# Information Theory

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These slides are based on slides from Le Song, Roni Rosenfeld, Chao Zhang, and Maneesh Sahani.

# Outline

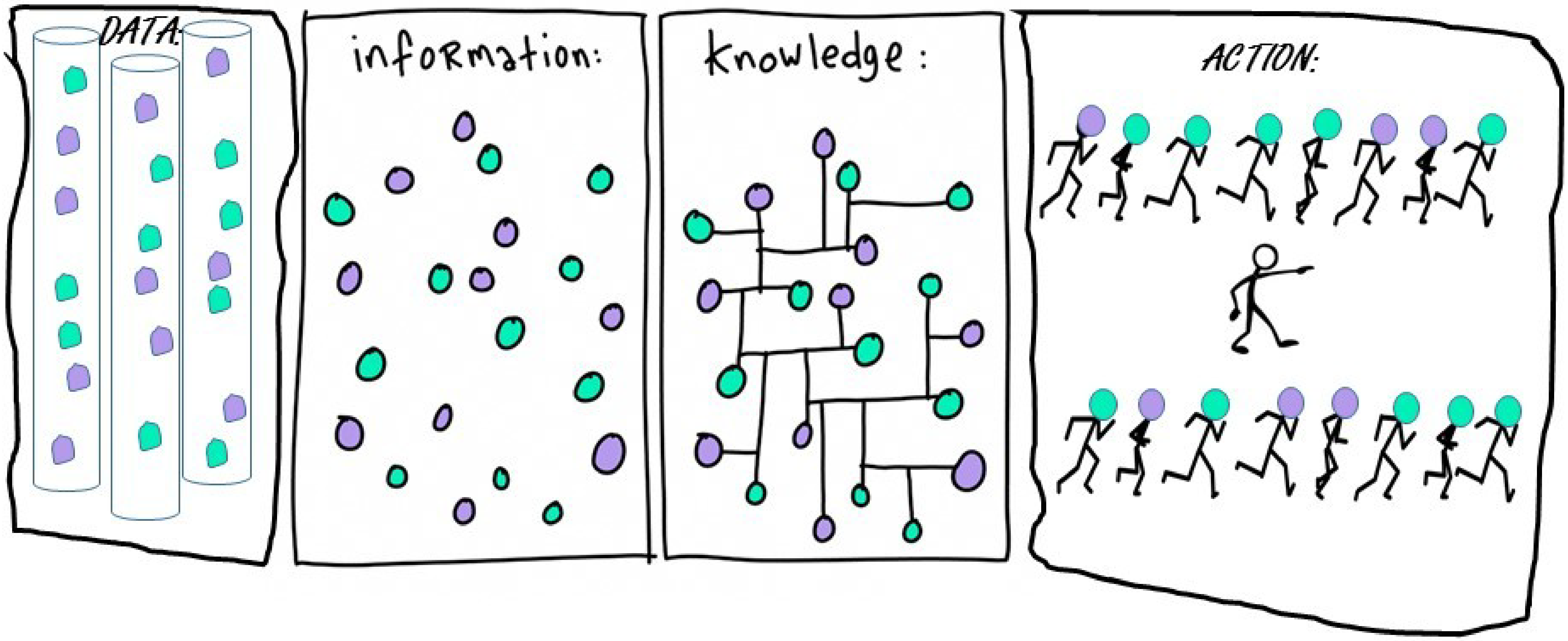
- Motivation
- Entropy
- Conditional Entropy and Mutual Information
- Cross-Entropy and KL-Divergence

