/2}\frac{\hat{\sigma}_U}{\sqrt{K_U}}]$$

# Point estimate and confidence interval

- Computing the lower bound and upper bound and the corresponding sample variances out point estimate will be equal to:

$$\frac{\hat{L} + \hat{U}}{2}$$

- The corresponding $1 - \alpha$ confidencer interval will be:

$$[\hat{L} - z_{\alpha/2} \frac{\hat{\sigma_L}}{\sqrt{K_L}}, \hat{U} + z_{\alpha/2} \frac{\hat{\sigma}_U}{\sqrt{K_U}}]$$

# Assumptions

- We consider a saving account together with $e \in \mathbb{N}$ financial securities as Hedging instruments.
- We fix a positive integer $M$ and introduce a time grid $0 = u_0 < u_1 < \ldots u_{NM}$ such that $u_{nM} = t_n$ for all $n = 0, 1, \ldots N$.
- We suppose that the information available at time $u_m$ is described by $H_m$ where $\mathbb{H} = (H_m)_{m=0}^{MN}$ is a filteration satisfying $H_{nM} = F_n$ for all $n$.
- If any of the financial securities pay dividends, they are immediately reinvested. We assume the resulting discounted value processes are of form $P_{u_m} = p_m(Y_m)$ for measurable functions $p_m : \mathbb{R}^d \to \mathbb{R}^e$ and an $H$-markov process $(Y_m)_{m=0}^{MN}$ such that $Y_{nM} = X_n$ for all $n = 0, \ldots N$.

# Assumptions

- A hedging strategy consists of a sequence $h = (h_m)_{m=0}^{NM-1}$ of functions $h_m : \mathbb{R}^d \to \mathbb{R}^e$ specifying the time $u_m$ holdings in $P^1_{u_m}, \dots, P^e_{u_m}$.
- As usual, money is dynamically deposited or borrowed from the saving account to make the strategy self-financing.
- The resulting discounted gain at time $u_m$ are given by:

$$(h.P)_{u_m} := \sum_{j=0}^{m-1} h_j(Y_j).(p_{j+1}(Y_{j+1}) - p_j(Y_j))$$

$$:= \sum_{j=0}^{m-1} \sum_{i=0}^{e} h^i_j(Y_j)(p^i_{j+1}(Y_{j+1}) - p^i_j(Y_j))$$

# Hedging Until the First Possible Exercise Time

- We assume $\tau^{\Theta}$ does not stop at time zero. Otherwise there is nothing to hedge.
- In a first step we compute the hedge until time $t_1$. If the option is still alive at time $t_1$ the hedge can then be computed until time $t_2$.
- We approximate the time $t_1$ value of the option with $V_{t_1}^{\Theta} = v^{\theta_1}(X_1)$ for the function $v^{\theta_1}(X_1) = g(1,x) \vee c^{\theta_1}(x)$, where $c^{\theta_1} : \mathbb{R}^d \to \mathbb{R}$ is the time $t_1$ continuation value function estimated in section 2.
- We search for hedging positions $h_m, m = 0, 1, \ldots, M-1$, that minimize the mean squared error:

$$\mathbb{E}[(\hat{V} + (h.P)_{t_1} - V_{t_1}^{\Theta})]$$

# Approximating using neural network

- Instead of searching the entire space for functions $h_m$ we approximate $h_m$ with neural networks $h^{\lambda} : \mathbb{R}^d \to \mathbb{R}^e$ and try to find parameters $\lambda_0, \ldots, \lambda_{M-1}$ that minimize:

$$\sum_{k=1}^{K_H} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - v^{\theta_1}(y_M^k))^2$$

  for Independent realizations of $(y_m^k)_{m=0}^M, k = 1, \ldots, K_H$ of $(Y_m)_{m=0}^M$.

- We train the networks $h_{\lambda_0}, \ldots, h_{\lambda_{M-1}}$ together using the Adam optimization method.

# Assessing the hedge quality

- We assess the quality of hedge by simulating new independent realizations $(y_m^k)_{m=0}^M$, $k = K_H + 1, \ldots, K_H + K_E$ of $(Y_m)_{m=0}^M$ and calculating the average hedging error.

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} \left(\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - v^{\theta_1}(y_M^k)\right)$$

and the empirical hedging shortfall:

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} \left(\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - v^{\theta_1}(y_M^k)\right)^-$$

over the time interval $[0, t_1]$

# Hedging Until the Exercise Time

- Alternatively we can precompute the whole hedging strategy from time 0 to $T$ and use it until the option is exercised.
- We introduce the functions:

$$v^{\theta_n}(x) = g(n, x) \vee c^{\theta_n}(x), C^{\theta_n}(x) := 0 \vee c^{\theta_n}(x), x \in \mathbb{R}^d$$

- We aim to hedge the difference $v^{\theta_n}(Y_{nM}) - C^{\theta_{n-1}}(Y_{(n-1)M})$ on each of the time intervals $[t_{n-1}, t_n]$ for $n = 1, \ldots, N$ separately.

# Approximating using neural network

- Again we seek for parameters $\lambda_{(n-1)M}, \dots, \lambda_{nM-1}$ for $n = 1, \dots, N$ for approximating the hedging functions.

- The loss function is defined as:

$$\sum_{k=1}^{K_H} \left(C^{\theta_{n-1}}(y^k_{(n-1)M}) + \sum_{m=0}^{M-1} h^{\lambda_m}(y^k_m) \cdot (p_{m+1}(y^k_{m+1}) - p_m(y^k_m)) - v^{\theta_n}(y^k_{nM})\right)^2$$

where $(y^k_m)_{m=0}^{NM}$, $k = 1, \dots, K_H$ are independent samples of $(Y_m)_{m=0}^{MN}$.

# Assessing the hedge quality

- In order to assess the hedge quality we generate independent samples $(y_m^k)m = 0^{MN}$, $k = K_H + 1, \ldots, K_H + K_E$, of $(Y_m)_{m=0}^{MN}$ and denote the realization of $\tau^\Theta$ along each sample path $(y_m^k)_{m=0}^{MN}$ by $\tau^k$.

- The corresponding hedging error is given by:

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - g(\tau^k, X_{\tau^k}))$$

and the empirical hedging shortfall by:

$$\frac{1}{K_E} \sum_{k=K_H+1}^{K_H+K_E} (\hat{V} + \sum_{m=0}^{M-1} h^{\lambda_m}(y_m^k) \cdot (p_{m+1}(y_{m+1}^k) - p_m(y_m^k)) - g(\tau^k, X_{\tau^k}))^-$$

# Thank You!

## Any Question?