Density Estimation

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The slides are inspired from Le Song.

Why we love exponential terms?

When I try to take derivative of Exp

Outline

- Overview
- Parametric Density Estimation
- Nonparametric Density Estimation

Continuous variable

Continuous probability distribution

Probability density function

Density value

Temperature (real number)

Gaussian Distribution

$$\int f_X(x)dx = 1$$

Discrete variable

Discrete probability distribution

Probability mass function

Probability value

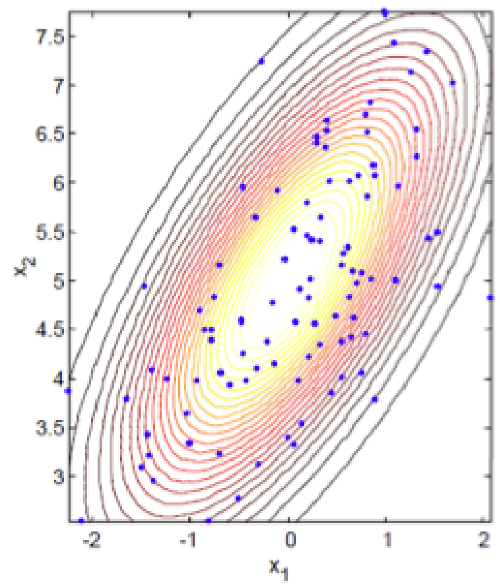
Coin flip (integer)

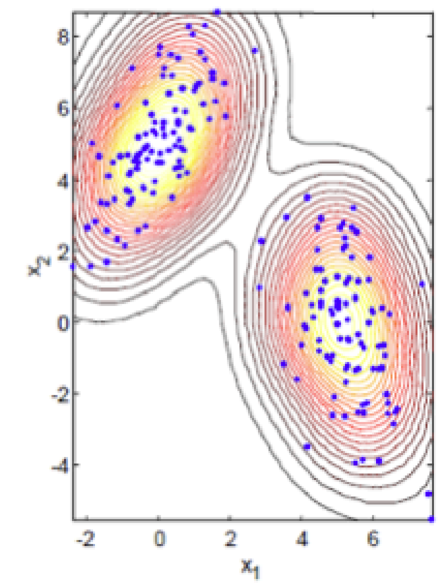
Bernoulli distribution

$$\sum_{x \epsilon A} f_X(x) = 1$$

Why Density Estimation?

- Learn more about the "shape" of the data cloud



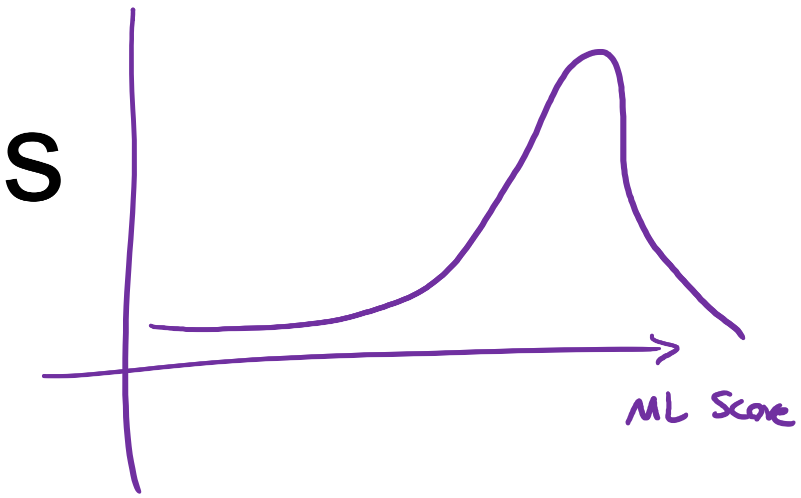


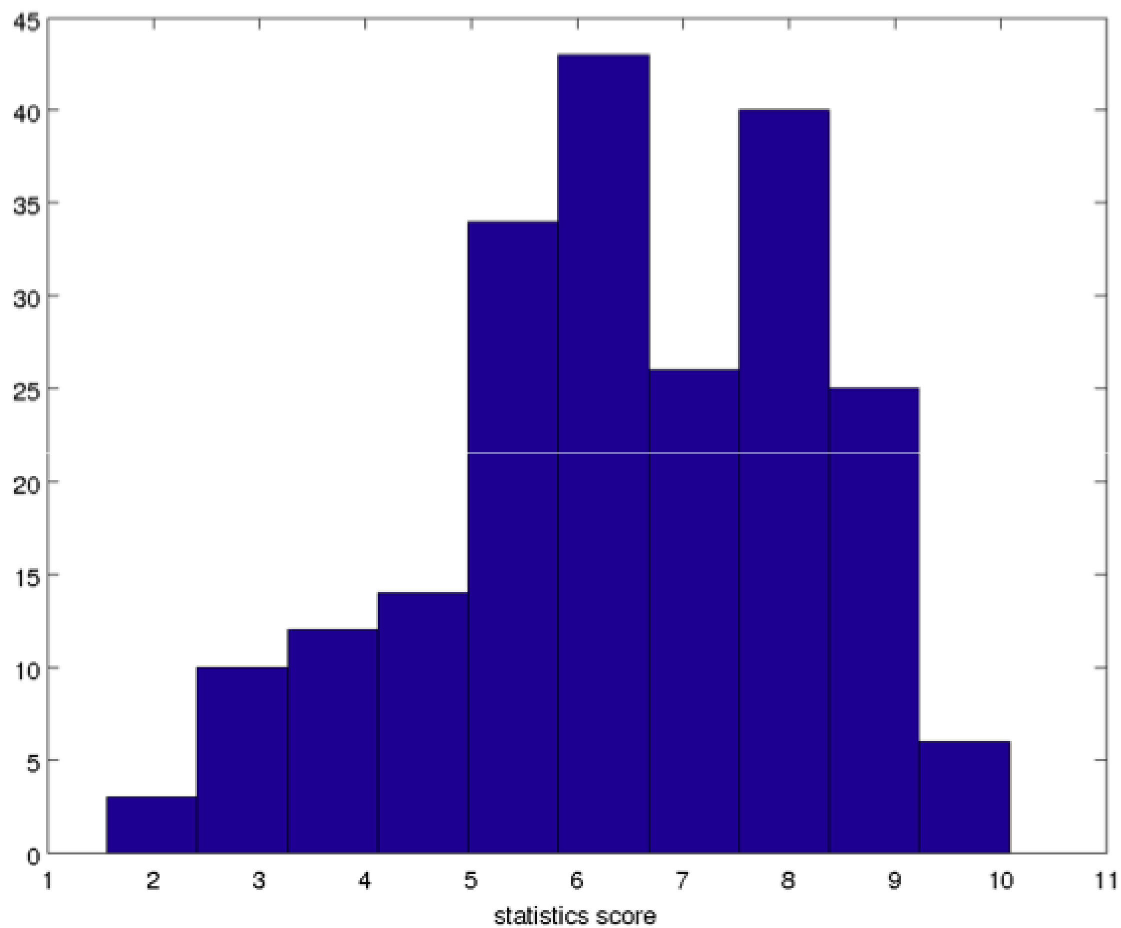
$$P(z_k \mid x) = \frac{P(x \mid z_k)\, P(z_k)}{P(x)}$$

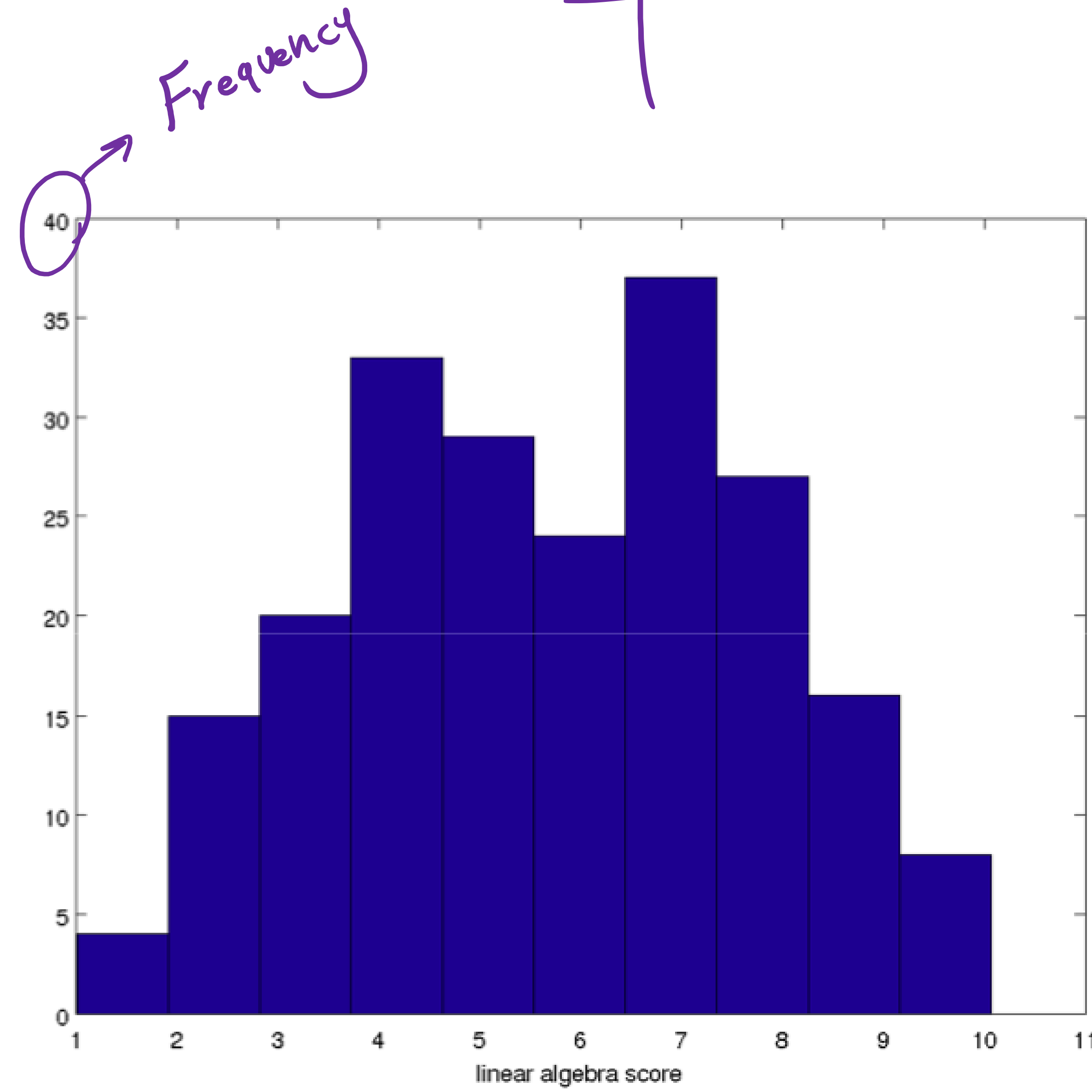
- Access the density of seeing a particular data point
  - Is this a typical data point? (high density value)
  - Is this an abnormal data point / outlier? (low density value)

- Building block for more sophisticated learning algorithms
  - Classification, regression, graphical models …
  - A simple recommendation system

Example: Test Scores

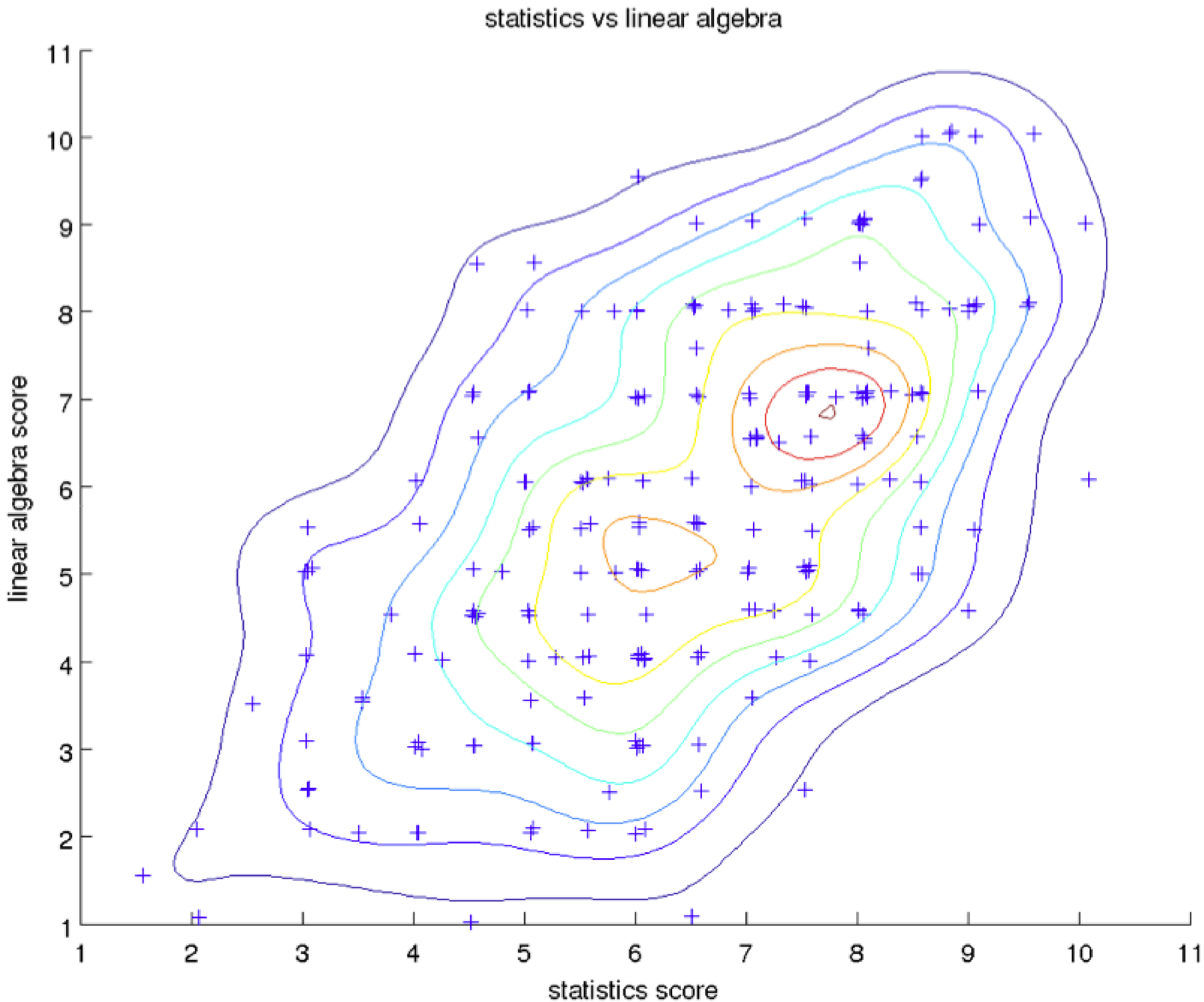






Histogram is an estimate of the probability distribution of a continuous variable

