Combinational & Sequential Circuit Design

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Definitions

- A product term of a function is said to be an **implicant**.
- A **Prime Implicant (PI)** is a product term obtained by combining the maximum possible number of adjacent 1-squares in the map.
- A **Prime Implicant** is a product that we cannot remove any of its literals.
- If a minterm is covered only by one prime implicant then this prime implicant is said to be **an Essential Prime Implicant (EPI).**

Definitions

- **A cover of a Boolean function is a set of implicants that covers its minterms.**
- **Minimum cover**
  - Cover of the function with minimum number of implicants.
  - Global optimum.
- **Minimal cover or irredundant cover**
  - Cover of the function that is not a proper superset of another cover.
  - No implicant can be dropped.
  - Local optimum.

Definitions

- Let $f(x_1,x_2,\ldots,x_n)$ be a Boolean function of n variables.
- The ***cofactor*** of $f(x_1,x_2,\ldots,x_i,\ldots,x_n)$ with respect to variable x_i is $f_{xi} = f(x_1,x_2,\ldots,x_i=1,\ldots,x_n)$
- The ***cofactor*** of $f(x_1,x_2,\ldots,x_i,\ldots,x_n)$ with respect to variable x_i' is $f_{xi'} = f(x_1,x_2,\ldots,x_i=0,\ldots,x_n)$
- **Theorem: Shannon's Expansion**

$$Let\ f : B^n \to B.Then\quad f(x_1,x_2,\ldots,x_i,\ldots,x_n) = x_i.f_{x_i} + x_i'.f_{x_i'}$$
$$= (x_i + f_{x_i'}).(x_i' + f_{x_i})\ \forall \mathrm{i} = 1,2,\ldots,\mathrm{n}$$

- Any function can be expressed as *sum of products* (*product of sums*) of n literals, **minterms** (**maxterms**), by recursive expansion.

A sample of Shanon

When we apply Shannon's expansion to a,

$$F(a,b,c,d,e,f) = (a+b+d)(a'+b+c)(b+c+d+e+f)$$

$$F(0,b,c,d,e,f) = (b+d)\cdot 1\cdot(b+c+d+e+f) = (b+d)(1+c+e+f) = b+d$$

$$F(1,b,c,d,e,f) = 1\cdot(b+c)\cdot(b+c+d+e+f) = (b+c)(1+d+e+f) = b+c$$

$$F(a,b,c,d,e,f) = \big(a+F(0,b,c,d,e,f)\big)\cdot\big(a'+F(1,b,c,d,e,f)\big)$$

$$= (a+b+d)(a'+b+c)$$

Definitions

- **Another example: f = ab + ac + bc**
  - $f_a = b + c$
  - $f_{a'} = bc$
  - $F = a\, f_a + a'\, f_{a'} = a\,(b + c) + a'\,(bc)$
- **A Boolean function can be interpreted as the set of its minterms.**
- **Operations and relations on Boolean functions can be viewed as operations on their minterm sets**
  - Sum of two functions is the Union ($\cup$) of their minterm sets
  - Product of two functions is the Intersection ($\cap$) of their minterm sets
  - Implication between two functions corresponds to containment ($\subseteq$) of their minterm sets
    - $f_1 \rightarrow f_2 \equiv f_1 \subseteq f_2 \equiv f_1' + f_2 = 1$

Boolean Expansion Based on Orthonormal Basis

- Let ϕ_i , i=1,2, …,k be a set of **Boolean functions** such that $\Sigma_{i=1 \text{ to } k} \ \phi_i = 1$ and $\phi_i \cdot \phi_j = 0$ for $\forall$ i $\neq$ j $\in${1,2,…,k}.
- An **Orthonormal Expansion** of a **function f** is

  $f = \Sigma_{i=1 \text{ to } k} \ f_{\phi i} \cdot \phi_i$
- $f_{\phi i}$ is called the **cofactor** of f w.r.t. ϕ_i $\forall$ i.
- **Example: f = ab+ac+bc; ϕ_1 = a; ϕ_2 = a';**
  - $f_{\phi 1}$ = b+c+bc=b+c
  - $f_{\phi 2}$ = bc
  - f = $\phi_1 \ f_{\phi 1} \cdot + \phi_2 \ f_{\phi 2}$ = a (b+c) + a'(bc)=ab+ac+a'bc= ab+ac+bc

Boolean Expansion Based on Orthonormal Basis

- **Theorem**
  - Let f, g, be two Boolean functions expanded with the same orthonormal basis ϕ_i , i=1,2, …,k
  - Let $\otimes$ be a binary operator on two Boolean functions

$$f \otimes g = \sum_{i=1}^{k} \Phi_i . (f_{\Phi_i} \otimes g_{\Phi_i})$$

- **Corollary**
  - Let f, g, be two Boolean functions with support variables $\{x_i, i=1,2, \ldots,n\}$.
  - Let $\otimes$ be a binary operator on two Boolean functions

$$f \otimes g = x_i . (f_{x_i} \otimes g_{x_i}) + x_i' . (f_{x_i'} \otimes g_{x_i'})$$

Boolean Expansion Based on Orthonormal Basis

- **Example:**
  - Let f = ab + c; g=a'c + b; Compute f $\oplus$g
  - Let ϕ_1=a'b'; ϕ_2=a'b; ϕ_3=ab'; ϕ_4=ab;
  - $f_{\phi 1}$ = c; $f_{\phi 2}$ = c; $f_{\phi 3}$ = c; $f_{\phi 4}$ = 1;
  - $g_{\phi 1}$ = c; $g_{\phi 2}$ = 1; $g_{\phi 3}$ = 0; $g_{\phi 4}$ = 1;
  - f = a'b' (c $\oplus$c) + a'b (c $\oplus$1) + ab' (c $\oplus$0) + ab (1 $\oplus$1)
    = a'bc' + ab'c
  - F= (ab+c) $\oplus$ (a'c+b)= (ab+c)(a+c')b' + (a'+b')c'(a'c+b)
    = (ab+ac)b' + (a'c+a'b)c' = ab'c +a'bc'

