Pricing and Hedging American Options using Deep Learning

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```
In [ ]: import numpy as np
        import matplotlib.pyplot as plt
        from collections import deque
        from itertools import product
        from tqdm import tqdm
        from gc import collect
        from scipy.stats import norm
        import seaborn as sns
        import pandas as pd
        import tensorflow as tf
        from tensorflow.keras.models import Sequential
        from tensorflow.keras.layers import Dense, BatchNormalization, Normalization
        from tensorflow.keras.optimizers import Adam, RMSprop
        from tensorflow.keras.callbacks import EarlyStopping, LearningRateScheduler
        from tensorflow.keras.losses import MeanSquaredError
        from tensorflow.keras.utils import plot_model
```

```
In [ ]: sns.set_theme()
```

1 Generating the data

```
In [ ]: # Set up the parameters
        parameters = {
          'd': 5,
          'r': 0.05,
          'delta': 0.1,
          'sigma': 0.2,
          's0': 90,
          'rho': 0,
          'K': 100,
          'T': 3,
```

```
        'N': 9,
        'M': 12,
        'batch size': 8120,
        'price steps': 100,
        'hedge steps': 8,
        'lower bound samples': 1_024_000,
        'upper bound repeats': 512,
        'upper bound samples': 512,
        'hedging samples': 8192,
        'hedging error samples': 40_960
    }
```

```
In [ ]: def modify_parameters(parameters):
            modified_parameters = parameters.copy()
            modified_parameters['delta'] = 0
            modified_parameters['r'] = 0
            modified_parameters['N'] = modified_parameters['N'] * modified_parameters['M']
            return modified_parameters
```

number_of_samples brownian motions in time interval $[0, t]$ are generated by sampling independent increments from a gaussian distribution and calculating partial sums of increments.

The resulting array will be of shape (number of samples, time points, dimension)

```
In [ ]: def brownian_motion(parameters, number_of_samples):
            samples = np.zeros((number_of_samples, parameters['N'] + 1, parameters['d']))
            time_delta = parameters['T'] / parameters['N']
            mean_vector = np.zeros(parameters['d'])
            correlation_matrix = np.ones((parameters['d'], parameters['d'])) * parameters['rho']
            increments = np.random.multivariate_normal(mean_vector, correlation_matrix, size=(nu
            return np.cumsum(np.concatenate([np.zeros((number_of_samples, 1, parameters['d'])), 
```

For each brownian motion sample generates num_repeats continuations from each time point of the sample the continuation will be same until some specified time point and will be sampled independently afterwards.

The resulting array will be of shape (num of samples, distinction time point, number of repeats, time point, dimension).

```
In [ ]: def continue_browian_motion(parameters, base_brownian_motion, num_repeats):
            number_of_samples = base_brownian_motion.shape[0]
            samples = np.zeros((number_of_samples, parameters['N'] + 1, num_repeats, parameters[
            time_delta = parameters['T'] / parameters['N']
            mean_vector = np.zeros(parameters['d'])
            correlation_matrix = np.ones((parameters['d'], parameters['d'])) * parameters['rho']
            samples = base_brownian_motion[:, np.newaxis, np.newaxis, :, :] + samples
            for i in range(1, parameters['N'] + 1):
              samples[:, :i, :, i] = samples[:, :i, :, i - 1] + np.random.multivariate_normal(mea
            return samples
```

Creates stock prices processes based on brownian motion process. The processes are computed via the formula in the paper:

$$S_t^i = s_0^i \exp([r - \delta_i - \sigma_i^2/2]t + \sigma_i W_t^i), \quad i = 1, \ldots, d$$

The resulting array will be of shape (number of samples, dimension).

```
In [ ]: def stock_price_processes(parameters, number_of_samples, underlying_brownian_motion=Nor
            if underlying_brownian_motion is None:
              underlying_brownian_motion = brownian_motion(parameters, number_of_samples)
            times = np.linspace(0, parameters['T'], underlying_brownian_motion.shape[1]).reshape
            return parameters['s0'] * np.exp((parameters['r'] - parameters['delta'] - parameters
```

Continues the stock prices in the same way as brownian motions.

```
In [ ]: def continue_stock_prices(parameters, base_brownian_motion, num_repeats):
            continued_brownian_motion = continue_browian_motion(parameters, base_brownian_motion
            times = np.linspace(0, parameters['T'], parameters['N'] + 1).reshape((-1, 1))
            return parameters['s0'] * np.exp((parameters['r'] - parameters['delta'] - parameters
```

Code to plot processes it takes an array of shape (time points, dimension) as input and plots each dimension separately in time.