Example: Test Scores



Parametric Density Estimation

- Models which can be described by a fixed number of parameters

- Discrete case: eg. Bernoulli distribution

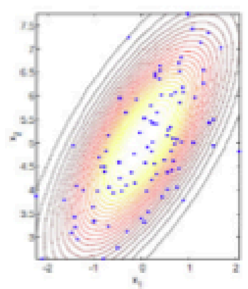
$$P(x|\theta) = \theta^x (1-\theta)^{1-x}$$

$$\boxed{\begin{array}{c} 1 \rightarrow Head \\ 0 \rightarrow Tails \end{array}}$$

one parameter, x $\in [0,1]$, which generate a family of models, $\mathcal{F} = \{P(x|\theta) \mid$ x $\in [0,1]\}$, $\boxed{\theta \; probability \; of \; possible \; outcome}$

- Continuous case: eg. Gaussian distribution in R^d

$$p(x|\mu,\Sigma) = \frac{1}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{d}{2}}} exp\left(-\frac{1}{2}(x-\mu)^{\top}\Sigma^{-1}(x-\mu)\right)$$



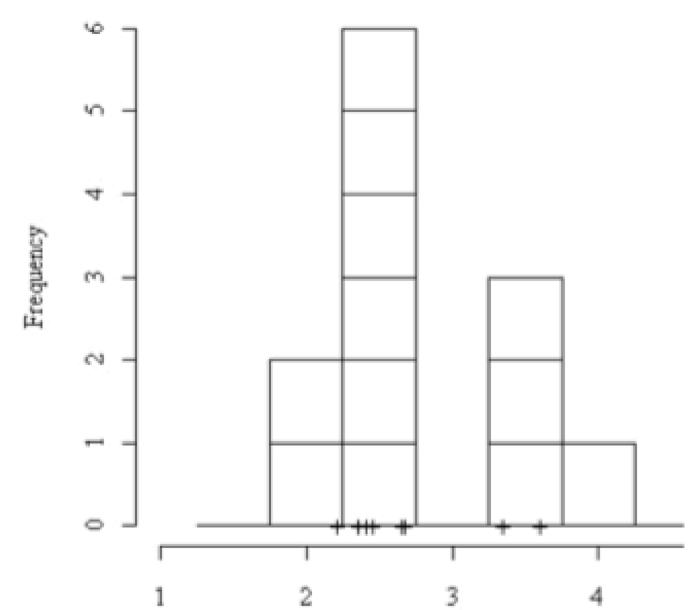
Two sets of parameters $\{\mu, \Sigma\}$, which again generate a family of models, $\mathcal{F} = \{p(x|\mu,\Sigma) \mid \mu \in R^d, \Sigma \in R^{d \times d} \; and \; PSD\}$,

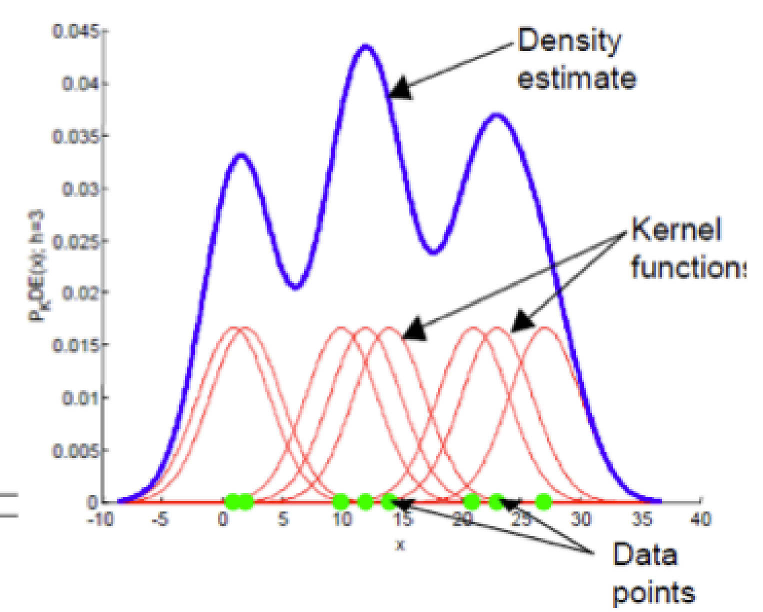
Nonparametric Density Estimation

- What are nonparametric models?
  - “nonparametric” does not mean there are no parameters
  - can not be described by a fixed number of parameters
  - one can think of there are many parameters