Sum of Product (SOP) Simplification Procedure

- **1. Identify all prime implicants covering 1's**
  - Example: For a function of 3 variables, group all possible groups of 4, then groups of 2 that are not contained in groups of 4, then minterms that are not contained in a group of 4 or 2.
- **2. Identify all essential prime implicants and select them.**
- **3. Check all minterms (1's) covered by essential prime implicants**
- **4. Repeat until all minterms (1's) are covered:**
  - Select the prime implicant covering the largest uncovered minterms (1's).

Don’t Care Conditions

- **In some cases, the function is not specified for certain combinations of input variables as 1 or 0.**
- **There are two cases in which it occurs:**
  - 1. The input combination never occurs.
  - 2. The input combination occurs but we do not care what the outputs are in response to these inputs because the output will not be observed.
- **In both cases, the outputs are called as unspecified and the functions having them are called as incompletely specified functions.**
- **In most applications, we simply do not care what value is assumed by the function for unspecified minterms.**

Don’t Care Conditions

- Unspecified minterms of a function are called as **don’t care conditions**. They provide further simplification of the function, and they are denoted by **X’s** to distinguish them from 1’s and 0’s.
- In choosing adjacent squares to simplify the function in a map, the don’t care minterms can be assumed either 1 or 0, depending on which combination gives the simplest expression.
- A don’t care minterm need not be chosen at all if it does not contribute to produce a larger implicant.

SOP Simplification Procedure using Don't Cares

- **1. Identify all prime implicants covering 1's & X's**
  - Each prime implicant must contain at least a single 1
- **2. Identify all essential prime implicants and select them.**
  - An essential prime implicant must be the only implicant covering at least a 1.
- **3. Check all 1's covered by essential prime implicants**
- **4. Repeat until all 1's are covered:**
  - Select the prime implicant covering the largest uncovered 1's.

Combinational Circuits Design Procedure

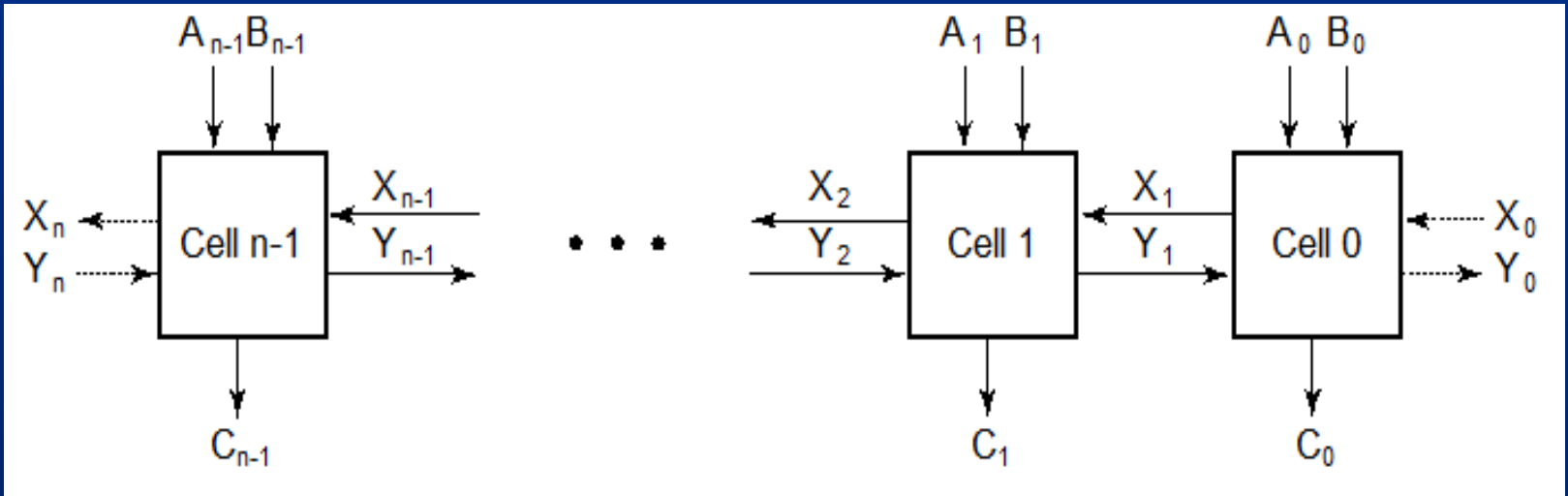
- **1. Specification (Requirement)**
  - Write a specification for what the circuit should do e.g. add two 4-bit binary numbers
  - Specify names for the inputs and outputs
- **2. Formulation**
  - Convert the Specification into a form that can be Optimized
  - Usually as a truth table or a set of Boolean equations that define the required relationships between the inputs and outputs
- **3. Logic Optimization**
  - Apply logic optimization (2-level & multi-level) to minimize the logic circuit
  - Provide a logic diagram or a netlist for the resulting circuit using ANDs, ORs, and inverters

Combinational Circuits Design Procedure

- **4. Technology Mapping and Design Optimization**
  - Map the logic diagram or netlist to the implementation technology and gate type selected, e.g. CMOS NANDs
  - Perform design optimizations of gate costs, gate delays, fan-outs, power consumption, etc.
  - Sometimes this stage is merged with stage 3
- **5. Verification**
  - Verify that the final design satisfies the original specification- Two methods:
    - Manual: Ensure that the truth table for the final technology-mapped circuit is identical to the truth table derived from specifications
    - By Simulation: Simulate the final technology-mapped circuit on a CAD tool and test it to verify that it gives the desired outputs at the specified inputs and meets delay specs etc.