# Uncertainty and Information

**Information** is processed data whereas **knowledge** is **information** that is modeled to be useful.

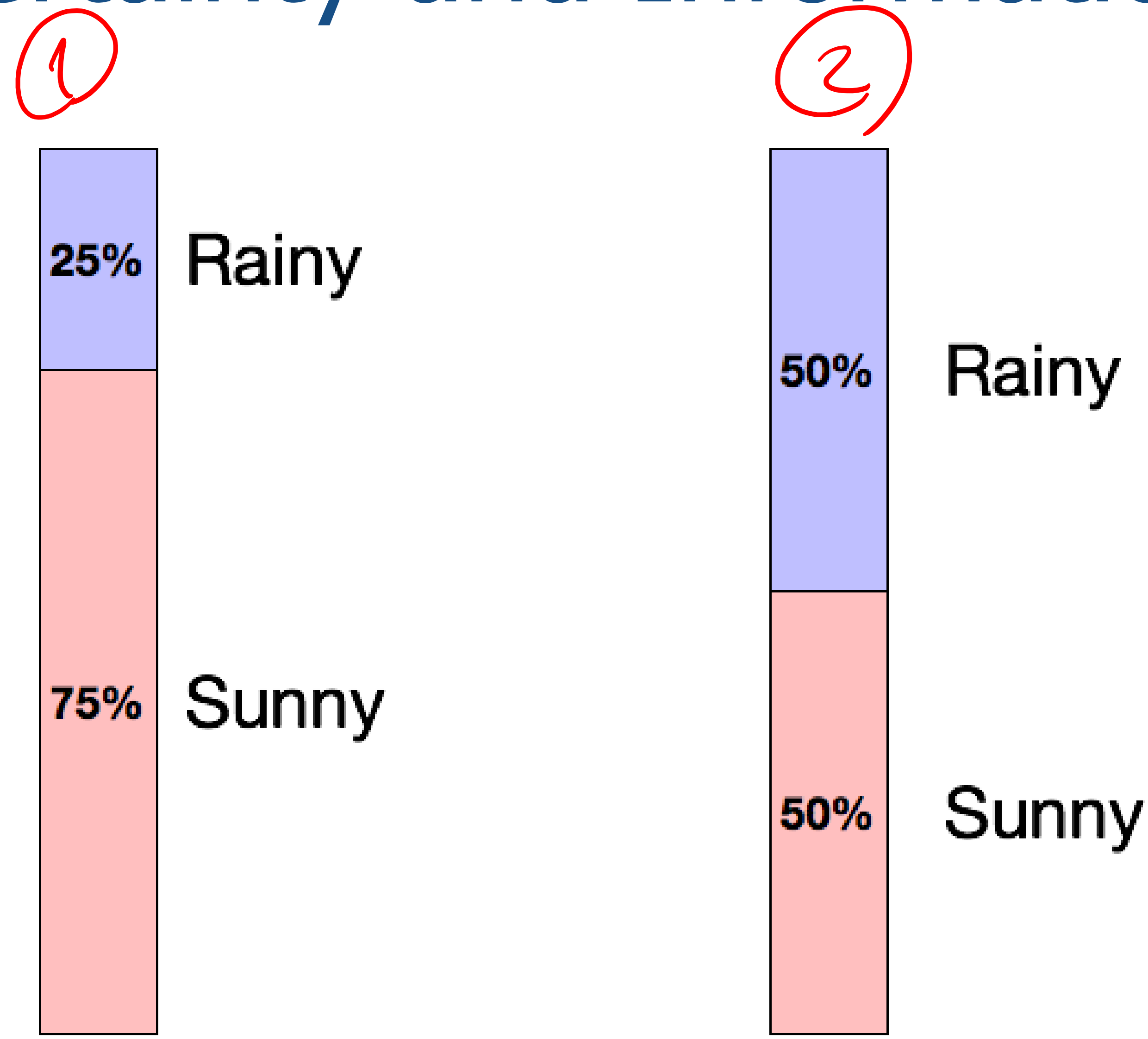
You need **information** to be able to get **knowledge**

- information $\neq$ knowledge
  Concerned with abstract possibilities, not their meaning




Created by Bruce Campbell: “DIKA – ancient Chinese saying for get up and DO! Data-Information-Knowledge-Action.”