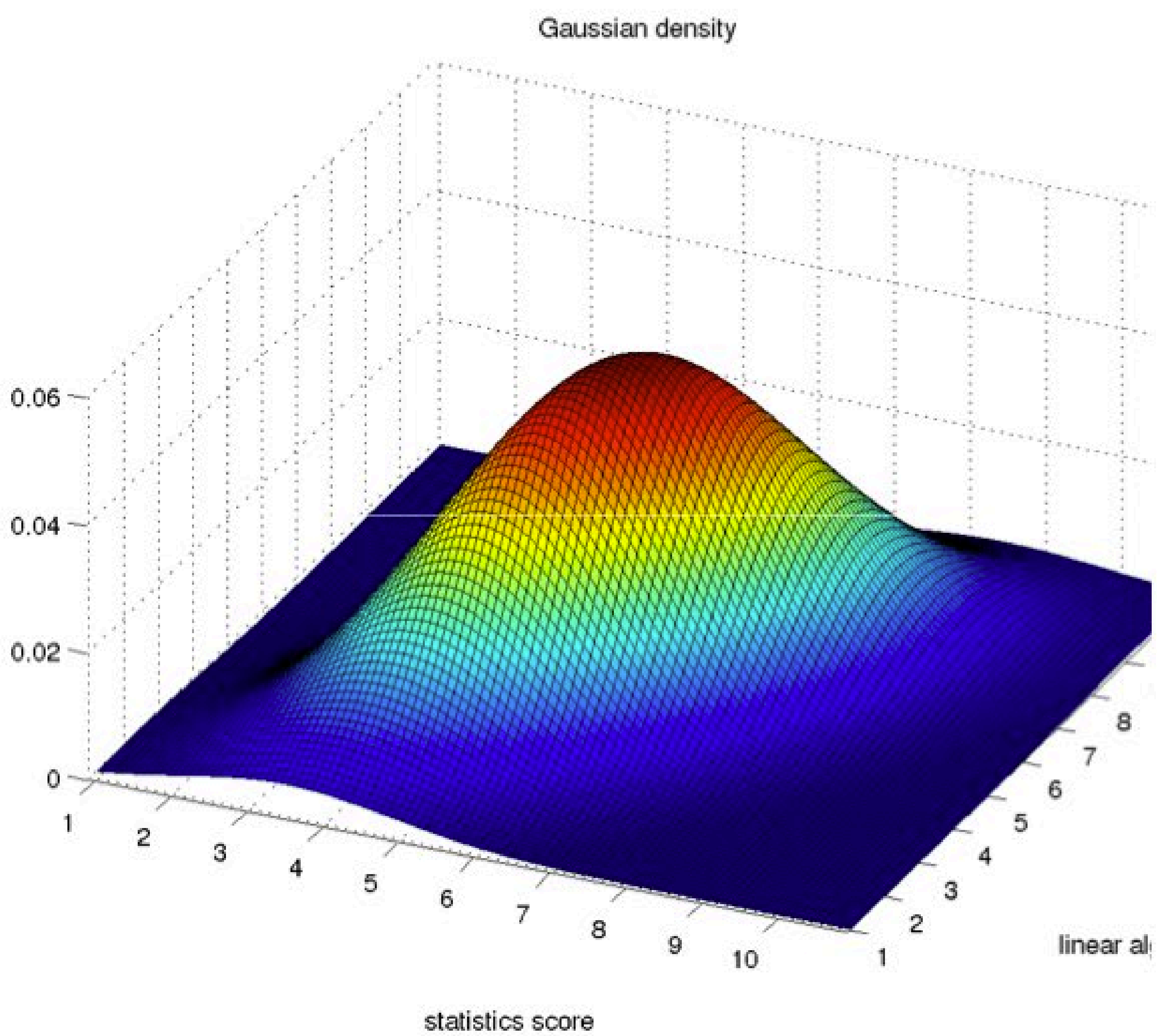
- Eg. Histogram

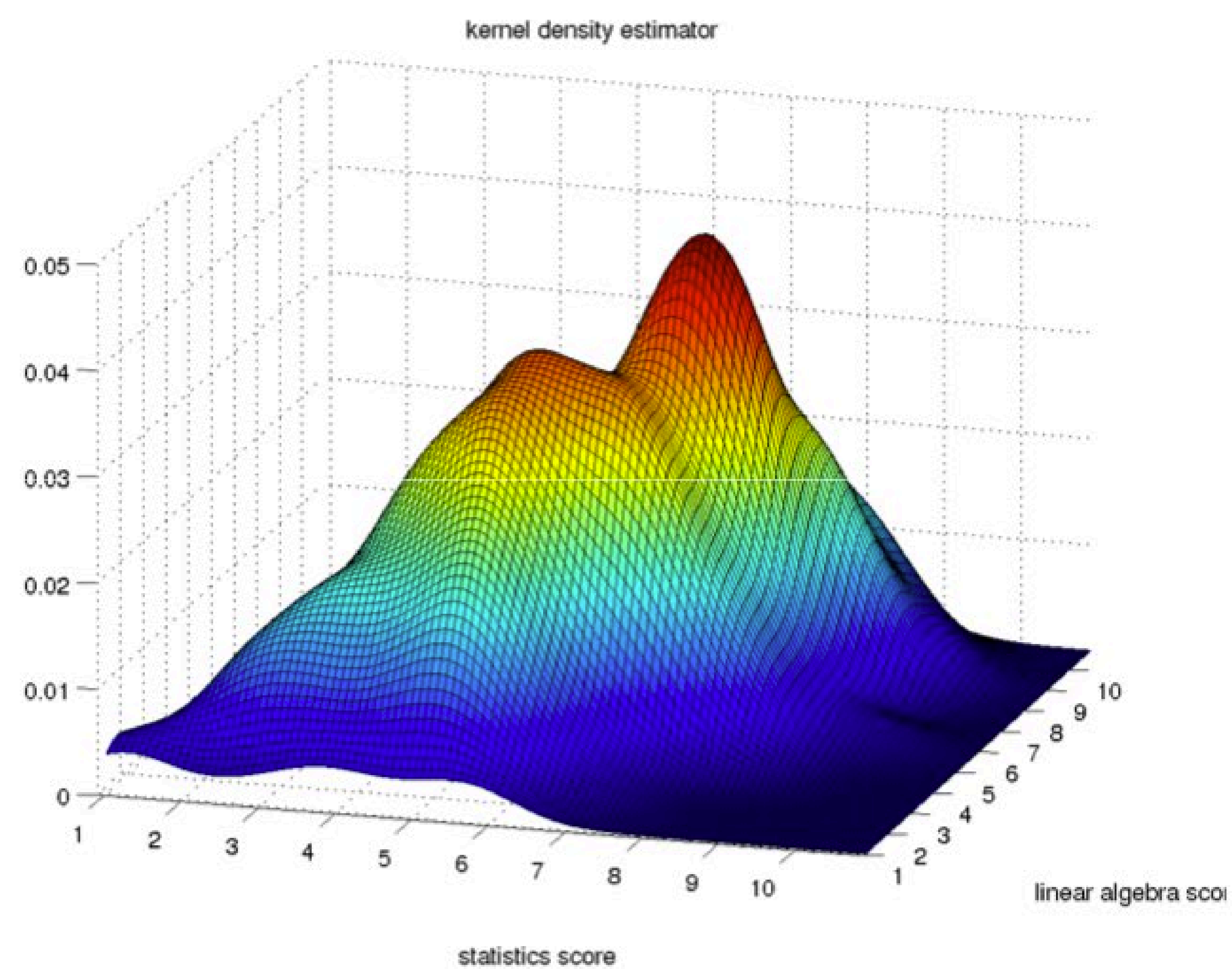
- Eg. Kernel density estimator



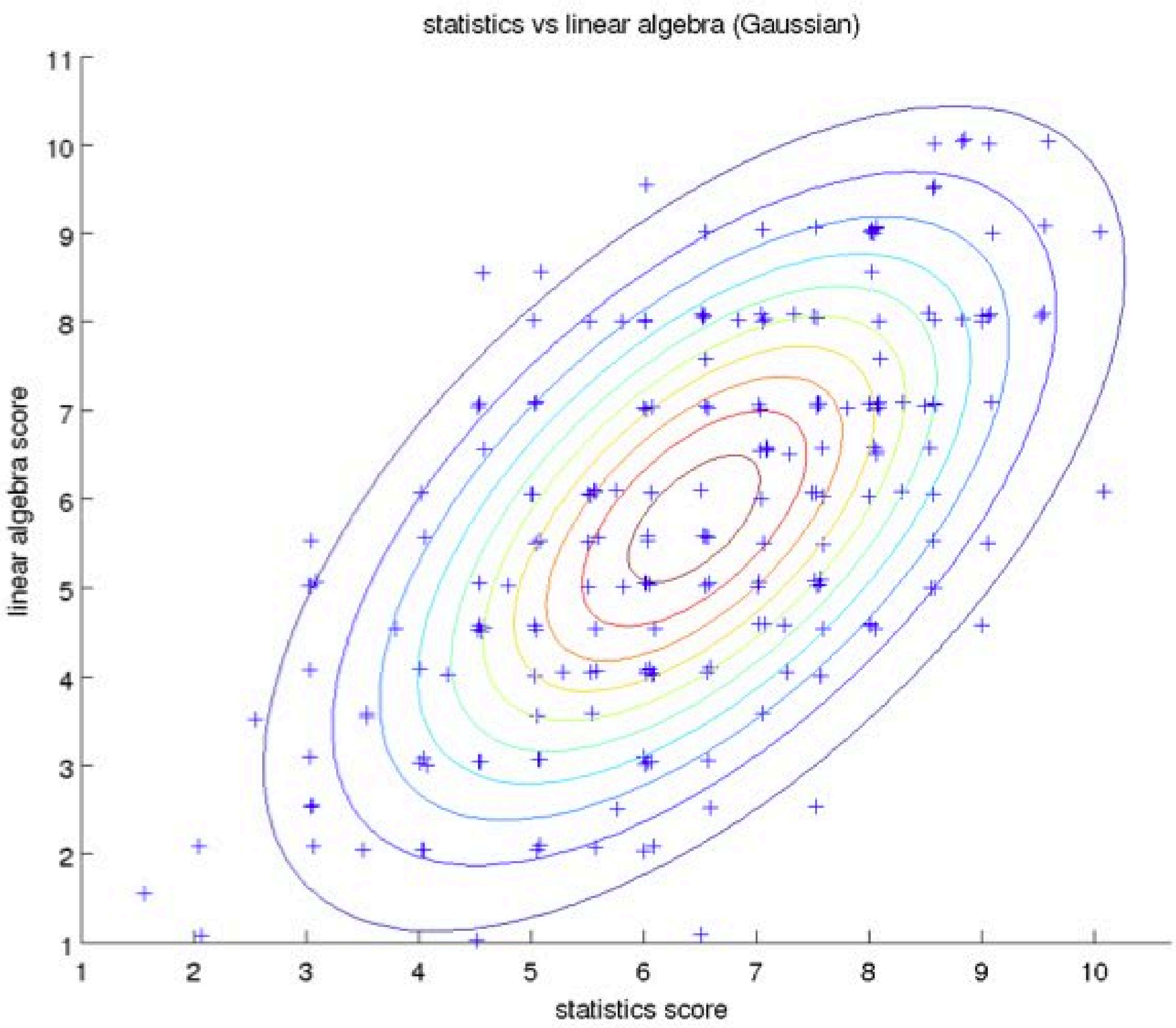


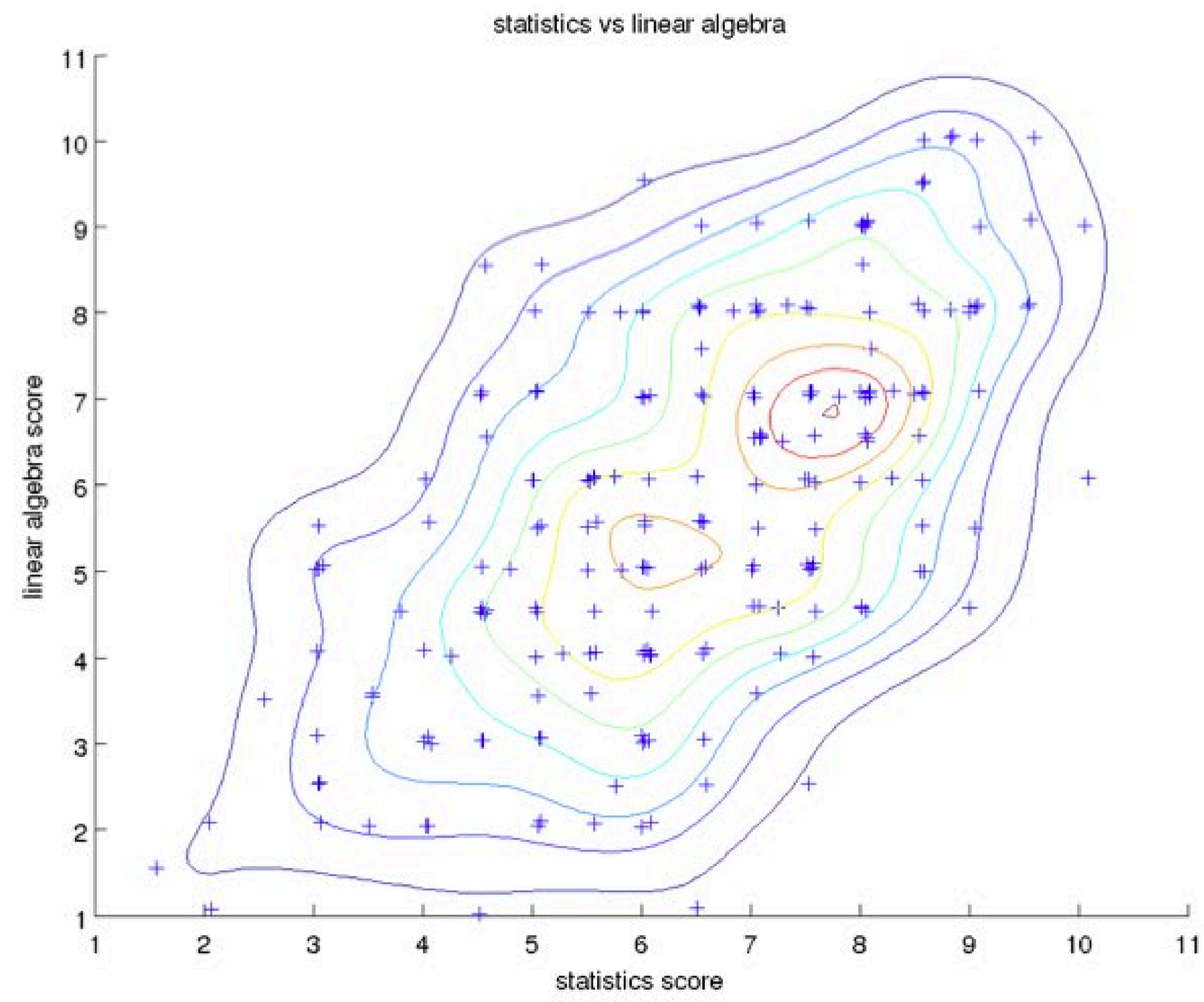
Parametric v.s. Nonparametric Density Estimation





Parametric v.s. Nonparametric Density Estimation





Outline

- Overview
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- Nonparametric Density Estimation

Estimating Parametric Models

- A very popular estimator is the maximum likelihood estimator (MLE), which is simple and has good statistical properties

- Assume that n data points $X = \{x_1, x_2, \dots, x_n\}$ drawn independently and identically (iid) from some distribution $P^*(x)$

$P(x_1, x_2, \dots, x_n) \overset{iid}{\equiv} \prod_{i=1}^{N} P(x_i) \rightarrow P(x_i \mid \theta)$

Using the parameters, we can estimate each data point

- Want to fit the data with a model $P(x|\theta)$ with parameter θ

$$\theta = argmax_\theta \log P(X \mid \theta) = \boxed{argmax_\theta \log \prod_{i=1}^{N} P(x_i \mid \theta)}$$

Example Problem

- Estimate the probability θ of landing in heads using a biased coin

- Given a sequence of n independently and identically distributed (iid) flips
  - Eg. $X = \{x_1, x_2, \ldots, x_n\} = \{1,0,1, \ldots, 0\}, x_i \in \{0,1\}$

- Model: $P(x|\theta) = \theta^x (1-\theta)^{1-x}$
  - $P(x|\theta) = \begin{cases} 1-\theta, for\ x = 0 \\ \theta, \quad for\ x = 1 \end{cases}$

- Likelihood of a single observation x_i ?
  - $L(\theta|x_n) = p(x_n|\theta) = \theta^{x_n}(1-\theta)^{1-x_n}$

MLE for Biased Coin

- Objective function, log-likelihood

$$l(\theta|\mathbf{X}) = \log L(\theta|\mathbf{X}) = \log \prod_{i=1}^{N} \theta^{x_i} (1-\theta)^{1-x_i} = \log(\theta^{N_H}(1-\theta)^{N_T})$$

$$= N_H \times \log\theta + N_T \times \log(1-\theta)$$

$N_H =$ number of heads, $N_T =$ number of tails

- Maximize $l(\theta|\mathbf{X})$ w.r.t. $\theta \rightarrow$ take derivative w.r.t. θ and set it to zero

$$\frac{\partial l(\theta|\mathbf{X})}{\partial\theta} = \frac{N_H}{\theta} - \frac{N-N_H}{1-\theta} = 0 \rightarrow \theta_{MLE} = \frac{N_H}{N}$$

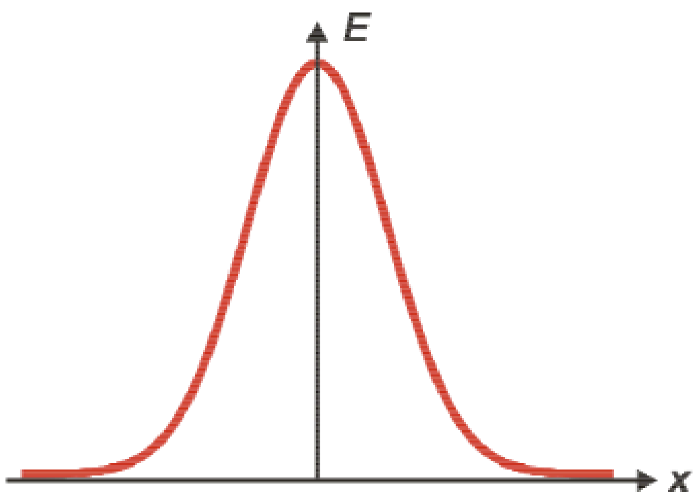
- Example: $N_H = 78$, $N_H = 22 \rightarrow \theta = 0.78$

Estimating Gaussian Distributions

- Gaussian distribution in R

$$p(x|\mu,\sigma) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

- Need to estimate two sets of parameters μ, σ



- Given n iid samples

$$X = \{x_1, x_2, \ldots, x_n\}, x_i \in R$$

- Density of a data point:

$$p(x_i\,|\mu,\sigma) \propto exp\left(-\frac{1}{2\sigma^2}(x_i-\mu)^2\right)$$

Estimating Gaussian Distributions

- Gaussian distribution in R

$$p(x|\mu,\sigma) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

$\theta \in \{\mu, \sigma\}$

$\prod_{i=1}^{N} p(x_i | \theta)$ $= \prod_{i=1}^{N} \frac{1}{\sqrt{(2\pi)}\,\sigma} exp \ldots$

- Mean

$$\mu = \frac{1}{\text{n}} \sum_{i=1}^{\text{n}} x_i$$

- Variance

$$\sigma^2 = \frac{1}{\text{n}} \sum_{i=1}^{\text{n}} (x_i - \mu)^2$$

MLE for Gaussian Distribution

log (exp (u)) = u

- Objective function, log likelihood

u

$$l(\mu,\sigma;X) = \log \prod_{i=1}^{N} \frac{1}{(2\pi)^{\frac{1}{2}}\sigma} exp\left(\left(-\frac{1}{2\sigma^2}(x_i - \mu)^2\right)\right)$$

$$= -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \sum_{i=1}^{N}\frac{(x_i - \mu)^2}{2\sigma^2}$$

- Maximize $l(\mu,\sigma;X)$ with respect to μ,σ

- Take derivatives w.r.t. μ, σ^2 $\rightarrow$ $\sigma^2 = u$ $\quad \frac{\partial l}{\partial u} = 0$

$$\frac{\partial l}{\partial \mu} = 0$$

$$\frac{\partial l}{\partial \sigma^2} = 0$$

MLE for Gaussian Distribution

$$\frac{\partial(\log u)}{\partial u} = \frac{u'}{u}$$

$$l(\mu, \sigma; X) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log \sigma^2 - \sum_{i=1}^{N} \frac{(x_i - \mu)^2}{2\sigma^2}$$

$$\frac{\partial l}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i - \mu) = 0$$

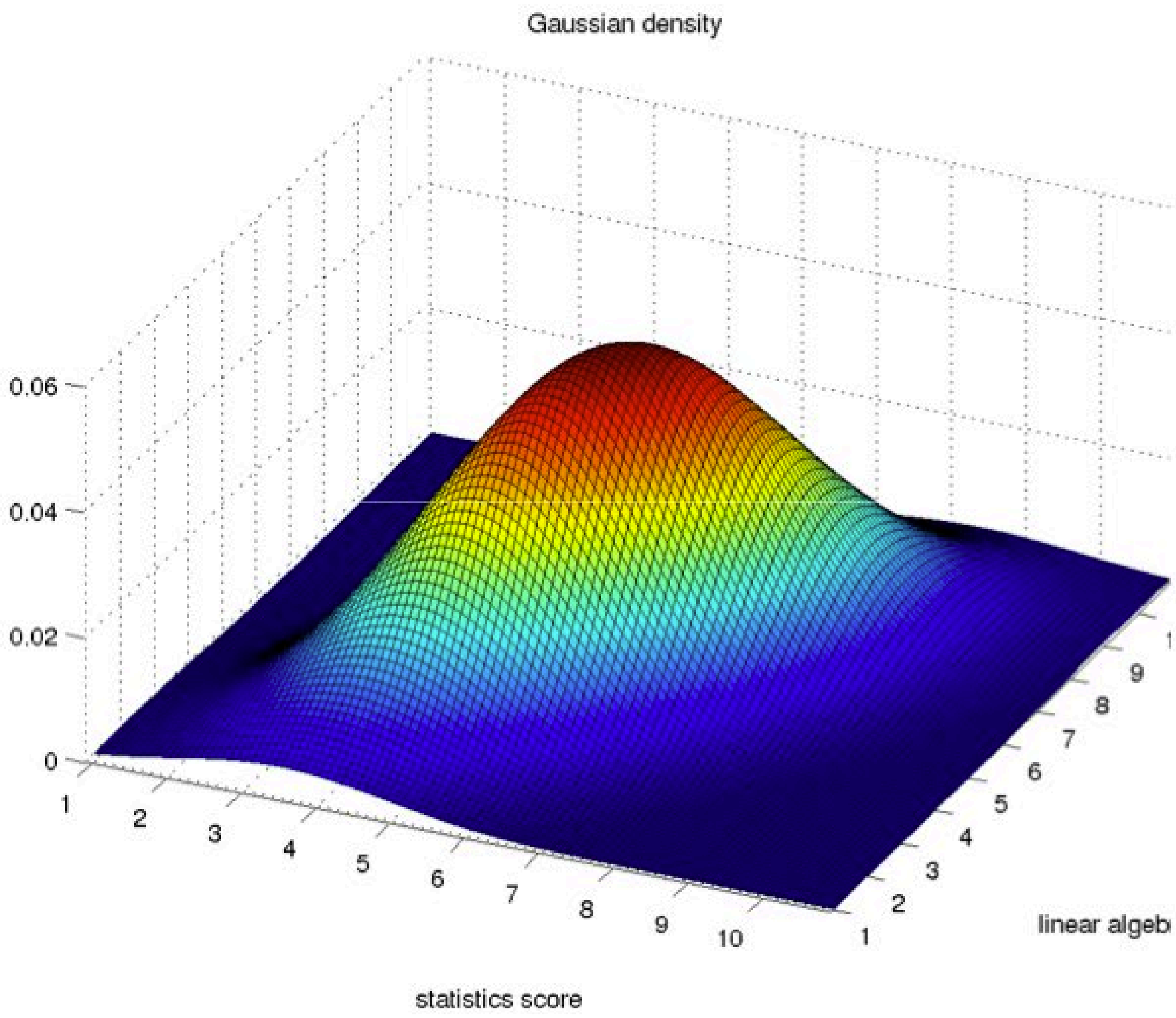
$$\Rightarrow \sum_{i}^{N} x_i = n\,\mu \Rightarrow \mu = \frac{1}{n}\sum_{i=1}^{N} x_i$$