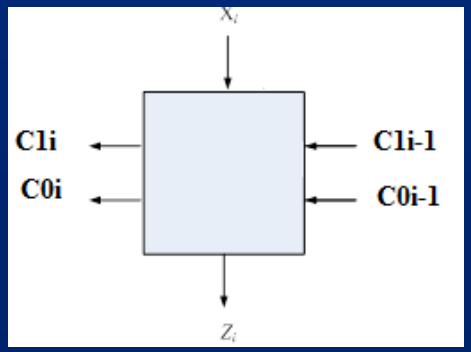
Iterative (Repetitive) Arithmetic Combinational Circuits

- An iterative array can be in a single dimension (1D) or multiple dimensions (spatially)
- Iterative array takes advantage of the regularity to make design feasible
- Block Diagram of a 1D Iterative Array



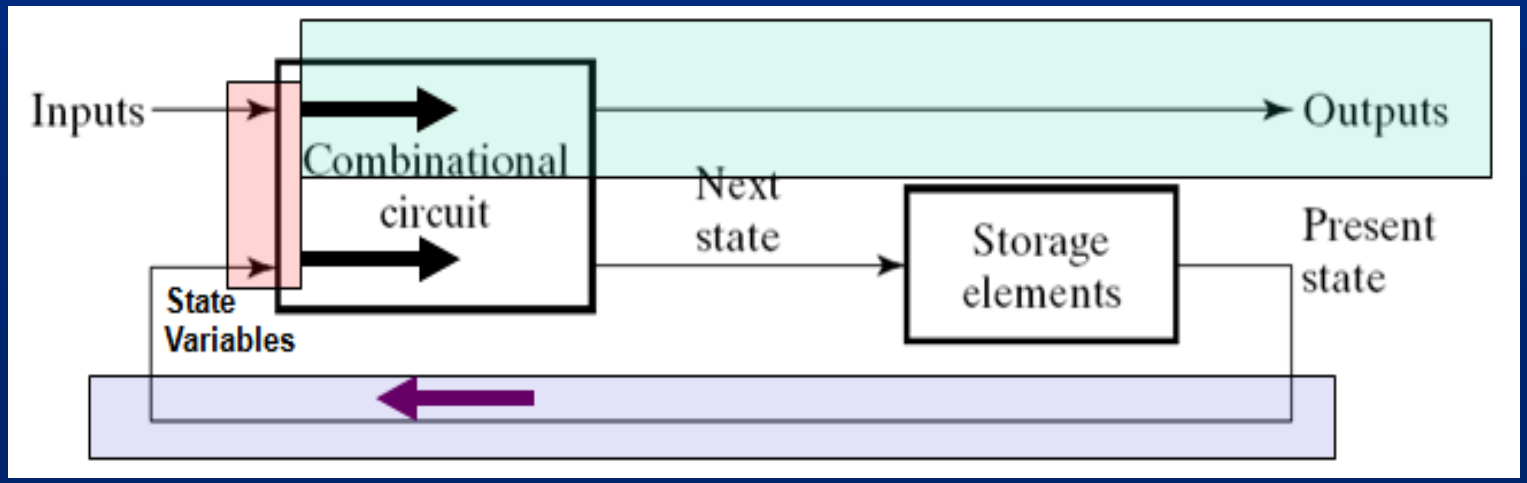
Iterative Design Example

- It is required to design a combinational circuit that computes the equation Y=3*X-1, where X is an n-bit signed 2's complement number
- This circuit can be designed by assuming that we have a borrow feeding first cell or by representing -1 in 2's complement as 11…11 and adding this 1 in each cell.
- We will follow the second approach. We need to represent carry-out values in the range 0 to 3. Thus, we need two signals to represent Carry out values.



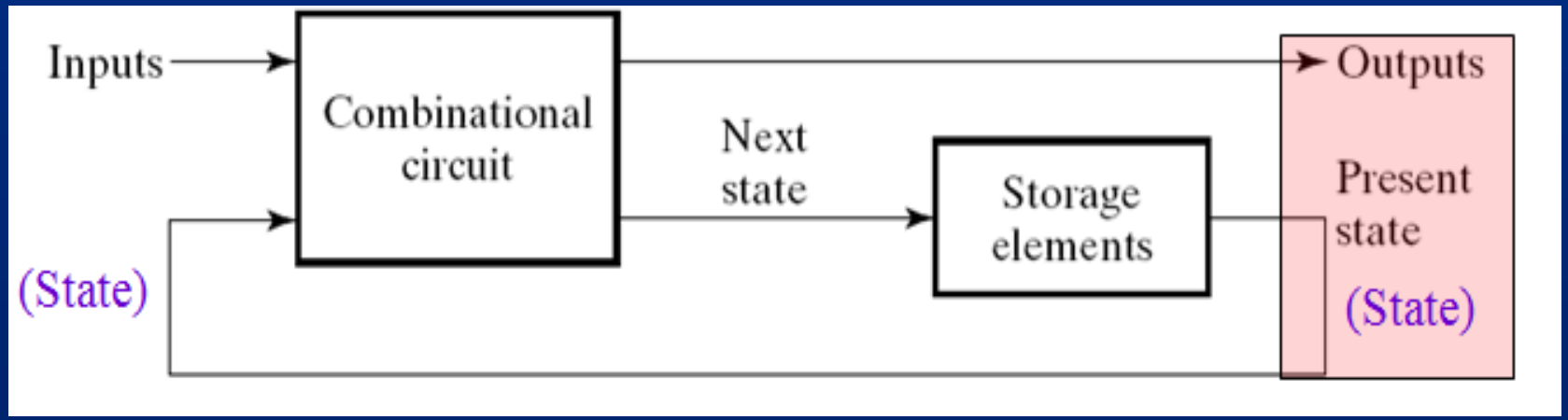
Sequential Circuit Model

- **A Sequential circuit consists of:**
  - Data Storage elements: (Latches / Flip-Flops)
  - Combinatorial Logic:
    - Implements a multiple-output function
    - Inputs are signals from the outside
    - Outputs are signals to the outside
    - State inputs (Internal): Present State from storage elements
    - State outputs, Next State are inputs to storage elements