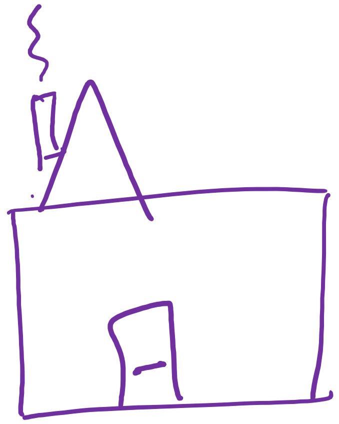
# Uncertainty and Information




**Which day is more uncertain?**

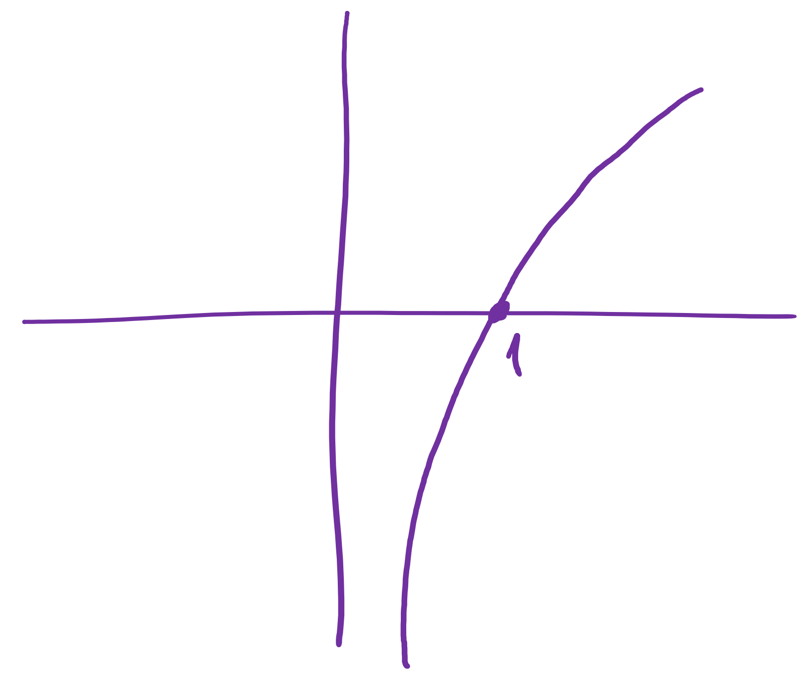
**How do we quantify uncertainty?**

High entropy correlates to high information or the more uncertain

 $\rightarrow$ [cat, cat, cat, dog] $\Longrightarrow$  $\rightarrow$

$P(X = cat) = 1 \quad \Longrightarrow \quad$ surprised information $\quad I(X=cat) = \log_2 \frac{1}{P(X=cat)}$

$P(X = dog) = \frac{1}{4} \qquad I(X=dog) = \log_2 \frac{1}{\frac{1}{2^2}} = 2$




$P(X = cat) = \frac{3}{4} \qquad I(X=cat) = \log_2 \frac{4}{3}$

$$E[g(x)] = \sum p(x)\, g(x)$$

$$\boxed{E[I(x)] = \sum p(x)\, I(x)} = H(x)$$

$$H(x) = P(x = cat)\, I(x = cat) + P(x = dog)\, I(x = dog)$$

$$H(x) = \frac{3}{4} \log_2 \frac{4}{3} + \frac{1}{4} \quad 2$$

# Information

Let X be a random variable with distribution p(x)

$$I(X) = \log_{2}(\frac{1}{p(x)})$$

# Motivation: Compression

- Suppose we observe a sequence of events:
  - Coin tosses
  - Words in a language
  - notes in a song
  - etc.
- We want to record the sequence of events in the smallest possible space.
- In other words we want the shortest representation which preserves all information.
- Another way to think about this: How much information does the sequence of events actually contain?

# MOTIVATION: COMPRESSION

To be concrete, consider the problem of recording coin tosses in unary.

$$T, T, T, T, H$$

Approach 1:

0 and 00

| H | T |
|---|---|
| 0 | 00 |

$$00, 00, 00, 00, 0$$

We used 9 characters

Which one has a higher probability: T or H?

Which one should carry more information: T or H?

# Motivation: Compression

To be concrete, consider the problem of recording coin tosses in unary.

$$T, T, T, T, H$$

Approach 2:

| H | T |
|---|---|
| 00 | 0 |

$$0, 0, 0, 0, 00$$

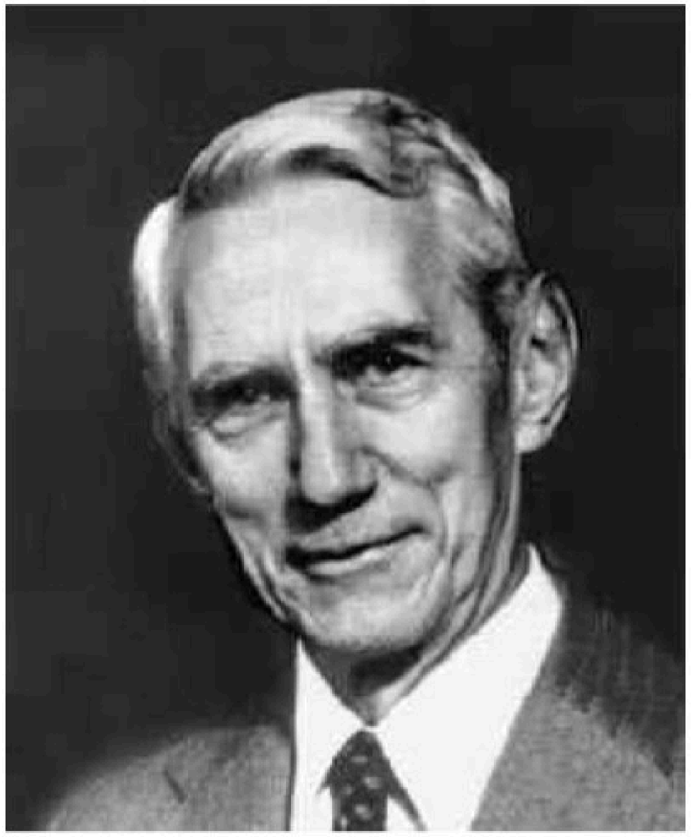
We used 6 characters

# Motivation: Compression

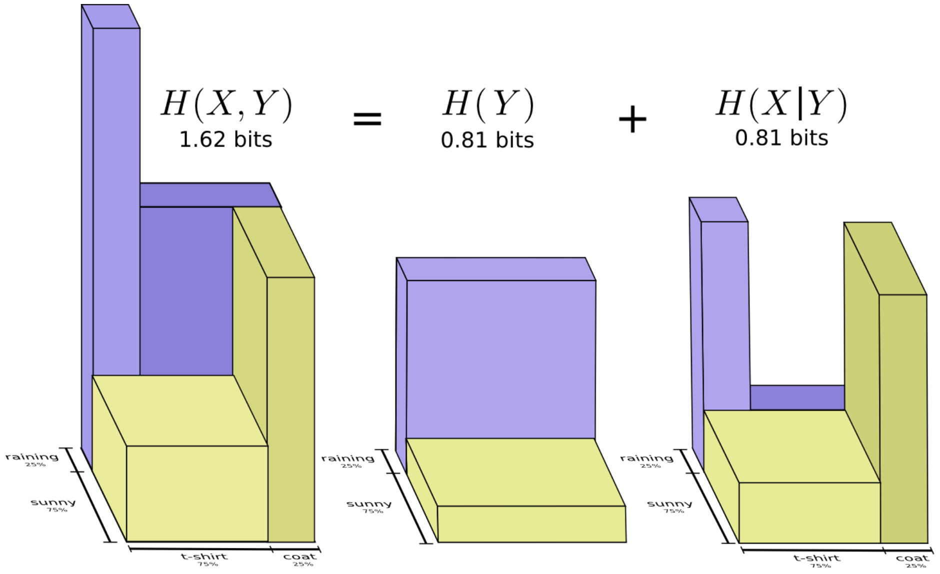
- Frequently occuring events should have short encodings
- We see this in english with words such as “a”, “the”, “and”, etc.
- We want to maximise the information-per-character
- seeing common events provides little information
- seeing uncommon events provides a lot of information

# Information Theory

- Information theory is a mathematical framework which addresses questions like:
  - How much information does a random variable carry about?
  - How efficient is a hypothetical code, given the statistics of the random variable?
  - How much better or worse would another code do?
  - Is the information carried by different random variables complementary or redundant?



Claude Shannon



# Outline

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- Entropy
- Conditional Entropy and Mutual Information
- Cross-Entropy and KL-Divergence