```
In [ ]: def plot_processes(samples, parameters):
            x = np.linspace(0, parameters['T'], parameters['N'] + 1)
            for i in range(samples.shape[1]):
              plt.plot(x, samples[:, i])
```

Plot of a 5 dimensonal brownian motion

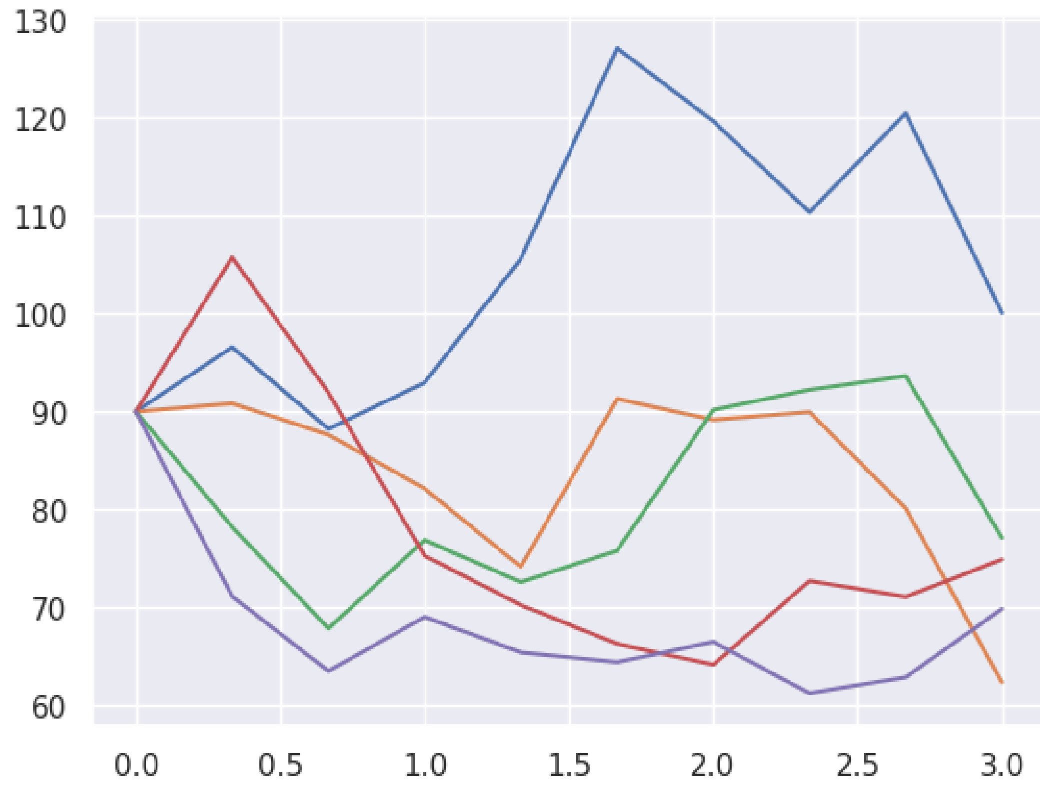
```
In [ ]: samples = brownian_motion(parameters, 1)
        plot_processes(samples[0], parameters)
        del samples
        collect();
```

Plot of single dimensional brownian motion continuations

```
In [ ]: samples = continue_browian_motion(parameters, brownian_motion(parameters, 1), 1)[0, :,
        plot_processes(samples, parameters)
        del samples
        collect();
```

Stock prices of 5 assets modeled by geometric brownian motion

```
In [ ]: prices = stock_price_processes(parameters, 1)
        plot_processes(prices[0], parameters)
        del prices
        collect();
```



Plot of continuations of a stock price

```
In [ ]: samples = continue_stock_prices(parameters, brownian_motion(parameters, 1), 1)[0, :, 0
        plot_processes(samples, parameters)
        del samples
        collect();
```

Plot of hedging instruments (Each interval is split into M subintervals).

In [ ]:
```
samples = stock_price_processes(modify_parameters(parameters), 1)
plot_processes(samples[0], modify_parameters(parameters))
del samples
collect()
```

Out[ ]: 3931

2 Continuation value approximation

As suggested by the paper training the continuations values work more effectively when we used option payoff as an aditional feature. So we add a last dimension to our data namely:

$$\hat{X}_n = (X_1^1, \ldots, X_n^d, X_n^{d+1})$$

where we have defined:

$$X_n^{d+1} = e^{-r\frac{nT}{n}} \left(\max_{1 \leq i \leq d} X_n^i - k \right)^+$$

```
In [ ]: def add_feature(samples, parameters):
            new_samples = np.zeros(samples.shape[:-1] + (samples.shape[-1] + 1,))
            times = np.linspace(0, parameters['T'], samples.shape[1])
            new_samples[..., :-1] = samples
            new_samples[..., -1] = np.maximum(samples.max(axis=-1) - parameters['K'], 0) * np.exp
            return new_samples
```

As suggested by the paper the model will be a neural network with two hidden layers of dimension $d + 50$ each and activation function of tanh followed by batch normalization.