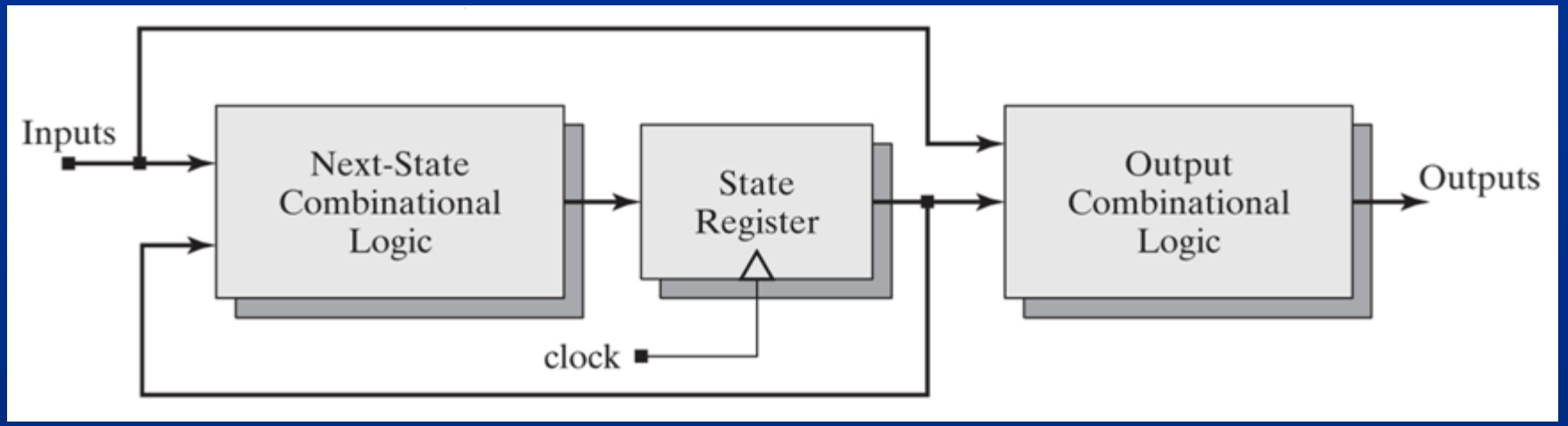
Sequential Circuit Model

- **Combinatorial Logic**
  - Next state function: Next State = f(Inputs, State)
  - 2 output function types : Mealy & Moore
  - Output function: Mealy Circuits Outputs = g(Inputs, State)
  - Output function: Moore Circuits Outputs = h(State)
- **Output function type depends on specification and affects the design significantly**