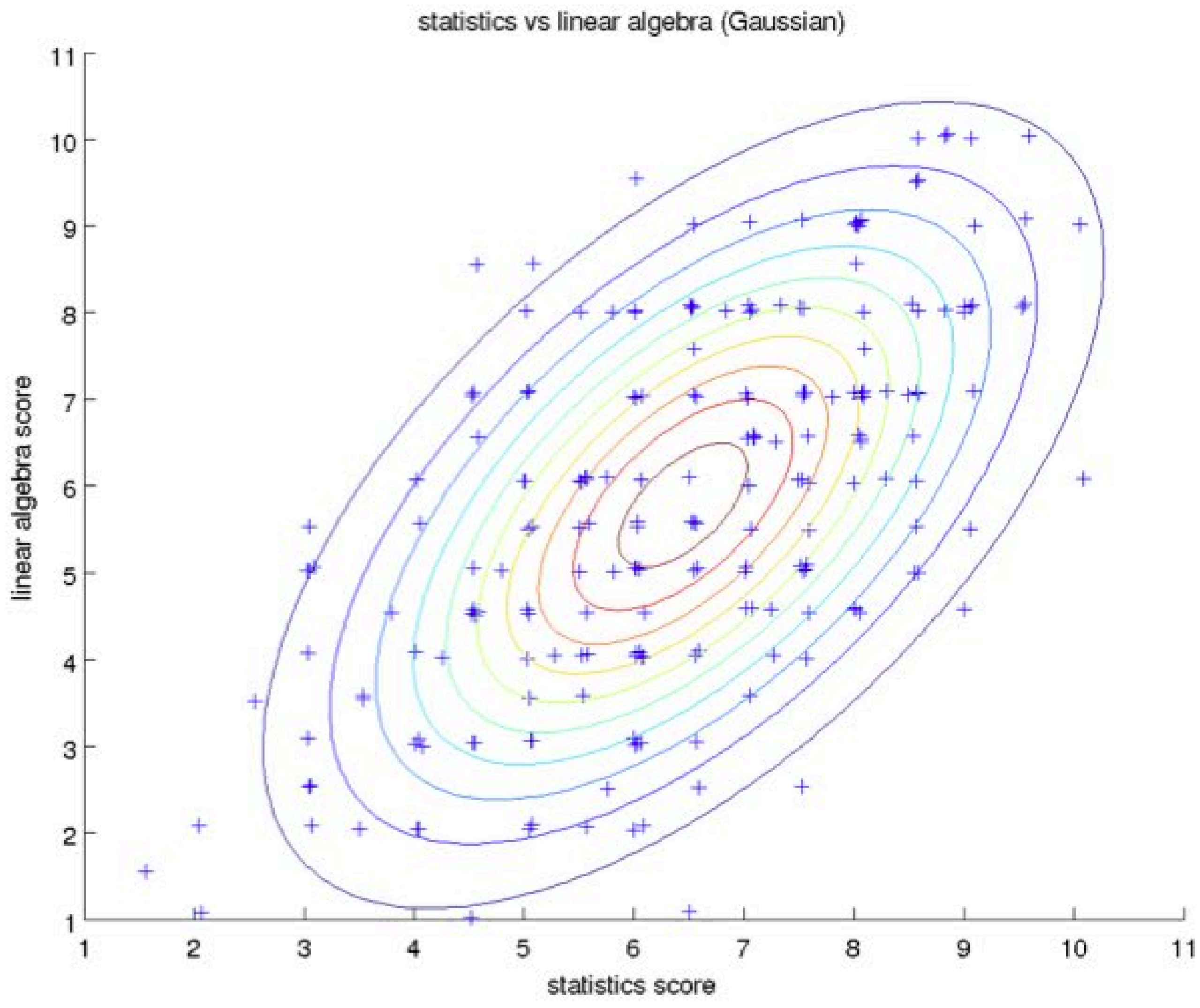
$$\frac{\partial l}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i}^{N}(x_i - \mu)^2 = 0$$

$$\Rightarrow \sum_{i}^{N}(x_i - \mu)^2 = n\,\sigma^2 \Rightarrow \frac{1}{n}\sum_{i=1}^{N}(x_i - \mu)^2 = \sigma^2$$

Variance

Example


Gaussian density
0.06
0.04
0.02
0
1
2
3
4
5
6
7
8
9
10
statistics score
linear algeb


statistics vs linear algebra (Gaussian)
linear algebra score
statistics score
1
2
3
4
5
6
7
8
9
10
11

Outline

- Overview
- Parametric Density Estimation
- Nonparametric Density Estimation

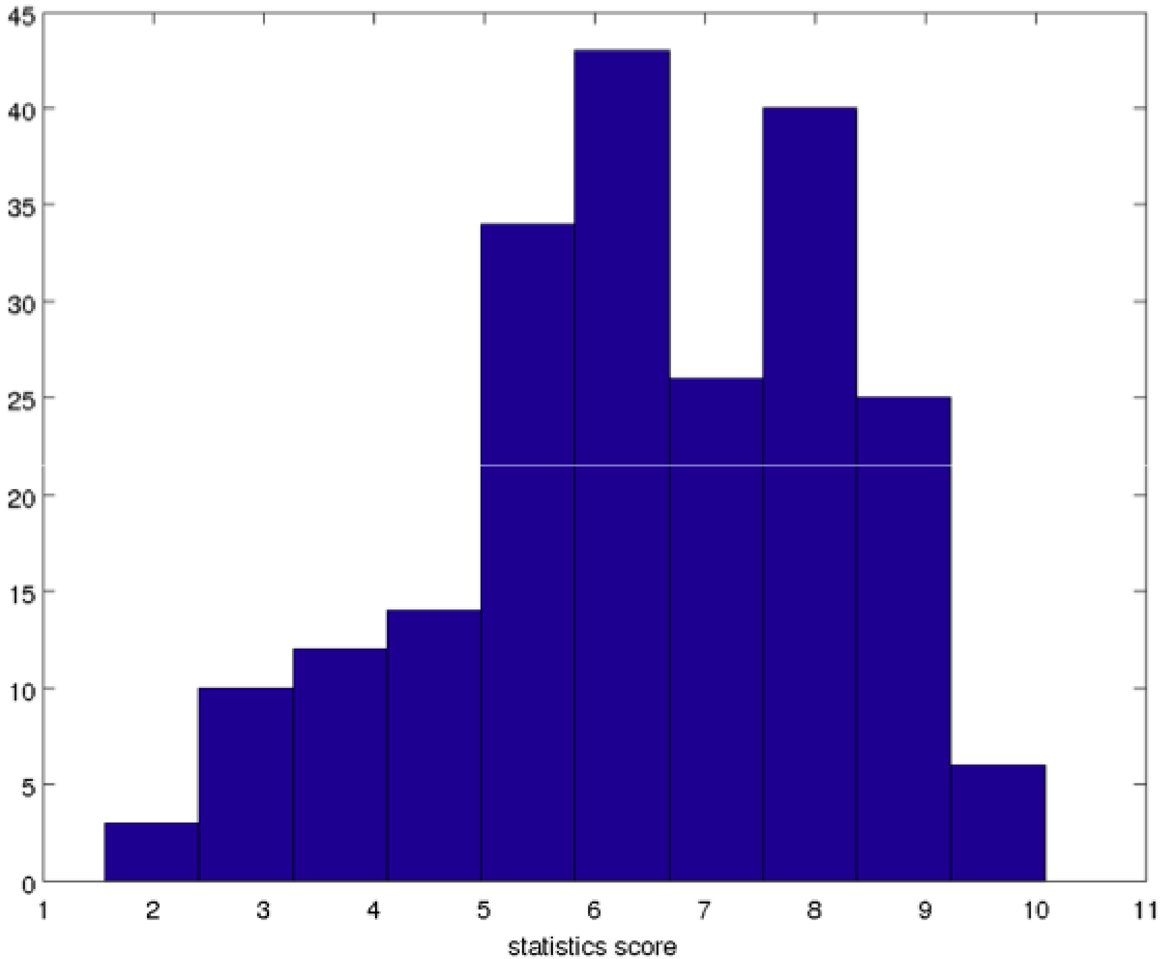
Can be used for:

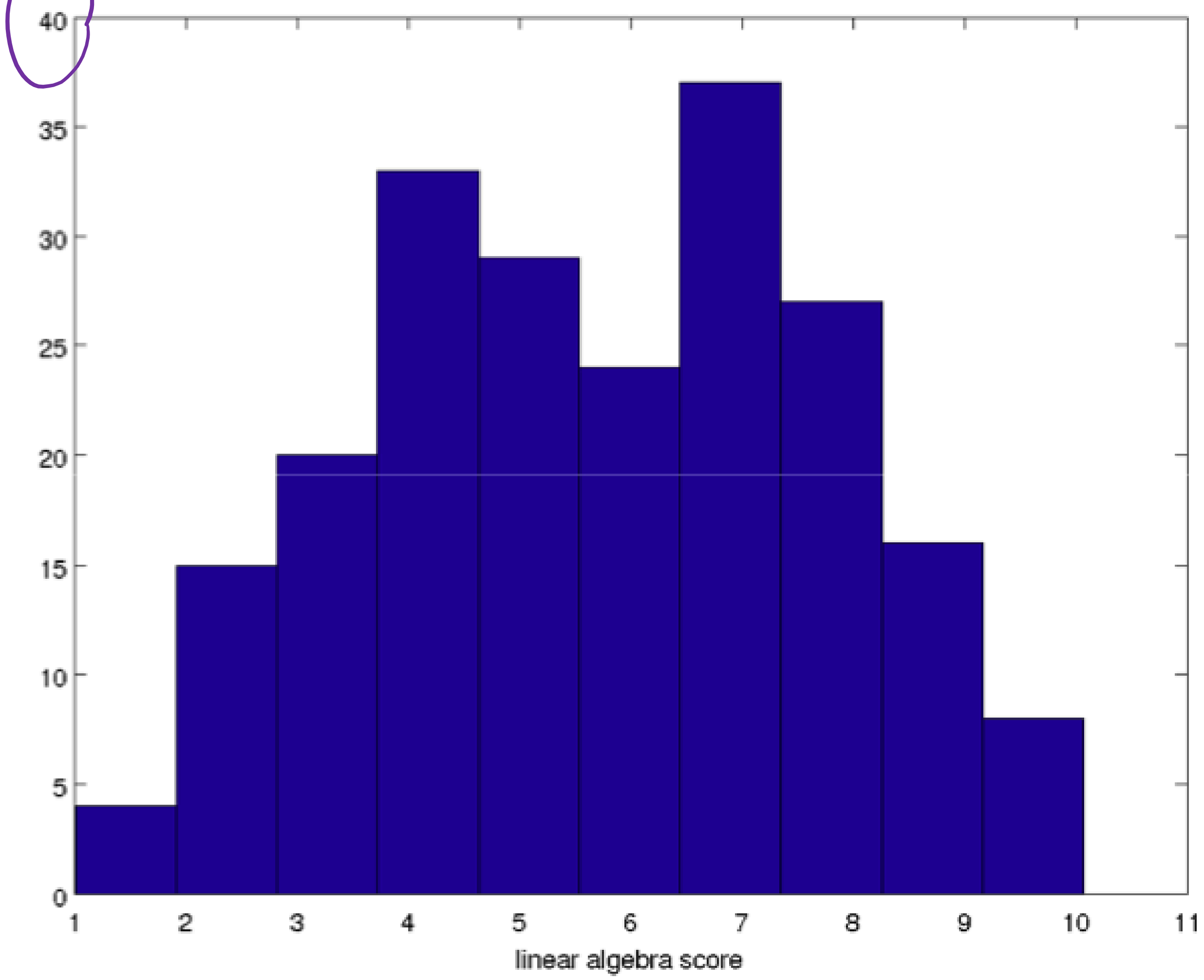
- Visualization
- Classification
- Regression

Example: Test Scores

- What is missing if we want density?