```
In [ ]: def get_continuation_value_model(parameters):
            model = Sequential([
                Dense(parameters['d'] + 50, activation='tanh', input_shape=(parameters['d'] + 1,)
                BatchNormalization(),
                Dense(parameters['d'] + 50, activation='tanh'),
                BatchNormalization(),
                Dense(1)
            ])
            return model
```

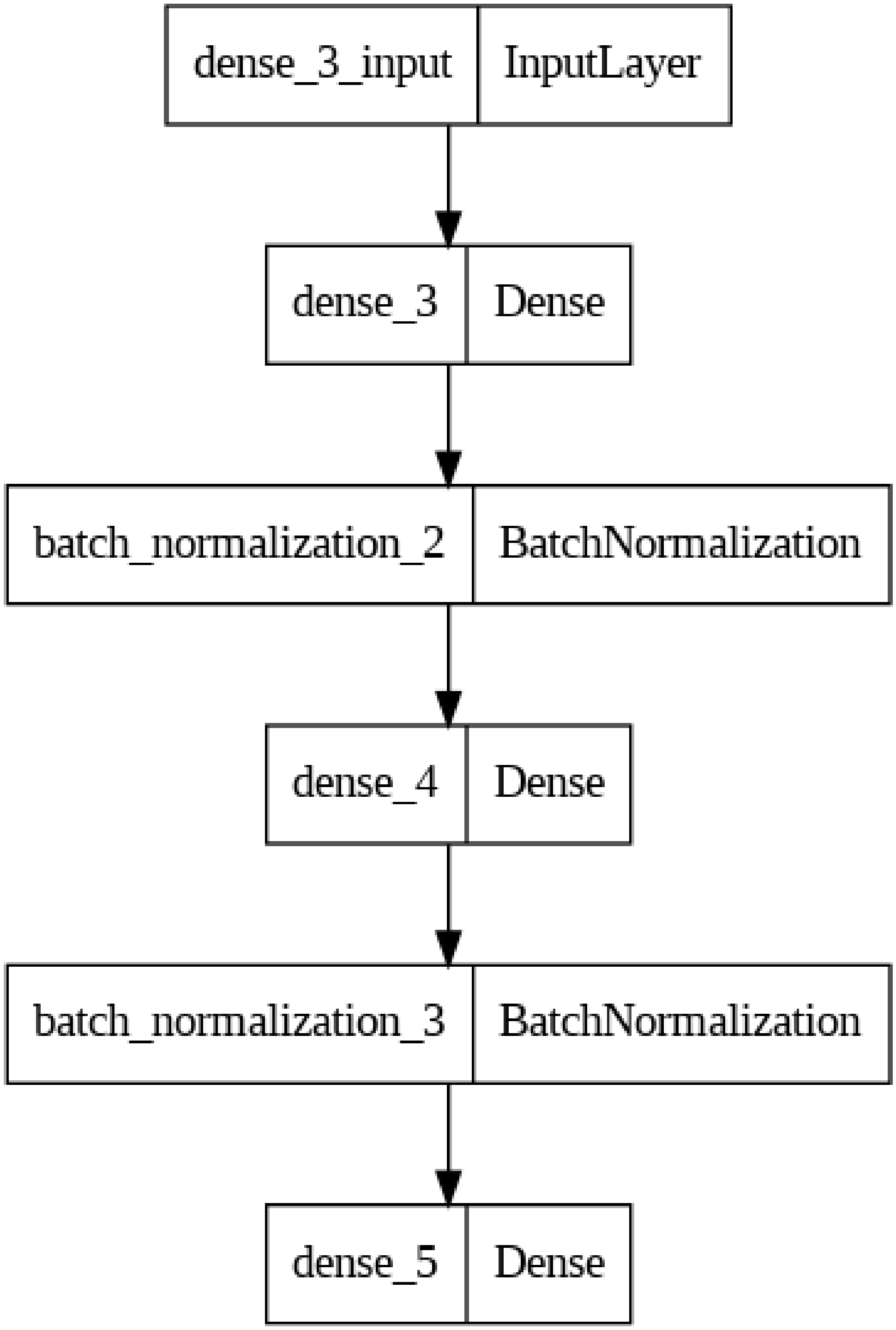
```
In [ ]: get_continuation_value_model(parameters).summary()
```

```
Model: "sequential"
_________________________________________________________________
 Layer (type)                Output Shape              Param #
=================================================================
 dense (Dense)               (None, 55)                385

 batch_normalization (BatchN  (None, 55)               220
 ormalization)

 dense_1 (Dense)             (None, 55)                3080

 batch_normalization_1 (Batc  (None, 55)               220
 hNormalization)

 dense_2 (Dense)             (None, 1)                 56

=================================================================
Total params: 3,961
Trainable params: 3,741
Non-trainable params: 220
_________________________________________________________________
```

A predictor class providing same functionality as keras model but always predicting a constant will be used for model 0 of the paper and an artificial last layer to make the code cleaner.

```
In [ ]: plot_model(get_continuation_value_model(parameters))
```

```
Out[None]:
```




```
In [ ]: class predictor():
            def __init__(self, average):
              self.average = average
```

```
    def predict(self, x, batch_size=None, verbose=False):
      return np.ones(x.shape[0]) * self.average

    def __call__(self, x):
      return self.predict(x)
```

Learning rate scheduler suggested by the paper. The structure is a bit different from paper it starts with learning rate 0.001 and divides it by 10 after 1500 batches.

```
In [ ]: def lr_scheduler(epoch, lr):
          if epoch in [15]:
            return lr / 10
          else:
            return lr
```

The code for learning continuation values as explained in the paper the following steps are taken in order to fit the neural networks:

1. Simulate paths $(x_n^k), k = 1, 2, \ldots, K$ of the underlying process $(X_n)_{n=0}^N$.
2. Set $s_N^k \equiv N$ for all k. s_n will server as the optimal stopping times when we stop after step n.
3. For $1 \leq n \leq N-1$, approximate $\mathbb{E}[G_{\tau_{n+1}} | X_n]$ with $c^{\theta_n}(X_n)$ by minimizing the sum:

$$\sum_{k=1}^{K} (g(s_{n+1}^k, x_{s_{n+1}^k}^k) - c^{\theta_n}(x_n^k))^2$$

4. Set

$$s_n^k = \begin{cases} n & \text{if } g(n, x_n^k) \geq c^{\theta_n}(x_n^k) \\ s_{n+1}^k & \text{otherwise} \end{cases}$$

5. Define $\theta_0 := \frac{1}{K} \sum_{k=1}^{K} g(s_1^k, x_{s_1^k}^k)$, and set c^θ constantly equal to θ_0.

An aditional model predicting constantly infinity is added for convient. (You should never continue after time n so the continuation value should be $-\infty$.