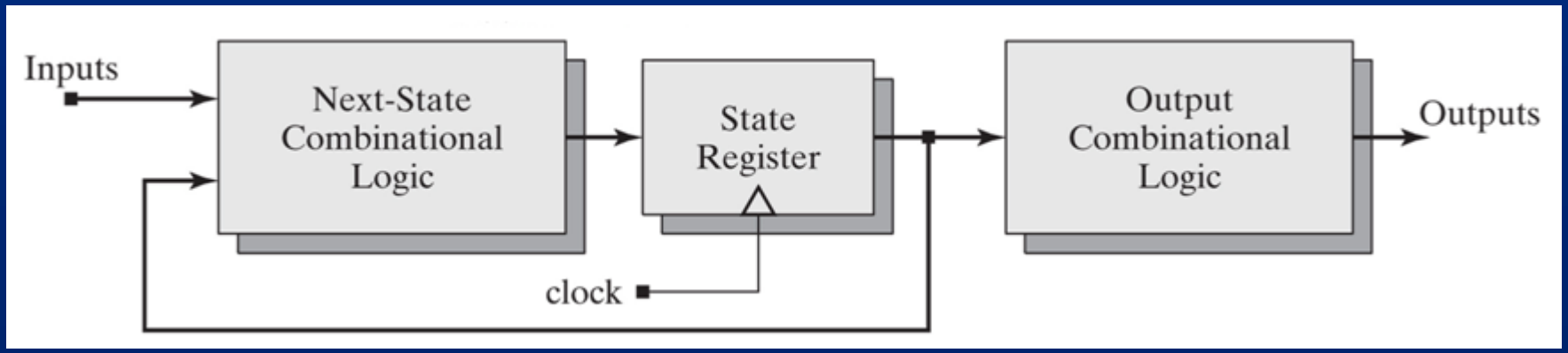


Sequential Circuit Model

Mealy Circuit

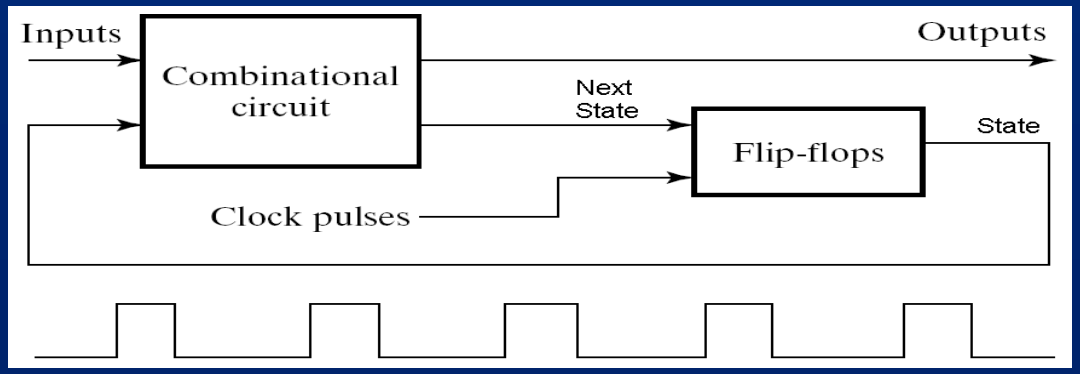


Moore Circuit



Timing of Sequential Circuits
Two Approaches

- Behavior depends on the times at which storage elements 'see' their inputs and change their outputs (next state → present state)
- Asynchronous
  - Behavior defined from knowledge of inputs at any instant of time and the order in continuous time in which inputs change
- Synchronous
  - Behavior defined from knowledge of signals at discrete instances of time
  - Storage elements see their inputs and change state only in relation to a timing signal (clock pulses from a clock)
  - The synchronous abstraction allows handling complex designs!

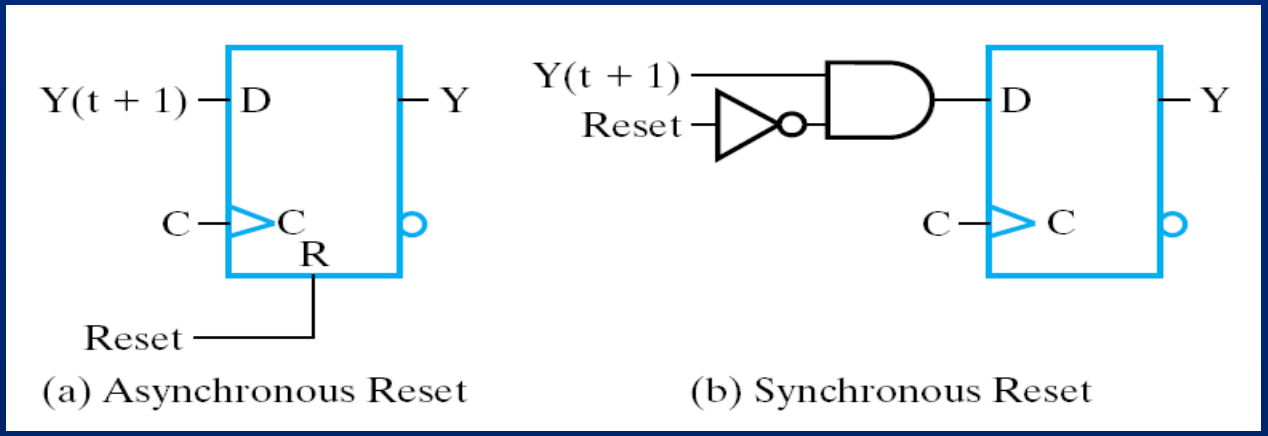


Sequential Circuit Design Procedure

- 1. **Specification** – e.g. Verbal description
- 2. **Formulation** – Interpret the specification to obtain a state diagram and a state table
- 3. **State Assignment** - Assign **binary** codes to symbolic states
- 4. **Flip-Flop Input Equation Determination** - Select flip-flop types and derive **flip-flop input equations** from **next state entries** in the state table
- 5. **Output Equation Determination** - Derive output equations from **output entries** in the state table
- 6. **Verification** - Verify correctness of final design

State Initialization

- When a sequential circuit is turned on, the state of the flip flops is unknown (Q could be 1 or 0)
- Before meaningful operation, we usually bring the circuit to an initial known state, e.g. by resetting all flip flops to 0's
- This is often done asynchronously through dedicated direct S/R inputs to the FFs
- It can also be done synchronously by going through the clocked FF inputs



State Minimization

- **Aims at reducing the number of machine states**
  - reduces the size of transition table.
- **State reduction may reduce**
  - the number of storage elements.
  - the combinational logic due to reduction in transitions
- **Completely specified finite-state machines**
  - No don't care conditions.
  - Easy to solve.
- **Incompletely specified finite-state machines**
  - Unspecified transitions and/or outputs.
  - Intractable problem.

State Minimization for Completely-Specified FSMs

- **Equivalent states**
  - Given any input sequence the corresponding output sequences match.
- **Theorem: Two states are equivalent iff**
  - they lead to identical outputs and
  - their next-states are equivalent.
- **Equivalence is *transitive***
  - Partition states into *equivalence classes*.
  - Minimum finite-state machine is unique.