# Entropy

- Entropy $H(Y)$ of a random variable $Y$

$H(y) = \sum p(x) I(x)$

$I(x) = \log_2 p(x)^{-1}$

$$H(Y) = -\sum_{k=1}^{K} P(y=k)\log_2 P(y=k)$$

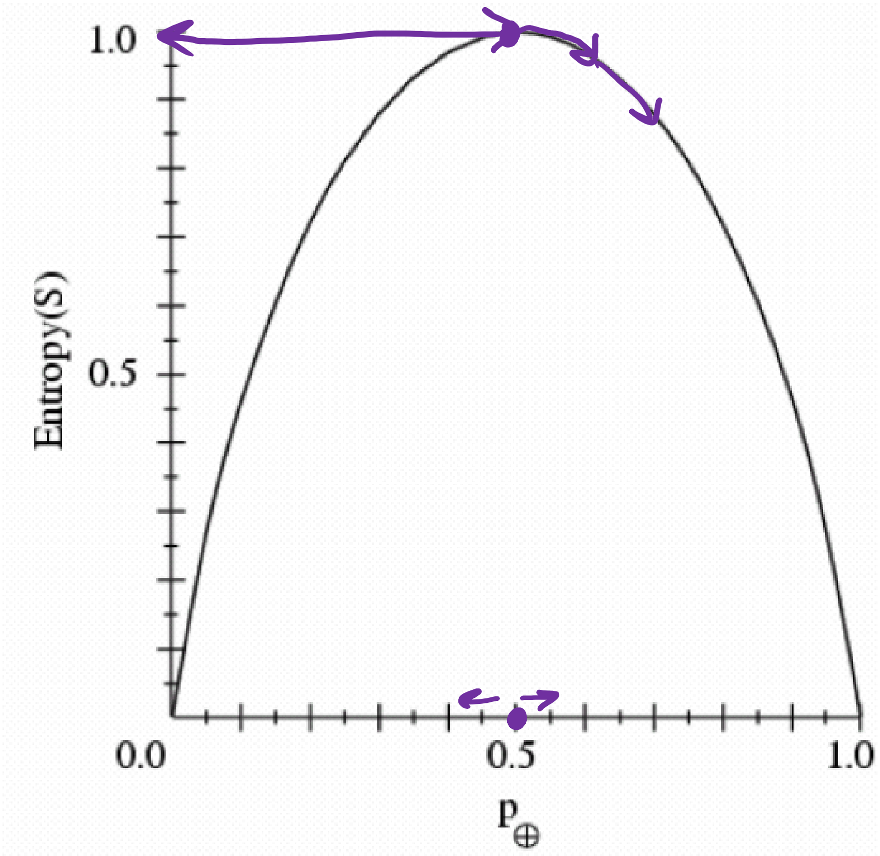
- $H(Y)$ is the expected number of bits needed to encode a randomly drawn value of $Y$ (under most efficient code)

- Information theory:

  Most efficient code assigns $-\log_2 P(Y=k)$ bits to encode the message $Y = k$, So, expected number of bits to code one random $Y$ is:

$$-\sum_{k=1}^{K} P(y=k)\log_2 P(y=k)$$

# Entropy




- $S$ is a sample of coin flips
- $p_+$ is the proportion of heads in $S$
- $p_-$ is the proportion of tails in $S$
- Entropy measure the uncertainty of $S$

$$H(S) \equiv -p_+ \log_2 p_+ - p_- \log_2 p_-$$

# Entropy Computation: An Example

$$H(S) \equiv -p_+ \log_2 p_+ - p_- \log_2 p_-$$

| | |
|---|---|
| head | 0 |
| tail | 6 |

**P(h) = 0/6 = 0 P(t) = 6/6 = 1**

**Entropy = – 0 log 0 – 1 log 1 = – 0 – 0 = 0**

| | |
|---|---|
| head | 1 |
| tail | 5 |

**P(h) = 1/6 P(t) = 5/6**

**Entropy = – (1/6) $\log_2$ (1/6) – (5/6) $\log_2$ (5/6) = 0.65**

| | |
|---|---|
| head | 2 |
| tail | 4 |

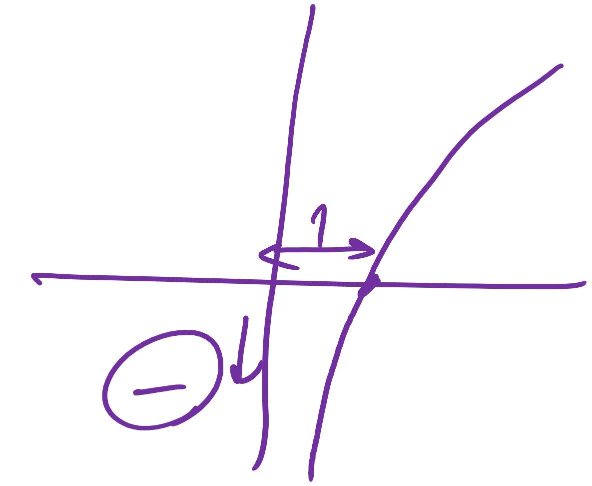
**P(h) = 2/6 P(t) = 4/6**

**Entropy = – (2/6) $\log_2$ (2/6) – (4/6) $\log_2$ (4/6) = 0.92**

actual cat

$p_i = \begin{bmatrix} [1\ 0\ 0] \\ [0\ 1\ 0] \text{ dog} \\ \vdots \end{bmatrix}$ $q_i = \begin{bmatrix} [0.8\ 0.1\ 0.1] \\ \vdots \end{bmatrix}$