```
In [ ]: def get_continuation_models(parameters):
          models = deque()
          optimal_stops = [parameters['N'] for _ in range(parameters['batch size'] * parameters
          samples = add_feature(stock_price_processes(parameters, parameters['batch size'] * pa
          for time in tqdm(range(parameters['N'] - 1, 0, -1)):
            X_train = samples[:, time, :]
            y_train = samples[range(parameters['batch size'] * parameters['price steps']), opti
            model = get_continuation_value_model(parameters)
            if models:
              model.set_weights(models[0].get_weights())
            model.compile(loss='mse', optimizer=Adam(learning_rate=0.001))
```

```
        model.fit(X_train, y_train, epochs=30 if models else 60, batch_size=parameters['ba
        models.appendleft(model)
        continuations = model.predict(X_train, batch_size=parameters['batch size'], verbose
        optimal_stops = [time if continuations[i] <= samples[i, time, -1] else optimal_stop
    average = np.array(samples[range(parameters['batch size'] * parameters['price steps']
    models.appendleft(predictor(average))
    models.append(predictor(-float('inf')))
    return list(models)
```

A function to get the stopping times based on continuation value models based on the recursion:

$$\tau_n = \begin{cases} n & \text{if } G_n \geq \mathbb{E}[G_{\tau_{n+1}}|X_n] \\ \tau_{n+1} & \text{otherwise} \end{cases}$$

we define stopping times as:

$$\tau^{\Theta} := \min\{n \in \{0, 1, \ldots, N-1\} : g(n, X_n) \geq c^{\theta_n}(X_n)\}$$

We take a mask in each step which indicates which samples we have not reached stopping time yet. This helps vectorizing the computation and each time we give teh complete data of each time step as input instead of iterating over the samples.

```
In [ ]: def get_stopping_times(parameters, models, samples):
    optimal_stops = np.array([parameters['N'] for _ in range(len(samples))])
    mask = np.ones(len(samples)).astype(np.bool8)
    for i in range(parameters['N']):
        bigger_mask = (samples[:, i, -1] >= models[i].predict(samples[:, i], batch_size=pa
        optimal_stops[np.logical_and(mask, bigger_mask)] = i
        mask = np.logical_and(mask, ~bigger_mask)
    return optimal_stops
```

3 Get Lower bound and Upper bounds

3.1 Lower bound

In this function we generate independent sample paths $(x_n^k)_{n=0}^N, k = K, K+1, \ldots, K+K_L$, of $(X_n)_{n=0}^N$ and approximate the lower bound L with the monte carlo average:

$$\hat{L} = \frac{1}{K_L} \sum_{k=K+1}^{K+K_L} g^k$$

Where g^k is the realization of $g(\tau^{\Theta}, X_{\tau^{\Theta}})$ along k'th sample.
Also by computing the sample standard deviation:

$$\hat{\sigma_L} = \sqrt{\frac{1}{K_L - 1} \sum_{k=K+1}^{K+K_L} (g^k - \hat{L})^2}$$

and using central limit theorem if we denote by $z_{\alpha/2}$ the $1-\alpha/2$ quantile of standard normal distribution we an asymptotically valid $1-\alpha/2$ confidence interval:

$$\left[\hat{L} - z_{\alpha/2}\frac{\hat{\sigma_L}}{\sqrt{K_L}}, \infty\right)$$

```
In [ ]: def get_lower_bound(parameters, models):
          samples = add_feature(stock_price_processes(parameters, parameters['lower bound samp
          values = samples[range(len(samples)), get_stopping_times(parameters, models, samples
          return values.mean(), values.std() / np.sqrt(parameters['lower bound samples'])
```

3.2 Upper bound

In order to compute the upper bound for the value of the option We need unbiased estimates of the continuation values based on computed stopping times. Taking exactly the result of model is obviously biased (because of using only one sample) In order to make it unbiased we generate several continuations for each path and time step and average the coresponding continuation values.