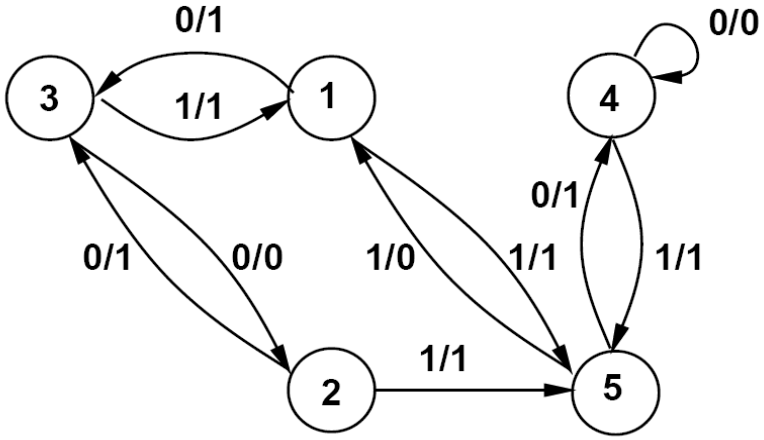
State Minimization Algorithm

- **Stepwise partition refinement.**
- **Initially**
  - $\prod_1$ = States belong to the same block when outputs are the same for any input.
- **Refine partition blocks: While further splitting is possible**
  - $\prod_{k+1}$ = States belong to the same block if they were previously in the same block and their next-states are in the same block of $\prod_k$ for any input.
- **At convergence**
  - Blocks identify equivalent states.

State Minimization Example

- Π_1 = {(s1, s2), (s3, s4), (s5)}.
- Π_2 = {(s1, s2), (s3), (s4), (s5)}.
- Π_2 = is a partition into equivalence classes
  - States (s1, s2) are equivalent.

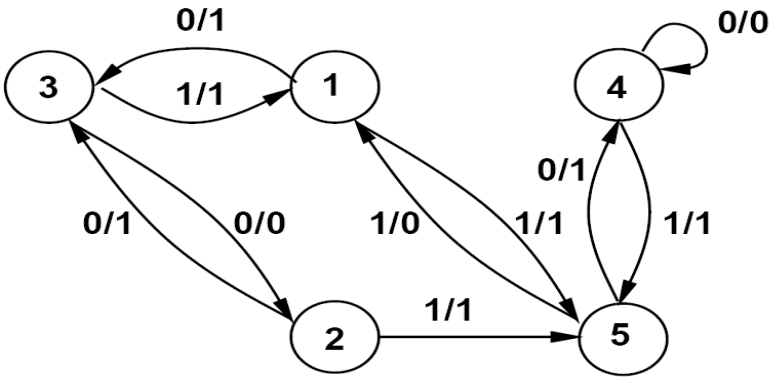
INPUT	STATE	N-STATE	OUTPUT
0	s_1	s_3	1
1	s_1	s_5	1
0	s_2	s_3	1
1	s_2	s_5	1
0	s_3	s_2	0
1	s_3	s_1	1
0	s_4	s_4	0
1	s_4	s_5	1
0	s_5	s_4	1
1	s_5	s_1	0



State Minimization Example

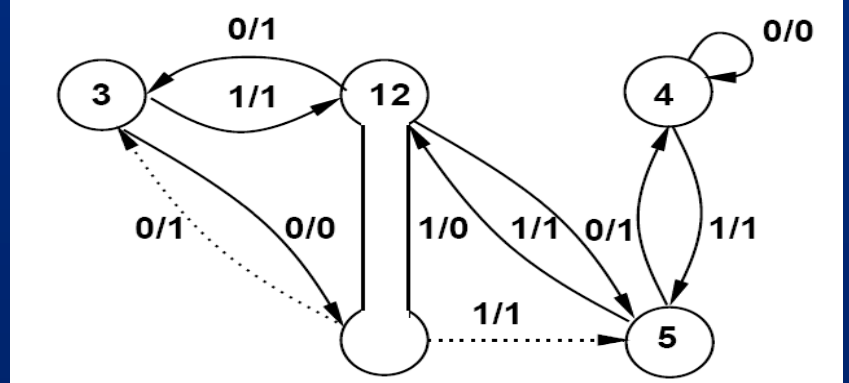
Original FSM

INPUT	STATE	N-STATE	OUTPUT
0	s_1	s_3	1
1	s_1	s_5	1
0	s_2	s_3	1
1	s_2	s_5	1
0	s_3	s_2	0
1	s_3	s_1	1
0	s_4	s_4	0
1	s_4	s_5	1
0	s_5	s_4	1
1	s_5	s_1	0



Minimal FSM

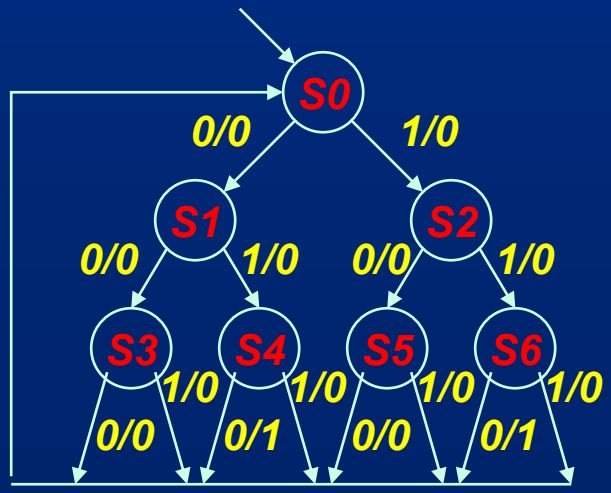
INPUT	STATE	N-STATE	OUTPUT
0	s_{12}	s_3	1
1	s_{12}	s_5	1
0	s_3	s_{12}	0
1	s_3	s_{12}	1
0	s_4	s_4	0
1	s_4	s_5	1
0	s_5	s_4	1
1	s_5	s_{12}	0



Another State Minimization Example

- Sequence Detector for codes of symbols 010 or 110 assuming that each symbol code is 3 bits in length

Input Sequence	Present State	Next State X=0	Next State X=1	Output X=0	Output X=1
Reset	S0	S1	S2	0	0
0	S1	S3	S4	0	0
1	S2	S5	S6	0	0
00	S3	S0	S0	0	0
01	S4	S0	S0	1	0
10	S5	S0	S0	0	0
11	S6	S0	S0	1	0