# Properties of Entropy



$$H(P) = \sum_i p_i \cdot \log \frac{1}{p_i}$$

1. Non-negative: $H(P) \geq 0$

$\sum p_i \log \frac{1}{p_i} - \sum p_i \log \frac{1}{q_i} < 0 \Rightarrow \sum p_i \left(\log \frac{1}{p_i} - \log \frac{1}{q_i}\right) < 0$

$\sum p_i \left(\log \frac{q_i}{p_i}\right) < 0$

2. Invariant wrt permutation of its inputs:
$$H(p_1, p_2, \ldots, p_k) = H(p_{\tau(1)}, p_{\tau(2)}, \ldots, p_{\tau(k)})$$

3. For any *other* probability distribution $\{q_1, q_2, \ldots, q_k\}$:

actual pd

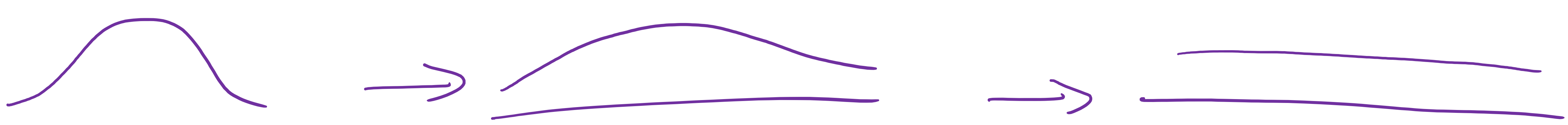
$$H(P) = \sum_i p_i \cdot \log \frac{1}{p_i} < \sum_i p_i \cdot \log \frac{1}{q_i}$$

log k

predicted pd

4. $H(P) \leq \log k$, with equality iff $p_i = 1/k \ \forall i$

5. The further $P$ is from uniform, the lower the entropy.



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# Joint Entropy

$p(a,b) = p(a|b)\,p(b)$

$H(a,b) = H(a|b) + H(b)$

Temperature

huMidity

| | cold | mild | hot | |
|---|---|---|---|---|
| low | 0.1 | 0.4 | 0.1 | 0.6 |
| high | 0.2 | 0.1 | 0.1 | 0.4 |
| | 0.3 | 0.5 | 0.2 | 1.0 |

- $H(T) = H(0.3, 0.5, 0.2) = 1.48548$ $\quad 0.3 \log_2 \frac{1}{0.3} + 0.5 \log_2 \frac{1}{0.5} + 0.2 \log_2 \frac{1}{0.2}$
- $H(M) = H(0.6, 0.4) = 0.970951$
- $\boxed{H(T) + H(M) = 2.456431}$
- **Joint Entropy**: consider the space of $(t, m)$ events $\boxed{H(T, M)} = \sum_{t,m} P(T = t, M = m) \cdot \log \frac{1}{P(T=t, M=m)}$
  $H(0.1, 0.4, 0.1, 0.2, 0.1, 0.1) = 2.32193$
  $0.1 \log_2 \frac{1}{0.1} - - - - +$

Notice that $H(T, M) \leqslant H(T) + H(M)$ !!!

$$H(T,M) = H(T|M) + H(M) = H(M|T) + H(T)$$

# Conditional Entropy

$$H(Y|X) = \sum_{x \in X} p(x) \underbrace{H(Y|X=x)}_{g(x)} = \sum_{x \in X, y \in Y} p(x,y) log \frac{p(x)}{p(x,y)}$$

$$P(T = t | M = m)$$

P(M=low) H(T|M=low) +
P(M=high) H(T|M=high)

| | cold | mild | hot | |
|---|---|---|---|---|
| low | 1/6 | 4/6 | 1/6 | 1.0 |
| high | 2/4 | 1/4 | 1/4 | 1.0 |

**Conditional Entropy**:

- $H(T|M = low) = H(1/6, 4/6, 1/6) = 1.25163$
- $H(T|M = high) = H(2/4, 1/4, 1/4) = 1.5$
- **Average Conditional Entropy** (aka equivocation):
  $H(T/M) = \sum_m P(M = m) \cdot H(T|M = m) =$
  $0.6 \cdot H(T|M = low) + 0.4 \cdot H(T|M = high) = 1.350978$