```
In [ ]: def get_values_by_averaging(parameters, models, underlying_brownian_motion=None):
          if underlying_brownian_motion is None:
            underlying_brownian_motion = brownian_motion(parameters, parameters['upper bound sa
          continued_samples = add_feature(continue_stock_prices(parameters, underlying_brownia
          values = continued_samples[..., -1, -1]
          for i in range(parameters['N']):
            time_i_values = values[:, i]
            mask = np.ones((parameters['upper bound samples'], parameters['upper bound repeats
            for j in range(i + 1, parameters['N']):
              bigger_mask = (continued_samples[:, i, :, j, -1].reshape((-1,)) >= models[j].pred
              time_i_values[np.logical_and(mask, bigger_mask)] = continued_samples[:, i][np.lo
              mask = np.logical_and(mask, ~bigger_mask)
            values[:, i, :] = time_i_values
          return values.mean(axis=2)
```

Uses unbiased estimates for the continuation values to compute martingale part of approximation for snell's envelope:

$$M_n^\theta - M_{n-1}^\theta = 1_{\{C_N^\Theta \leq g(n,X_n)\}} g(n, X_n) + 1_{\{C_N^\Theta > g(n,X_n)\}} C_n^\theta - C_{n-1}$$

```
In [ ]: def get_martingales(parameters, models, underlying_brownian_motion=None):
          if underlying_brownian_motion is None:
            underlying_brownian_motion = brownian_motion(parameters, parameters['upper bound sa
          samples = add_feature(stock_price_processes(parameters, parameters['upper bound samp
          values = get_values_by_averaging(parameters, models, underlying_brownian_motion=unde
          martingales = np.zeros((parameters['upper bound samples'], parameters['N'] + 1))
          for i in range(1, parameters['N'] + 1):
            if i == parameters['N']:
              martingales[:, i] = martingales[:, i - 1] + samples[:, i, -1] - values[:, i - 1]
            else:
              decisions = (samples[:, i, -1] >= models[i].predict(samples[:, i], batch_size=pa
              martingales[:, i] = martingales[:, i - 1] + decisions * samples[:, i, -1] + (1 -
          return martingales
```

Computes the upper bound based on the relation:

$$V_0 \leq \mathbb{E}\left[\max_{0\leq n\leq N}(g(n, X_n) - M_n - \epsilon_n)\right]$$

by generating independent relizations of m_n^k of $M_n^\theta + \epsilon$ along independent relizations $(x_n^k)_{n=0}^N, k = 1, \ldots, K$ and estimates U as:

$$\hat{U} = \frac{1}{K_u}\sum_{k=K+K_L+1}^{K+K_L+K_U}\max_{0\leq n\leq N}(g(n, x_n^k) - m_n^k)$$

based on the sample standard deviation:

$$\hat{U} = \sqrt{\frac{1}{K_u - 1}\sum_{k=K+K_L+1}^{K+K_L+K_U}(\max_{0\leq n\leq N}(g(n, x_n^k) - m_n^k) - \hat{U})^2}$$

We get the $1 - \alpha/2$ confidence interval:

$$\left(-\infty, \hat{U} + z_{\alpha/2}\frac{\hat{\sigma_U}}{\sqrt{K_u}}\right]$$

```
In [ ]: def get_upper_bound(parameters, models):
          underlying_brownian_motion = brownian_motion(parameters, parameters['upper bound samp
          samples = add_feature(stock_price_processes(parameters, parameters['upper bound sampl
          martingales = get_martingales(parameters, models, underlying_brownian_motion)
          max_difference = (samples[..., -1] - martingales).max(axis=1)
          return max_difference.mean(), max_difference.std() / np.sqrt(len(max_difference))
```

The point estimate is given by:

$$\hat{V} = \frac{\hat{L} + \hat{U}}{2}$$

and $1 - \alpha$ confidence interval is given by:

$$\left[\hat{L} - z_{\alpha/2}\frac{\hat{\sigma_L}}{\sqrt{K_L}}, \hat{U} + z_{\alpha/2}\frac{\hat{\sigma_U}}{\sqrt{K_u}}\right]$$

```
In [ ]: def get_point_estimate_and_confindence_interval(parameters, models):
          lower_bound_mean, lower_bound_std = get_lower_bound(parameters, models)
          upper_bound_mean, upper_bound_std = get_upper_bound(parameters, models)
          return (lower_bound_mean + upper_bound_mean) / 2, (lower_bound_mean + norm.ppf(0.125
```

Estimates the price with a confidence interval based on the parameters in the paper.

```
In [ ]: ds = [5, 10]
        s0s = [90, 100, 110]
        to_dataframe_list = []
        for d, s0 in product(ds, s0s):
          parameters['s0'] = s0
          parameters['d'] = d
```

```
        continuation_models = get_continuation_models(parameters)
        point_estimate, confidence_interval = get_point_estimate_and_confindence_interval(pa
        to_dataframe_list.append([d, s0, point_estimate, f'[{confidence_interval[0]}, {confi
        del continuation_models, point_estimate, confidence_interval
        collect()
    price_estimation_results = pd.DataFrame(to_dataframe_list, columns=['d', 's0', 'Point 
```

```
100%|| 8/8 [03:03<00:00, 22.97s/it]
```

The results are a bit less accurate than the paper due to shorter tarining time and using smaller samples because of RAM and GPU limits in colab but they are accurate enought to ensure the correctness of implementation.

```
In [ ]: price_estimation_results
```

```
Out[ ]:     d   s0  Point Estimate                       95% confidence interval
        0   5   90       16.583213  [16.328291815860617, 16.925829869544604]
        1   5  100       26.066984   [25.878481168374357, 26.33529173631315]
        2   5  110       36.656044    [36.25839442892457, 37.16202627668781]
        3  10   90       26.087416   [25.85291989546661, 26.413441582477905]
        4  10  100       38.092223     [37.72118684121251, 38.5852557650039]
        5  10  110       50.621243  [50.287022719046576, 51.062966013000384]
```

3.3 Hedging

The model we use for modeling amount of hedge there are two hidden layers with $d + 50$ layers and tanh activation function and batch normalization.