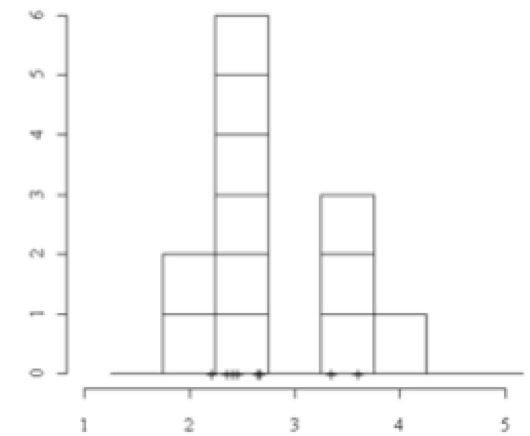
convert this into density value





1-D Histogram

- One the simplest nonparametric density estimator



- Given N iid samples $X = \{x_1, x_2, \dots, x_n\} = x_i \in [0,1)$

- Split [0,1) into M bins

$$B_1 = \left[0, \frac{1}{M}\right), B_2 = \left[\frac{1}{M}, \frac{2}{M}\right), \dots, B_l = \left[\frac{l-1}{M}, \frac{l}{M}\right), \dots, B_M = \left[\frac{M-1}{M}, 1\right)$$

- Count the number of points, c_1 within B_1, c_2 within B_2...

$$\frac{1}{M}$$

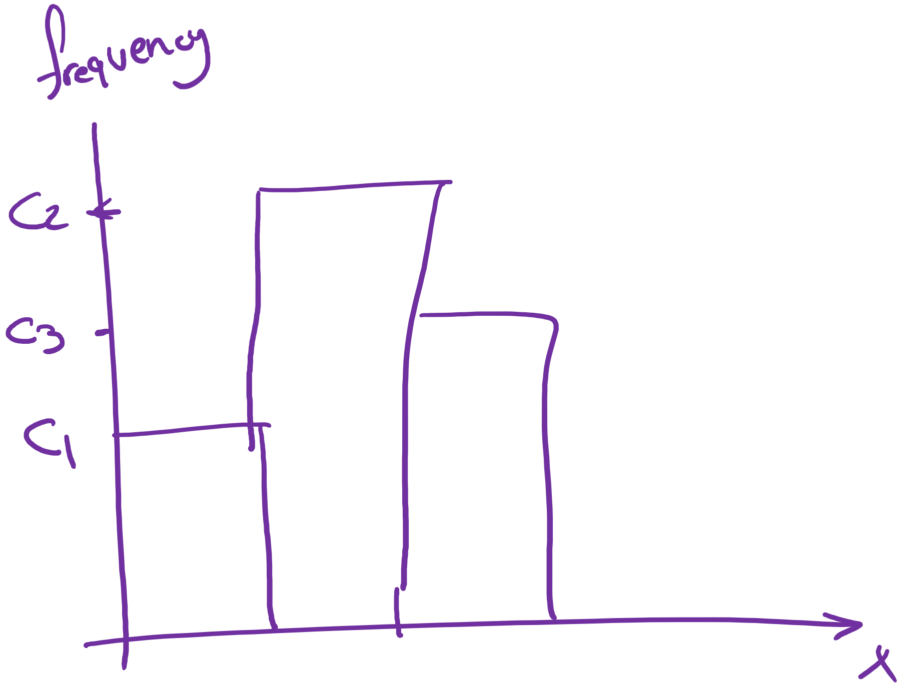
- For a new test data point x which belongs to B_l

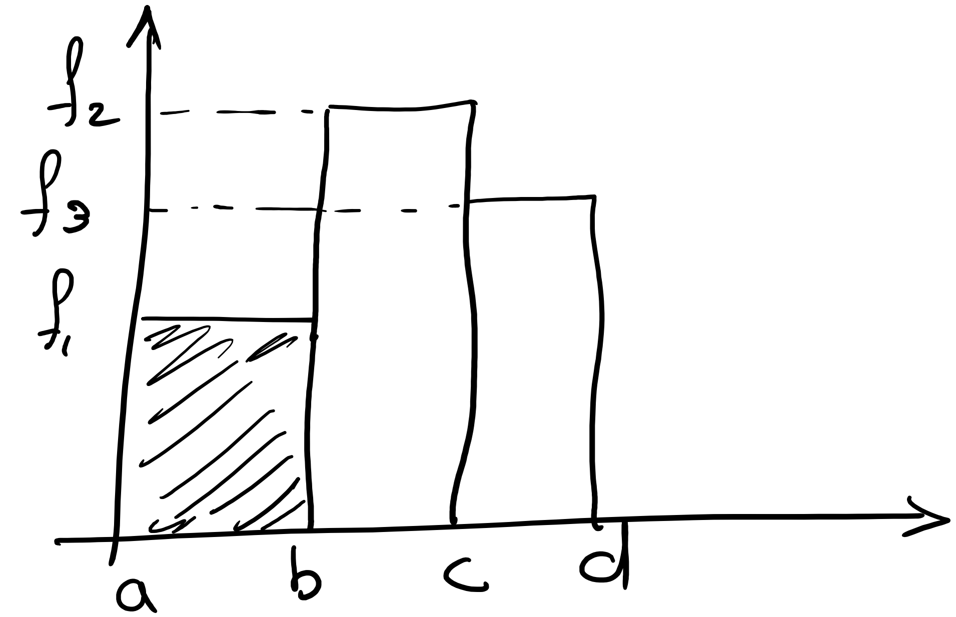
$$p(x) = \boxed{\frac{M}{N} \sum_{i=1}^{N} 1(x_i \epsilon B_l)} = \frac{number\ of\ points\ in\ bin\ B_l\ (c_l)}{total\ number\ of\ data\ points\ \times bin\ width}$$

$$P = \int p(x)dx$$

The probability that point x is drawn from a distribution p(x)

?



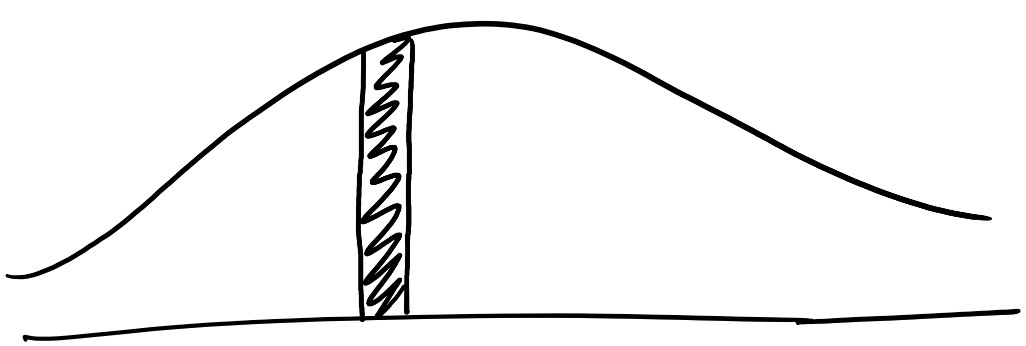


$$c_1 + c_2 + c_3 = N$$

$$P_1 = \frac{c_1}{N} \qquad P_2 = \frac{c_2}{N}$$

$$f_1 (b-a) = f_1 \frac{1}{M} = P_1$$

$$\frac{c_1}{N} = f_1 \frac{1}{M} \Rightarrow f_1 = \frac{c_1}{N \times \frac{1}{M}}$$



Why is Histogram Valid?

- Requirement for density $p(x)$
- $p(x) \geq 0, \int_\Omega p(x)dx = 1$
- For histogram, $\int_{[0,1)} p(x)dx = \int_0^1 \boxed{\frac{M}{N}\sum_{i=1}^{N} 1(x_i \,\epsilon\, B_l)dx}$

$$= \int_0^{\frac{1}{M}} \overbrace{\frac{M}{N}\sum_{i=1}^{N} 1(x_i \,\epsilon\, B_l)}^{c_1} dx + \int_{\frac{1}{M}}^{\frac{2}{M}} \overbrace{\frac{M}{N}\sum_{i=1}^{N} 1(x_i \,\epsilon\, B_l)}^{c_2} dx + \ldots + \int_{\frac{l-1}{M}}^{\frac{l}{M}} \overbrace{\frac{M}{N}\sum_{i=1}^{N} 1(x_i \,\epsilon\, B_l)}^{c_l} dx =$$

$$= \frac{M}{N}\left[\int_0^{\frac{1}{M}} c_1\, dx + \int_{\frac{1}{M}}^{\frac{2}{M}} c_2\, dx + \cdots + \int_{\frac{l-1}{M}}^{\frac{l}{M}} c_l\, dx + \cdots + \int_{\frac{M-1}{M}}^{1} c_M\, dx\right] =$$

$$\frac{c_1+c_2+\cdots+c_M}{N} = 1$$

$$= \frac{M}{N}\sum_{l=1}^{M}\int_{\frac{l-1}{M}}^{\frac{l}{M}} c_l dx = \frac{M}{N}\sum_{j=1}^{M} c_l\left[\frac{l}{M} - \frac{l-1}{M}\right] = \sum_{l=1}^{M}\frac{c_l}{N} = 1$$

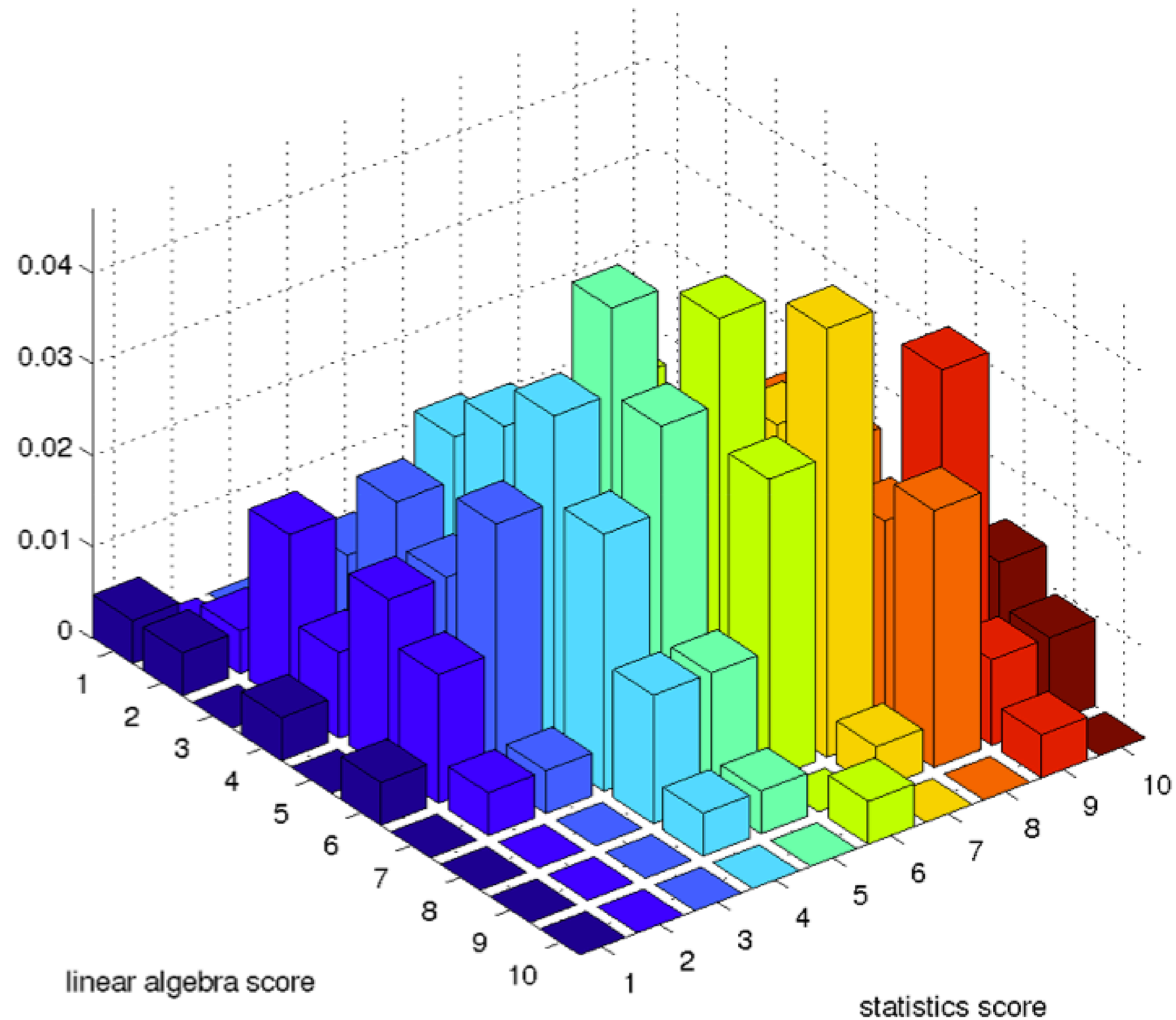
Higher-Dimensional Histogram

- Given n iid samples $X = \{x_1, x_2, \dots, x_n\}, x_i \in [0,1)^d$
- Split $[0,1)^d$ evenly into M^d bins
- Bin size is $h = \frac{1}{M}$

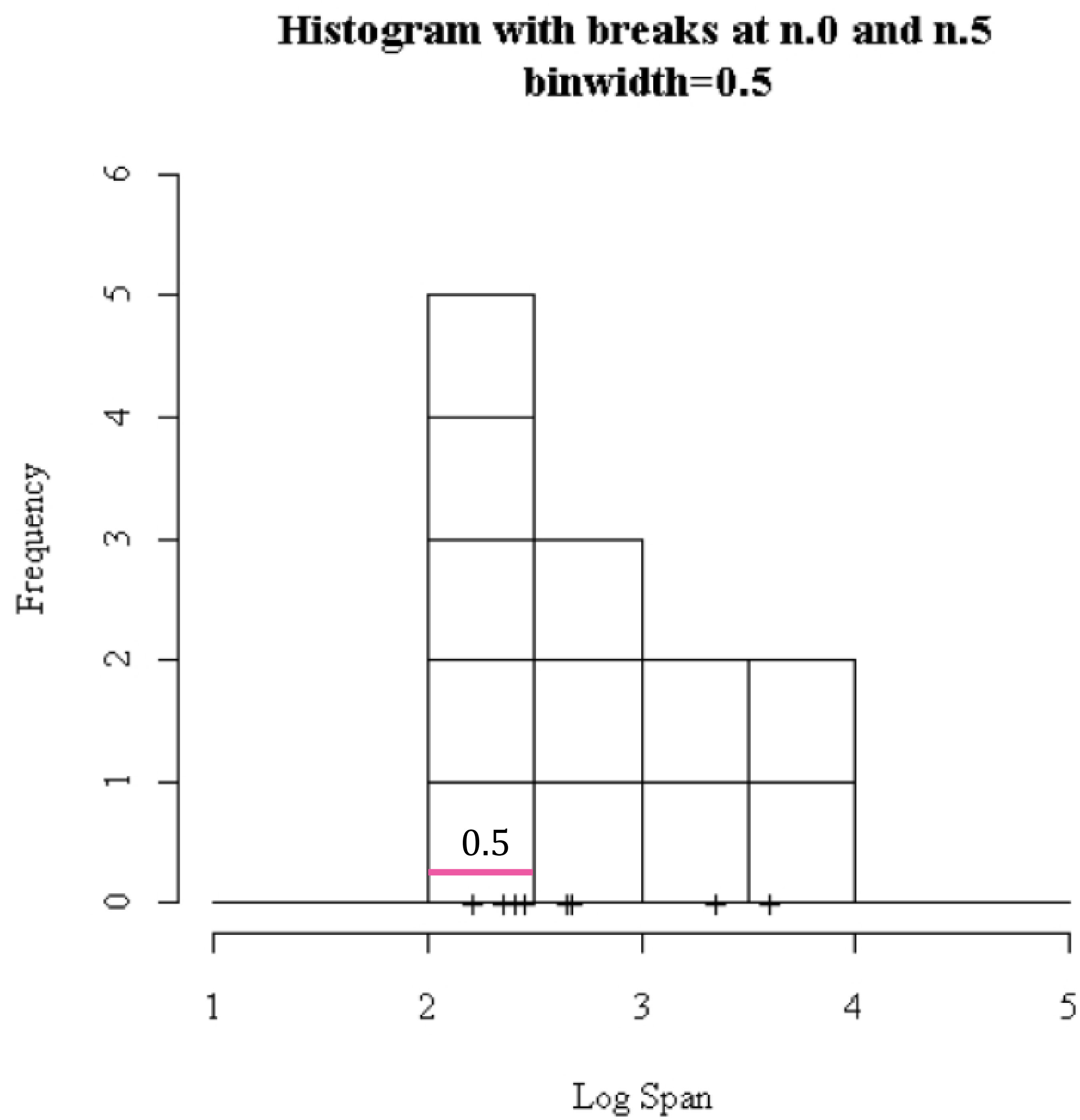
$Two\ Dimensional\ data$:

$M = 10$ (number of bins in each dimension)

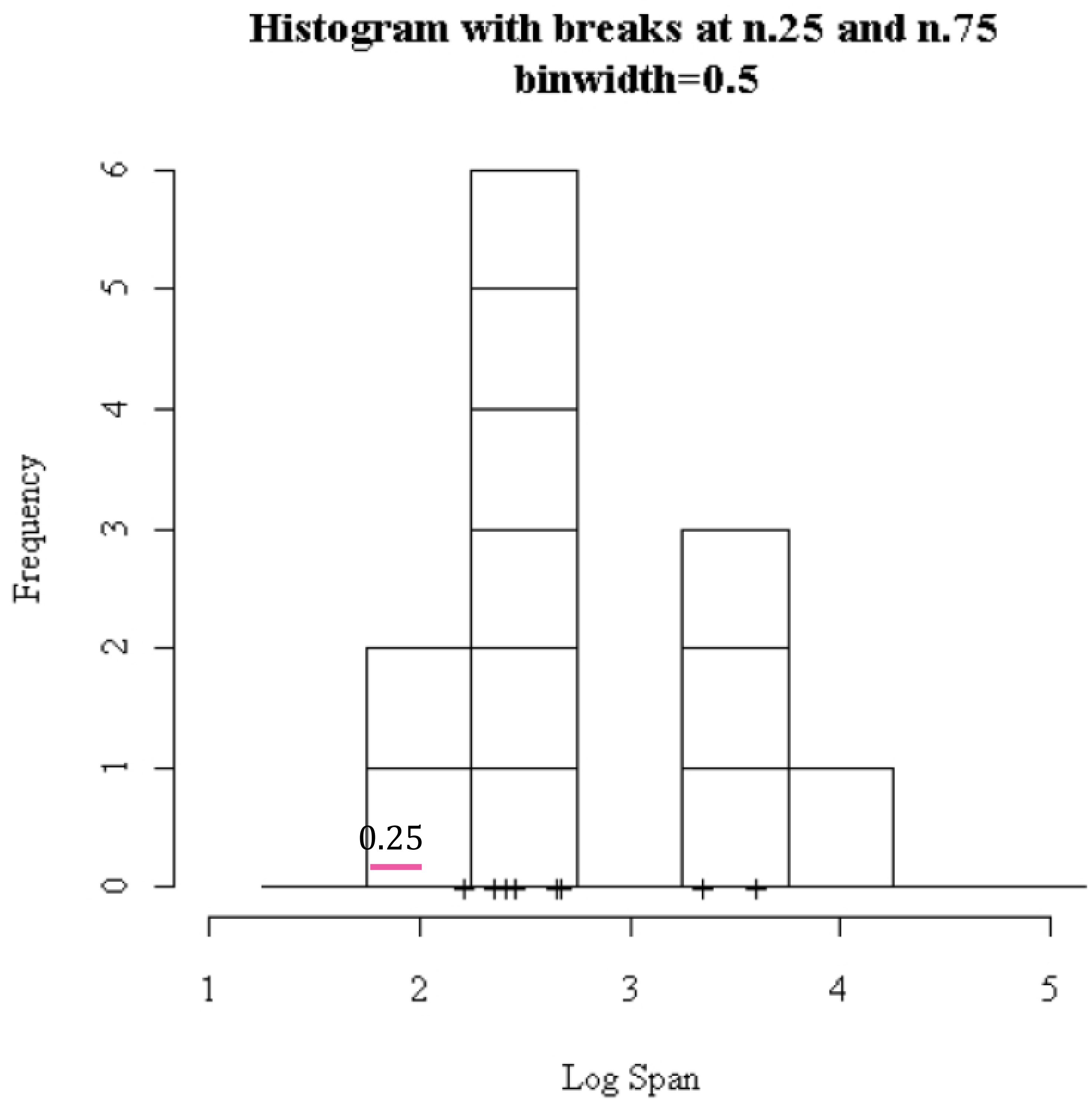
$M^2 = 100$ (total number of bins for two dimensional data)


0.04
0.03
0.02
0.01
0
1
2
3
4
5
6
7
8
9
10
linear algebra score
1
2
3
4
5
6
7
8
9
10
statistics score

Output Depends on Where You Put the Bins



Output Depends on Where You Put the Bins



Kernel Density Estimation

- Kernel density estimator

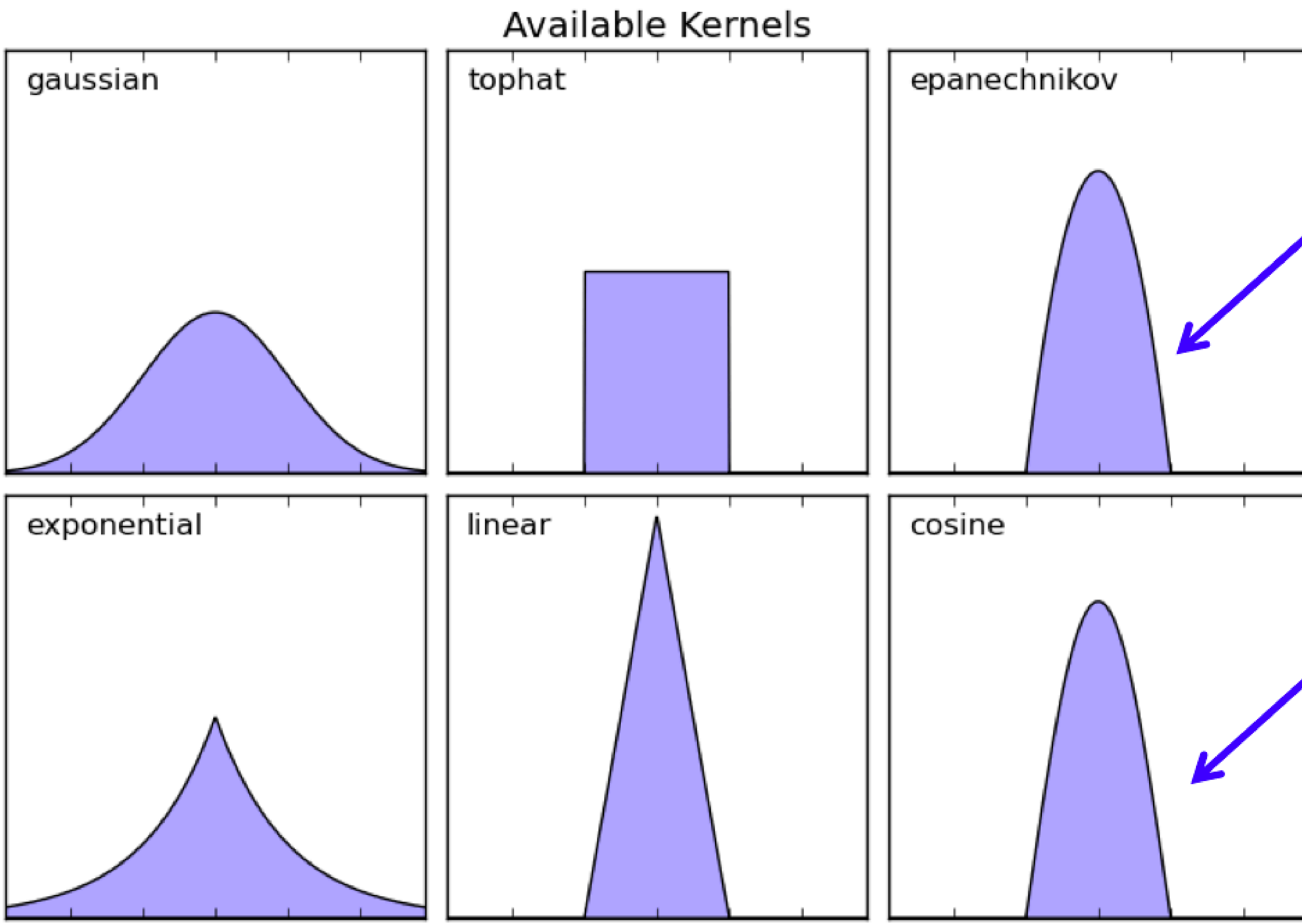
$$p(x) = \frac{1}{N}\sum_{i}^{N}\frac{1}{h}K\left(\frac{x_l - x_i}{h}\right) \qquad x_l = x_{gridline}$$

- Smoothing kernel function
  - $K(u) \geq 0,$
  - $\int K(u)du = 1,$
  - $\int uK(u) = 0,$
  - $\int u^2K(u)du \leq \infty$

- An example: Gaussian kernel $K(u) = \frac{1}{\sqrt{2\pi}}e^{-u^2/2}$

Smoothing Kernel Functions

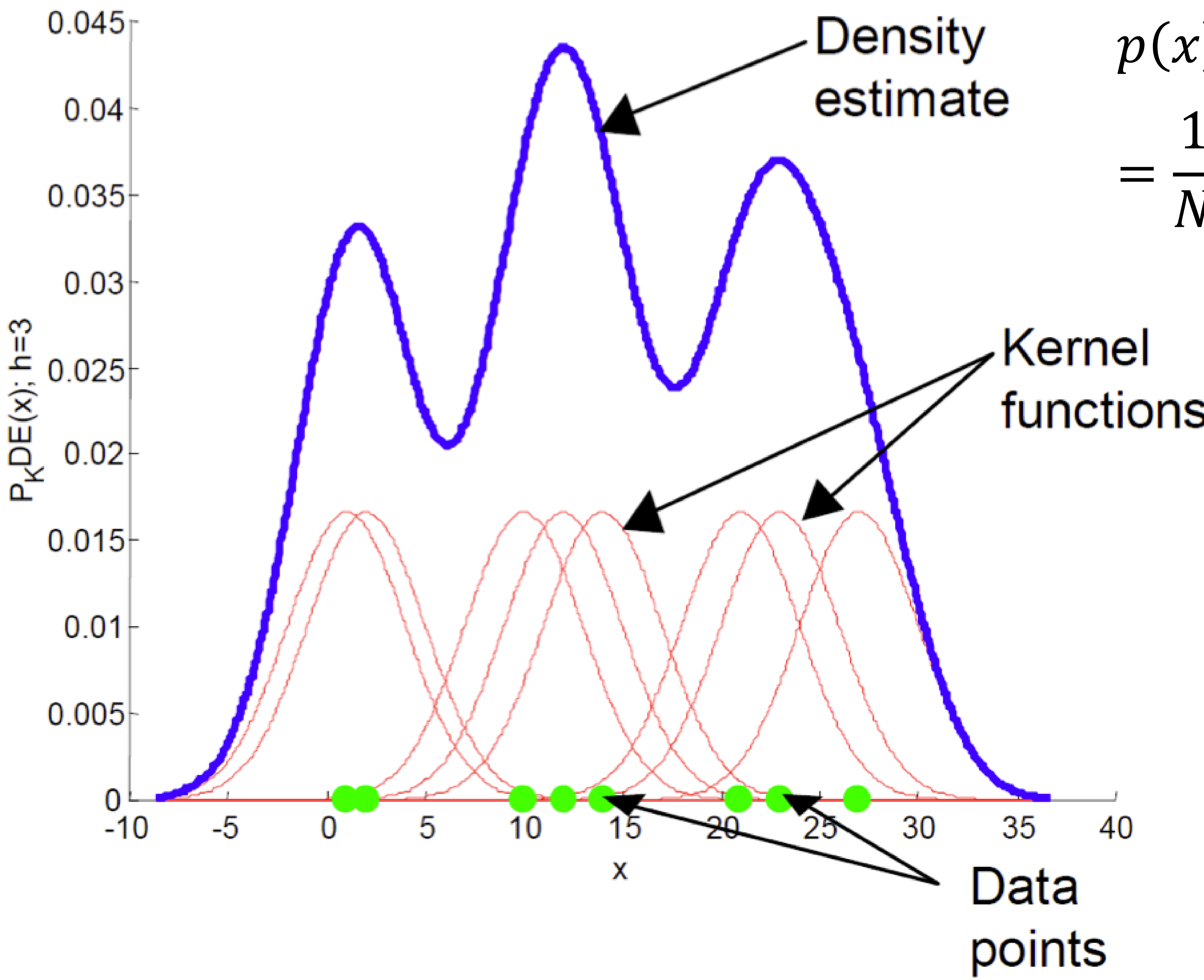
- An example: Gaussian kernel $K(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$