# Conditional Entropy

$$P(M = m|T = t)$$

| | cold | mild | hot |
|---|---|---|---|
| low | 1/3 | 4/5 | 1/2 |
| high | 2/3 | 1/5 | 1/2 |
| | 1.0 | 1.0 | 1.0 |

Conditional Entropy:

- $H(M|T = cold) = H(1/3, 2/3) = 0.918296$
- $H(M|T = mild) = H(4/5, 1/5) = 0.721928$
- $H(M|T = hot) = H(1/2, 1/2) = 1.0$
- Average Conditional Entropy (aka Equivocation):
  $H(M/T) = \sum_t P(T = t) \cdot H(M|T = t) =$
  $0.3 \cdot H(M|T = cold) + 0.5 \cdot H(M|T = mild) + 0.2 \cdot H(M|T = hot) = 0.8364528$

# Conditional Entropy

- Conditional entropy $H(Y|X)$ of a random variable $Y$ given $X_i$

Discrete random variables:

$$H(Y|X) = \sum_{x \in X} p(x_i) H(Y|X = x_i) = \sum_{x \in X, y \in Y} p(x_i, y_i) log \frac{p(x_i)}{p(x_i, y_i)}$$

Continuous:

$$H(Y|X) \; = -\int \left( \sum_{k=1}^{K} P(y = k|x_i) \log_2 P(y = k) \right) p(x_i) dx_i$$

# Mutual Information → information gain

- Mutual information: quantify the reduction in uncertainty in $Y$ after seeing feature $X_i$

$$I(X_i, Y) = H(Y) - H(Y|X_i)$$

- The more the reduction in entropy, the more informative a feature.

- Mutual information is symmetric
  - $I(X_i, Y) = I(Y, X_i) = H(X_i) - H(X_i|Y)$
  - $I(Y|X) = \int \sum_k^K p(x_i, y = k) \log_2 \frac{p(x_i, y=k)}{p(x_i)p(y=k)} \; dx_i$
  - $= \int \sum_k^K p(x_i|y = k)p(y = k) \log_2 \frac{p(x_i|y = k)}{p(x_i)} \; dx_i$

# Properties of Mutual Information

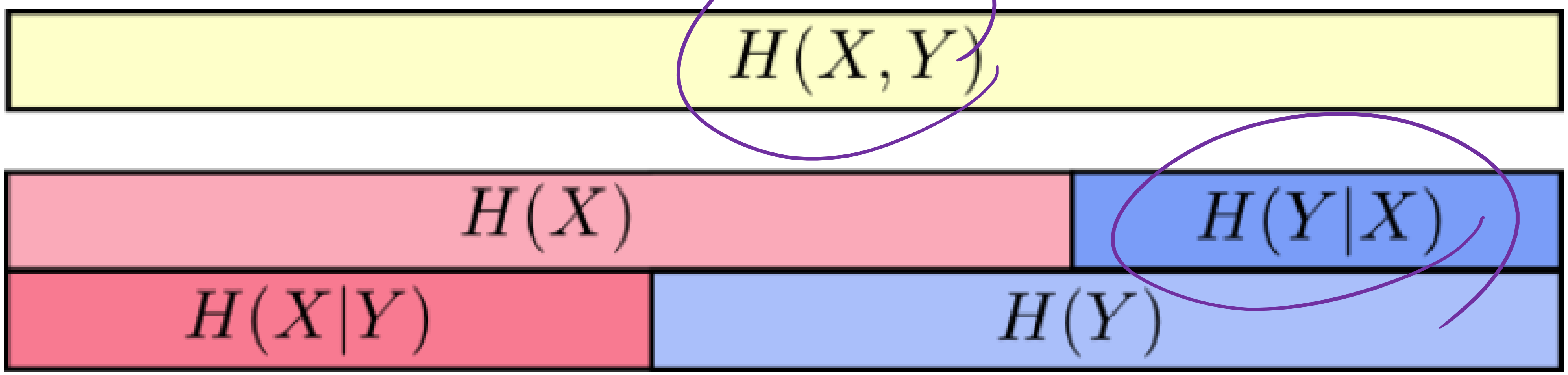
$$
\begin{aligned}
I(X,Y) = H(X) - H(X|Y) \\
&= \sum_x P(x) \cdot \log \frac{1}{P(x)} - \sum_{x,y} P(x,y) \cdot \log \frac{1}{P(x|y)} \\
&= \sum_{x,y} P(x,y) \cdot \log \frac{P(x|y)}{P(x)} \\
&= \boxed{\sum_{x,y} P(x,y) \cdot \log \frac{P(x,y)}{P(x)P(y)}}
\end{aligned}
$$