```
In [ ]: def get_hedging_model(parameters):
            return Sequential([
                Dense(parameters['d'] + 50, activation='tanh', input_shape=(parameters['d'],)),
                BatchNormalization(),
                Dense(parameters['d'] + 50, activation='tanh'),
                BatchNormalization(),
                Dense(parameters['d'])
            ])
```

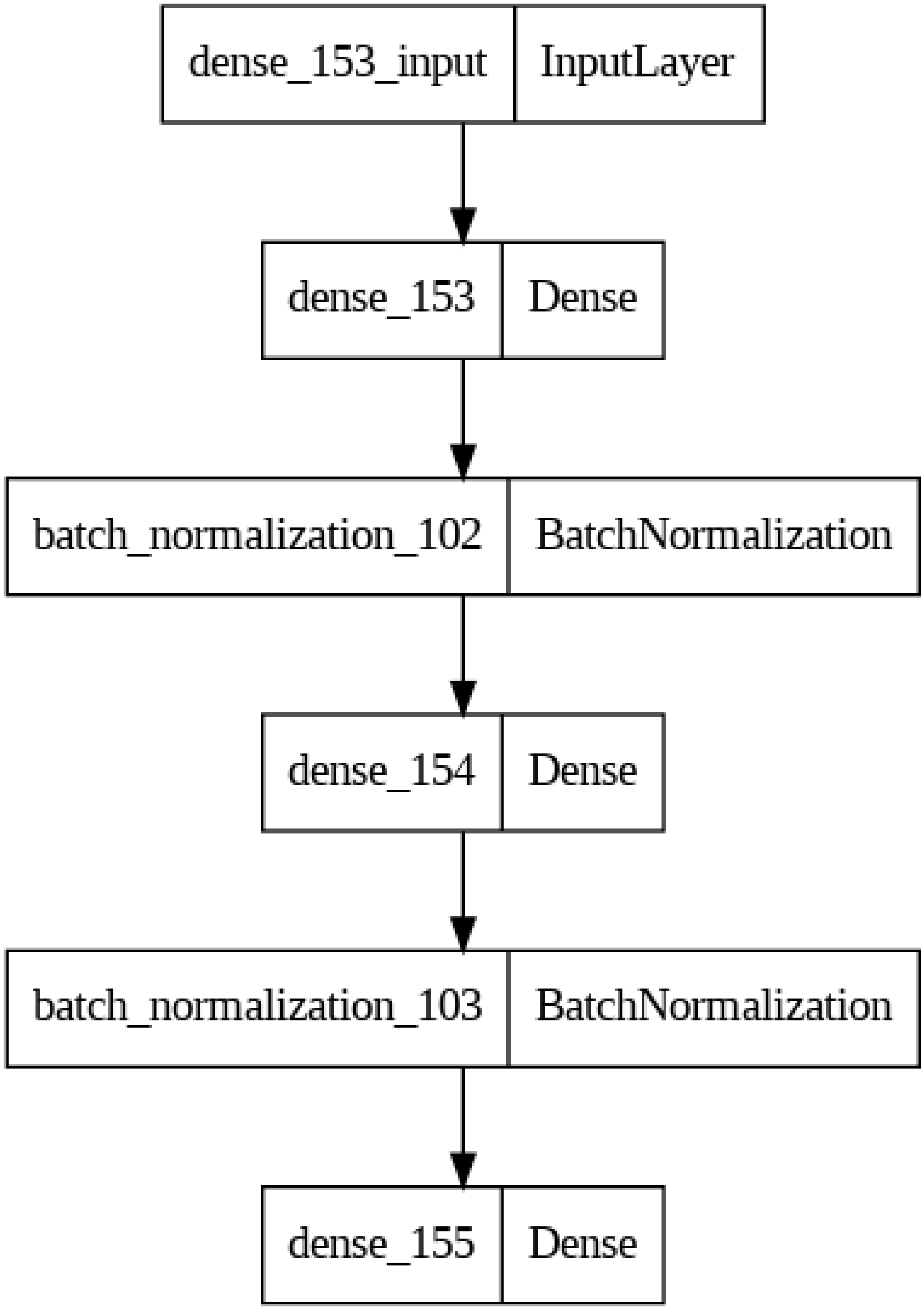
```
In [ ]: get_hedging_model(parameters).summary()
```

```
Model: "sequential_50"
_________________________________________________________________
 Layer (type)                Output Shape              Param #
=================================================================
 dense_150 (Dense)           (None, 60)                660


 batch_normalization_100 (Ba  (None, 60)               240
```

```
 tchNormalization)

 dense_151 (Dense)           (None, 60)                3660

 batch_normalization_101 (Ba  (None, 60)               240
 tchNormalization)

 dense_152 (Dense)           (None, 10)                610

=================================================================
Total params: 5,410
Trainable params: 5,170
Non-trainable params: 240
_________________________________________________________________
```