$K(u) =$
$\frac{3}{4}(1 - u^2) I(|u| \leq 1)$

$K(u) =$
$\frac{\pi}{4} \cos\left(\frac{\pi}{2} u\right) I(|u| \leq 1)$

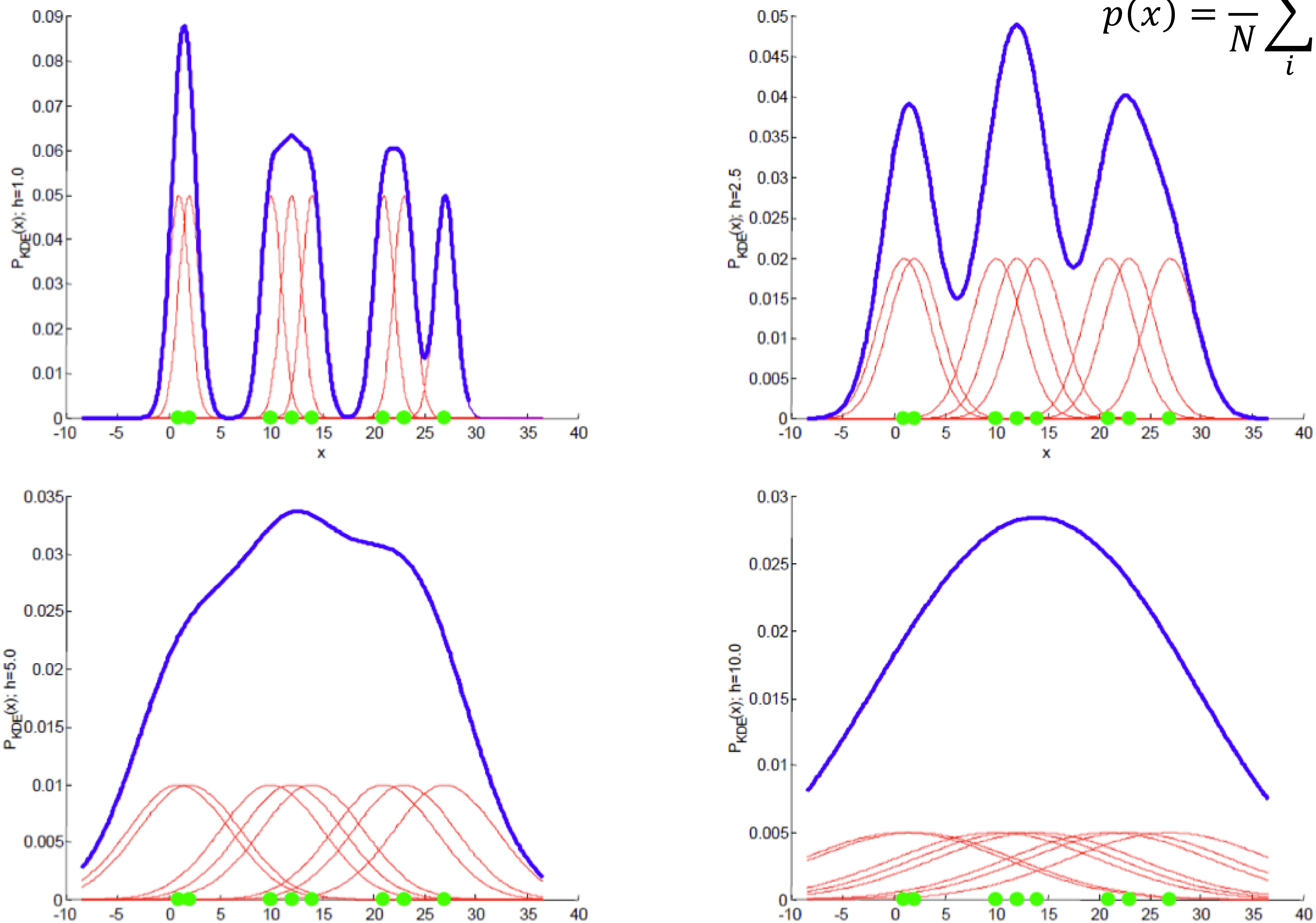
Example



$$p(x) = \frac{1}{N}\sum_{i}^{N}\frac{1}{h}K\left(\frac{x_l - x_i}{h}\right)$$

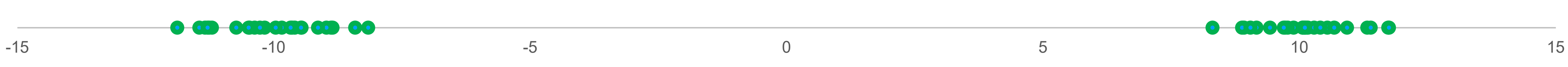
Effect of the Kernel Bandwidth (h)

$$p(x) = \frac{1}{N}\sum_{i}^{N}\frac{1}{h}K\left(\frac{x_l - x_i}{h}\right)$$



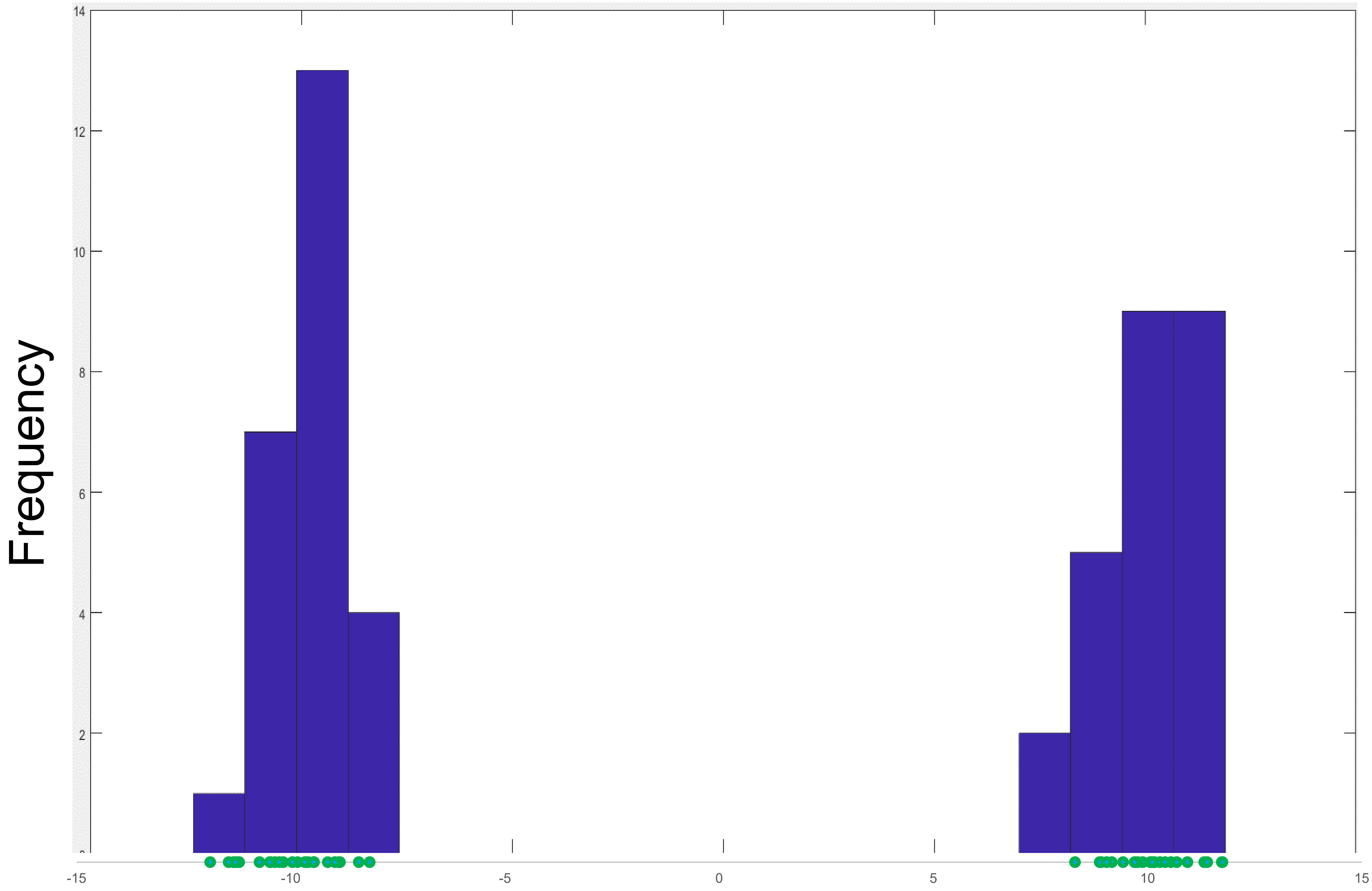
Visual Example

50 datapoints are given to us



Visual Example

Let’s implement 20 bins histogram



Visual Example

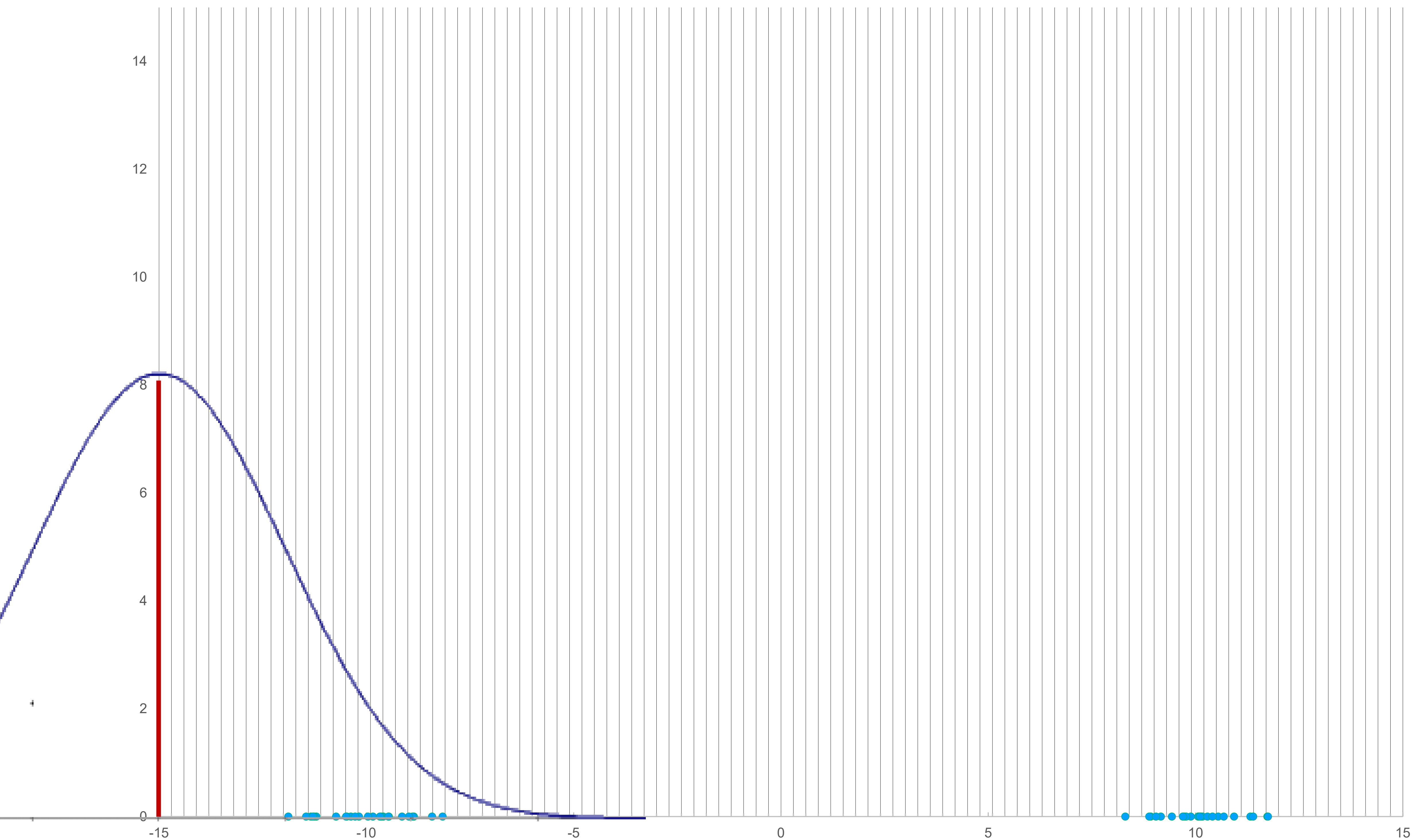
Let's create 200 uniform gridlines (x_l) to have a smoother density function **OR** simply you can just implement this on each datapoint

For **each** linearly spaced gridline x_l, let's calculate the Gaussian kernel value over the given 50 points

$$p(x) = \frac{1}{N} \sum_{i}^{N} \frac{1}{h} K(u_i)$$

$$u_i = \frac{x_l - x_i}{h}$$

$$K(u_i) = \frac{1}{\sqrt{2\pi}} e^{-u_i^2/2}$$



Density value

As an example of kernel heights for line at 0

Linearly spaced lines

$$\begin{array}{r} x_l = -15 \\ \vdots \\ x_l = 0 \\ \vdots \\ x_l = 15 \end{array}
\begin{pmatrix} \\ \\ 0.000\ 0.0001\ 0.0002\ 0.0003\ 0.0004 \ldots\ldots 0.0\ 0.0\ 0.0\ 0.0 \ldots\ldots\ 0.0004\ 0.0003\ 0.0002\ 0.0001\ 0.0000 \\ \\ \\ \end{pmatrix}_{200\mathrm{X}50}$$

200X50

Density value

Linearly spaced lines

$x_l = -15$

$\vdots$

$x_l = 0$ 0.000 0.0001 0.0002 0.0003 0.0004……………………0 0 0 0…………………… 0.0004 0.0003 0.0002 0.0001 0.0000

Density at L = 0

$$P(x_l = 0) = mean(\boldsymbol{L0})$$

$\vdots$

$x_l = 15$

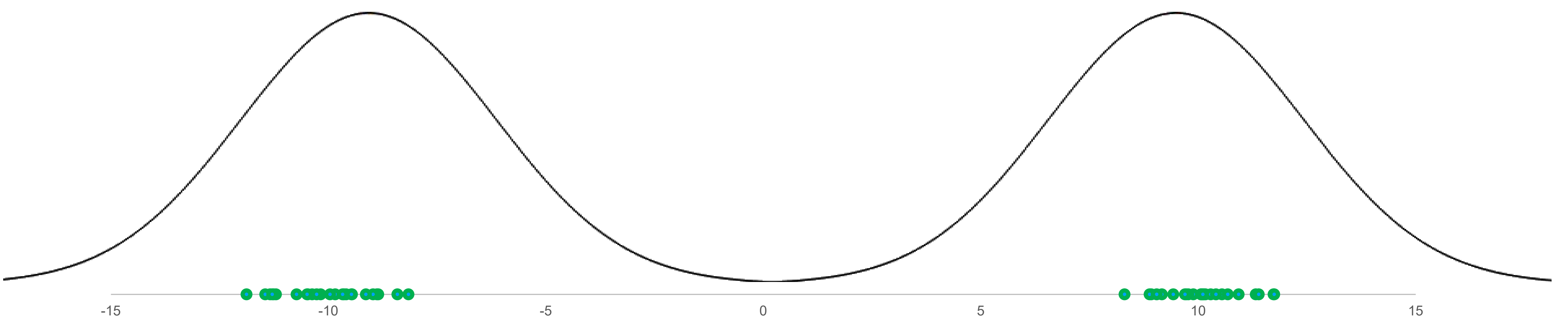
$$p(x) = \frac{1}{N}\sum_{i}^{N}\frac{1}{h}K\left(\frac{x_l - x_i}{h}\right)$$