Properties of Average Mutual Information:

- Symmetric
- Non-negative
- Zero iff $X, Y$ independent

# CE and MI: Visual Illustration

Joint Entropy



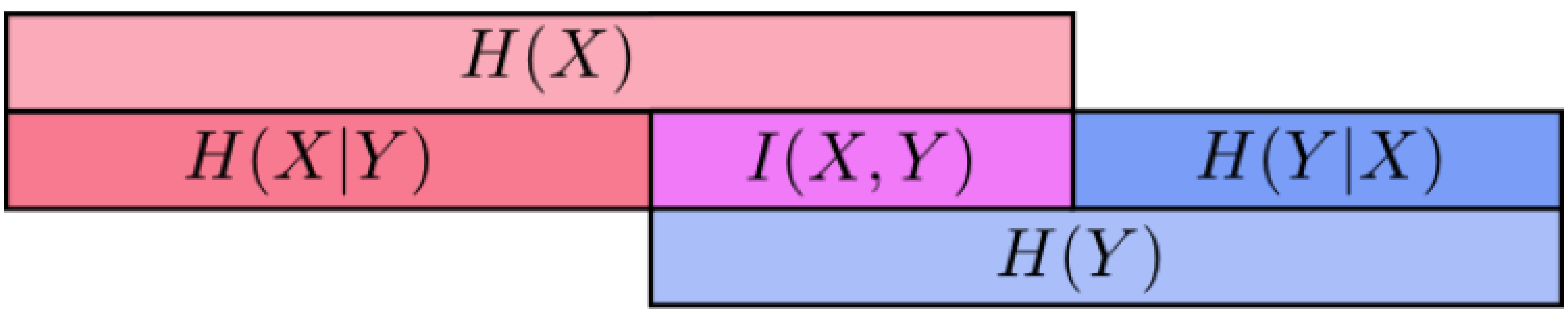




Image Credit: Christopher Olah.

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Let’s work on this subject in our Optimization lecture

# Cross Entropy

**Cross Entropy**: The expected number of bits when a wrong distribution Q is assumed while the data actually follows a distribution P

$$H(p,q) = -\sum_{x\in\mathcal{X}} p(x)\,\log q(x) = H(P) + KL[P][Q]$$

This is because:

$$H(p,q) = \mathrm{E}_p[l_i] = \mathrm{E}_p\left[\log \frac{1}{q(x_i)}\right]$$

$$H(p,q) = \sum_{x_i} p(x_i)\,\log \frac{1}{q(x_i)}$$

$$H(p,q) = -\sum_x p(x)\,\log q(x).$$

# Kullback-Leibler Divergence

Another useful information theoretic quantity measures the difference between two distributions.

$$\mathbf{KL}[P(S)\|Q(S)] = \sum_s P(s)\log\frac{P(s)}{Q(s)}$$

$$= \underbrace{\sum_s P(s)\log\frac{1}{Q(s)}}_{\text{cross entropy}} - \mathbf{H}[P] = H(P,Q) - H(P)$$

Excess cost in bits paid by encoding according to $Q$ instead of $P$.

KL Divergence is a **KIND OF** distance measurement

$$-\mathbf{KL}[P\|Q] = \sum_s P(s)\log\frac{Q(s)}{P(s)}$$

log function is concave or convex?

$$\sum_s P(s)\log\frac{Q(s)}{P(s)} \leq \log\sum_s P(s)\frac{Q(s)}{P(s)}$$

By Jensen Inequality

$$= \log\sum_s Q(s) = \log 1 = 0$$

So $\mathbf{KL}[P\|Q] \geq 0$. Equality iff $P = Q$

When $P = Q,\ KL[P||Q] = 0$

# Take-Home Messages

- Entropy
  - A measure for uncertainty
  - Why it is defined in this way (optimal coding)
  - Its properties
- Joint Entropy, Conditional Entropy, Mutual Information
  - The physical intuitions behind their definitions
  - The relationships between them
- Cross Entropy, KL Divergence
  - The physical intuitions behind them
  - The relationships between entropy, cross-entropy, and KL divergence