```
In [ ]: plot_model(get_hedging_model(parameters))
```

```
Out[None]:
```



Code to train simultaniously M neural network to minimize hedging error in one subinterval.

The computation does not completely fit into a keras model so we create manually a training loop at each epoch the training loss is evaluated by summing up heding results in each interval and seeing how much we missed the on the hedge.

The weights of neural networks trained on previous subinterval are used to initialize the wights if they are given in previous_models argument.

The network parameters h_{λ_m} are trained to minimize the error:

$$\sum_{k=1}^{K_H} (C^{\theta_{n-1}}(y^k_{(n-1)M}) + \sum_{m=(n-1)M}^{nM-1} h^{\lambda_m}(y^k_m) \cdot (p_{m+1}(y^k_{m+1}) - p_m(y^k_m)) - v^{\theta_n}(y^k_{nM}))^2$$

Where p_m is the value of hedging instruments at time m and:

$$v^{\theta_n}(x) := g(n,x) \vee c^{\theta_n}(x)$$

The hedging instruments are also of the form:

$$P^i_{u_m} = s_0 \exp(\sigma W^i_{u_m} - \sigma^2 u_m/2)$$

i.e stocks with dividends continuasly reinvested in the same stock.

```
In [ ]: def train_single_interval(parameters, stock, hedging_instruments, continuation_models,
          models = [get_hedging_model(parameters) for _ in range(parameters['M'])]
          if previous_models is not None:
            for model, copy_model in zip(models, previous_models):
              model.set_weights(copy_model.get_weights())
          weights = []
          for model in models:
            weights.extend(model.trainable_weights)
          optimizer = Adam()
          value_at_future = np.maximum(continuation_models[1].predict(stock[:, -1], verbose=Fa
          pbar = tqdm(range(epoch))
          for _ in pbar:
            loss = 0
            with tf.GradientTape() as tape:
              predictions = tf.squeeze(continuation_models[0](stock[:, 0]))
              for i in range(parameters['M']):
                predictions = tf.add(tf.cast(predictions, tf.float32), tf.reduce_sum(models[i]
              loss = MeanSquaredError()(value_at_future, predictions)
            grads = tape.gradient(loss, weights)
            optimizer.apply_gradients(zip(grads, weights))
            pbar.set_description(f'loss: {float(loss.numpy())}')
          return models
```

Code to train the neural network for each sub interval.

```
In [ ]: def get_hedging_models(parameters):
          modified_parameters = modify_parameters(parameters)
          underlying_brownian_motion = brownian_motion(modified_parameters, parameters['hedge s
          stock = add_feature(stock_price_processes(parameters, parameters['hedge steps'] * pa
          hedging_instruments = add_feature(stock_price_processes(modified_parameters, paramete
          models = []
          for i in range(parameters['N']):
            models.extend(train_single_interval(parameters, stock[:, i * parameters['M']: (i +
          return models
```

Calculate hedging error and hedging shortfall by creating independent samples and caculating the averaging hedging error and hedging shortfall for them.

```
In [ ]: def get_hedging_error(parameters, hedging_models, continuation_models, one_step=False)
            modified_parameters = modify_parameters(parameters)
            underlying_brownian_motion = brownian_motion(modified_parameters, parameters['hedging
            stock = add_feature(stock_price_processes(parameters, parameters['hedging error sampl
            hedging_instruments = add_feature(stock_price_processes(modified_parameters, paramete
            stopping_times = get_stopping_times(parameters, continuation_models, stock[:, [i * pa
            errors = np.ones(parameters['hedging error samples']) * continuation_models[0].averag
            if one_step:
              errors -= np.maximum(continuation_models[1].predict(stock[:, parameters['M']]).sque
            else:
              errors -= stock[range(len(stock)), stopping_times, -1]
            for time in range(parameters['M'] if one_step else parameters['M'] * parameters['N']
              errors[stopping_times > time] += (hedging_models[time].predict(stock[:, time, :-1]
            return errors
```

Compute the hedging results on parameters specified in the paper.

```
In [ ]: ds = [5]
        s0s = [90, 100, 110]
        ms = [12]
        to_dataframe_list = []
        for d, s0, m in product(ds, s0s, ms):
          collect()
          parameters['d'] = d
          parameters['s0'] = s0
          parameters['M'] = m
          continuation_models = get_continuation_models(parameters)
          hedging_models = get_hedging_models(parameters)
          one_step_errors = get_hedging_error(parameters, hedging_models, continuation_models,
          total_errors = get_hedging_error(parameters, hedging_models, continuation_models, one
          to_dataframe_list.append([d, s0, m, one_step_errors.mean(), np.abs(one_step_errors[on
                                    total_errors.mean(), np.abs(total_errors[one_step_errors <
          del continuation_models, hedging_models, one_step_errors, total_errors
          collect()
        hedging_results = pd.DataFrame(to_dataframe_list, columns=['d', 's0', 'M', 'IHE', 'IHS

100%|| 8/8 [02:41<00:00, 20.16s/it]
  0%|          | 0/300 [00:00<?, ?it/s]WARNING:tensorflow:5 out of the last 5 calls to <functio
WARNING:tensorflow:6 out of the last 6 calls to <function _BaseOptimizer._update_step_xla at 0x
loss: 2.066514015197754: 100%|| 300/300 [02:05<00:00,  2.40it/s]
loss: 6.703769207000732: 100%|| 100/100 [00:53<00:00,  1.87it/s]
loss: 7.10136604309082: 100%|| 100/100 [00:55<00:00,  1.82it/s]
loss: 6.418933868408203: 100%|| 100/100 [00:55<00:00,  1.80it/s]
loss: 6.267137050628662: 100%|| 100/100 [00:56<00:00,  1.76it/s]
loss: 7.104688167572021 5: 100%|| 100/100 [00:57<00:00,  1.75it/s]
loss: 11.12924575805664: 100%|| 100/100 [00:56<00:00,  1.78it/s]
```