200X50

Visual Example

Based on Gaussian kernel estimator

Interactive Example



Numerical Example

$For\ \sigma = 1;$

```
% Data ; There are 200 data points (-13~<data<~13)
randn('seed',1)                                    % Used for reproducibility
x = [randn(100,1)-10; randn(100,1)+10]; % Two Normals mixed (GROUND TRUTH)
```

Silverman's rule of thumb: If using the Gaussian kernel, a good choice for is

$$h = \left(\frac{4\hat{\sigma}^2}{3N}\right)^{\frac{1}{5}} \approx 1.06\hat{\sigma}N^{-\frac{1}{5}}$$

```
h = std(x)*(4/3/numel(x))^(1/5);      % Bandwidth estimated by Silverman's Rule of Thumb
```

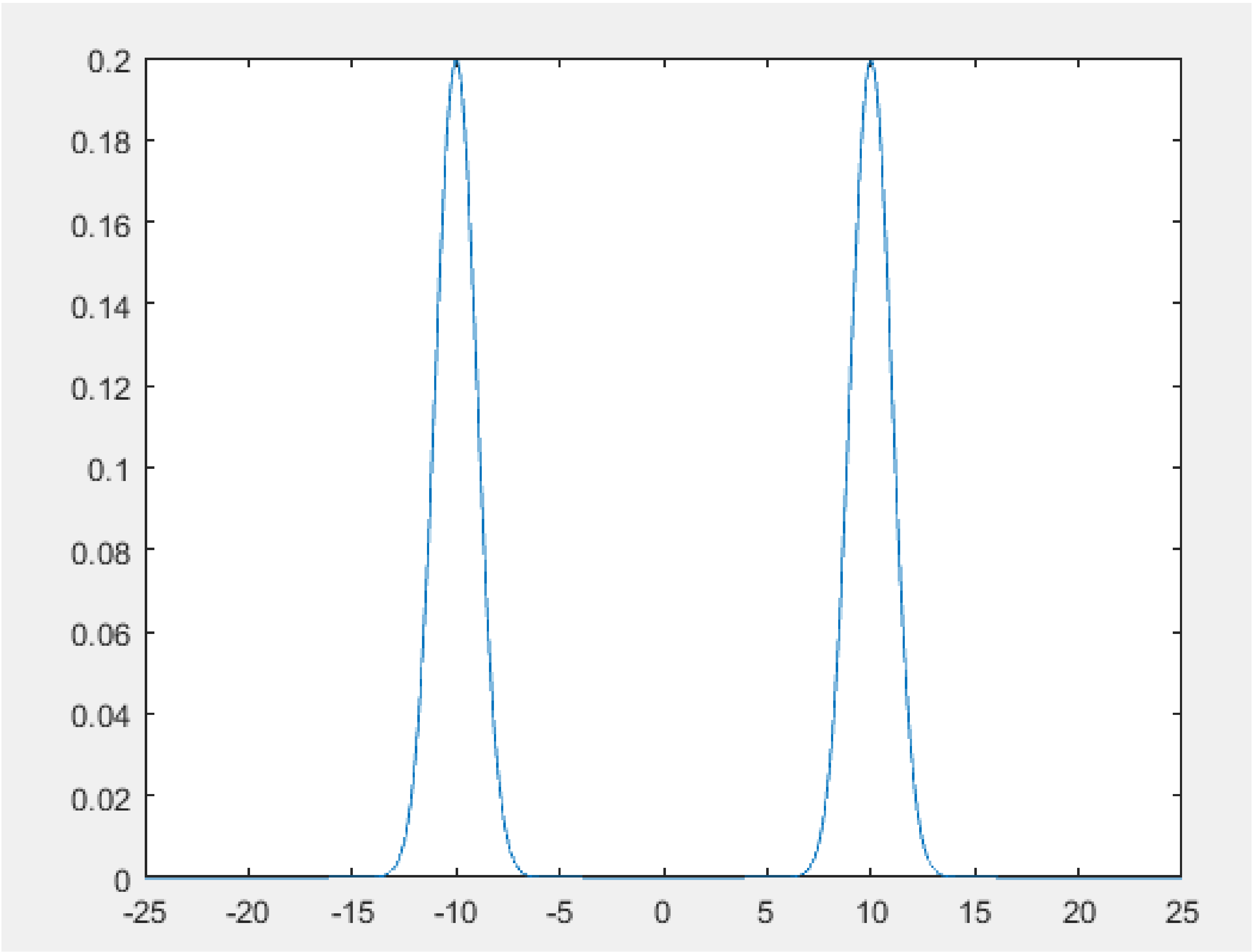
% Let's create apply density estimation over 1000 linearly spaced points (x_l)

```
xl = linspace(-25,+25,1000);    % gridlines
```

% Let's generate a "TRUE" density over all the bins given the "Ground Truth" information.

```
truepdf_firstnormal = exp(-.5*(xl-10).^2)/sqrt(2*pi);
truepdf_secondnormal = exp(-.5*(xl+10).^2)/sqrt(2*pi);
truepdf = truepdf_firstnormal/2 + truepdf_secondnormal/2;
% divided down by 2, because we are adding density value two times
```

```
plot(x,truepdf)                              % Plot True Density
```



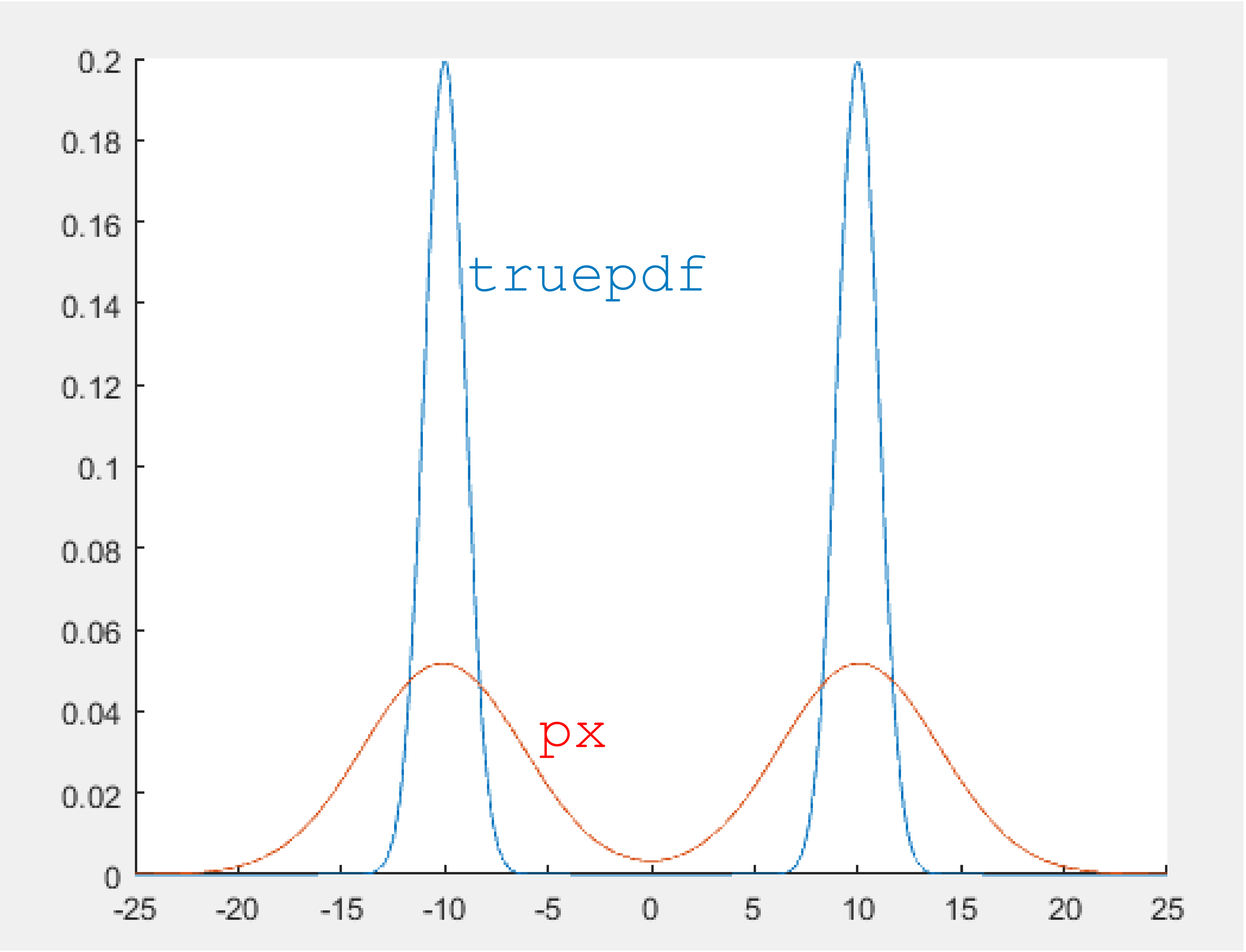
% Let’s calculate Gaussian kernel density for each linearly spaced point over 200 Given data points

$$p(x) = \frac{1}{N}\sum_{i}^{N}\frac{1}{h}K(u_i) \qquad u_i = \frac{x_l - x_i}{h}$$

Gaussian kernel $$K(u_i) = \frac{1}{\sqrt{2\pi}}e^{-u_i^2/2}$$

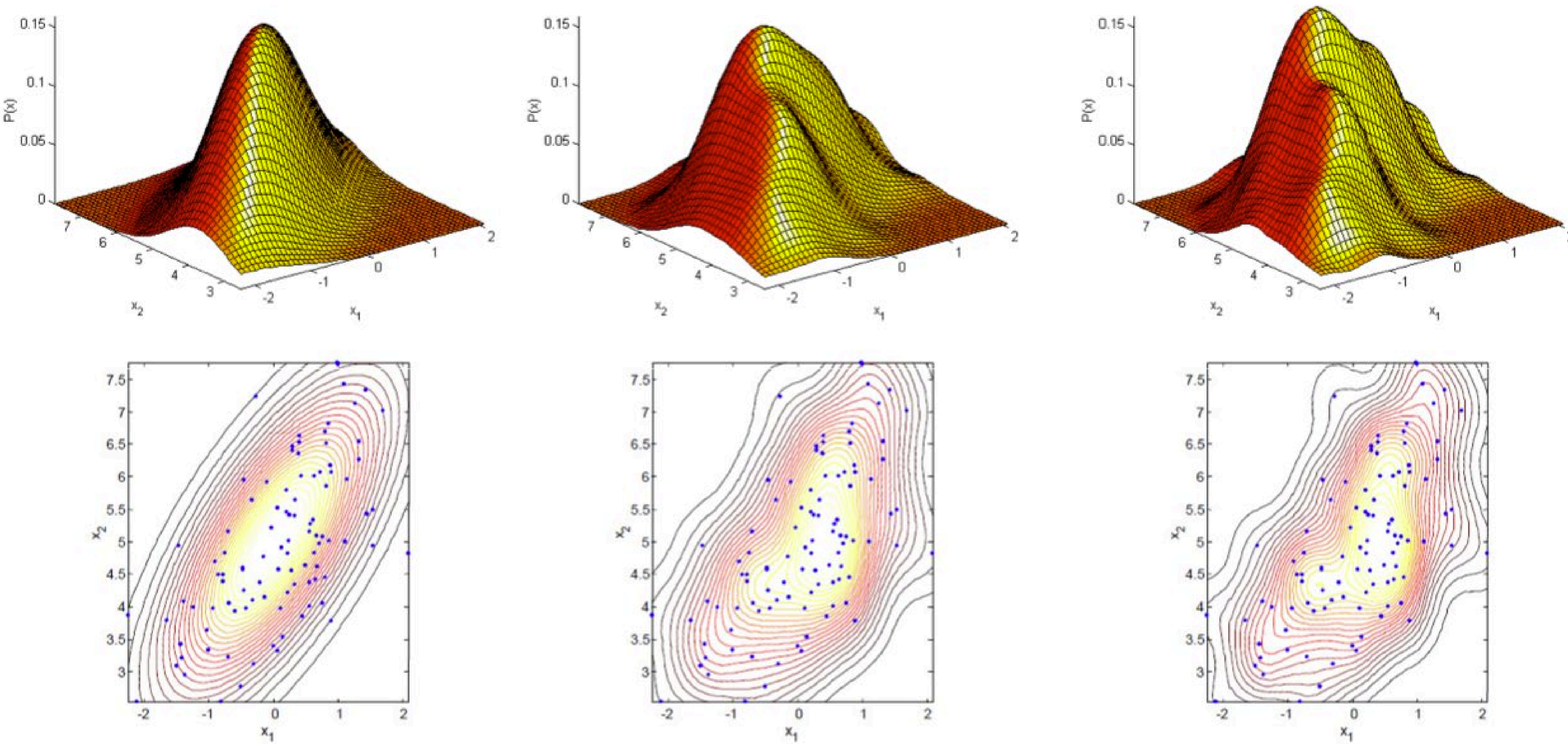
```
for l=1:size(xl,1) % let’s loop over grid lines (x_l)
   u = (xl(l) - x)./h; % length of u is 200
   Ku = exp(-.5*u.^2)/sqrt(2*pi);
   Ku = Ku./h;
   px(l) = mean(Ku);
end
```

```
plot(x,truepdf)
plot(x,px)
```



Two-Dimensional Examples

- This example shows the product KDE of a bivariate <u>unimodal</u> Gaussian
  - 100 data points were drawn from the distribution
  - The figures show the true density (left) and the estimates using $h = 1.06\sigma N^{-1/5}$ (middle) and $h = 0.9AN^{-1/5}$ (right)



Choosing the Kernel Bandwidth

- Silverman's rule of thumb: If using the Gaussian kernel, a good choice for is

$$h \approx 1.06\hat{\sigma}N^{-\frac{1}{5}}$$

where is the standard deviation of the samples

- A better but more computational intensive approach:
  - Randomly split the data into two sets
  - Obtain a kernel density estimate for the first
  - Measure the likelihood of the second set
  - Repeat over many random splits and average

Non-parametric vs parametric

Summary

- Parametric density estimation
  - Maximum likelihood estimation
  - Different parametric forms
- Nonparametric density estimation
  - Histogram
  - Kernel density estimation