Another State Minimization Example

Input Sequence	Present State	Next State X=0	Next State X=1	Output X=0	Output X=1
Reset	S0	S1	S2	0	0
0	S1	S3	S4	0	0
1	S2	S5	S6	0	0
00	S3	S0	S0	0	0
01	S4	S0	S0	1	0
10	S5	S0	S0	0	0
11	S6	S0	S0	1	0

(S0 S1 S2 S3 S4 S5 S6)

(S0 S1 S2 S3 S5) (S4 S6)

(S0 S3 S5) (S1 S2) (S4 S6)

(S0) (S3 S5) (S1 S2) (S4 S6)

S1 is equivalent to S2

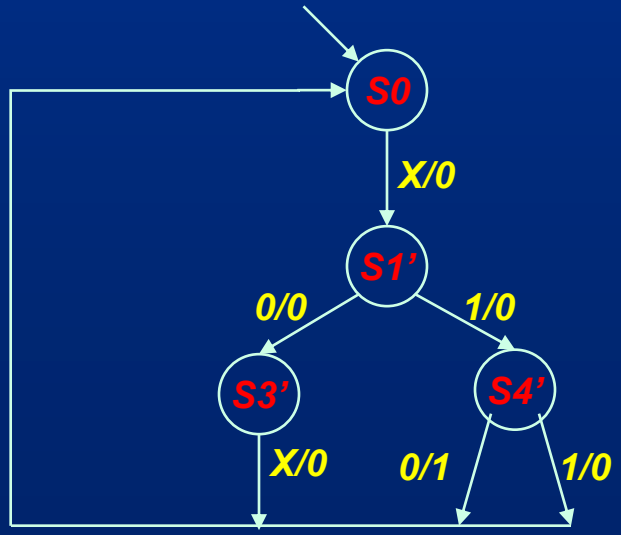
S3 is equivalent to S5

S4 is equivalent to S6

Another State Minimization Example

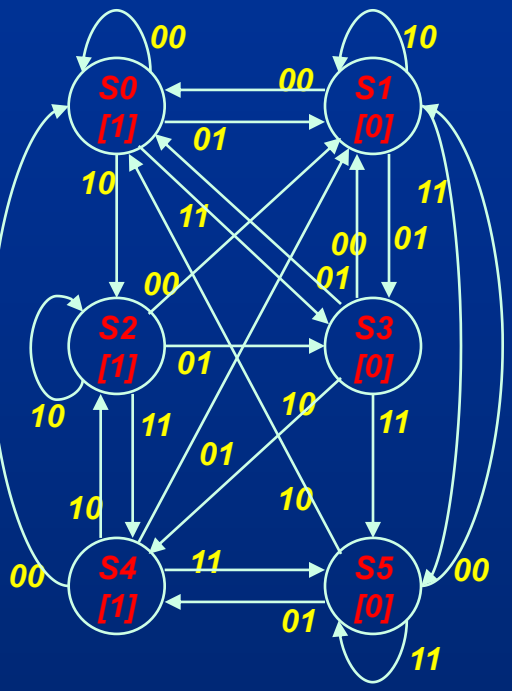
- State minimized sequence detector for 010 or 110

Input Sequence	Present State	Next State X=0	Next State X=1	Output X=0	Output X=1
Reset	S0	S1'	S1'	0	0
0 + 1	S1'	S3'	S4'	0	0
X0	S3'	S0	S0	0	0
X1	S4'	S0	S0	1	0



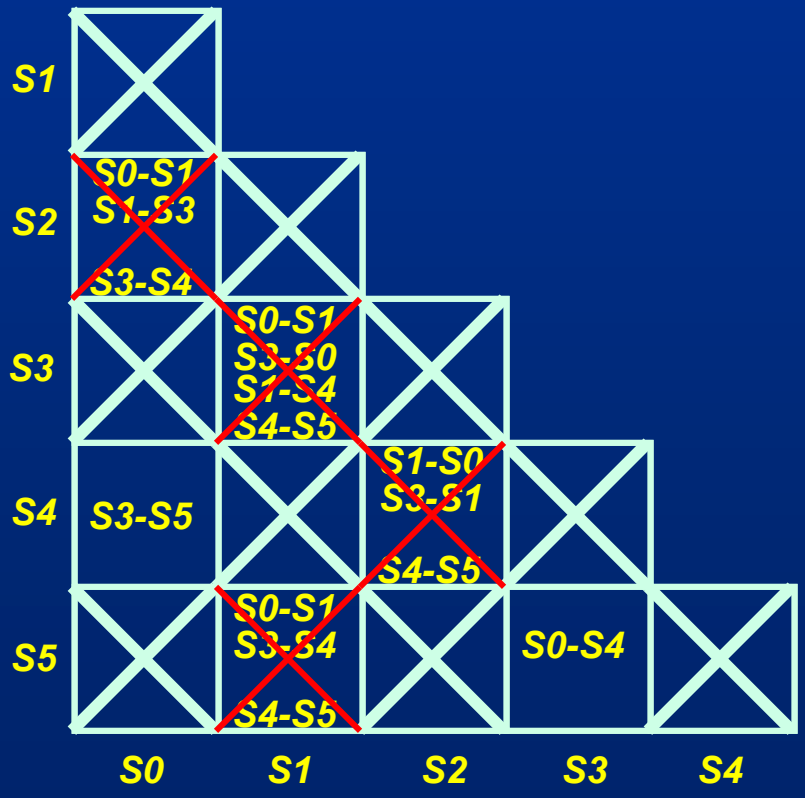
Multiple Input Example

present state	next state 00	01	10	11	output
S0	S0	S1	S2	S3	1
S1	S0	S3	S1	S4	0
S2	S1	S3	S2	S4	1
S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0