```
loss: 11.276144027709961: 100%|| 100/100 [00:55<00:00,  1.81it/s]
loss: 13.22627067565918: 100%|| 100/100 [00:57<00:00,  1.75it/s]
```

Result for higher dimensions could not fit into RAM due to RAM limit of colab. The Heding Error and Hedging shortfall are small enough to ensure correctness of implementation.

In [ ]:
```
hedging_results
```

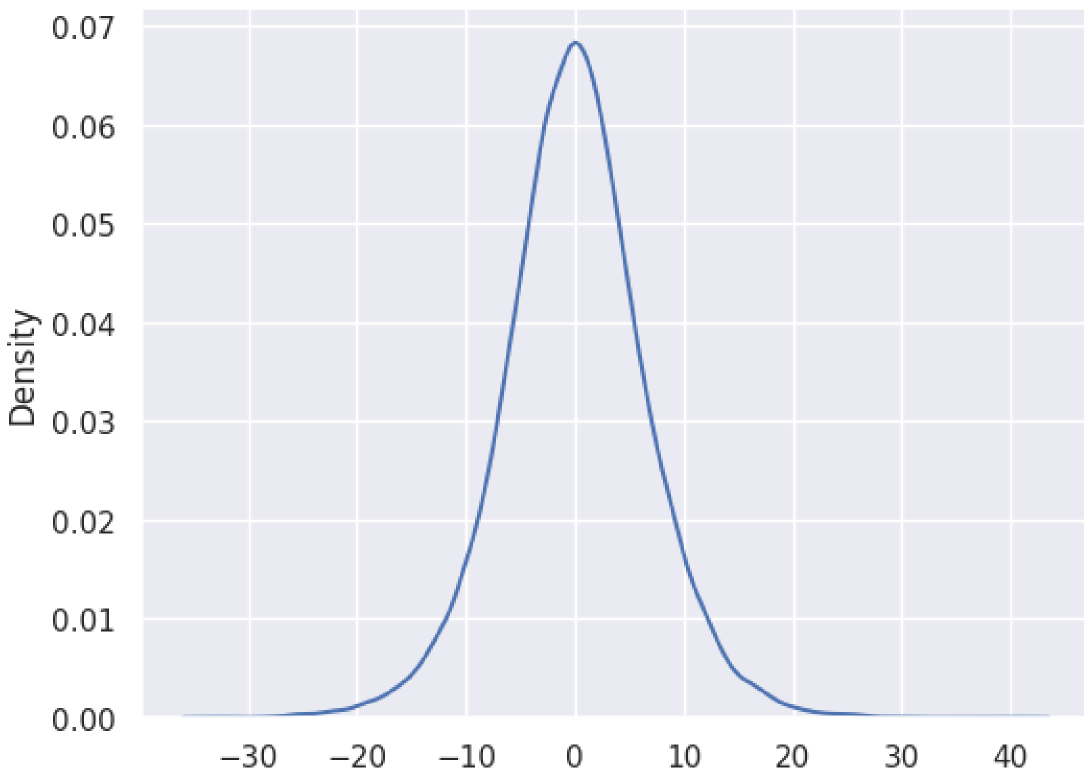
Out[ ]:

	d	s0	M	IHE	IHS	IHS / V	HE	HS	HS / V
0	5	90	12	0.712886	1.020717	0.061939	0.028708	4.896300	0.297115
1	5	100	12	0.513142	1.223778	0.047181	0.105874	4.658804	0.179612
2	5	110	12	-0.318863	1.517249	0.041512	0.104633	4.959441	0.135689

In [ ]:
```
parameters['d'] = 5
parameters['s'] = 90
parameters['m'] = 12
continuation_models = get_continuation_models(parameters)
hedging_models = get_hedging_models(parameters)
total_errors = get_hedging_error(parameters, hedging_models, continuation_models, one_s
```

```
100%|| 8/8 [03:08<00:00, 23.60s/it]
loss: 3.21195650100708: 100%|| 300/300 [02:12<00:00,  2.27it/s]
loss: 12.125414848327637: 100%|| 100/100 [01:02<00:00,  1.60it/s]
loss: 9.646745681762695: 100%|| 100/100 [00:59<00:00,  1.69it/s]
loss: 7.639955043792725: 100%|| 100/100 [00:56<00:00,  1.77it/s]
loss: 7.48165225982666: 100%|| 100/100 [00:55<00:00,  1.80it/s]
loss: 7.587880611419678: 100%|| 100/100 [00:56<00:00,  1.76it/s]
loss: 9.872846603393555: 100%|| 100/100 [00:58<00:00,  1.70it/s]
loss: 13.4998140335083: 100%|| 100/100 [00:55<00:00,  1.82it/s]
loss: 18.091604232788086: 100%|| 100/100 [00:58<00:00,  1.71it/s]
```

In [ ]:
```
sns.kdeplot(total_errors);
```




```
In [ ]:
```