Implication Chart Method

- Cross out incompatible states based on outputs
- Then cross out more cells if indexed chart entries are already crossed out



present state	next state 00	01	10	11	output
S0	S0	S1	S2	S3	1
S1	S0	S3	S1	S4	0
S2	S1	S3	S2	S4	1
S3	S1	S0	S4	S5	0
S4	S0	S1	S2	S5	1
S5	S1	S4	S0	S5	0

present state	next state 00	01	10	11	output
S0'	S0'	S1	S2	S3'	1
S1	S0'	S3'	S1	S0'	0
S2	S1	S3'	S2	S0'	1
S3'	S1	S0'	S0'	S3'	0

minimized state table (S0==S4) (S3==S5)

State Minimization Computational Complexity

- **Polynomially-bound algorithm.**
- **There can be at most |S| partition refinements.**
- **Each refinement requires considering each state**
  - Complexity $O(|S|^2)$.
- **Actual time may depend upon**
  - Data-structures.
  - Implementation details.

State Encoding

- **Determine a binary encoding of the states ($|S|=n_s$) that optimize machine implementation**
  - Area
  - Cycle-time
  - Power dissipation
  - Testability
- **Assume D-type registers.**
- **Circuit complexity is related to**
  - Number of storage bits n_b used for state representation
  - Size of combinational component
- **There are** $2^{n_b}!/(2^{n_b}-n_s)!$ **possible encodings**
- **Implementation Modeling**
  - Two-level circuits.
  - Multiple-level circuits.

State Encoding Example

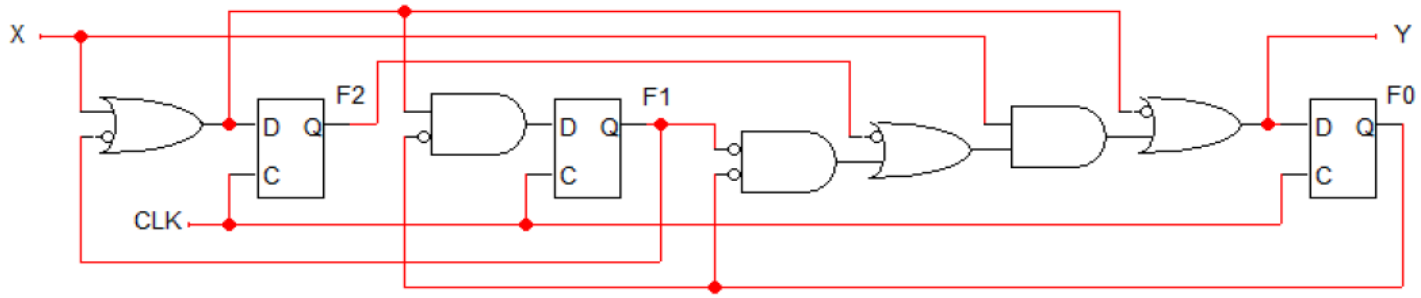
Table 1. An example of an FSM.

Present State	Next State		Output	
	X=0	X=1	X=0	X=1
S0	S2	S2	0	0
S1	S2	S0	0	1
S2	S4	S3	0	1
S3	S1	S2	1	0
S4	S1	S4	1	0

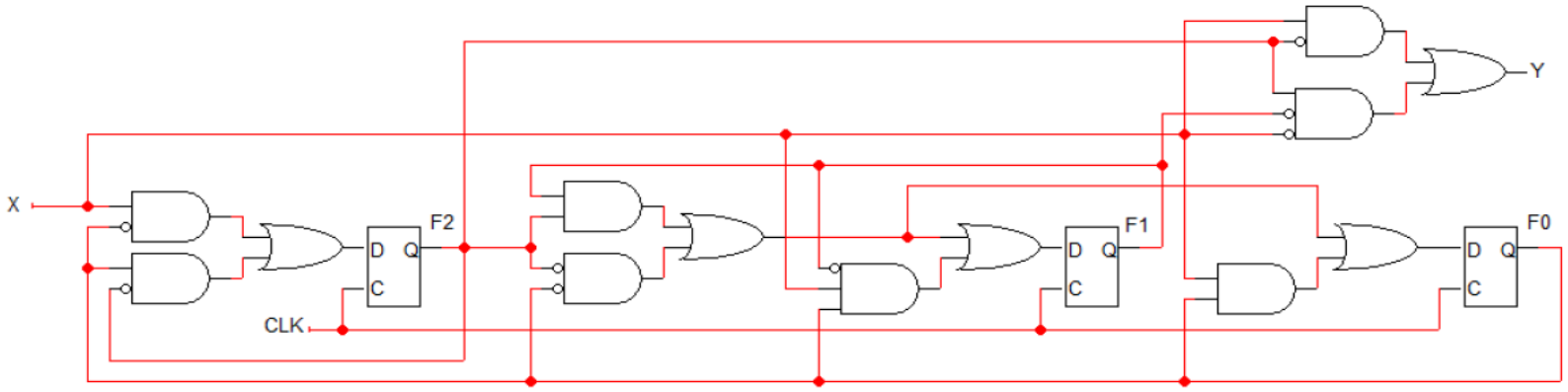
Table 2. State assignments with resulting area cost.

State	Ass. 1	Ass. 2
S0	101	111
S1	001	000
S2	100	011
S3	111	101
S4	110	100
Area (No. of Literals)	10	20

State Encoding Example



(a) *Circuit resulting from "Ass. 1".*



(b) *Circuit resulting from "Ass. 2".*